LECTURE 3: Quantization and QFT

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Outline

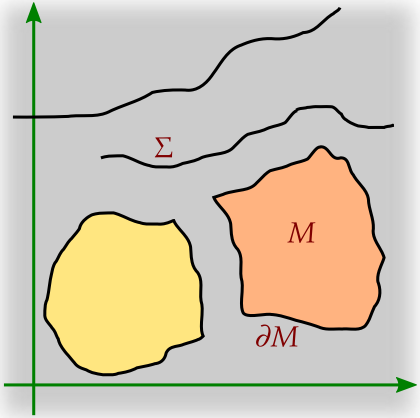
The quantization problem

Quantum theories are often constructed by applying a **quantization scheme** to a **classical theory**. Standard quantization schemes are designed to output the ingredients of the **standard formulation**, i.e., a Hilbert space and operators on it. Instead we need quantization schemes that output the ingredients of the GBF: a Hilbert space per hypersurface, an amplitude map per region, and possibly observable maps (not considered in this talk).

Recall from last lecture that the axiomatic framework of the GBF is inspired by the **Feynman path integral**. As a consequence, the **Schrödinger-Feynman quantization** scheme can be easily adapted to the GBF.

Recall: Spacetime

Spacetime is modeled by a collection of **hypersurfaces** and **regions**.



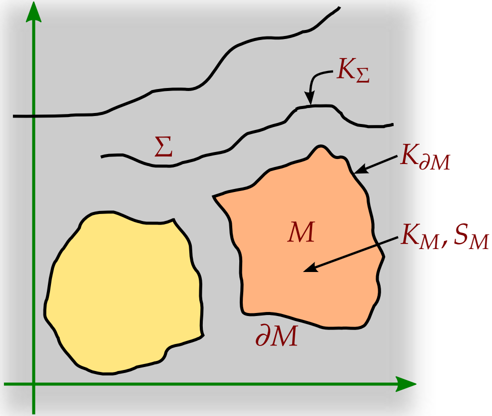
Spacetime regions are the arena for **local physics**.

"Holography"

Information about local physics is **communicated** between adjacent regions through **interfacing hypersurfaces** (channels).

Classical data (I)

A **classical field theory** needs to be encoded in terms of data associated to these geometric structures (**hypersurfaces** and **regions**).



per hypersurface Σ :

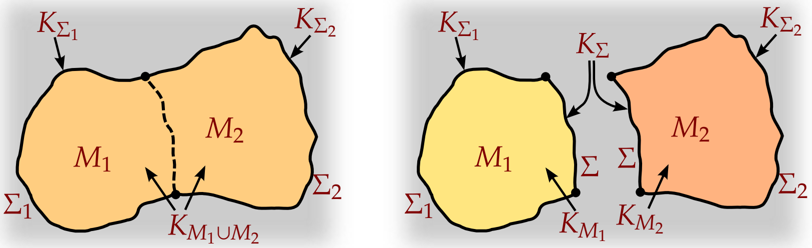
a space K_Σ of **field configurations** on Σ,

per region M :

a space K_M of **field configurations** in M and a function $S_M : K_M \to \mathbb{R}$, the **action**.

Classical data (II)

The **classical data** satisfy a number of **axioms** concerning decomposition of hypersurfaces, gluing of regions etc. These follow "automatically" from the field theoretic setup.



For example, given a **gluing** we have the **exact sequence**:

$$K_{M_1 \cup M_2} \to K_{M_1} \times K_{M_2} \rightrightarrows K_\Sigma$$

We also have **additivity of the action** under **composition of spacetime regions** (recall last lecture) etc.

Schrödinger-Feynman quantization: hypersurfaces

In the **Schrödinger representation** states are **wave functions** on field configurations. The state space $\mathcal{H}_\Sigma$ for the hypersurface Σ is the space of complex functions on K_Σ with inner product,

$$\langle \psi', \psi \rangle_\Sigma = \int_{K_\Sigma} \mathcal{D}\varphi \, \overline{\psi'(\varphi)} \psi(\varphi).$$

Here, $\mathcal{D}\varphi$ is a **translation invariant measure** on K_Σ.

Such a measure does not really exist in most cases. As usual, it is fruitful to proceed pretending that it does.

Schrödinger-Feynman quantization: regions

The **Feynman path integral** serves to define the **field propagator** $Z_M : K_{\partial M} \to \mathbb{C}$ in a spacetime region M,

$$Z_M(\varphi) = \int_{\phi \in K_M, \phi|_{\partial M} = \varphi} \mathcal{D}\phi \, e^{\mathrm{i} S_M(\phi)}.$$

Here, $\mathcal{D}\phi$ is a **translation invariant measure** on K_M.

The **amplitude map** ρ_M is then,

$$\rho_M(\psi) = \int_{K_{\partial M}} \mathcal{D}\varphi \, \psi(\varphi) Z_M(\varphi).$$

These quantum data "automatically" satisfy the axioms of the GBF.

Again, the measure $\mathcal{D}\phi$ does not actually exist in most cases.

We consider a **real scalar field** theory in **Minkowski spacetime** with the action

$$S_M(\phi) = \frac{1}{2}\int \mathrm{d}^4x \left((\partial_\mu\phi)\partial^\mu\phi - m^2\phi^2\right).$$

The equations of motion are given by the **Klein-Gordon equation**:

$$(\Box + m^2)\phi = 0.$$

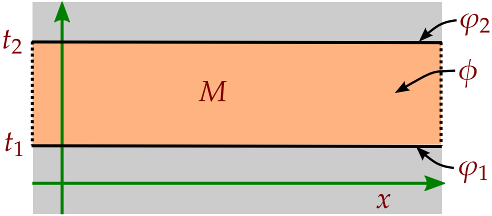
We take the **configuration spaces** associated to **hypersurfaces** and **regions** to be vector spaces of real valued functions:

- $K_\Sigma := \{\varphi : \Sigma \to \mathbb{R}\}$
- $K_M := \{\phi : M \to \mathbb{R}\}$

Klein-Gordon Theory

Standard Geometry (I)

I. Consider **constant-time hypersurfaces** and **time-interval regions** as in the **standard formulation**.



Consider an **constant-time hypersurface** at time t. Expanding in **Fourier modes**, elements of K_t are conveniently parametrized in terms of functions η on **momentum space**,

$$\varphi(x) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3 2E} \left(\eta(k) e^{-\mathrm{i}(Et - kx)} + \overline{\eta(k)} e^{\mathrm{i}(Et - kx)}\right).$$

$\mathcal{H}_t$ is the space of **wave functions** over K_t.

Klein-Gordon Theory

Standard Geometry (II)

Consider a region $[t_1, t_2] \times \mathbb{R}^3$ determined by a time interval. The path integral can be formally solved there, yielding the **field propagator**,

$$Z_{[t_1,t_2]}(\varphi_1, \varphi_2) = N_{[t_1,t_2]} \exp\left(-\frac{1}{2}\int \mathrm{d}^3x \begin{pmatrix}\varphi_1 & \varphi_2\end{pmatrix} W \begin{pmatrix}\varphi_1 \\ \varphi_2\end{pmatrix}\right),$$

$$W = \frac{-\mathrm{i}\omega}{\sin\omega\Delta}\begin{pmatrix}\cos\omega\Delta & -1 \\ -1 & \cos\omega\Delta\end{pmatrix}, \quad \Delta = t_2 - t_1, \quad \omega = \sqrt{-\sum_i \partial_i^2 + m^2}.$$

This can be obtained by

1. using the **stationary phase method** in the path integral with a **classical solution** matching boundary conditions
2. formally solving the **inverse problem** of determining the classical solution **in terms of boundary data** through an operator equation
3. **inserting** the solution into the path integral

The propagator allows to calculate amplitudes. Demanding time-translation invariance allows to deduce the **vacuum wave function**,

$$\psi_0(\varphi) = C \exp\left(-\frac{1}{2}\int \mathrm{d}^3x\, \varphi\omega\varphi\right).$$

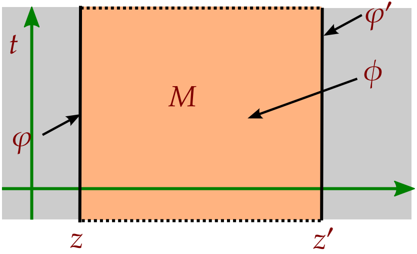
Multi-particle states are now wave functions of the form $p\psi_0$, where p is a **polynomial**. It is then straightforward to calculate the inner product on $\mathcal{H}_t$. This also allows the identification of $\mathcal{H}_t$ with the **Fock space** over K_t.

In particular, a particle state may be characterized by **3 quantum numbers**: the 3 momenta k_i.

Klein-Gordon Theory

Timelike Hypersurfaces (I)

II. Consider hypersurfaces with constant $z = x_1$ coordinate and corresponding **space-interval regions**.



Parametrize configurations on **constant** x_1 **hypersurface** analogous to the spacelike case,

$$\varphi(t, \tilde{x}) = \int_{|E|>m} \frac{\mathrm{d}E\,\mathrm{d}^2\tilde{k}}{(2\pi)^3 2k_1} \left(\eta(E, \tilde{k}) e^{-\mathrm{i}(Et - \tilde{k}\tilde{x} - k_1 x_1)} + \text{c.c.} \right),$$

where $\tilde{x} := (x_2, x_3)$, $\tilde{k} := (k_2, k_3)$ and $k_1 := \sqrt{|E^2 - \tilde{k}^2 - m^2|}$.
Note that the sign of E can be negative!

Klein-Gordon Theory

Timelike Hypersurfaces (II)

Quantization can be performed similarly to the spacelike case. We omit the details.

In particular, a particle state may be characterized by **3 quantum numbers**: the momenta k_2, k_3 and the energy E. Recall that E may be negative. This yields the **same degrees of freedom** as in the spacelike case.

But, in contrast to the spacelike case there is no notion of **in-state** or **out-state**. Rather each particle in a multi-particle state might individually be either **in-going** or **out-going**. This is what the sign of the energy E encodes.

[RO 2005]

Klein-Gordon Theory

Timelike Hypersurfaces (III)

Additional field configurations exist with $|E| < m$,

$$\varphi(t, \tilde{x}) = \int_{|E|<m} \frac{\mathrm{d}E\, \mathrm{d}^2\tilde{k}}{(2\pi)^3 2k_1} \left(\eta(E, \tilde{k})(\cosh(k_1 x_1) + \mathrm{i}\sinh(k_1 x_1))e^{-\mathrm{i}(Et - \tilde{k}\tilde{x})} + \text{c.c.}\right).$$

Configurations correspond to two classes of solutions:

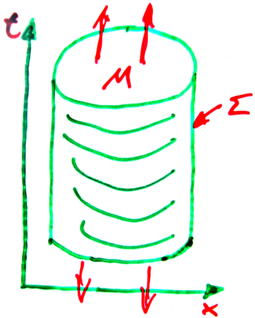
- **Propagating waves**: $E^2 > \tilde{k}^2 + m^2$, oscillate in space
- **Evanescent waves**: $E^2 < \tilde{k}^2 + m^2$, exponential in space

The space of configurations decomposes as a direct sum $K_{x_1} = K^{\mathrm{p}}_{x_1} \oplus K^{\mathrm{e}}_{x_1}$.
The space of wave functions is a tensor product $\mathcal{H}_{x_1} = \mathcal{H}^{\mathrm{p}}_{x_1} \otimes \mathcal{H}^{\mathrm{e}}_{x_1}$.

A vacuum can be defined in $\mathcal{H}^{\mathrm{e}}_{x_1}$ [D. Colosi, RO 2007], but the existence and properties of particle states remain unclear...

Klein-Gordon Theory

The Hypercylinder



To go beyond standard transition amplitudes, consider an example with a connected boundary. [RO 2005]

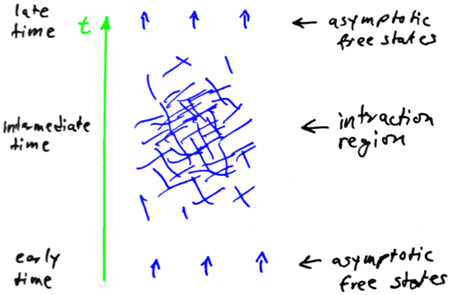
- $M = \mathbb{R} \times B^3_R$.
- $\partial M = \Sigma_R = \mathbb{R} \times S^2_R$.

(Consider propagating waves only.)

- The state space $\mathcal{H}_{\Sigma_R}$ is again a **Fock space**.
- A particle can be characterized by three quantum numbers: **energy** E and **angular momentum** l, m.
- The sign of the energy determines if a particle is **in-going** or **out-going**. The state space decomposes as $\mathcal{H}_{\Sigma_R} = \mathcal{H}_{\text{in}} \otimes \mathcal{H}_{\text{out}}$.
- This decomposition is neither geometrical nor temporal.

S-matrix

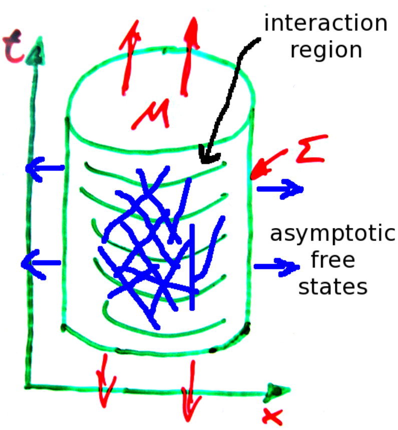
Usually, interacting QFT is described via the S-matrix:



Assume interaction is relevant only after the initial time t_1 and before the final time t_2. The S-matrix is the asymptotic limit of the amplitude between free states at early and at late time:

$$\langle\psi_2|S|\psi_1\rangle = \lim_{\substack{t_1\to-\infty \\ t_2\to+\infty}} \langle\psi_2|U_{\text{int}}(t_1,t_2)|\psi_1\rangle$$

Spatially asymptotic S-matrix (I)



Similarly, we can describe interacting QFT via a spatially asymptotic amplitude. Assume interaction is relevant only within a radius R from the origin in space (but at all times). Consider then the asymptotic limit of the amplitude of a free state on the hypercylinder when the radius goes to infinity:

$$\boxed{S(\psi) = \lim_{R\to\infty} \rho_R(\psi)}$$

[D. Colosi, RO 2007–2008]

Spatially asymptotic S-matrix (II)

Results:

- The **perturbative description of interactions** works as in the standard path integral and S-matrix picture. Technically, the interactions are introduced via **sources**. In the hypercylinder geometry, this involves **evanescent modes** in an essential way, even if they vanish asymptotically.
- The S-matrices are equivalent when the interaction is confined in space and time. This equivalence is realized through an **isomorphism of** the asymptotic state spaces.
- In the standard formulation, **crossing symmetry** is an emergent feature of the S-matrix. In the hypercylinder setting of the GBF crossing symmetry is manifest.

QFT in curved spacetime

The methods presented so far can be generalized for applications to QFT in **curved spacetime**. Some applications so far:

- Certain spacetimes do not admit asymptotic states at timelike infinity. The usual S-matrix cannot be defined there. A famous example is **Anti-deSitter spacetime**. Here the **spatially asymptotic S-matrix** can be defined and yield a physically meaningful description of quantum scattering. [M. Dohse, work in progress]
- A natural application would also be to **static black holes**.
- The spatially asymptotic S-matrix can also be defined in other curved spacetimes such as **deSitter spacetime**. [D. Colosi, 2009]
- **Unitary quantum evolution** can be shown to hold in certain classes of curved spacetime. [D. Colosi, RO 2009]
- The **Unruh effect** has been analyzed with these tools. [D. Colosi, D. Rätzel 2012; E. Bianchi, H. M. Haggard, C. Rovelli 2013]

QFT with compact regions

All applications mentioned involve hypersurfaces and spacetime regions that are infinitely extended. However, an aim of the GBF program is to achieve an **explicitly local description** of QFT. At least in **Riemannian spacetime** this is relatively easy to achieve.

Consider a real scalar field theory (in n dimensions) with action

$$S_M(\phi) = \frac{1}{2} \int \mathrm{d}^n x \left((\partial_i \phi) \partial_i \phi - m^2 \phi^2 \right).$$

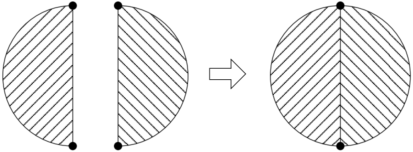
The equations of motion are the **Helmholtz equation**

$$\left(\partial_i \partial_i + m^2 \right) \phi = 0$$

It turns out, that there is no difficulty in obtaining the boundary Hilbert spaces and amplitudes for spacetime regions that have for example the **shape of an n-ball** and may be taken to be **arbitrarily small**.
[D. Colosi, RO 2008]

Why corners?

We would like to obtain any compact region by gluings of just one topological type of elementary region: the ball.



Example: Two ball-shaped regions are glued to another ball-shaped region. This requires gluing along parts of boundaries.

This introduces (virtual) corners where boundaries are split. In a sense these are boundaries of boundaries.

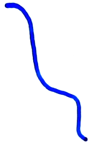
- If regions are topological, the corners are “invisible” and only become apparent in the process of gluing.
- If regions are differentiable, corners may become actual corners, i.e., visible in the differentiable structure as points with a neighborhood diffeomorphic to $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^{n-1}$. This requires to extend the notion of manifold.

2-dimensional quantum Yang-Mills theory

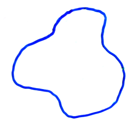
The problem of corners has been solved within the GBF in **2-dimensional quantum Yang-Mills theory**. At the same time this gave an example of implementing **gauge symmetries** into the GBF. In the following we exhibit some of the geometric aspects of this. (Slides taken from another talk.)
[RO 2006]

Two dimensions: hypersurfaces

There are two types of elementary hypersurfaces (A):



an open string with state space $\mathcal{H}_\mathrm{O}$



a closed string with state space $\mathcal{H}_\mathrm{C}$

We also assign $\iota_\mathrm{O} : \mathcal{H}_\mathrm{O} \to \mathcal{H}_{\bar{\mathrm{O}}}$ and $\iota_\mathrm{C} : \mathcal{H}_\mathrm{C} \to \mathcal{H}_{\bar{\mathrm{C}}}$ (b.1).

Two dimensions: hypersurfaces

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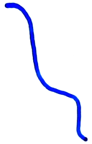
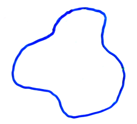


an open string with state space $\mathcal{H}_{\mathrm{O}}$

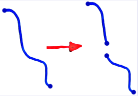


a closed string with state space $\mathcal{H}_{\mathrm{C}}$

We also assign $\iota_{\mathrm{O}} : \mathcal{H}_{\mathrm{O}} \to \mathcal{H}_{\bar{\mathrm{O}}}$ and $\iota_{\mathrm{C}} : \mathcal{H}_{\mathrm{C}} \to \mathcal{H}_{\bar{\mathrm{C}}}$ (b.1).

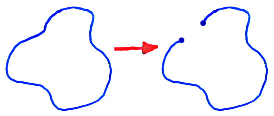
There are two types of elementary decompositions (a):

open string to two open strings



$\tau_{\mathrm{OO}} : \mathcal{H}_{\mathrm{O}} \otimes \mathcal{H}_{\mathrm{O}} \to \mathcal{H}_{\mathrm{O}}$

closed string to an open string



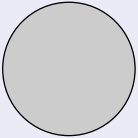
$\tau_{\mathrm{OC}} : \mathcal{H}_{\mathrm{O}} \to \mathcal{H}_{\mathrm{C}}$

Remarks: τ_{OO} must be associative, $\tau_{\mathrm{OC}} \circ \tau_{\mathrm{OO}}$ must be commutative

Two dimensions: regions

Any connected region is a Riemann surface with holes. It is characterized by two non-negative integers, the genus g and the number of holes n.

There is only one type of elementary region, the disc D with amplitude $\rho_D : \mathcal{H}_C \to \mathbb{C}$ (B).



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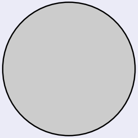
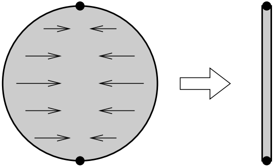


The slice region associated with an open string can be thought of as a squeezed disc:



This gives rise to a bilinear pairing $\mathcal{H}_O \otimes \mathcal{H}_O \to \mathbb{C}$ defined by

$$(\cdot,\cdot)_O = \hat{\rho}_D := \rho_D \circ \tau_{OC} \circ \tau_{OO}$$

By axiom (b.2) this is related to the inner product on $\mathcal{H}_O$ via $\langle\cdot,\cdot\rangle_O = (\iota_O(\cdot),\cdot)_O$.