REAL ANALYSIS – Semester 2026-2

Contents

1 Measurable spaces and topology

The basic idea behind integration theory via measures may be roughly described as follows: Given a space (set) we want to associate "sizes" to "pieces" of the space. To do this we first have to make precise what we mean by a "piece", i.e., what subsets we admit as "pieces". This is the purpose of the concept of a σ-algebra and a measurable space. Given that we know what a piece is, we want to assign a number to it, its "size", in such a way that sizes add up appropriately when we join pieces. This is provided by the concept of a measure. Then, we can declare the integral for the characteristic function on a piece to be the size of the piece. Approximating more arbitrary functions by linear combinations of characteristic functions for pieces then yields a general notion of integral.

Before embarking on the theory of measurable spaces we recall basic notions of topolgy.

1.1 Basic Definitions

Definition 1.1 (Topology)**.** Let S be a set. A subset $\mathcal{T}$ of the set $\mathfrak{P}(S)$ of subsets of S is called a *topology* iff it has the following properties:

- $\emptyset \in \mathcal{T}$ and $S \in \mathcal{T}$.
- Let $\{U_i\}_{i\in I}$ be a family of elements in $\mathcal{T}$. Then $\bigcup_{i\in I} U_i \in \mathcal{T}$.
- Let $U, V \in \mathcal{T}$. Then $U \cap V \in \mathcal{T}$.

A set equipped with a topology is called a *topological space*. The elements of $\mathcal{T}$ are called the *open* sets in S. A complement of an open set in S is called a *closed* set.

Definition 1.2. Let S be a topological space and $x \in S$. Then a subset $U \subseteq S$ is called a *neighborhood* of x iff it contains an open set which in turn contains x. We denote the set of neighborhoods of x by $\mathcal{N}_x$.

Definition 1.3. Let S be a topological space and U a subset. The *closure* $\overline{U}$ of U is the smallest closed set containing U. The *interior* $\mathring{U}$ of U is the largest open set contained in U. U is called *dense* in S iff $\overline{U} = S$.

Definition 1.4 (base)**.** Let $\mathcal{T}$ be a topology. A subset $\mathcal{B}$ of $\mathcal{T}$ is called a *base* of $\mathcal{T}$ iff the elements of $\mathcal{T}$ are precisely the unions of elements of $\mathcal{B}$. It is called a *subbase* iff the elements of $\mathcal{T}$ are precisely the finite intersections of unions of elements of $\mathcal{B}$.

Proposition 1.5. *Let S be a set and $\mathcal{B}$ a subset of $\mathfrak{P}(S)$. $\mathcal{B}$ is the base of a topology on S iff it satisfies all of the following properties:*

- *$\emptyset \in \mathcal{B}$.*
- *For every $x \in S$ there is a set $U \in \mathcal{B}$ such that $x \in U$.*

- *Let $U, V \in \mathcal{B}$. Then there exists a family $\{W_\alpha\}_{\alpha\in A}$ of elements of $\mathcal{B}$ such that $U \cap V = \bigcup_{\alpha\in A} W_\alpha$.*

Proof. **Exercise.** □

Definition 1.6. Let S be a topological space and p a point in S. We call a family $\{U_\alpha\}_{\alpha\in A}$ of open neighborhoods of p a *neighborhood base* at p iff for any neighborhood V of p there exists $\alpha \in A$ such that $U_\alpha \subseteq V$.

Definition 1.7 (Continuity)**.** Let S, T be topological spaces. A map $f : S \to T$ is called *continuous at* $p \in S$ iff $f^{-1}(\mathcal{N}_{f(p)}) \subseteq \mathcal{N}_p$. f is called *continuous* iff it is continuous at every $p \in S$. We denote the space of continuous maps from S to T by $\mathrm{C}(S, T)$.

Proposition 1.8. *Let S, T be topological spaces and $f : S \to T$ a map. Then, f is continuous iff for every open set $U \in T$ the preimage $f^{-1}(U)$ in S is open.*

Proof. **Exercise.** □

Proposition 1.9. *Let S, T, U be topological spaces, $f \in \mathrm{C}(S, T)$ and $g \in \mathrm{C}(T, U)$. Then, the composition $g \circ f : S \to U$ is continuous.*

Proof. Immediate. □

Definition 1.10. Let S, T be topological spaces. A bijection $f : S \to T$ is called a *homeomorphism* iff f and f^{-1} are both continuous. If such a homeomorphism exists S and T are called *homeomorphic*.

Definition 1.11. Let $\mathcal{T}_1$, $\mathcal{T}_2$ be topologies on the set S. Then, $\mathcal{T}_1$ is called *finer* than $\mathcal{T}_2$ and $\mathcal{T}_2$ is called *coarser* than $\mathcal{T}_1$ iff all open sets of $\mathcal{T}_2$ are also open sets of $\mathcal{T}_1$.

Definition 1.12 (Induced Topology)**.** Let S be a topological space and U a subset. Consider the topology given on U by the intersection of each open set on S with U. This is called the *induced topology* on U.

Definition 1.13 (Product Topology)**.** Let S be the Cartesian product $S = \prod_{\alpha\in I} S_\alpha$ of a family of topological spaces. Consider subsets of S of the form $\prod_{\alpha\in I} U_\alpha$ where finitely many U_α are open sets in S_α and the others coincide with the whole space $U_\alpha = S_\alpha$. These subsets form the base of a topology on S which is called the *product topology*.

Exercise 1. Show that alternatively, the product topology can be characterized as the coarsest topology on $S = \prod_{\alpha\in I} S_\alpha$ such that all projections $S \twoheadrightarrow S_\alpha$ are continuous.

Proposition 1.14. *Let S, T, X be topological spaces and $f \in \mathrm{C}(S \times T, X)$, where $S \times T$ carries the product topology. Then the map $f_x : T \to X$ defined by $f_x(y) = f(x, y)$ is continuous for every $x \in S$.*

Proof. Fix $x \in S$. Let U be an open set in X. We want to show that $W := f_x^{-1}(U)$ is open. We do this by finding for any $y \in W$ an open neighborhood of y contained in W. If W is empty we are done, hence assume that this is not so. Pick $y \in W$. Then $(x, y) \in f^{-1}(U)$ with $f^{-1}(U)$ open by continuity of f. Since $S \times T$ carries the product topology there must be open sets $V_x \subseteq S$ and $V_y \subseteq T$ with $x \in V_x$, $y \in V_y$ and $V_x \times V_y \subseteq f^{-1}(U)$. But clearly $V_y \subseteq W$ and we are done. □

Definition 1.15 (Quotient Topology)**.** Let S be a topological space and $\sim$ an equivalence relation on S. Then, the *quotient topology* on $S/\sim$ is the finest topology such that the quotient map $S \twoheadrightarrow S/\sim$ is continuous.

1.2 Some properties of topological spaces

In a topological space it is useful if two distinct points can be distinguished by the topology. A strong form of this distinguishability is the *Hausdorff property.*

Definition 1.16 (Hausdorff)**.** Let S be a topological space. Assume that given any two distinct points $x, y \in S$ we can find open sets $U, V \subset S$ such that $x \in U$ and $y \in V$ and $U \cap V = \emptyset$. Then, S is said to have the *Hausdorff property.* We also say that S is a *Hausdorff space.*

Definition 1.17. Let S be a topological space. S is called *first-countable* iff there exists a countable neighborhood base at each point of S. S is called *second-countable* iff the topology of S admits a countable base.

Definition 1.18 (open cover)**.** Let S be a topological space and $U \subseteq S$ a subset. A family of open sets $\{U_\alpha\}_{\alpha \in A}$ is called an *open cover* of U iff $U \subseteq \bigcup_{\alpha \in A} U_\alpha$.

Proposition 1.19. *Let S be a second-countable topological space and $U \subseteq S$ a subset. Then, every open cover of U contains a countable subcover.*

Proof. **Exercise.** □

Definition 1.20 (compact)**.** Let S be a topological space and $U \subseteq S$ a subset. U is called *compact* iff every open cover of U contains a finite subcover.

Proposition 1.21. *A closed subset of a compact space is compact. A compact subset of a Hausdorff space is closed.*

Proof. **Exercise.** □

Proposition 1.22. *The image of a compact set under a continuous map is compact.*

Proof. **Exercise.** □

Definition 1.23. A topological space is called *locally compact* iff every point has a compact neighborhood.

Exercise 2 (One-point compactification)**.** Let S be a locally compact Hausdorff space. Let $\tilde{S} := S \cup \{\infty\}$ to be the set S with an extra element ∞ adjoint. Define a subset U of $\tilde{S}$ to be open iff either U is an open subset of S or U is the complement of a compact subset of S. Show that this makes $\tilde{S}$ into a compact Hausdorff space.

1.3 σ-Algebras and Measurable Spaces

Definition 1.24 (Boolean Algebra)**.** Let A be a set equipped with three operations: $\wedge : A \times A \to A$, $\vee : A \times A \to A$ and $\neg : A \to A$ and two special elements $0, 1 \in A$. Suppose these satisfy the following properties:

- $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ and $(x \vee y) \vee z = x \vee (y \vee z) \quad \forall x, y, z \in A$. (associativity)
- $x \wedge y = y \wedge x$ and $x \vee y = y \vee x \quad \forall x, y \in A$. (commutativity)
- $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \quad \forall x, y, z \in A$. (distributivity)
- $x \wedge (x \vee y) = x$ and $x \vee (x \wedge y) = x \quad \forall x, y \in A$. (absorption)
- $x \wedge \neg x = 0$ and $x \vee \neg x = 1 \quad \forall x \in A$. (complement)

Then, A is called a *Boolean algebra.*

Proposition 1.25. *Let A be a Boolean algebra. Then, the following properties hold:*

$$x \wedge x = x,\ x \vee x = x,\ x \wedge 0 = 0,\ x \wedge 1 = x,\ x \vee 0 = x,\ x \vee 1 = 1 \quad \forall x \in A.$$

Proof. **Exercise.** □

Exercise 3. Show that the set with two elements $0, 1$ forms a Boolean algebra. This is important in logic, where 0 stands for ”false” and 1 for ”true”.

Exercise 4. Let S be a set. Show that the set $\mathfrak{P}(S)$ of subsets of S forms a Boolean algebra, where $\vee = \cup$ is the union, $\wedge = \cap$ is the intersection and $\neg$ is the complement of sets.

Definition 1.26 (Algebra of sets)**.** Let S be a set. A subset $\mathcal{M}$ of the set $\mathfrak{P}(S)$ of subsets of S is called an *algebra* of sets iff it is a Boolean subalgebra of $\mathfrak{P}(S)$.

Proposition 1.27. *Let S be a set and $\mathcal{M}$ a subset of the set $\mathfrak{P}(S)$ of subsets of S. Then $\mathcal{M}$ is an algebra of sets iff it contains the empty set and is closed under complements, finite unions, and finite intersections.*

Proof. Immediate. □

<u>Exercise</u> 5. Show that the above proposition remains true if we erase either the requirement for closedness under finite unions or the requirement for closedness under finite intersections.

Definition 1.28. Let S be a set and $\mathcal{M}$ an algebra of subsets of S. We call $\mathcal{M}$ a *σ-algebra* of sets iff it is closed under countable unions and countable intersections.

<u>Exercise</u> 6. Show that the above definition remains unchanged if we remove either the requirement for closedness under countable unions or closedness under countable intersections.

Definition 1.29. Let S be a set and $\mathcal{B}$ a subset of the set $\mathfrak{P}(S)$ of subsets of S. Then, the smallest σ-algebra $\mathcal{M}$ on S containing $\mathcal{B}$ is called the σ-algebra *generated* by $\mathcal{B}$.

<u>Exercise</u> 7. Justify the above definition by showing that the smallest σ-algebra in the sense of the definition always exists.

Definition 1.30. Let S be a set and $\mathcal{B}$ a subset of $\mathfrak{P}(S)$. Then, $\mathcal{B}$ is called *monotone* iff it satisfies the following properties:

- Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of elements of $\mathcal{B}$ such that $A_n \subseteq A_{n+1}$. Then, $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{B}$.
- Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of elements of $\mathcal{B}$ such that $A_n \supseteq A_{n+1}$. Then, $\bigcap_{n\in\mathbb{N}} A_n \in \mathcal{B}$.

Proposition 1.31. *1. A σ-algebra is monotone. 2. An algebra that is monotone is a σ-algebra.*

Proof. **<u>Exercise</u>.** □

Proposition 1.32 (Monotone Class Theorem)**.** *Let S be a set and $\mathcal{N}$ an algebra of subsets of S. Then, the smallest set $\mathcal{M}$ of subsets of S which contains $\mathcal{N}$ and is monotone is the σ-algebra generated by $\mathcal{N}$.*

Proof. For each $A \in \mathcal{M}$ and consider

$$\mathcal{M}_A := \{B \in \mathcal{M} : A \cap B \in \mathcal{M}, A \cap \neg B \in \mathcal{M}, \neg A \cap B \in \mathcal{M}\}.$$

It is easy to see that $\mathcal{M}_A$ is monotone. [**<u>Exercise</u>.**Show this!] Furthermore, if $A \in \mathcal{N}$, then $\mathcal{N} \subseteq \mathcal{M}_A$ since $\mathcal{N}$ is an algebra. So in this case $\mathcal{M} \subseteq \mathcal{M}_A$ by minimality of $\mathcal{M}$ and consequently $\mathcal{M} = \mathcal{M}_A$. Thus, for $B \in \mathcal{M}$ we have $B \in \mathcal{M}_A$ and hence $A \in \mathcal{M}_B$ if $A \in \mathcal{N}$. So, $\mathcal{N} \subseteq \mathcal{M}_B$ and by minimality we conclude $\mathcal{M} = \mathcal{M}_B$ for any $B \in \mathcal{M}$. But this means that $\mathcal{M}$ is an algebra. Thus, by Proposition 1.31.2, $\mathcal{M}$ is a σ-algebra. Furthermore, by minimality and Proposition 1.31.1, it is the σ-algebra generated by $\mathcal{N}$. □

Definition 1.33. Let S be a set and $\mathcal{M}$ a σ-algebra of subsets of S. Then, we call the pair $(S, \mathcal{M})$ a *measurable space* and the elements of $\mathcal{M}$ *measurable sets*.

Definition 1.34. Let S be a measurable space and U a subset of S. Then, the σ-algebra on S intersected with U is called the *induced* σ-algebra on U.

Definition 1.35. Let S be a topological space. Then, the σ-algebra generated by the topology of S is called the algebra of *Borel* sets. Its elements are called *Borel measurable*.

1.4 Sequences and convergence

Definition 1.36 (Convergence of sequences)**.** Let $x := \{x_n\}_{n\in\mathbb{N}}$ be a sequence of points in a topological space S. We say that x has an *accumulation point (or limit point)* p iff for every neighborhood U of p we have $x_k \in U$ for infinitely many $k \in \mathbb{N}$. We say that x *converges* to a point p iff for any neighborhood U of p there is a number $n \in \mathbb{N}$ such that for all $k \geq n : x_k \in U$.

Proposition 1.37. *Let S, T be topological spaces and $f : S \to T$. If f is continuous, then for any $p \in S$ and sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to p, the sequence $\{f(x_n)\}_{n\in\mathbb{N}}$ in T converges to $f(p)$. Conversely, if S is first-countable and for any $p \in S$ and sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to p, the sequence $\{f(x_n)\}_{n\in\mathbb{N}}$ in T converges to $f(p)$, then f is continuous.*

Proof. **Exercise**. □

Proposition 1.38. *Let S be Hausdorff space and $\{x_n\}_{n\in\mathbb{N}}$ a sequence in S which converges to a point $p \in S$. Then, $\{x_n\}_{n\in\mathbb{N}}$ does not converge to any other point in S.*

Proof. **Exercise**. □

Definition 1.39. Let S be a topological space and $U \subseteq S$ a subset. Consider the set B_U of sequences of elements of U. Then the set $\overline{U}^s$ consisting of the points to which some element of B_U converges is called the *sequential closure* of U.

Proposition 1.40. *Let S be a topological space and $U \subseteq S$ a subset. Let x be a sequence of points in U which has an accumulation point $p \in S$. Then, $p \in \overline{U}$.*

Proof. Suppose $p \notin \overline{U}$. Since $\overline{U}$ is closed $S \setminus \overline{U}$ is an open neighborhood of p. But $S \setminus \overline{U}$ does not contain any point of x, so p cannot be accumulation point of x. This is a contradiction. □

Corollary 1.41. *Let S be a topological space and U a subset. Then, $U \subseteq \overline{U}^s \subseteq \overline{U}$.*

Proof. Immediate. □

Proposition 1.42. *Let S be a first-countable topological space and U a subset. Then, $\overline{U}^s = \overline{U}$.*

Proof. **<u>Exercise</u>.** □

Definition 1.43. Let S be a topological space and $U \subseteq S$ a subset. U is said to be *limit point compact* iff every sequence in U has an accumulation point (limit point) in U. U is called *sequentially compact* iff every sequence of elements of U contains a subsequence converging to a point in U.

Proposition 1.44. *Let S be a first-countable topological space and $x = \{x_n\}_{n\in\mathbb{N}}$ a sequence in S with accumulation point p. Then, x has a subsequence that converges to p.*

Proof. By first-countability choose a countable neighborhood base $\{U_n\}_{n\in\mathbb{N}}$ at p. Now consider the family $\{W_n\}_{n\in\mathbb{N}}$ of open neighborhoods $W_n := \bigcap_{k=1}^n U_k$ at p. It is easy to see that this is again a countable neighborhood base at p. Moreover, it has the property that $W_n \subseteq W_m$ if $n \geq m$. Now, Choose $n_1 \in \mathbb{N}$ such that $x_{n_1} \in W_1$. Recursively, choose $n_{k+1} > n_k$ such that $x_{n_{k+1}} \in W_{k+1}$. This is possible since W_{k+1} contains infinitely many points of x. Let V be a neighborhood of p. There exists some $k \in \mathbb{N}$ such that $U_k \subseteq V$. By construction, then $W_m \subseteq W_k \subseteq U_k$ for all $m \geq k$ and hence $x_{n_m} \in V$ for all $m \geq k$. Thus, the subsequence $\{x_{n_m}\}_{m\in\mathbb{N}}$ converges to p. □

Proposition 1.45. *Sequential compactness implies limit point compactness. In a first-countable space the converse is also true.*

Proof. **<u>Exercise</u>.** □

Proposition 1.46. *A compact set is limit point compact.*

Proof. Consider a sequence x in a compact set S. Suppose x does not have an accumulation point. Then, for each point $p \in S$ we can choose an open neighborhood U_p which contains only finitely many points of x. However, by compactness, S is covered by finitely many of the sets U_p. But their union can only contain a finite number of points of x, a contradiction. □

1.5 Metric and pseudometric spaces

Definition 1.47. Let S be a set and $d : S \times S \to \mathbb{R}_0^+$ a map with the following properties:

- $d(x,y) = d(y,x) \quad \forall x,y \in S$. (symmetry)
- $d(x,z) \leq d(x,y) + d(y,z) \quad \forall x,y,z \in S$. (triangle inequality)
- $d(x,x) = 0 \quad \forall x \in S$.

Then d is called a *pseudometric* on S. S is also called a *pseudometric space*. Suppose d also satisfies

- $d(x,y) = 0 \implies x = y \quad \forall x, y \in S$. (definiteness)

Then d is called a *metric* on S and S is called a *metric space*.

Definition 1.48. Let S be a pseudometric space, $x \in S$ and $r > 0$. Then the set $B_r(x) := \{y \in S : d(x,y) < r\}$ is called the *open ball* of radius r centered around x in S. The set $\overline{B}_r(x) := \{y \in S : d(x,y) \leq r\}$ is called the *closed ball* of radius r centered around x in S.

Proposition 1.49. *Let S be a pseudometric space. Then, the open balls in S together with the empty set form the basis of a topology on S. This topology is first-countable and such that closed balls are closed. Moreover, the topology is Hausdorff iff S is metric.*

Proof. **Exercise.** □

Definition 1.50. A topological space is called *(pseudo)metrizable* iff there exists a (pseudo)metric such that the open balls given by the (pseudo)metric are a basis of its topology.

Proposition 1.51. *In a pseudometric space any open ball can be obtained as the countable union of closed balls. Similarly, any closed ball can be obtained as the countable intersection of open balls.*

Proof. **Exercise.** □

Proposition 1.52. *Let S be a set equipped with two pseudometrics d^1 and d^2. Then, the topology generated by d^2 is finer than the topology generated by d^1 iff for all $x \in S$ and $r_1 > 0$ there exists $r_2 > 0$ such that $B^2_{r_2}(x) \subseteq B^1_{r_1}(x)$. In particular, d^1 and d^2 generate the same topology iff the condition holds both ways.*

Proof. **Exercise.** □

Proposition 1.53 (epsilon-delta criterion)**.** *Let S, T be pseudometric spaces and $f : S \to T$ a map. Then, f is continuous at $x \in S$ iff for every $\epsilon > 0$ there exists $\delta > 0$ such that $f(B_\delta(x)) \subseteq B_\epsilon(f(x))$.*

Proof. **Exercise.** □

1.6 Measurable Functions

As we see the concept of a measurable space is very similar to the concept of a topological space. Both are based on a set of subsets closed under certain operations. We can push this analogy further and consider the analog of a continuous function: a measurable function.

Definition 1.54. Let S, T be measurable spaces. Then a map $f : S \to T$ is called *measurable* iff the preimage of every measurable set of T is a measurable set of S. If either T or S or T and S are topological spaces instead we call f measurable iff it is measurable with respect to the generated σ-algebra(s) of Borel sets.

Proposition 1.55. *Let S, T, U be measurable spaces, $f : S \to T$ and $g : T \to U$ measurable. Then, $g \circ f : S \to U$ is measurable.*

Proof. Immediate. □

Proposition 1.56. *Let S be a measurable space, T a topological space and $f : S \to T$. Then, f is measurable iff the preimage of every open set is measurable. Also, f is measurable iff the preimage of every closed set is measurable.*

Proof. **Exercise.** □

Corollary 1.57. *Let S and T be topological spaces and $f : S \to T$ a continuous map. Then, f is measurable.*

Proposition 1.58. *Let S be a measurable space, T and U topological spaces, $f : S \to T \times U$. Denote by $f_T : S \to T$ and $f_U : S \to U$ the component functions. If the product $f : S \to T \times U$ is measurable, then both f_T and f_U are measurable. Conversely, if T and U are second-countable and f_T and f_U are measurable, then f is measurable.*

Proof. First suppose that f is measurable. Then, $f_T = p_T \circ f$, where p_T is the projection $T \times U \to T$. Since p_T is continuous, it is measurable by Corollary 1.57 and the composition f_T is measurable by Proposition 1.55. In the same way it follows that f_U is measurable.

Conversely, suppose now that f_T and f_U are measurable. If $V \subseteq T$ and $W \subseteq U$ are open sets, then $f_T^{-1}(V)$ and $f_U^{-1}(W)$ are measurable in S and so is their intersection $f^{-1}(V \times W) = f_T^{-1}(V) \cap f_U^{-1}(W)$. Since T and U are second-countable, every open set in $T \times U$ can be written as a countable union of products of open sets in T and U [**Exercise.**Show this!]. But the preimage of such a countable union in S under f^{-1} can be written as a countable union of preimages. Since these are measurable, their countable union is also measurable. It follows then from Proposition 1.56 that f is measurable. □

In the following $\mathbb{K}$ denotes either the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$.

Proposition 1.59. *Let S be a measurable space, $f, g : S \to \mathbb{K}$ measurable and $\lambda \in \mathbb{K}$. Then:*

1. $|f| : x \mapsto |f(x)|$ *is measurable.*
2. $f + g : x \mapsto f(x) + g(x)$ *is measurable.*

3. *$\lambda f : x \mapsto \lambda f(x)$ is measurable.*

4. *$fg : x \mapsto f(x)g(x)$ is measurable.*

5. *If $\mathbb{K} = \mathbb{R}$ then $\sup(f, g)$ and $\inf(f, g)$ are measurable.*

Proof. **Exercise.** □

This shows in particular that measurable functions with values in $\mathbb{R}$ or $\mathbb{C}$ form an algebra. Another important property of the set of measurable maps is its closedness under pointwise limits. This can be formulated for the more general case when the values are taken in a metric space.

Theorem 1.60 (adapted from S. Lang)**.** *Let S be a measurable space and T a metric space. Suppose $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of measurable functions $f_n : S \to T$ which converges pointwise to the function $f : S \to T$. Then, f is measurable.*

Proof. Let U be an open set in T. Suppose $x \in f^{-1}(U)$. Since $\{f_n(x)\}_{n\in\mathbb{N}}$ converges to $f(x)$ there exists $m \in \mathbb{N}$ such that $x \in f_n^{-1}(U)$ for all $n > m$. In particular, $x \in \bigcup_{n=k}^{\infty} f_n^{-1}(U)$ for any $k \in \mathbb{N}$ and so also $x \in \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} f_n^{-1}(U)$. Since this is true for any $x \in f^{-1}(U)$ we get

$$f^{-1}(U) \subseteq \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} f_n^{-1}(U).$$

Consider now for all $l \in \mathbb{N}$ the open sets

$$U_l := \{x \in U : \inf_{y\notin U} d(x, y) > 1/l\}.$$

Then, $U = \bigcup_{l=1}^{\infty} U_l$ and applying the above reasoning to each U_l we get,

$$f^{-1}(U) \subseteq \bigcup_{l=1}^{\infty}\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} f_n^{-1}(U_l).$$

Suppose now that $x \notin f^{-1}(U)$ and fix $l \in \mathbb{N}$. Since $B_{1/l}(f(x)) \cap U_l = \emptyset$ there exists $m \in \mathbb{N}$ such that $x \notin f_n^{-1}(U_l)$ for all $n > m$. In particular, $x \notin \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} f_n^{-1}(U_l)$. Since this is true for any $l \in \mathbb{N}$ we get $x \notin \bigcup_{l=1}^{\infty}\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} f_n^{-1}(U_l)$. Since this is true for any $x \notin f^{-1}(U)$ we get, combining with the above result,

$$f^{-1}(U) = \bigcup_{l=1}^{\infty}\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} f_n^{-1}(U_l).$$

Since f_n is measurable for all $n \in \mathbb{N}$ the right hand side is measurable. We have thus shown that preimages of open sets are measurable. By Proposition 1.56 this is sufficient for f to be measurable. □

Definition 1.61. Let S be a measurable space. A map $f : S \to \mathbb{K}$ is called a *simple map* iff it is measurable and takes only finitely many values.

Proposition 1.62. *Let S be a measurable space and $f : S \to \mathbb{K}$ a map that takes only finitely many values. Then f is a simple map (i.e., is measurable) iff the preimage of each of the values of f is measurable.*

Proof. **Exercise.** □

Proposition 1.63. *The simple functions with values in $\mathbb{K}$ form a subalgebra of the algebra of measurable functions with values in $\mathbb{K}$.*

Proof. **Exercise.** □

Theorem 1.64 (adapted from S. Lang)**.** *Let S be a measurable space and $f : S \to \mathbb{K}$ measurable. Then, f is the pointwise limit of a sequence of simple maps. If, moreover, f takes values in $\mathbb{R}^+_0$, then the sequence can be chosen to increase monotonically.*

Proof. Consider first the case $\mathbb{K} = \mathbb{R}$. Fix $n \in \mathbb{N}$. For each $k \in \{1, \dots, 2^{n+1}n\}$ define the interval $I_k := [-n + \frac{k-1}{2^n}, -n + \frac{k}{2^n})$. Also, define $I_0 := (-\infty, -n)$ and $I_{2^{n+1}n+1} := [n, \infty)$. Notice that $\mathbb{R}$ is the disjoint union of the measurable intervals I_k for $k \in \{0, \dots, 2^{n+1}n+1\}$. Now set $X_k := f^{-1}(I_k)$ for all $k \in \{0, \dots, 2^{n+1}n+1\}$. Since the intervals I_k are measurable so are the sets X_k. Define the function $f_n : S \to \mathbb{R}$ by $f_n(X_k) := -n + \frac{k-1}{2^n}$ for all $k \in \{1, \dots, 2^{n+1}n + 1\}$ and $f_n(X_0) := -n$. It is easy to see that $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of simple functions that converge pointwise to f. [**Exercise.**Show this!] Moreover, if f takes values in $\mathbb{R}^+_0$ only, the sequence is monotonically increasing. [**Exercise.**Show this!] To treat the case $\mathbb{K} = \mathbb{C}$ we decompose f into its real and imaginary part. The sum of simple sequences for each part is again a simple sequence. □

2 Measures

2.1 Positive Measures

Definition 2.1. Let $\{a_n\}_{n\in\mathbb{N}}$ be a monotonously increasing sequence of real numbers. Then we say that $\lim_{n\to\infty} a_n = \infty$ iff for any $a \in \mathbb{R}$ there exists $m \in \mathbb{N}$ such that $a_n > a$ for all $n > m$. (See Exercise 2.)

Definition 2.2 (Positive Measure)**.** Let S be a set with an algebra $\mathcal{M}$ of subsets. Then, a map $\mu : \mathcal{M} \to [0, \infty]$ is called a *(positive) measure* iff it is countably additive, i.e., satisfies the following properties:

- $\mu(\emptyset) = 0$.
- Let $\{U_n\}_{n\in\mathbb{N}}$ be a sequence of elements of $\mathcal{M}$ such that $U_n \cap U_m = \emptyset$ if $n \neq m$ and such that $\bigcup_{n\in\mathbb{N}} U_n \in \mathcal{M}$. Then,

$$\mu\left(\bigcup_{n\in\mathbb{N}} U_n\right) = \sum_{n\in\mathbb{N}} \mu\left(U_n\right).$$

If $U \in \mathcal{M}$, then $\mu(U)$ is called its *measure*. Moreover, a measurable space S with σ-algebra $\mathcal{M}$ and positive measure $\mu : \mathcal{M} \to [0, \infty]$ is called a *measure space*.

We shall mostly be interested in the case where $\mathcal{M}$ actually is a σ-algebra. However, it will turn out convenient to keep the definition more general when we consider constructing measures.

Proposition 2.3. *Let S be a set, $\mathcal{M}$ an algebra of subsets of S and $\mu : \mathcal{M} \to [0, \infty]$ a measure. Then, the following properties hold:*

1. *Let $A, B \in \mathcal{M}$ and $A \subseteq B$. Then, $\mu(A) \leq \mu(B)$.*
2. *Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of elements of $\mathcal{M}$ such that $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{M}$. Then,*

$$\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) \leq \sum_{n\in\mathbb{N}} \mu(A_n).$$

3. *Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of elements of $\mathcal{M}$ such that $A_n \subseteq A_{n+1}$ for all $n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{M}$. Then,*

$$\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \lim_{n\to\infty} \mu(A_n).$$

4. *Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of elements of $\mathcal{M}$ such that $A_n \supseteq A_{n+1}$ for all $n \in \mathbb{N}$ and $\bigcap_{n\in\mathbb{N}} A_n \in \mathcal{M}$. If furthermore, $\mu(A_n) < \infty$ for some $n \in \mathbb{N}$ then,*

$$\mu\left(\bigcap_{n\in\mathbb{N}} A_n\right) = \lim_{n\to\infty} \mu(A_n).$$

Proof. **Exercise.** □

Exercise 8. Check whether the following examples are measures.

- Let S be a set and consider the σ-algebra of all subsets of S. If $A \subseteq S$ is finite define $\mu(A)$ to be its number of elements. If $A \subseteq S$ is infinite define $\mu(A) = \infty$. μ is called the *counting measure*.
- Let S be a set and consider the σ-algebra of all subsets of S. If $A \subseteq S$ is finite define $\mu(A) = 0$. If $A \subseteq S$ is infinite define $\mu(A) = \infty$.
- Let S be a set and consider the σ-algebra of all subsets of S. If $A \subseteq S$ is countable define $\mu(A) = 0$. If $A \subseteq S$ is not countable define $\mu(A) = \infty$.
- Let S be a set and consider the σ-algebra of all subsets of S. Let $x \in S$. For $A \subseteq S$ define $\mu(A) = 1$ if $x \in A$ and $\mu(A) = 0$ otherwise. μ is called the *Dirac measure* with respect to x.

Definition 2.4. Let S be a measure space and $A \subseteq S$ a measurable subset. We say that A is *σ-finite* iff it is equal to some countable union of measurable sets with finite measure. We say that a measure is *finite* respectively *σ-finite* iff the measure space is finite respectively σ-finite with respect to the measure.

Exercise 9. Which of the examples of measures above are σ-finite?

Definition 2.5. Let $(S, \mathcal{M}, \mu)$ be a measure space. If every subset of any set of measure 0 is measurable, then we call $(S, \mathcal{M}, \mu)$ *complete*.

Proposition 2.6. *Let $(S, \mathcal{M}, \mu)$ be a measure space. Then, there exists a unique complete measure space $(S, \mathcal{M}^*, \mu^*)$ such that $\mathcal{M}^*$ is a σ-algebra containing $\mathcal{M}$ and $\mu^*|_{\mathcal{M}} = \mu$ and $\mathcal{M}^*$ is smallest with these properties. $(S, \mathcal{M}^*, \mu^*)$ is called the* completion *of $(S, \mathcal{M}, \mu)$. Moreover, the element of $\mathcal{M}^*$ are precisely the sets of the form $A \cup N$, where $A \in \mathcal{M}$ and N is a subset of a set of measure 0 in $\mathcal{M}$.*

Proof. **Exercise.** □

Proposition 2.7. *Let $(S, \mathcal{M}, \mu)$ be a measure space and $f : S \to \mathbb{K}$ measurable with respect to $\mathcal{M}^*$. Then, there exists a function $g : S \to \mathbb{K}$ such that g is measurable with respect to $\mathcal{M}$ and g does not differ from f outside of a subset $N \in \mathcal{M}$ of measure 0.*

Proof. By Theorem 1.64 there exists a sequence $\{f_n\}_{n\in\mathbb{N}}$ of simple maps with respect to $\mathcal{M}^*$ that converges pointwise to f. For each f_n we can find a set $N_n \in \mathcal{M}$ of measure 0 such that the function $k_n : S \to \mathbb{K}$ defined by $k_n(p) = f_n(p)$ if $p \in S \setminus N_n$ and $k_n(p) = 0$ otherwise, is simple with respect to $\mathcal{M}$. (**Exercise.**Show this!) The set $N := \bigcup_{n=1}^{\infty} N_n \in \mathcal{M}$ has measure zero. Moreover, $g_n : S \to \mathbb{K}$ defined by $g_n(p) = f_n(p)$ if $p \in S \setminus N$ and $g_n(p) = 0$ otherwise, is simple with respect to $\mathcal{M}$. Moreover, the sequence $\{g_n\}_{n\in\mathbb{N}}$ converges pointwise to $g : S \to \mathbb{K}$ defined by $g(p) = f(p)$ if $p \in S \setminus N$ and $g(p) = 0$ otherwise. Thus, by Theorem 1.60, g is measurable with respect to $\mathcal{M}$. □

2.2 Extension of Measures

We now turn to the question of how to construct measures. We will focus here on the method of extension. That is, we consider a measure that is merely defined on an algebra of subsets and extend it to a measure on a σ-algebra.

Definition 2.8. Let S be a set and $\mathcal{M}$ a σ-algebra of subsets of S. Then, a map $\lambda : \mathcal{M} \to [0, \infty]$ is called an *outer measure* on $\mathcal{M}$ iff it satisfies the following properties:

- $\lambda(\emptyset) = 0$.
- Let $A, B \in \mathcal{M}$ and $A \subseteq B$. Then, $\lambda(A) \le \lambda(B)$. (monotonicity)
- Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of elements of $\mathcal{M}$. Then,
$$\lambda\left(\bigcup_{n\in\mathbb{N}} A_n\right) \le \sum_{n\in\mathbb{N}} \lambda\left(A_n\right). \quad \text{(countable subadditivity)}$$

Lemma 2.9. *Let S be a set, $\mathcal{N}$ an algebra of subsets of S and μ a measure on $\mathcal{N}$. On the σ-algebra $\mathfrak{P}(S)$ of all subsets of S define the function $\lambda : \mathfrak{P}(S) \to [0, \infty]$ given by*
$$\lambda(X) = \inf\left\{\sum_{n\in\mathbb{N}} \mu(A_n) : A_n \in \mathcal{N}\, \forall n \in \mathbb{N} \,\mathit{and}\, X \subseteq \bigcup_{n\in\mathbb{N}} A_n\right\}.$$
Then, λ is an outer measure on $\mathfrak{P}(S)$. Moreover, it extends μ, i.e., $\lambda(A) = \mu(A)$ for all $A \in \mathcal{N}$.

Proof. **Exercise.** □

Definition 2.10. Let S be a set and λ an outer measure on the σ-algebra $\mathfrak{P}(S)$ of all subsets of S. Then, $A \subseteq S$ is called *λ-measurable* iff $\lambda(X) = \lambda(X \cap A) + \lambda(X \cap \neg A)$ for all $X \subseteq S$.

Lemma 2.11. *Let S be a set and λ an outer measure on the σ-algebra $\mathfrak{P}(S)$ of all subsets of S. Let $\mathcal{M}$ be the set of subsets of S that are λ-measurable. Then, $\mathcal{M}$ is a σ-algebra and λ is a complete measure on $\mathcal{M}$.*

Proof. It is clear that $\mathcal{M}$ contains the empty set and S. Also, from the definition it is clear that a set is λ-measurable iff its complement is. Let now $A, B \in \mathcal{M}$. We proceed to show that $A \cap B \in \mathcal{M}$. Let $C \subseteq S$ be arbitrary. Since B is λ-measurable we have,

$$\lambda(C \cap A \cap B) + \lambda(C \cap A \cap \neg B) = \lambda(C \cap A).$$

Adding $\lambda(C \cap \neg A)$ we get,

$$\lambda(C \cap A \cap B) + \lambda(C \cap A \cap \neg B) + \lambda(C \cap \neg A) = \lambda(C),$$

since A is λ-measurable. The λ-measurability of $A \cap B$ follows if we can show,

$$\lambda(C \cap A \cap \neg B) + \lambda(C \cap \neg A) = \lambda(C \cap \neg(A \cap B)).$$

But this equation can be rewritten term-wise as,

$$\lambda(C \cap \neg(A \cap B) \cap A) + \lambda(C \cap \neg(A \cap B) \cap \neg A) = \lambda(C \cap \neg(A \cap B)),$$

which follows from the λ-measurability of A. Thus $\mathcal{M}$ is an algebra.

Now consider a sequence $\{A_n\}_{n\in\mathbb{N}}$ of disjoint λ-measurable sets and let $A = \bigcup_{n\in N} A_n$ be their union. Let $C \subseteq S$ be arbitrary. By iteration we find that for any $n \in \mathbb{N}$ we have,

$$\lambda(C \cap (A_1 \cup \cdots \cup A_n)) = \sum_{k=1}^{n} \lambda(C \cap A_k).$$

On the other hand $C \cap \neg A \subseteq C \cap \neg(A_1 \cup \cdots \cup A_n)$ and so $\lambda(C \cap \neg A) \le \lambda(C \cap \neg(A_1 \cup \cdots \cup A_n))$ since λ is an outer measure. With λ-measurability of $A_1 \cup \cdots \cup A_n$ we get thus,

$$\begin{aligned}\lambda(C) &= \lambda(C \cap (A_1 \cup \cdots \cup A_n)) + \lambda(C \cap \neg(A_1 \cup \cdots \cup A_n)) \\ &\ge \sum_{k=1}^{n} \lambda(C \cap A_k) + \lambda(C \cap \neg A).\end{aligned}$$

But this is true for any $n \in \mathbb{N}$, so,

$$\lambda(C) \ge \sum_{k=1}^{\infty} \lambda(C \cap A_k) + \lambda(C \cap \neg A) \ge \lambda(C \cap A) + \lambda(C \cap \neg A).$$

as λ is an outer measure. The same property of λ gives us directly the converse inequality,

$$\lambda(C) \le \lambda(C \cap A) + \lambda(C \cap \neg A).$$

Thus, A is λ-measurable and consequently $\mathcal{M}$ is a σ-algebra. What is more, setting $C = A$ we see that λ is countably additive on $\mathcal{M}$, i.e., defines a positive measure on it.

Now let $A \in \mathcal{M}$ with $\lambda(A) = 0$ and $B \subseteq A$. Also let $C \subseteq S$ be arbitrary. Then we have,

$$\lambda(C \cap B) \le \lambda(C \cap A) \le \lambda(A) = 0.$$

We also get,

$$\lambda(C) = \lambda(C \cap A) + \lambda(C \cap \neg A) = \lambda(C \cap \neg A) \le \lambda(C \cap \neg B) \le \lambda(C).$$

In particular,

$$\lambda(C \cap B) = 0 \quad \text{and} \quad \lambda(C \cap \neg B) = \lambda(C).$$

Thus, B is λ-measurable. This shows completeness of $(S, \mathcal{M}, \lambda)$. $\square$

Theorem 2.12 (Hahn)**.** *Let S be a set, $\mathcal{N}$ an algebra of subsets of S and μ a measure on $\mathcal{N}$. Then, μ can be extended to a σ-algebra $\mathcal{M}$ containing $\mathcal{N}$ such that μ is a complete measure on $\mathcal{M}$ and for all $X \in \mathcal{M}$ we have*

$$\mu(X) = \inf \left\{ \sum_{n\in\mathbb{N}} \mu(A_n) : A_n \in \mathcal{N}\, \forall n \in \mathbb{N} \,\textit{and}\, X \subseteq \bigcup_{n\in\mathbb{N}} A_n \right\}.$$

Proof. **Exercise.** $\square$

Proposition 2.13 (Uniqueness of Extension)**.** *Let S be a measurable space with σ-algebra $\mathcal{M}$ and measures μ_1, μ_2. Suppose there is an algebra $\mathcal{N} \subseteq \mathcal{M}$ generating $\mathcal{M}$ and such that $\mu(A) := \mu_1(A) = \mu_2(A)$ for all $A \in \mathcal{N}$. Furthermore, assume that μ is σ-finite with respect to $\mathcal{N}$. Then, $\mu_1 = \mu_2$ also on $\mathcal{M}$.*

Proof. Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of elements of $\mathcal{N}$ such that $S = \bigcup_{n\in\mathbb{N}} X_n$ and $X_n \subseteq X_{n+1}$ and $\mu(X_n) < \infty$ for all $n \in \mathbb{N}$. (By σ-finiteness, there is a sequence $\{Y_k\}_{k\in\mathbb{N}}$ with $S = \bigcup_{k\in\mathbb{N}} Y_k$ and $\mu(Y_k) < \infty$ for all $k \in \mathbb{N}$. Now set $X_n := \bigcup_{k=1}^n Y_k$.) Define the finite measures $\mu_{1,n}(A) := \mu_1(A \cap X_n)$ and $\mu_{2,n}(A) := \mu_2(A \cap X_n)$ on $\mathcal{M}$ for all $n \in \mathbb{N}$. Now, let $\mathcal{B}_n$ be the subsets of $\mathcal{M}$ where $\mu_{1,n}$ and $\mu_{2,n}$ agree. By construction, $\mathcal{N} \subseteq \mathcal{B}_n$ for all $n \in \mathbb{N}$. We show that the $\mathcal{B}_n$ are monotone.

Fix $n \in \mathbb{N}$. Let $\{A_k\}_{k\in\mathbb{N}}$ be a sequence of elements of $\mathcal{B}_n$ such that $A_k \subseteq A_{k+1}$ for all $k \in \mathbb{N}$ and set $A := \bigcup_{k\in\mathbb{N}} A_k$. Then, using Proposition 2.3,

$$\mu_{1,n}(A) = \lim_{k\to\infty} \mu_{1,n}(A_k) = \lim_{k\to\infty} \mu_{2,n}(A_k) = \mu_{2,n}(A).$$

So, $A \in \mathcal{B}_n$. Now, let $\{A_k\}_{k\in\mathbb{N}}$ be a sequence of elements of $\mathcal{B}_n$ such that $A_k \supseteq A_{k+1}$ for all $k \in \mathbb{N}$ and set $A := \bigcap_{k\in\mathbb{N}} A_k$. Again using Proposition 2.3 we get (note that the finiteness of the measure is essential here),

$$\mu_{1,n}(A) = \lim_{k\to\infty} \mu_{1,n}(A_k) = \lim_{k\to\infty} \mu_{2,n}(A_k) = \mu_{2,n}(A).$$

So, $A \in \mathcal{B}_n$. Hence, $\mathcal{B}_n$ is monotone and by Proposition 1.32 we must have $\mathcal{M} \subseteq \mathcal{B}_n$ and hence $\mathcal{M} = \mathcal{B}_n$.

Thus, $\mu_{1,n} = \mu_{2,n}$ for all $n \in \mathbb{N}$. But then, $\mu_1 = \lim_{n\to\infty} \mu_{1,n} = \lim_{n\to\infty} \mu_{2,n} = \mu_2$. This completes the proof. □

Proposition 2.14. *Let $(S, \mathcal{M}, \mu)$ be a σ-finite measure space. Let $\mathcal{N}$ be an algebra of subsets of S that generates $\mathcal{M}$. Denote the completion of $\mathcal{M}$ with respect to μ by $\mathcal{M}^*$. Then, for any $X \in \mathcal{M}^*$ with finite measure and any $\epsilon > 0$ there exists $A \in \mathcal{N}$ such that*

$$\mu((X \setminus A) \cup (A \setminus X)) < \epsilon.$$

Proof. Let $X \in \mathcal{M}^*$ with finite measure. By Hahn's Theorem 2.12 there exists a sequence $\{A_n\}_{n\in\mathbb{N}}$ of disjoint elements of $\mathcal{N}$ such that $X \subseteq \bigcup_{n\in\mathbb{N}} A_n$ and

$$\sum_{n=1}^{\infty} \mu(A_n) < \mu(X) + \epsilon/2.$$

Now fix $k \in \mathbb{N}$ such that

$$\sum_{n=k+1}^{\infty} \mu(A_n) < \epsilon/2.$$

Set $A := \bigcup_{n=1}^{k} A_n$. Then, on the one hand,

$$\mu(A \setminus X) \le \mu\left(\left(\bigcup_{n=1}^{\infty} A_n\right) \setminus X\right) < \epsilon/2,$$

while on the other hand,

$$\mu(X \setminus A) \le \mu\left(\left(\bigcup_{n=1}^{\infty} A_n\right) \setminus A\right) = \mu\left(\bigcup_{n=k+1}^{\infty} A_n\right) < \epsilon/2.$$

This implies the statement. □

2.3 The Lebesgue Measure

In the following we are going to construct the Lebesgue measure. This is the unique (as we shall see) measure on the real numbers assigning to an interval its length. The construction proceeds in various stages.

Lemma 2.15. *The finite disjoint unions of intervals of the type $[a, b)$, $(-\infty, a)$, and $[a, \infty)$ together with $\emptyset$ form an algebra $\mathcal{N}$ of subsets of the real numbers.*

Proof. **Exercise.** □

Lemma 2.16. *The prescription $\mu([a,b)) = b - a$ determines uniquely a finitely additive function $\mu : \mathcal{N} \to [0,\infty]$ on the algebra $\mathcal{N}$ considered above.*

Proof. **Exercise.** $\square$

Lemma 2.17. *The function $\mu : \mathcal{N} \to [0,\infty]$ defined above is countably additive and thus a measure.*

Proof. Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of pairwise disjoint elements of $\mathcal{N}$ such that $A := \bigcup_{n\in\mathbb{N}} A_n \in \mathcal{N}$. We wish to show that

$$\mu(A) = \sum_{n\in\mathbb{N}} \mu(A_n).$$

By finite additivity we have $\mu(A) \geq \mu(\bigcup_{n=1}^m A_n) = \sum_{n=1}^m \mu(A_n)$ for all $m \in \mathbb{N}$ and hence

$$\mu(A) \geq \sum_{n\in\mathbb{N}} \mu(A_n).$$

It remains to show the opposite inequality.

Assume at first that A is a finite interval $[a,b)$. Then, A is the disjoint union of a sequence of intervals $\{I_k\}_{k\in\mathbb{N}}$ with $I_k = [a_k, b_k)$ in such a way that each A_n is the finite union of some I_k. (We also allow the degenerate case $a_k = b_k$ in which case $I_k = \emptyset$.) Fix now $\epsilon > 0$ (with $\epsilon < b-a$) and define $I'_k := (a_k - 2^{-(k+1)}\epsilon, b_k)$ for all $k \in \mathbb{N}$. Then, the open sets $\{I'_k\}_{k\in\mathbb{N}}$ cover the compact interval $[a, b-\epsilon/2]$. Thus, there is a finite set of indices $I \subset \mathbb{N}$ such that $[a, b-\epsilon/2] \subset \bigcup_{k\in I} I'_k$. Then clearly also $[a, b-\epsilon/2) \subset \bigcup_{k\in I} I''_k$, where $I''_k := [a_k - 2^{-(k+1)}\epsilon, b_k)$. By finite additivity of μ we get

$$\begin{aligned}\mu([a, b-\epsilon/2)) \leq \mu\left(\bigcup_{k\in I} I''_k\right) &\leq \sum_{k\in I} \mu\left(I''_k\right) \\ &= \sum_{k\in I}\left(\mu(I_k) + \epsilon 2^{-(k+1)}\right) \leq \epsilon/2 + \sum_{k\in I}\mu(I_k).\end{aligned}$$

But since $\mu(A) = \mu([a, b-\epsilon/2)) + \epsilon/2$, we find $\mu(A) \leq \epsilon + \sum_{k\in I}\mu(I_k)$. Thus, there exists $m \in \mathbb{N}$ such that $\mu(A) \leq \epsilon + \sum_{n=1}^m \mu(A_n)$. But since ϵ was arbitrary we can conclude $\mu(A) \leq \sum_{n\in\mathbb{N}} \mu(A_n)$ and hence equality.

Exercise. Complete the proof. $\square$

Proposition 2.18. *Consider the real numbers with its σ-algebra $\mathcal{B}$ of Borel sets. Then, the prescription $\mu([a,b)) := b - a$ uniquely extends to a measure $\mu : \mathcal{B} \to [0,\infty]$.*

Proof. By Lemmas 2.15, 2.16 and 2.17 the prescription uniquely defines a measure μ on the algebra $\mathcal{N}$ of unions of intervals of the type $[a,b)$, $(-\infty,a)$, and $[a,\infty)$. By Theorem 2.12

μ extends to a σ-algebra $\mathcal{M}$ containing $\mathcal{N}$. But the σ-algebra generated by $\mathcal{N}$ is the σ-algebra $\mathcal{B}$ of Borel sets. (**<u>Exercise</u>.**Show this!) So, in particular, we get a measure on $\mathcal{B}$. By Proposition 2.13 this is unique since μ is σ-finite on $\mathcal{N}$. (**<u>Exercise</u>.**Show this latter statement!) ☐

Definition 2.19. The measure defined in the preceding Proposition is called the *Lebesgue measure* on $\mathbb{R}$.

<u>Exercise</u> 10. Consider the real numbers with the Lebesgue measure. Determine $\mu(\mathbb{Q})$ and $\mu(\mathbb{R} \setminus \mathbb{Q})$.

<u>Exercise</u> 11. The Cantor set C is a subset of the interval $[0,1]$. It can be described for example as

$$C = \bigcap_{n=0}^{\infty} \bigcup_{k=0}^{(3^n-1)/2} \left[\frac{2k}{3^n}, \frac{2k+1}{3^n}\right].$$

Show that $\mu(C) = 0$.

Proposition 2.20. *The Lebesgue measure is translation invariant, i.e.,* $\mu(A + c) = \mu(A)$ *for any measurable* A *and* $c \in \mathbb{R}$*.*

Proof. Straightforward. ☐

<u>Exercise</u> 12. Consider the following equivalence relation on $\mathbb{R}$: Let $x \sim y$ iff $x - y \in \mathbb{Q}$. Now choose (using the axiom of choice) one representative out of each equivalence class, such that this representative lies in $[0,1]$. Call the set obtained in this way A.

1. Show that $(A + r) \cap (A + s) = \emptyset$ if r and s are distinct rational numbers. Supposing that A is Lebesgue measurable, conclude that $\mu(A) = 0$.

2. Show that $\mathbb{R} = \bigcup_{q \in \mathbb{Q}} (A + q)$. Supposing that A is Lebesgue measurable, conclude that $\mu(A) > 0$.

We obtain a contradiction showing that A is not Lebesgue measurable.

We can define the Lebesgue measure more generally for $\mathbb{R}^n$. The intervals of the type $[a, b)$ are replaced by products of such intervals. Otherwise the construction proceeds in parallel.

Proposition 2.21. *Consider* $\mathbb{R}^n$ *with its* σ*-algebra* $\mathcal{B}$ *of Borel sets. Then, the prescription* $\mu([a_1, b_1) \times \cdots \times [a_n, b_n)) = (b_1 - a_1) \cdots (b_n - a_n)$ *uniquely extends to a measure* $\mu : \mathcal{B} \to [0, \infty]$*.*

<u>Exercise</u> 13. Sketch the proof by explaining the changes with respect to the one-dimensional case.

3 The integral

3.1 The integral of positive functions

Let X be a set and H a vector space of functions on X with values in $\mathbb{R}$. Denote by H^+ the subset of functions with values in $[0, \infty)$.

Definition 3.1. We say that a map $F : H \to \mathbb{R}$ is *positive* iff for all $f, g \in H$ with $f \geq g$ we have $F(f) \geq F(g)$.

Proposition 3.2. *A linear map $F : H \to \mathbb{R}$ is positive iff $f \in H$ with $f \geq 0$ implies $F(f) \geq 0$.*

The set $[0, \infty]$ carries a total order by declaring $a \leq \infty$ for all $a \in [0, \infty]$. When viewed as a topological space we will consider the topology of the one-point compactification of $[0, \infty)$. We declare the following rules for addition and multiplication, additional to the usual ones in $[0, \infty)$:

- $a + \infty = \infty$ for all $a \in [0, \infty]$.
- $0 \cdot \infty = 0$
- $a \cdot \infty = \infty$ for all $a \in (0, \infty]$.

Definition 3.3. We say that a map $F : H^+ \to [0, \infty]$ is *positive* iff for all $f, g \in H^+$ with $f \geq g$ we have $F(f) \geq F(g)$.

Definition 3.4. We say that a map $F : H^+ \to [0, \infty]$ is *semilinear* iff

- $F(\lambda f) = \lambda F(f)$ for $\lambda \in [0, \infty)$ and $f \in H^+$, and,
- $F(f + g) = F(f) + F(g)$ for $f, g \in H^+$.

Proposition 3.5. *A semilinear map $F : H^+ \to [0, \infty]$ is positive.*

Proof. **Exercise.** □

Proposition 3.6. *Consider a semilinear map $F : H^+ \to [0, \infty]$. Set $K^+ := F^{-1}([0, \infty))$ and $K := K^+ - K^+$ as a subset of H. Then, K is a vector subspace of H. Also, there is a unique positive linear map $F' : K \to \mathbb{R}$ such that $F'|_{K^+} = F|_{K^+}$.*

Proof. **Exercise.** □

Proposition 3.7. *Suppose that $f, g \in H$ implies $\sup(f, g) \in H$ and (equivalently) $\inf(f, g) \in H$. Then, $f, g \in K$ implies $\sup(f, g) \in K$ and $\inf(f, g) \in K$.*

Proof. Note that the equivalence between the conditions follows from $\inf(f,g) = -\sup(-f,-g)$. Moreover, note that the condition that $f,g \in H$ implies $\sup(f,g) \in H$ is equivalent to the apparently weaker condition that $f \in H$ implies $\sup(f,0) \in H$. This is because $\sup(f,g) = \sup(f-g,0)+g$. Now let $f \in K$. By definition there exist $f_1, f_2 \in K^+$ such that $f = f_1 - f_2$. But, as is easy to see $\sup(f,0) \le f_1$. So $\sup(f,0) \in K^+ \subset K$. This completes the proof. □

Let $(X,\mathcal{M})$ be a measurable space. We denote the vector space of measurable functions on X with values in $\mathbb{K}$ by $\mathcal{L}(X,\mathbb{K})$ and the subspace of simple functions by $\mathcal{S}(X,\mathbb{K})$. In the case $\mathbb{K} = \mathbb{R}$ we also simply write $\mathcal{L}(X)$ and $\mathcal{S}(X)$. We denote the subsets of functions with values in $[0,\infty)$ by $\mathcal{L}^+(X)$ and $\mathcal{S}^+(X)$ respectively. We call such functions *positive*. Note that in this sense 0 is a positive function.

Let $(X,\mathcal{M},\mu)$ be a measure space. We define in the following the *μ-integral*, or simply *integral*, which associates to certain measurable functions $f : X \to \mathbb{K}$ a value, denoted

$$f \mapsto \int_X f \,\mathrm{d}\mu.$$

When it is clear with respect to which measure the integral is taken, the symbol $\mathrm{d}\mu$ may be omitted. When the integral is taken with respect to the whole measure space and it is clear which measure space this is, the subscript indicating the set over which is integrated may be omitted.

Definition 3.8. The *integral* for positive simple functions is the map $\mathcal{S}^+(X) \to [0,\infty]$ defined as follows. Given $f \in \mathcal{S}^+(X)$ let $f(X) = \{a_1,\dots,a_n\}$ and $X_i := f^{-1}(a_i)$. Then,

$$\int_X f \,\mathrm{d}\mu := \sum_{i=1}^n a_i \mu(X_i).$$

Proposition 3.9. *The integrals $\mathcal{S}^+(Y) \to [0,\infty]$ for $Y \in \mathcal{M}$ are the unique collection of maps with the following properties:*

1. *They are semilinear maps.*
2. *$\int_Y 1 \,\mathrm{d}\mu = \mu(Y)$, where 1 is the constant function with value 1.*
3. *Let $Y_1, Y_2 \in \mathcal{M}$ such that $Y_1 \cap Y_2 = \emptyset$ and $Y = Y_1 \cup Y_2$. Then,*

$$\int_Y f \,\mathrm{d}\mu = \int_{Y_1} f \,\mathrm{d}\mu + \int_{Y_2} f \,\mathrm{d}\mu.$$

Proof. We first show unicity. Thus, we suppose that we are given an integral with the described properties. Let $f \in \mathcal{S}^+(X)$ and set $f(X) = \{a_1,\dots,a_n\}$ and $X_i := f^{-1}(a_i)$.

Iterating property (3), then applying semilinearity (1), then property (2), we recover the previous definition of the integral,

$$\int_X f\,\mathrm{d}\mu = \sum_{i=1}^n \int_{X_i} a_i \cdot 1\,\mathrm{d}\mu = \sum_{i=1}^n a_i \int_{X_i} 1\,\mathrm{d}\mu = \sum_{i=1}^n a_i \mu(X_i).$$

We turn to the proof of the different properties for the defined integral. Property (2) is immediate from the definition. We proceed to demonstrate property (3). Thus, let $f \in \mathcal{S}^+(Y)$ and set $f(Y) = \{a_1, \ldots, a_n\}$ and $X_i := f^{-1}(a_i)$. Then, the restrictions $f|_{Y_1}$ and $f|_{Y_2}$ take values in subsets of $\{a_1, \ldots, a_n\}$ and we have $X_i \cap Y_j = f|_{Y_j}^{-1}(a_i)$. Thus, we get from the definition of the integral,

$$\int_{Y_j} f\,\mathrm{d}\mu = \sum_{i=1}^n a_i \mu(X_i \cap Y_j).$$

Strictly speaking we should only sum over the values a_i actually occuring in Y_j. However, the summands for the additional values vanish since for these $X_i \cap Y_j = \emptyset$ and thus $\mu(X_i \cap Y_j) = 0$, so including them does not modify the sum. We then have,

$$\begin{aligned}\int_{Y_1} f\,\mathrm{d}\mu + \int_{Y_2} f\,\mathrm{d}\mu &= \sum_{i=1}^n a_i \left(\mu(X_i \cap Y_1) + \mu(X_i \cap Y_2)\right)\\ &= \sum_{i=1}^n a_i \mu((X_i \cap Y_1) \cup (X_i \cap Y_2)) = \sum_{i=1}^n a_i \mu(X_i) = \int_Y f\,\mathrm{d}\mu.\end{aligned}$$

We proceed to demonstrate property (1). We start with the multiplicative property of a semilinear map. Let $f \in \mathcal{S}^+(X)$, $\lambda \in [0, \infty)$ and $g = \lambda f$. If $\lambda = 0$ we have immediately $\int_X g\,\mathrm{d}\mu = 0$, as required. Suppose thus $\lambda \neq 0$. Set $f(X) = \{a_1, \ldots, a_n\}$ and $X_i := f^{-1}(a_i)$. Then $g(X) = \{\lambda a_1, \ldots, \lambda a_n\}$. Note that the values λa_i are all distinct. By definition we then have, as required,

$$\int_X g\,\mathrm{d}\mu = \sum_{i=1}^n \lambda a_i \mu(X_i) = \lambda \sum_{i=1}^n a_i \mu(X_i) = \lambda \int_X f\,\mathrm{d}\mu.$$

It remains to show additivity of the integral. Thus, let $f, g \in \mathcal{S}^+(X)$. Set $f(X) = \{a_1, \ldots, a_n\}$ and $X_i := f^{-1}(a_i)$ as well as $g(X) = \{b_1, \ldots, b_m\}$ and $Y_j := g^{-1}(b_j)$. Define $Z_{ij} = X_i \cap Y_j$. Note that the Z_{ij} form a disjoint partition of X. Moreover the function $f + g$ takes the constant value $a_i + b_j$ on Z_{ij}. Using property (3) and the definition of the integral we get,

$$\begin{aligned}\int_X (f+g)\,\mathrm{d}\mu &= \sum_{i,j}\int_{Z_{ij}}(f+g)\,\mathrm{d}\mu = \sum_{i,j}(a_i+b_j)\mu(Z_{ij})\\ &= \sum_{i,j} a_i\mu(Z_{ij}) + \sum_{i,j} b_j\mu(Z_{ij}) = \sum_{i=1}^n a_i\mu(X_i) + \sum_{j=1}^m b_j\mu(Y_j)\\ &= \int_X f\,\mathrm{d}\mu + \int_X g\,\mathrm{d}\mu.\end{aligned}$$

This completes the proof. □

Definition 3.10. The *μ-integral* for positive measurable functions is the map $\mathcal{L}^+(X) \to [0,\infty]$ defined as follows. Given $f \in \mathcal{L}^+(X)$,

$$\int_X f\,\mathrm{d}\mu := \sup\left\{\int_X g\,\mathrm{d}\mu : 0 \le g \le f \text{ and } g \in \mathcal{S}^+(X)\right\}$$

The coincidence of this definition with the previous one in the case of simple maps is implied by the following result.

Proposition 3.11. *The integrals $\mathcal{L}^+(Y) \to [0,\infty]$ for $Y \in \mathcal{M}$ are a collection of maps with the following properties:*

1. *They coincide with Definition 3.8 for simple maps.*
2. *They are multiplicative, i.e. $\int_Y \lambda f \mathrm{d}\mu = \lambda \int_Y f \mathrm{d}\mu$ for $\lambda \in [0,\infty)$.*
3. *They are positive, i.e. $\int_Y f\,\mathrm{d}\mu \le \int_Y g\,\mathrm{d}\mu$ if $f \le g$.*
4. *Let $Y_1, Y_2 \in \mathcal{M}$ such that $Y_1 \cap Y_2 = \emptyset$ and $Y = Y_1 \cup Y_2$. Then,*

$$\int_Y f\,\mathrm{d}\mu = \int_{Y_1} f\,\mathrm{d}\mu + \int_{Y_2} f\,\mathrm{d}\mu.$$

Proof. **Exercise.** □

Theorem 3.12 (Monotone Convergence Theorem)**.** *Let $\{f_n\}_{n\in\mathbb{N}}$ be an increasing sequence of positive measurable functions on X that converges pointwise to a function $f : X \to [0,\infty)$. Then, f is measurable and*

$$\lim_{n\to\infty}\int_X f_n\,\mathrm{d}\mu = \int_X f\,\mathrm{d}\mu.$$

Proof. The measurability of f follows from Theorem 1.60. We denote,

$$b := \lim_{n\to\infty}\int_X f_n\,\mathrm{d}\mu.$$

Since $f_n \leq f$ for all $n \in \mathbb{N}$ positivity of the integral implies,

$$b \leq \int_X f \, \mathrm{d}\mu.$$

Let $g \in \mathcal{S}^+(X)$ such that $0 \leq g \leq f$ and choose $0 < c < 1$. Define $E_n \in \mathcal{M}$ as,

$$E_n := \{x \in X : f_n(x) \geq cg(x)\}$$

Then $\{E_n\}_{n\in\mathbb{N}}$ is an increasing sequence of measurable subsets of X with $X = \bigcup_{n \in N} E_n$. Moreover, for any $n \in \mathbb{N}$,

$$\int_X f_n \, \mathrm{d}\mu \geq \int_{E_n} f_n \, \mathrm{d}\mu \geq c \int_{E_n} g \, \mathrm{d}\mu.$$

The limit $n \to \infty$ exits on both sides. On the left hand side this is b. To see the limit on the right hand side set $g(X) = \{a_1, \ldots, a_m\}$ and $X_i := g^{-1}(\{a_i\})$. Then, we have

$$\int_{E_n} g \, \mathrm{d}\mu = \sum_{i=1}^{m} a_i \mu(E_n \cap X_i).$$

But $\mu(E_n \cap X_i) \to \mu(X_i)$ as $n \to \infty$ by Proposition 2.3.3. We obtain,

$$b \geq c \int_X g \, \mathrm{d}\mu.$$

But c was arbitrary, so the inequality is valid without c. On the other hand, by definition of the integral of f as a supremum of intgerals of simple functions g we obtain,

$$b \geq \int_X f \, \mathrm{d}\mu.$$

Combining both inequalities yields the desired equality. □

Proposition 3.13. *The integral $\mathcal{L}^+(Y) \to [0, \infty]$ is a semilinear map.*

Proof. It remains to show additivity. **Exercise.**Hint: Use approximability from below by simple functions (Theorem 1.64) and apply the Monotone Convergence Theorem 3.12. □

Lemma 3.14 (Fatou's Lemma)**.** *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of positive measurable functions on X such that $f := \liminf_{n\to\infty} f_n$ takes only finite values. Then, f is measurable and*

$$\int f \leq \liminf_{n\to\infty} \int f_n.$$

Proof. Define $g_n := \inf_{k \geq n} f_k$. Note that g_n is the limit $m \to \infty$ of the decreasing sequence of measurable functions $\{h_{n,m}\}_{m \geq n}$ defined as $h_{n,m} := \inf\{f_k : n \leq k \leq m\}$, so it is measurable. The sequence $\{g_n\}_{n \in \mathbb{N}}$ is increasing. Moreover, $f = \lim_{n \to \infty} g_n$. So, by the Monotone Convergence Theorem 3.12 we have,

$$\lim_{n \to \infty} \int g_n = \int f.$$

On the other hand, by definition of g_n and positivity of the integral we have,

$$\int g_n \leq \int f_k \quad \forall k \geq n.$$

This implies,

$$\int g_n \leq \inf_{k \geq n} \int f_k.$$

Taking the limit yields,

$$\lim_{n \to \infty} \int g_n \leq \liminf_{n \to \infty} \int f_n.$$

This completes the proof. $\square$

3.2 Integrable functions

Let $\mathcal{L}^{1+}(X, \mu)$ denote the subset of $\mathcal{L}^+(X)$ such that the integral is finite,

$$\mathcal{L}^{1+}(X, \mu) := \left\{ f \in \mathcal{L}^+(X) : \int_X f \, \mathrm{d}\mu < \infty \right\}.$$

Define now $\mathcal{L}^1(X, \mu) := \mathcal{L}^{1+}(X, \mu) - \mathcal{L}^{1+}(X, \mu)$. By Proposition 3.6 $\mathcal{L}^1(X, \mu)$ is a vector space and we obtain a uniquely defined positive linear map

$$\int_X : \mathcal{L}^1(X, \mu) \to \mathbb{R}.$$

We call $\mathcal{L}^1(X, \mu)$ the space of *integrable* functions. Note also that given $f, g \in \mathcal{L}^1(X, \mu, \mathbb{R})$, $\sup(f, g)$ and $\inf(f, g)$ are measurable by Proposition 1.59 and integrable by Proposition 3.7.

We may now extend the notion of integral to functions that take values in the complex numbers rather than the real numbers. A further extension to functions taking values in Banach spaces over $\mathbb{R}$ or $\mathbb{C}$ is also straightforward, but we shall not consider this here.

We define the complex vector space $\mathcal{L}^1(X, \mu, \mathbb{C}) := \mathcal{L}^1(X, \mu) + \mathrm{i}\mathcal{L}^1(X, \mu)$ of *integrable* complex valued functions. The integral is extended from $\mathbb{R}$ to $\mathbb{C}$ by complex linearity. For $f = f_R + \mathrm{i} f_I$ with $f_R, f_I \in \mathcal{L}^1(X, \mu)$ we define,

$$\int_X f \, \mathrm{d}\mu := \int_X f_R \, \mathrm{d}\mu + \mathrm{i} \int_X f_I \, \mathrm{d}\mu.$$

We also write $\mathcal{L}^1(X, \mu, \mathbb{R}) := \mathcal{L}^1(X, \mu)$ and $\mathcal{L}^1(X, \mu, \mathbb{K})$ if we want to make statements valid for both cases $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$.

Proposition 3.15. *We summarize basic properties of the integral.*

1. *The integral is a positive linear function.*

2. $f, g \in \mathcal{L}^1(X, \mu, \mathbb{R})$ *implies* $\sup(f, g), \inf(f, g) \in \mathcal{L}^1(X, \mu, \mathbb{R})$.

3. *Let* $Y_1, Y_2 \in \mathcal{M}$ *such that* $Y_1 \cap Y_2 = \emptyset$ *and* $Y = Y_1 \cup Y_2$. *Then, for* $f \in \mathcal{L}^1(Y, \mu, \mathbb{K})$,

$$\int_Y f \, \mathrm{d}\mu = \int_{Y_1} f \, \mathrm{d}\mu + \int_{Y_2} f \, \mathrm{d}\mu.$$

Proof. It remains to demonstrate the validity of (3). This follows from linearity upon decomposing f into a linear combination of positive integrable functions and Proposition 3.11.(3). □

Theorem 3.16. *Let* $f \in \mathcal{L}(X, \mathbb{K})$. *Then* f *is integrable iff* $|f| \in \mathcal{L}^+(X)$ *is integrable.*

Proof. Without loss of generality take $\mathbb{K} = \mathbb{C}$. Suppose that $f \in \mathcal{L}(X, \mathbb{C})$. Let $f_{\mathrm{R}} = \Re(f)$ and $f_{\mathrm{I}} = \Im(f)$. Define $f_{\mathrm{R}}^+ = \sup(f_{\mathrm{R}}, 0)$ and $f_{\mathrm{R}}^- = \sup(-f_{\mathrm{R}}, 0)$. Similarly, $f_{\mathrm{I}}^+ = \sup(f_{\mathrm{I}}, 0)$ and $f_{\mathrm{I}}^- = \sup(-f_{\mathrm{I}}, 0)$. Note that all these component functions are positive and $f = f_{\mathrm{R}}^+ - f_{\mathrm{R}}^- + \mathrm{i} f_{\mathrm{I}}^+ - \mathrm{i} f_{\mathrm{I}}^-$. Now suppose that f is integrable. Then, by definition both f_{R} and f_{I} are integrable. Moreover, f_{R}^+, f_{R}^-, f_{I}^+, f_{I}^- are all integrable and so is their sum. The inequality

$$|f| \le |f_{\mathrm{R}}| + |f_{\mathrm{I}}| = f_{\mathrm{R}}^+ + f_{\mathrm{R}}^- + f_{\mathrm{I}}^+ + f_{\mathrm{I}}^-$$

implies the integrability of $|f|$. Conversely, suppose that $|f|$ is integrable. But all of f_{R}^+, f_{R}^-, f_{I}^+, f_{I}^- are smaller or equal to $|f|$, so they are all integrable. So is thus their linear combination f. □

Proposition 3.17. *The space of integrable simple functions* $\mathcal{S}^1(X, \mu, \mathbb{K})$ *is precisely the space of simple functions that vanish outside of a set of finite measure.*

Proof. **Exercise.** □

Theorem 3.18. *For* $f \in \mathcal{L}^1(X, \mu, \mathbb{K})$, $|\int f| \le \int |f|$.

Proof. Let $c \in \mathbb{K}$ with $|c| = 1$ such that $|\int f| = c \int f$. Then,

$$\left| \int f \right| = c \int f = \int c f = \int \Re(c f) \le \int |c f| = \int |f|.$$

□

We say "almost everywhere" to mean "outside a set of measure zero".

Theorem 3.19 (Averaging Theorem)**.** *Let X be a measure space with σ-finite measure μ. Let $S \subseteq \mathbb{K}$ be a closed subset and $f \in \mathcal{L}^1(X, \mu, \mathbb{K})$. If for every measurable set A of finite and positive measure we have*

$$\frac{1}{\mu(A)} \int_A f \mathrm{d}\mu \in S,$$

then $f(x) \in S$ for almost all $x \in X$.

Proof. Let $C := \{x \in X : f(x) \notin S\}$. We need to show that $\mu(C) = 0$. Assume the contrary, i.e., $\mu(C) > 0$. Write $\mathbb{K} \setminus S = \bigcup_{n\in\mathbb{N}} B_n$ as a countable union of closed balls $\{B_n\}_{n\in\mathbb{N}}$. (Use second countability of $\mathbb{K}$ and recall Proposition 1.51.) Their preimages are measurable and cover C. There is at least one closed ball B_n such that $\mu(f^{-1}(B_n)) > 0$. Say this closed ball has center x and radius r. Furthermore, there is a measurable subset $D \subseteq f^{-1}(B_n)$ such that $0 < \mu(D) < \infty$. Then,

$$\left|\frac{1}{\mu(D)} \int_D f \,\mathrm{d}\mu - x\right| = \frac{1}{\mu(D)} \left|\int_D (f - x) \,\mathrm{d}\mu\right|$$
$$\leq \frac{1}{\mu(D)} \int_D |f - x| \,\mathrm{d}\mu \leq \frac{1}{\mu(D)} \int_D r \,\mathrm{d}\mu = r.$$

In particular, $\frac{1}{\mu(D)} \int_D f \,\mathrm{d}\mu \in B_n$. But $B_n \cap S = \emptyset$, so we get a contradiction with the assumptions. ☐

Exercise 14. 1. Explain where in the above proof σ-finiteness was used. 2. Extend the proof to the case where μ is not σ-finite by replacing $f(x) \in S$ with $f(x) \in S \cup \{0\}$ in the statement of the Theorem.

Exercise 15 (Fundamental Theorem of Differentiation and Integration)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and $a, b \in \mathbb{R}$ with $a \leq b$. Then,

$$\int_a^b f' \,\mathrm{d}\mu = f(b) - f(a),$$

where μ is the Lebesgue measure. [Hint: Note that f' is integrable on $[a, b]$. Consider the map $g : \mathbb{R} \to \mathbb{R}$ given by $g(y) := \int_a^y f' \,\mathrm{d}\mu$. Show that g is continuously differentiable and that $g' = f'$. Apply the fact that a function with vanishing derivative is constant to the difference $f - g$ to conclude the proof.]

Exercise 16 (Partial Integration)**.** Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and $a, b \in \mathbb{R}$ with $a \leq b$. Show that,

$$\int_a^b f g' \,\mathrm{d}\mu = f g|_a^b - \int_a^b f' g \,\mathrm{d}\mu,$$

where $\mathrm{d}\mu$ is the Lebesgue measure.