The Group-theoretic Analog of Kuratowski's Closure-complement Theorem

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Abstract. We state and prove a result which we regard as the group-theoretic analog of Kuratowski's famed closure-complement theorem.

1. INTRODUCTION. Kuratowski's closure-complement theorem (a.k.a. Kuratowski's 14-set theorem) asserts that, if (X, τ) is a topological space and A is any subset of X, then at most 14 distinct subsets of X can be obtained from A by taking closures and complements successively in any order we choose. This result is attributed to the Polish mathematician Kazimierz Kuratowski (1896–1980) because it was in the first part of his Ph.D. dissertation that it first saw the light of day; see [**14**] or [**20**, pp. 37–54]. A proof of Kuratowski's closure-complement theorem can be found in the more readily available paper [**7**]; yet, according to what we read at the end of the opening paragraph of that article, the proof supplied therein is "similar to, though not identical with, Kuratowski's original."

Ever since its advent, Kuratowski's closure-complement theorem has been a regular source of interest in mathematics. To give an idea of its recurrence in the literature, we note that it appears as an exercise in a number of books on general topology (see, for instance, [**6**, p. 56] or [**17**, p. 102]) and that it has been featured prominently in this MONTHLY (having been the subject matter of at least four papers—viz., [**1**], [**13**], [**15**], and [**19**]—and a nonnegligible number of entries in its outstanding "Problems and Solutions" section). Furthermore, there is even a website [**4**], maintained by Mark Bowron, devoted solely to "chronicle the abundance of literature related to ... Kuratowski's closure-complement theorem."

It is more or less clear from even a cursory glance at the aforementioned website that Kuratowski's closure-complement theorem has lent itself to all manners of generalizations and/or variations throughout the years. It is our aim in this article to state and prove what we regard as the group-theoretic analog of it:

Theorem 1. *Let $(G, \cdot)$ be a group and $A \in \mathcal{P}(G)$. Then, by repeatedly applying the operations of complementation and span to A (in any order we choose), no more than 8 different subsets of G are obtained.*

As far as we are aware, the very explicit variation on the Kuratowski closure-complement theme that we are considering here has not appeared in print before. In establishing it, we are going to resort to a seemingly innocuous corollary of a classical exercise in group theory which we first encountered in [**18**].

Let us elaborate on the statement and significance of Theorem 1. The phrase *the operation of span* is a shorthand we will be using throughout the article for the function $\mathcal{P}(G) \xrightarrow{\langle\cdot\rangle} \mathcal{P}(G)$ that maps every $S \in \mathcal{P}(G)$ to the subgroup spanned (or generated) by S. Now then, the reason that we claim that, in the context of groups, this theorem is the analog of Kuratowski's closure-complement theorem is the parallelism that exists between the notions of closure in a topological space (X, τ) and that of spanned subgroup in a group $(G, \cdot)$. On the one hand, the closure of $A \subseteq X$ can be defined as the intersection $\mathrm{cl}(A)$ of all the closed subsets of X that contain A; on the other hand, the subgroup spanned by $S \subseteq G$, denoted hereafter as $\langle S \rangle$, is defined as the intersection of all the subgroups of G that contain S. Thus, while the former is nothing but the smallest closed subset of X wherein A is contained, the latter is the smallest subgroup of G wherein S is contained. The analogy can be further strengthened by considering the four *fundamental* properties of the *topological closure operator* $\mathrm{cl}\colon \mathcal{P}(X) \to \mathcal{P}(X)$ that sends every $A \in \mathcal{P}(X)$ to $\mathrm{cl}(A)$:

(C.1) $\mathrm{cl}(\emptyset) = \emptyset$.
(C.2) $\forall\, A \subseteq X\, (A \subseteq \mathrm{cl}(A))$. (Extensiveness)
(C.3) $\forall\, A, B \subseteq X\, (\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B))$.
(C.4) $\forall\, A \subseteq X\, (\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A))$. (Idempotence)

We regard these properties as fundamental because in a sense, as the following proposition attests, they completely characterize closure operators in topology.

Proposition 1. *Suppose we are given a set X and a function $\mathbf{cl}\colon \mathcal{P}(X) \to \mathcal{P}(X)$ assigning to every $A \in \mathcal{P}(X)$ a set $\mathbf{cl}(A) \subseteq X$ in such a way that the conditions (C.1)–(C.4) are satisfied. Then the family $\tau^* := \{X \setminus A\colon A = \mathbf{cl}(A)\}$ is a topology on X; moreover, for every $A \in \mathcal{P}(X)$, the set $\mathbf{cl}(A)$ is the closure of A in the topological space (X, τ^*).*

It is not at all difficult to prove this proposition and, in any event, its proof can be consulted in [**6**, p. 22]. In case the reader is already wondering how it all relates to the analogy in question, we proceed to substantiate our point: on any given group $(G, \cdot)$, the operation of span $S \mapsto \langle S \rangle$ fulfills properties that are analogous to those we deem *fundamental* to the topological closure operator, to wit:

(S.1) $\langle \emptyset \rangle = \{e\}$.
(S.2) $\forall\, S \subseteq G\, (S \subseteq \langle S \rangle)$. (Extensiveness)
(S.3) $\forall\, S, T \subseteq G\, (\langle S \cup T \rangle \supseteq \langle S \rangle \cup \langle T \rangle)$.
(S.4) $\forall\, S \subseteq G\, (\langle \langle S \rangle \rangle = \langle S \rangle)$. (Idempotence)

The resemblance between properties (C.i) and (S.i) for each $i \in \{1, 2, 3, 4\}$ is simply undeniable!

This paper is organized as follows: in Section 2 we prove the facts in group theory on which the proof of our main result (Theorem 1) depends. Section 3 is devoted to the proof of Theorem 1; we comment about the optimality of the theorem in Section 4. Finally, in Section 5 we discuss an alternative way to reinforce our point that Theorem 1 is the analog for groups of Kuratowski's famed closure-complement theorem: the standpoint in this section is somewhat different from the intrinsic (totally group-theoretic) approach followed in the first sections of the article.

2. AUXILIARY PROPOSITIONS.

Lemma 1. *The union of two subgroups of a group is itself a subgroup if and only if one of the subgroups contains the other.*

Proof. Let A and B be subgroups of a group G. If $A \subseteq B$, then $A \cup B = B$ is a subgroup; if $B \subseteq A$, then $A \cup B = A$ is a subgroup.

Now, let us suppose that $A \cup B$ is a subgroup of G (where both A and B are subgroups of G). If neither $A \subseteq B$ nor $B \subseteq A$ holds, then there exist elements $a \in A \setminus B$ and $b \in B \setminus A$. Since $A \cup B$ is a subgroup, we have that $ab \in A \cup B$ and thus $ab \in A$ or $ab \in B$. In the former case, we get that $b = a^{-1}(ab) \in A$, which is a contradiction; in the latter, we get that $a = (ab)b^{-1} \in B$, which is also a contradiction. ■

A straightforward consequence of this lemma is the following result.

Corollary 1. *No group is the union of two of its proper subgroups.*

Both Lemma 1 and Corollary 1 have appeared in the pages of the MONTHLY on several occasions. For instance, Lemma 1 was listed as problem E1592 in the May 1963 issue of the MONTHLY; a solution to this problem was published approximately one year later—see [**2**]. For its part, Corollary 1 has appeared as theorem number one in at least three MONTHLY papers: see [**9**, p. 492], [**5**, p. 52], and [**3**, p. 413]. In addition, Corollary 1 was one of the two items that constituted problem B2 of the 30th William Lowell Putnam Competition (which took place on December 6, 1969); it can also be found as the antepenultimate exercise on page 118 of [**11**].

The ensuing corollary will be one of the key ingredients in our proof of Theorem 1.

Corollary 2 ([18, p. 23]). *If S is a proper subgroup of a group G, then $\langle G \setminus S\rangle = G$.*

Proof. It is apparent that $G = S \cup \langle G \setminus S\rangle$. Then, since $S \subsetneq G$, it follows from Corollary 1 that $\langle G \setminus S\rangle$ cannot be a proper subgroup of G. Hence, $\langle G \setminus S\rangle = G$ and we are done. ■

For some reason, the following slightly more general version of the previous corollary didn't make it into [**18**]. We make room for it below because it is also going to come in handy in the proof of Theorem 1; nevertheless, we omit its proof because it is practically the same as that of Corollary 2.

Corollary 3. *If S is a subset of a group G, then either $\langle S\rangle = G$ or $\langle G \setminus S\rangle = G$.*

3. PROOF OF THE MAIN THEOREM. Let $(G, \cdot)$ be a group and $A \in \mathcal{P}(G)$. Since the assertion is trivially true if $|G| \leq 3$, we assume in what follows that $|G| > 3$. We wish to determine an upper bound for the number of subsets of G that can be obtained from A by taking complements and spans successively. Given that the operation of complementation is involutory and the operation of span is idempotent, it suffices to determine an upper bound for the number of different subsets of G that we get from A when we apply these operations alternately.

The subsets we get from A by applying the operation of span in the first place and taking complements and spans alternately afterwards are:

$$A,\ \langle A\rangle,\ G \setminus \langle A\rangle,\ \langle G \setminus \langle A\rangle\rangle,\ \ldots. \tag{1}$$

If $\langle A\rangle \subsetneq G$, then Corollary 2 allows us to ascertain that $\langle G \setminus \langle A\rangle\rangle = G$. This implies that the sequence in (1) continues in the following way:

$$A,\ \langle A\rangle,\ G \setminus \langle A\rangle,\ G,\ \emptyset,\ \langle\emptyset\rangle = \{e\},\ G \setminus \{e\},\ \cancel{\langle G \setminus \{e\}\rangle = G},\ \ldots. \tag{2}$$

From the reappearance of G in the above sequence, we gather that by alternately applying the operations of span and complementation—in that order—to any $A \in \mathcal{P}(G)$, we can produce at most **six** different subsets of G. (In the case in which $\langle A\rangle = G$, the number of distinct subsets of G, apart from A, that are comprised in (2) is at most **four**.)

On the other hand, the subsets we get from A by applying the operation of complementation in the first place and taking spans and complements alternately afterwards are:

$$A,\ G \setminus A,\ \langle G \setminus A \rangle,\ G \setminus \langle G \setminus A \rangle,\ \langle G \setminus \langle G \setminus A \rangle \rangle,\ \ldots. \tag{3}$$

If $\langle A \rangle \subsetneq G$, then Corollary 3 implies that $\langle G \setminus A \rangle = G$; we infer in this case that the maximum number of distinct subsets of G that may appear in (3) but not in (2) is **one**. If $\langle A \rangle = G$ and $\langle G \setminus A \rangle \subsetneq G$, then the maximum number of distinct subsets of G that may appear in (3) but not in (2) is **three**. If $\langle A \rangle = G$ and $\langle G \setminus A \rangle = G$, then the maximum number of distinct subsets of G that may appear in (3) but not in (2) is **one**.

It follows from all this that, starting with any $A \in \mathcal{P}(G)$, by repeatedly applying the operations of complementation and span to A, we can produce at most $1 + 6 + 1 = 1 + 4 + 3 = 8$ different subsets of G. ■

For a better understanding of our argument, we illustrate by means of a tree diagram the analysis of the case in which $\langle A \rangle \subsetneq G$; see Figure 1. A squiggly arrow in Figure 1 indicates that we take the span of the subset of G appearing in the node on the left, whereas a dashed arrow indicates that we apply complementation to the subset of G appearing in the node on the left.

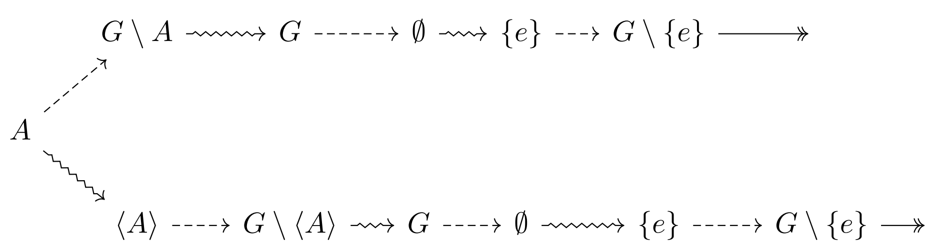

Figure 1. Illustrating the proof for an $A \in \mathcal{P}(G)$ such that $\langle A \rangle \subsetneq G$.

The two-headed arrows indicate that no additional subsets of G are produced from the nodes from which they emanate by taking successive spans and complements. Thus, the maximum number of different subsets of G that may be obtained from A by alternately taking complements and spans or spans and complements is given by the actual number of different subsets of G in the diagram!

4. OPTIMALITY OF THE THEOREM. The result we have just established is optimal (in the absence of additional information on G). Indeed, given that the first seven sets in (2) are different from one another when A is not a subgroup of G, $|A| > 1$, and $\langle A \rangle \subsetneq G$, we conclude that if neither A nor $G \setminus A$ is a subgroup of G and the easily verifiable conditions

$$|A| > 1,\ \langle A \rangle \subsetneq G, \tag{4}$$

hold, then the eight sets

$$A,\ \langle A \rangle,\ G \setminus \langle A \rangle,\ G,\ \emptyset,\ \{e\},\ G \setminus \{e\},\ G \setminus A \tag{5}$$

are all different from one another. Armed with this criterion for the sets in (5) to be distinct, we proceed to illustrate that there are groups with subsets from which it is possible to produce exactly eight subsets by applying the operations of complementation and span successively and in any way we choose.

Example 1. Let G be a group of order 6 and a be an element of order 3 in G. If $A = \{e, a\}$, where e is the identity element of G, then we have that neither A nor $G \setminus A$ is a subgroup of G and that $|A| > 1$, and $\langle A \rangle = \{e, a, a^2\} \subsetneq G$; the criterion in the previous paragraph allows us to ascertain that, by repeatedly applying the operations of complementation and span to A, we get exactly 8 different subsets of G. A remark which the diligent reader may want to verify later on: an example of the sharpness of our main theorem in which $|G| < 6$ cannot be given!

Example 2. We might start from an infinite group too. Although it is not difficult to come up with examples in which G is both infinite and cyclic, we have opted to base this second example upon an important noncyclic infinite abelian group about which we first read in [**12**, p. 4].

For every $n \in \mathbb{Z}^+$, let $H_n := \{z \in \mathbb{C} \colon z^{2^n} = 1\}$. Clearly enough, every H_n is a subgroup of the multiplicative subgroup of the field of complex numbers. Since the inclusion

$$H_n \subseteq H_{n+1}$$

is valid for every $n \in \mathbb{Z}^+$, it follows that $\mathbb{G} := \bigcup_{n=0}^{\infty} H_n$ is a group in its own right. This group $\mathbb{G}$ is known as the Prüfer 2-group and it is commonly denoted by $\mathbb{Z}(2^\infty)$. Then if $A := \{1, -1, i\}$, we readily observe that

i) A is not a subgroup of $\mathbb{G}$,
ii) $|A| > 1$,
iii) $\langle A \rangle = \{1, -1, i, -i\} = H_2 \subsetneq \mathbb{G}$, and
iv) $\mathbb{G} \setminus A$ is not subgroup of $\mathbb{G}$.

The criterion allows us to conclude that, by repeatedly applying the operations of complementation and span to A, we get exactly 8 distinct subsets of $\mathbb{G}$. It is noteworthy that we can make this example work in $\mathbb{Z}(p^\infty)$, for any given prime number p, by selecting a suitable $A := A(p) \in \mathcal{P}(\mathbb{Z}(p^\infty))$.

5. FINAL REMARKS. In view of the fact that Lemma 1 is also valid, *mutatis mutandis*, for vector spaces (see [**8**, p. 19]) and rings (see [**16**, p. 1035]), it is clear that, emulating what we have done in Sections 2 and 3 of this paper, we can settle analogs of Kuratowski's closure-complement theorem in those categories as well. The interested reader will surely have no difficulties formulating and proving the corresponding results.

On a different note, it may be fitting to comment about a more holistic way to support our claim that Theorem 1 is the analog for groups of Kuratowski's celebrated closure-complement theorem.

Preston C. Hammer, in his investigation of the "causes" of the closure-complement phenomenon, proved that Kuratowski's theorem may be derived from a weaker set of requirements on the topological closure operator. Specifically, he proved in [**10**, Thm. A, pp. 75-76] the following theorem.

Theorem 2. *Let X be a set and $c\colon \mathcal{P}(X) \to \mathcal{P}(X)$ be a function that satisfies*

(C'.1) $\forall\, A \subseteq X\, (A \subseteq c(A))$. (Extensiveness)
(C'.2) $\forall\, A, B \subseteq X\, (A \subseteq B \Rightarrow c(A) \subseteq c(B))$.
(C'.3) $\forall\, A \subseteq X\, (c(c(A)) = c(A))$. (Idempotence)

Let $d \colon \mathcal{P}(X) \to \mathcal{P}(X)$ *be the function that maps any* $A \in \mathcal{P}(X)$ *to* $X \setminus A$*. Then for every* $A \in \mathcal{P}(X)$ *the following collection of subsets of* X *is closed under the functions* c *and* d*:*

$$\begin{matrix} dA, & cdA, & dcdA, & cdcdA, & dcdcdA, & cdcdcdA, & dcdcdcdA, \\ A, & cA, & dcA, & cdcA, & dcdcA, & cdcdcA, & dcdcdcA. \end{matrix} \tag{6}$$

(Here and below we omit the parentheses when evaluating the functions c *and* d*; so,* cA *is to be understood as* $c(A)$*,* dA *as* $d(A)$*,* cdA *as* $c(d(A))$*, and so forth.)*

The fact that the function $c \colon \mathcal{P}(X) \to \mathcal{P}(X)$ is idempotent and the function $d \colon \mathcal{P}(X) \to \mathcal{P}(X)$ is involutory implies that, in order to prove Theorem 2, it suffices to show that, when we alternately apply the functions d and c to any $A \in \mathcal{P}(X)$, there is an absolute threshold after which no new subsets of X are produced. In particular, in [**10**] Hammer settles this by means of the identity $cdcdcdcA = cdcA$ which holds true for every $A \in \mathcal{P}(X)$; thus, it does make sense to schematize the process of alternately applying the functions c and d to an arbitrary $A \in \mathcal{P}(X)$ as follows:

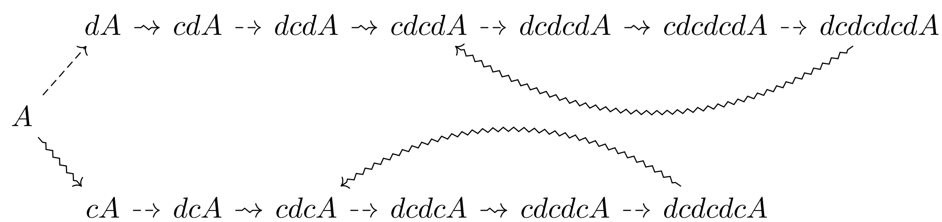

Figure 2. The alternate application of the functions c and d to an $A \in \mathcal{P}(X)$. In the top row, d is applied first and then c. In the bottom row, we apply c first. Both of the arrows that go backwards come as a result of the identity $cdcdcdcA = cdcA$. The maximum number of distinct subsets that may be produced during the process is fourteen!

That Hammer's theorem encompasses both Kuratowski's original closure-complement theorem and a *coarser* version of our Theorem 1 is evident, for both the closure operator on a topological space (X, τ) and the operation of span on a group $(G, \cdot)$ satisfy the conditions (C'.1)–(C'.3). Besides, since the function $h \colon \mathcal{P}(\mathbb{R}^n) \to \mathcal{P}(\mathbb{R}^n)$ that sends any given $A \in \mathcal{P}(\mathbb{R}^n)$ to its convex hull $h(A)$ also satisfies (C'.1)–(C'.3), Hammer's theorem implies at once a weaker form of William Koenen's main result in [**13**]: namely, there exists a natural number n such that if we successively take complements and convex hulls of any given $A \subseteq \mathbb{R}^n$ (in any order we choose), then the maximum number of distinct subsets of $\mathbb{R}^n$ that can be produced is at most n.

The bound 14 is sharp in the case of the original closure-complement problem; that is, it is possible to provide examples of topological spaces (X, τ) in which, by picking an appropriate $A \in \mathcal{P}(X)$ and successively taking closures and complements of it, one ends up with exactly 14 different subsets of X (e.g., see [**1**, p. 871]). Koenen managed to show that, in the variant of Kuratowski's closure-complement problem he considered in his aforementioned work, the bound $n = 10$ is sharp.

In the context of groups, we have reckoned the immediate application of Theorem 2 as coarse because the bound it guarantees for the maximum number of different subsets that can be obtained by repeatedly applying the operations of complementation and span to any fixed subset of G is 14, whereas our first principles approach in Sections 2–4 of the article has yielded 8 as the sharp bound for the number of distinct subsets of G that can be so obtained. It was by paying close attention to what Corollary 2 really meant that we realized that, in the category of groups, the bound 14 could be significantly brought down.

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