LECTURE 4: Towards Quantum Gravity

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Towards Quantum Gravity

In last week's lecture we have seen the GBF in the context of **quantum field theory with a fixed metric background**. However, part of the appeal of the GBF lies in its applicability in the **absence of a metric background**. Thus, today's lecture will be about some aspects of doing **quantum gravity** based on the GBF.

1. Extracting observable quantities
   - Transition amplitudes?
   - Graviton scattering
2. Approaches to quantum gravity
   - The spin foam approach
   - A top-down approach

Transition amplitudes (I)

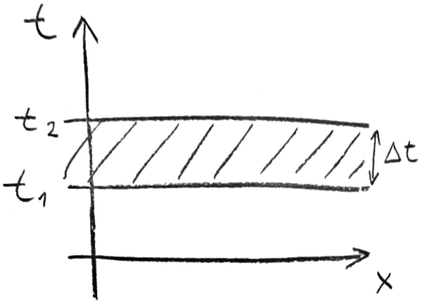
Do transition amplitudes between "spacelike hypersurfaces" make sense in quantum gravity?

Try to uphold the following assumptions:

- There is a **semiclassical regime**:
  This appears to be necessary to reproduce known physics.
- **Transition amplitudes** do make sense:
  This is problematic because of the locality problem.

Transition amplitudes (II)

Consider transition amplitudes in a background spacetime.



We prepare a state ψ at t_1, wait for a time Δt, then measure if we obtain the state η at t_2. The probability for this depends on Δt:

$$P = |\langle \eta | U(\Delta t) | \psi \rangle|^2$$

Recall properties of U:

- Composition: $U(\Delta t) U(\Delta t') = U(\Delta t + \Delta t')$
- Unitarity: $U^\dagger = U^{-1}$

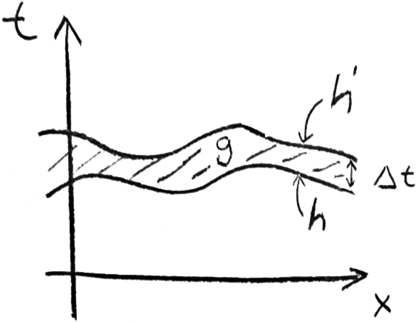
Transition amplitudes (III)

In quantum gravity no background time is available on which the "evolution operator" U can depend. By composition, we expect $U^2 = U$. Then unitarity yields $U = 1$. The transition probability from ψ to η should hence be merely the inner product:

$$P = |\langle\eta|\psi\rangle|^2$$

For this to make sense, the operational information about the time difference Δt must be contained in the states η, ψ.

In the **classical** theory this looks reasonable.

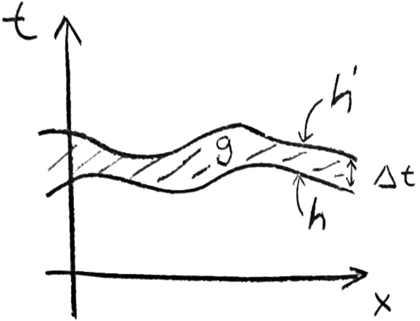


Thick sandwich conjecture

Given an initial 3-metric h and a (similar) final 3-metric h' there is generically one interpolating 4-metric g (up to equivalence).

Transition amplitudes (V)

In the **semiclassical** theory this no longer makes sense.



Suppose ψ_h and $\psi_{h'}$ are semiclassical states associated to h and h'.
The transitions

- $\psi_h \longrightarrow \psi_{h'}$
- $\psi_h \longrightarrow \psi_h$

are both **certain**.

So, the transition probabilities are

$$|\langle\psi_{h'}|\psi_h\rangle|^2 = |\langle\psi_h|\psi_h\rangle|^2 = 1$$

This implies $\psi_h = \psi_{h'}$ up to a phase. But, h and h' are generally physically different (not related by a 3-diffeomorphism).

Transition amplitudes (VI)

Conclusion

Transition amplitudes between semiclassical states in quantum gravity do not seem to make sense.

At least one of the underlying assumptions must be wrong.

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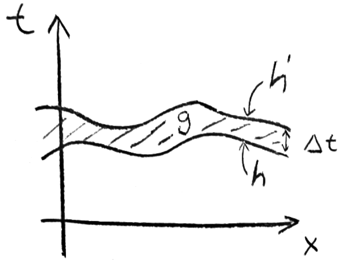
Can the GBF help here?

Yes, because the meaning of **“semiclassical sector”** is different.

Transition amplitudes (VII)



We suppose that the boundary state space $\mathcal{H}_{\partial M} = \mathcal{H}_{\text{initial}} \otimes \mathcal{H}_{\text{final}}$ has a sector $\mathcal{H}_{\text{class}}$ that approximately describes a fixed classical spacetime g with weak perturbations, i.e., $\mathcal{H}_{\partial M} = \mathcal{H}_{\text{class}} \oplus \mathcal{H}_{\text{rest}}$.

$\mathcal{H}_{\text{class}}$ "knows" about g. But, it does not factorize with respect to $\mathcal{H}_{\text{initial}} \otimes \mathcal{H}_{\text{final}}$. The previous contradiction is avoided.

Conclusion

The assumption that the state spaces $\mathcal{H}_{\text{initial}}$ and $\mathcal{H}_{\text{final}}$ individually carry **semiclassical sectors** is wrong. Only the **joint boundary state space** $\mathcal{H}_{\partial M}$ might carry such a sector $\mathcal{H}_{\text{class}}$. States in it will generically be complicated linear combinations of tensor products of initial and final states.

On graviton scattering (I)

As an example of meaningful probabilities in quantum gravity we consider a generic **graviton scattering** problem. [RO 2006]

Consider a ball shaped region in spacetime, $M = B^4$, with boundary $\partial M = S^3$. Suppose the state space $\mathcal{H}_{\partial M}$ has a **sector** $\mathcal{H}_{\text{lin}}$ that (approximately) describes **gravitons** on **Minkowski spacetime**, i.e., $\mathcal{H}_{\partial M} = \mathcal{H}_{\text{lin}} \oplus \mathcal{H}_{\text{rest}}$.

Supposing that the dynamics in $\mathcal{H}_{\text{lin}}$ is near to that of a free field theory we can decompose $\mathcal{H}_{\text{lin}} = \mathcal{H}_{\text{lin,in}} \otimes \mathcal{H}_{\text{lin,out}}$. Consider then a scattering process with $\psi_{\text{in}} \in \mathcal{H}_{\text{lin,in}}$ describing the **in-particles** (prepared) and $\psi_{\text{out}} \in \mathcal{H}_{\text{lin,out}}$ describing the **out-particles** (to be measured). Then, the scattering probability p is,

$$p = \frac{\sum_i |\rho_M(P_{\text{lin,in+out}}\xi_i)|^2}{\sum_i |\rho_M(P_{\text{lin,in}}\xi_i)|^2} \sim |\rho_M(\psi_{\text{in}} \otimes \psi_{\text{out}})|^2 \sim \text{``}|\langle \psi_{\text{out}}, \psi_{\text{in}} \rangle|^2\text{''}.$$

On graviton scattering (II)

Remarks

- The present setting may provide a physical interpretation to a "graviton propagator" in spin foam models. [C. Rovelli 2005; E. Bianchi, L. Modesto, C. Rovelli, S. Speziale 2006; . . .]
- It is important to remember that $|\rho_M(\psi)|^2$ for some $\psi \in \mathcal{H}_{\partial M}$ does not in general have the interpretation of a probability. This is true here only due to special circumstances and only approximately.
- The detailed results will depend on how exactly we choose $\mathcal{H}_{\text{lin}}$ in $\mathcal{H}_{\partial M}$, in which way it approximates a **Fock space**, up to which energies, etc. These ambiguities might be related to the renormalization ambiguities of perturbative quantum gravity.

Approaches to quantum gravity

Approaches to quantum gravity naturally based on the GBF:

1. Perturbative quantum gravity: This depends on the integration of QFT with the GBF. If finite regions can be described successfully, this might yield new insight into this approach. But there is no reason to expect improvement of the **non-renormalizability** issue.
2. Spin foam quantum gravity: **Spin foam models** arise naturally from a **path integral** picture. Also, they naturally describe **finite regions** of spacetime. This suggests their interpretation as background independent quantum theories in terms of the GBF.
3. A functorial top-down approach: The mathematical structure of TQFT that is part of the GBF also suggests a top-down approach: Guided by **axiomatics**, **functoriality**, and **representation theory** and with a minimum of assumptions explore the **theory space**.

We start with the Palatini action of gravity,

$$S_M^{\mathrm{Palatini}}(e, A) = \int_M \mathrm{tr}(e \wedge e \wedge F).$$

- A – connection with gauge group $\mathrm{Spin}(1,3) = \mathrm{SL}(2, \mathbb{C})$
- F – curvature 2-form of the connection A
- e – 4-bein frame field

To simplify this theory we replace $e \wedge e$ with the Lie algebra valued 2-form field B. This yields BF theory,

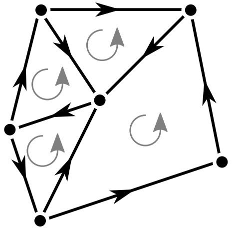
$$S_M^{\mathrm{BF}}(B, A) = \int_M \mathrm{tr}(B \wedge F).$$

This is not gravity, but becomes gravity if we add certain **constraints**.

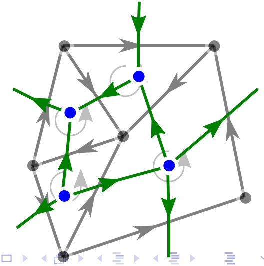
Discretized connections I

BF theory is much simpler than gravity and can be quantized explicitly. It tuns out that the B-field can be integrated out so we only need to consider configurations of the connection field A.

To make the "space of connections" on the hypersurface Σ more manageable, we discretize Σ via a **cellular decomposition**.

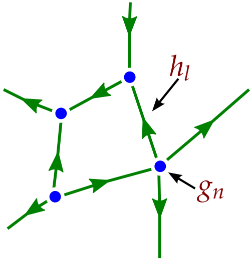


Given a "gauge" (local trivialization), connections give rise to **holonomies** along paths. We choose paths dual to the cellular decomposition. We call them **links** (green lines). Their end points are **nodes** (blue dots).

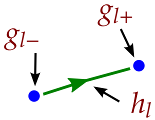


Discretized connections II

The holonomies associate one element h_l of the structure group G to each link l. We denote this space by $K^1_\Sigma = G^L$, where L is the number of links in Σ.



A gauge transformation consists of the assignment of one element g_n of G to each node n. The gauge group is thus $K^0_\Sigma = G^N$, where N is the number of nodes.



A gauge transformation $g \in K^0_\Sigma$ acts on $h \in K^1_\Sigma$ via $(g \triangleright h)_l := g_{l+} h_l g_{l-}^{-1}$. The **configuration space** is the quotient $K_\Sigma := K^1_\Sigma / K^0_\Sigma$.

State space

Supposing that G is compact for simplicity, there is a unique normalized biinvariant measure on G, the **Haar measure** μ. This allows to define a Hilbert space $\mathrm{L}^2(G)$ of complex functions on G with the inner product,

$$\langle \psi, \eta \rangle = \int_G \overline{\psi(g)} \eta(g) \, \mathrm{d}\mu(g).$$

By putting the same inner product on each copy of G, we obtain a Hilbert space $\mathcal{H}^1_\Sigma := \mathrm{L}^2(K^1_\Sigma)$. The action of the gauge group K^0_Σ on K^1_Σ induces an action on $\mathcal{H}^1_\Sigma$. The subspace $\mathcal{H}_\Sigma \subseteq \mathcal{H}^1_\Sigma$ of invariant functions on K^1_Σ can be identified with a space of functions on the configuration space K_Σ. This Hilbert space is our **state space**.

Propagator

Recall that in Schrödinger-Feynman quantization amplitudes are determined by propagators.

$$\rho_M(\psi) = \int_{K^1_\Sigma} \psi(h)\, Z_M(h^{-1})\, \mathrm{d}\mu(h)$$

Here, it is simpler to think of the **propagator** as a function $Z_M : K^1_{\partial M} \to \mathbb{C}$ rather than a function $K_{\partial M} \to \mathbb{C}$.

For BF theory the propagator turns out to be,

$$\tilde{Z}^{\mathrm{BF}}_M(h) = \prod_{l \in \partial M} \delta(h_l).$$

In gauge invariant form this is,

$$Z^{\mathrm{BF}}_M(h) = \int_{K^0_{\partial M}} \prod_{l \in \partial M} \delta(g_{l-} h_l g_{l+}^{-1})\, \mathrm{d}\mu(g).$$

Other models

If we want to get closer to gravity and implement **constraints** it is useful to discretize also the interior of M via a cellular decomposition. We may then think of each cell in the interior as an "elementary" spacetime region, all glued together according to the **gluing axioms** of the GBF. That is, to specify a model we only need to specify the **cell propagator** for one single cell.

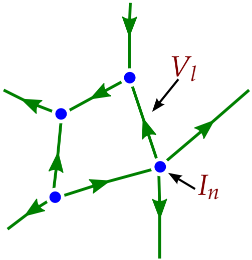
A famous model for implementing the constraints is the **Barrett-Crane model**. In this model $G = \mathrm{SU}(2) \times \mathrm{SU}(2)$ and we write $g = (g^{\mathrm{L}}, g^{\mathrm{R}})$. The cell propagator for (a version of) this model is,

$$Z_C^{\mathrm{BC}}(h) = \int_{K^0_{\partial C}} \prod_{l \in \partial C} \left(\int_{\mathrm{SU}(2)\times\mathrm{SU}(2)} \delta(g^{\mathrm{L}}_{l-} k h^{\mathrm{L}}_l k' (g^{\mathrm{L}}_{l+})^{-1}) \delta(g^{\mathrm{R}}_{l-} k h^{\mathrm{R}}_l k' (g^{\mathrm{R}}_{l+})^{-1}) \, \mathrm{d}\mu(k) \mathrm{d}\mu(k') \right) \mathrm{d}\mu(g).$$

The dual picture: spin networks

Elements of the Hilbert space $\mathcal{H}_\Sigma$ on the discretized hypersurface Σ can be constructed explicitly in terms of **spin networks**.

- Associate to each link l a finite-dimensional irreducible representation V_l of G.
- Associate to each node n an intertwiner $I_n \in \mathrm{Inv}\left(\bigotimes_{l \in \partial n} V_l^{\pm}\right)$ between the representations of the adjacent nodes.



Spin networks yield a complete description of $\mathcal{H}_\Sigma$:

$$\mathcal{H}_\Sigma = \bigoplus_{V_l} \bigotimes_{n \in \Sigma} \mathrm{Inv}\left(\bigotimes_{l \in \partial n} V_l^{\pm} \right).$$

The dual picture: spin foams

In order to obtain the amplitude for a region M composed of many elementary regions (cells) we need to sum over a complete ON-basis for each hypersurface where cells are glued together. (Recall GBF gluing rule.) Taking basis consisting of spin networks, each summand will by labeled by an assignment of a spin network to each of these interior hypersurfaces. We can think of those spin networks as extended through all the interior of M. Links then become surfaces and nodes become lines where the surfaces meet. Surfaces are labeled by irreducible representations and lines by intertwiners. This picture is what is usually called a **spin foam**. The vertices where the lines meet are dual to the cells. Thus, the cell amplitudes ρ_C, evaluated on spin networks, are usually called **vertex amplitudes**.

Spin foam summary

- Spacetime **hypersurfaces** are certain 3-dimensional cell complexes (e.g. **triangulations**)
- Spacetime **regions** are certain 4-dimensional cell complexes (e.g. **triangulations**)
- **Gauge fields** on hypersurfaces are encoded in terms of **holonomies** between cells
- The **state spaces** on hypersurfaces can be described in terms of **spin networks** (with ends!)
- A simple spin foam model is completely determined by its **cell (vertex) amplitudes**
- Spin foam **partition functions**, **amplitudes** etc. then follow from the **GBF axioms**
- Complicated spin foam models might involve additional data on hypersurfaces corresponding to additional (non-gauge) fields

A top-down approach to quantum gravity

List properties expected of a quantum theory of gravity and **construct/classify** models with these properties.

- In GR the metric is dynamical, but the differentiable structure may be fixed. To describe local physics, we should at least admit ball-shaped regions.

→ Consider a class of oriented compact **differentiable 4-manifolds** with boundary. The class must include 4-balls and be closed under gluing. These are our admissible **regions**.

→ Admissible **hypersurfaces** are boundaries of regions and their connected components. (These hypersurfaces carry in addition the structure of an "infinitesimal 4-manifold neighborhood".)

→ To each hypersurface Σ we must associate a **separable Hilbert space** $\mathcal{H}_\Sigma$ and to each region M an **amplitude map** $\rho_M : \mathcal{H}_{\partial M} \to \mathbb{C}$.

→ These structures have to satisfy the **axioms**. Gluings have to be compatible with the extra structure of the 3-manifolds.

First refinement: Symmetry

- **Diffeomorphisms** are **gauge symmetries** of GR.
- On each **region** M acts its group of orientation preserving diffeomorphisms G_M.
- On each **hypersurface** Σ acts its group of orientation preserving diffeomorphisms G_Σ. (Note that this must also act on the neighborhood structure.) This induces $i_M : G_M \to G_{\partial M}$.
- Let $G_M^{\text{int}} \subseteq G_M$ be the subgroup that acts identically on the boundary. We have the exact sequence

$$G_M^{\text{int}} \to G_M \to G_{\partial M}$$

- For each hypersurface Σ, G_Σ must act on $\mathcal{H}_\Sigma$ by unitary transformations, i.e., $\mathcal{H}_\Sigma$ is a **unitary representation** of G_Σ. This action must be compatible with decompositions.
- For each region M, ρ_M must be **invariant** under $i_M(G_M)$. That is, $\rho_M(g \triangleright \psi) = \rho_M(\psi)$ for any $\psi \in \mathcal{H}_{\partial M}$ and $g \in i_M(G_M) \subseteq G_{\partial M}$.

Second refinement: Corners

We admit more general 3-manifolds as hypersurfaces which may themselves have boundaries. To this end we introduce a generalized notion of the decomposition of a hypersurface: [RO 2006]

generalized decomposition

Let Σ be a hypersurface. A decomposition of Σ is a presentation as a finite union of hypersurfaces $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_n$ such that: (1) each Σ_i is closed in Σ and (2) the intersection of any Σ_i with any Σ_j is contained in their boundaries.

We correspondingly extend the notion of gluing of regions in such a way that regions may be glued along components of boundaries. This has the advantage that all amplitude maps are completely determined if the amplitude maps for ball-shaped regions are given.

Third refinement: Projectivity

It is well know that representations of symmetry groups on the Hilbert space in quantum mechanics only have to be **projective representations**. This is related to the fact that what has to be preserved under symmetries are only measurable quantities like probabilities and expectation values. The same is true in the general boundary formulation. In light of this the previously mentioned implementation of symmetries may be relaxed accordingly.