LECTURE 2: Learning from quantum field theory – basics of the general boundary formulation

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Outline

Recap

We recall that the standard formulation of quantum theory has severe deficiencies that impede its application in a general relativistic context, notably:

- Dependence on a predetermined notion of time
- Non-locality in space

How to proceed?

How do we obtain a better foundation of quantum theory?

Learn from nature! For a theorist this means: Take the best description of nature at a fundamental level that we have available. This is **quantum field theory**. Analyze its **operational core** and look for clues of an underlying structure.

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This leads us here to a formulation of the foundations such that

- there is no reference to time
- locality is manifest
- the standard formulation is recovered (when applicable)

This is called the **general boundary formulation** and it is based on

- The mathematical framework of **topological quantum field theory**. (A branch of modern algebraic topology.)
- A generalization of the **Born rule**.

Lessons from quantum field theory

Various important structural features of quantum field theory as it is practically used are awkward from the point of view of the standard formulation. We focus on a few:

1. The **Feynman path integral**. This turns out to be much more suitable to describe the dynamics of quantum field theory than Hamiltonian or time-evolution operators.
2. **Crossing symmetry**. This property of the S-matrix is completely unmotivated from the point of view of the standard formulation.
3. The **time-ordered product** of fields. This rather than the operator product is the relevant structure to extract physical predictions.

Taking the listed structures seriously from a foundational point of view gives valuable clues towards a reformulation.

The dynamics of quantum field theory is efficiently described using the Feynman path integral [Feynman 1948]. In particular, the **transition amplitudes** describing time-evolution can be recovered from the path integral.

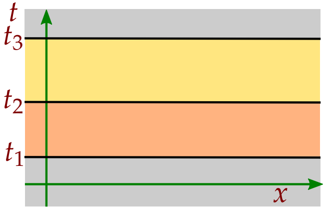
$$\langle \psi_2, U_{[t_1,t_2]}\psi_1 \rangle = \int_{K_{t_1}\times K_{t_2}} \mathcal{D}\varphi_1 \mathcal{D}\varphi_2\ \psi_1(\varphi_1)\overline{\psi_2(\varphi_2)} Z_{[t_1,t_2]}(\varphi_1,\varphi_2)$$

$$Z_{[t_1,t_2]}(\varphi_1,\varphi_2) := \int_{K_{[t_1,t_2]},\phi|_{t_i}=\varphi_i} \mathcal{D}\phi\ e^{\mathrm{i}S(\phi)}$$

$K_{[t_1,t_2]}$ – space of field configurations in the spacetime region $[t_1,t_2]\times\mathbb{R}^3$.
K_{t_i} – instantaneous space of field configurations at t_i.

Consider the composition of time-evolutions

- in operator form: $U_{[t_1,t_3]} = U_{[t_2,t_3]} \circ U_{[t_1,t_2]}$
- in terms of matrix elements:
  $\langle \psi_3, U_{[t_1,t_3]}\psi_1 \rangle = \sum_{i\in N} \langle \psi_3, U_{[t_2,t_3]}\xi_i \rangle \langle \xi_i, U_{[t_1,t_2]}\psi_1 \rangle$



In the path integral picture this arises from a temporal **composition property** of the path integral.

$$Z_{[t_1,t_3]}(\varphi_1, \varphi_3) = \int_{K_{t_2}} \mathcal{D}\varphi_2 \, Z_{[t_1,t_2]}(\varphi_1, \varphi_2) Z_{[t_2,t_3]}(\varphi_2, \varphi_3)$$

Composition in spacetime I

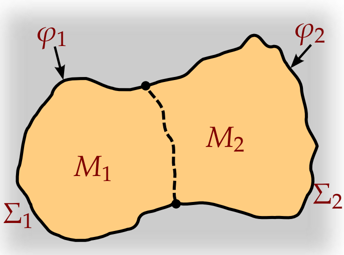
1 – From the path integral to TQFT

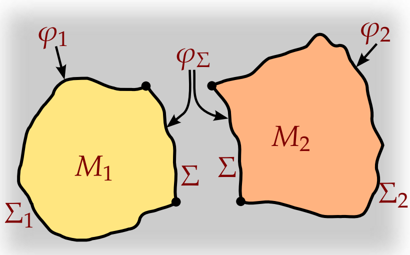
The path integral satisfies a much more general **composition property** in spacetime. This comes from:

- The locality of the integral over field configurations in spacetime
- The additivity of the action in spacetime: Say M_1 and M_2 are non-overlapping spacetime regions, then,

$$S_{M_1 \cup M_2} = S_{M_1} + S_{M_2} \quad \text{and so} \quad e^{\mathrm{i}S_{M_1 \cup M_2}} = e^{\mathrm{i}S_{M_1}} e^{\mathrm{i}S_{M_2}}.$$

Composition in spacetime II



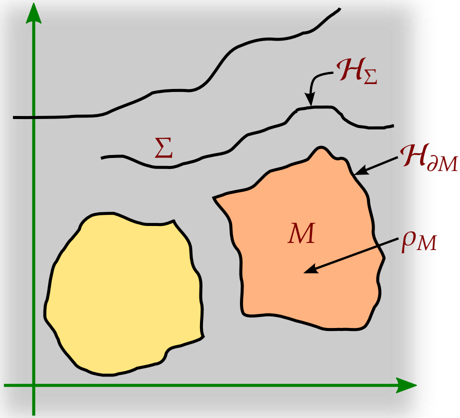


$$Z_{M_1 \cup M_2}(\varphi_1, \varphi_2) = \int_{K_\Sigma} \mathcal{D}\varphi_\Sigma \, Z_{M_1}(\varphi_1, \varphi_\Sigma) Z_{M_2}(\varphi_\Sigma, \varphi_2)$$

Lesson

This suggests that quantum (field) theory itself should incorporate such a generalized composition property.

This property of the path integral motivated the notion of **topological quantum field theory** [E. Witten, G. Segal, M. Atiyah etc. ca. 1988].



To geometric structures (pieces of spacetime)

- **hypersurfaces** Σ: oriented manifolds of dim. $d-1$
- **regions** M: oriented manifolds of dim. d with boundary

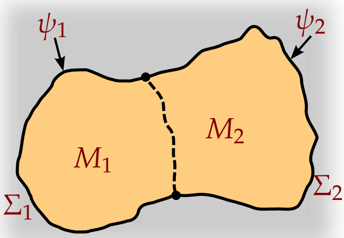
associate algebraic structures

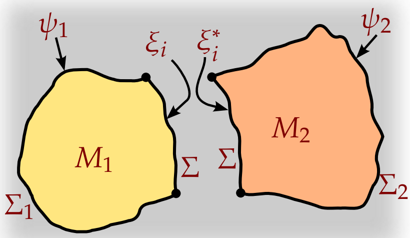
- to Σ a Hilbert space $\mathcal{H}_\Sigma$
- to M an **amplitude map** $\rho_M : \mathcal{H}_{\partial M} \to \mathbb{C}$

Core axioms

1 – From the path integral to TQFT

- Let $\overline{\Sigma}$ denote Σ with opposite orientation. Then $\mathcal{H}_{\overline{\Sigma}} = \mathcal{H}_{\Sigma}^*$.
- **(Decomposition rule)** Let $\Sigma = \Sigma_1 \cup \Sigma_2$ be a disjoint union of hypersurfaces. Then $\mathcal{H}_{\Sigma} = \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\Sigma_2}$.
- **(Gluing rule)** If M_1 and M_2 are adjacent regions, then:



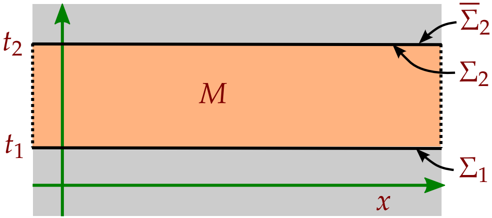


$$\rho_{M_1 \cup M_2}(\psi_1 \otimes \psi_2) = \rho_{M_1} \diamond \rho_{M_2}(\psi_1 \otimes \psi_2) := \sum_{i \in N} \rho_{M_1}(\psi_1 \otimes \xi_i)\rho_{M_2}(\xi_i^* \otimes \psi_2)$$

Here, $\psi_1 \in \mathcal{H}_{\Sigma_1}$, $\psi_2 \in \mathcal{H}_{\Sigma_2}$ and $\{\xi_i\}_{i \in N}$ is an ON-basis of $\mathcal{H}_{\Sigma}$.

Recovering transition amplitudes

1 – From the path integral to TQFT



- region: $M = [t_1, t_2] \times \mathbb{R}^3$
- boundary: $\partial M = \Sigma_1 \cup \overline{\Sigma}_2$
- state space: $\mathcal{H}_{\partial M} = \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\overline{\Sigma}_2} = \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}^*_{\Sigma_2}$

Via time-translation symmetry identify $\mathcal{H}_{\Sigma_1} \cong \mathcal{H}_{\Sigma_2} \cong \mathcal{H}$. Then,

$$\rho_{[t_1,t_2]}(\psi_1 \otimes \psi_2^*) = \langle \psi_2, U_{[t_1,t_2]}\psi_1 \rangle.$$

Recovering transition amplitudes

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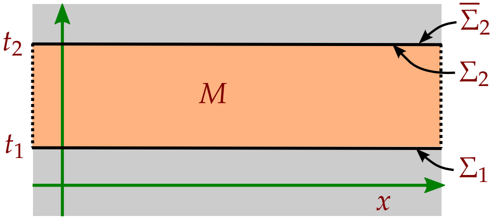

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$$\rho_{[t_1,t_2]}(\psi_1 \otimes \psi_2^*) = \langle \psi_2, U_{[t_1,t_2]} \psi_1 \rangle.$$

- But, does it make sense do go beyond this example?
- Does the boundary Hilbert space $\mathcal{H}_{\partial M}$ have a useful physical interpretation in general?

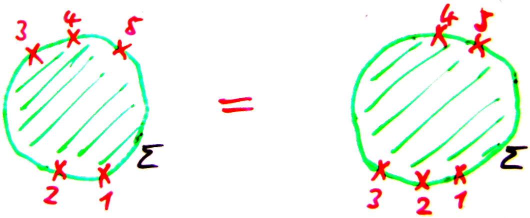
Crossing symmetry

2 – Crossing symmetry and the joint state space

Quantum field theory satisfies **crossing symmetry**. That is, transition amplitudes remain (essentially) invariant when individual particles are moved between the in- and the out-state spaces.

Thus, particles might reasonably thought of as living in a joint product Hilbert space $\mathcal{H}_{\text{in}} \otimes \mathcal{H}_{\text{out}}$, distinguished merely by quantum numbers.

The analogous picture for a connected boundary looks like this:



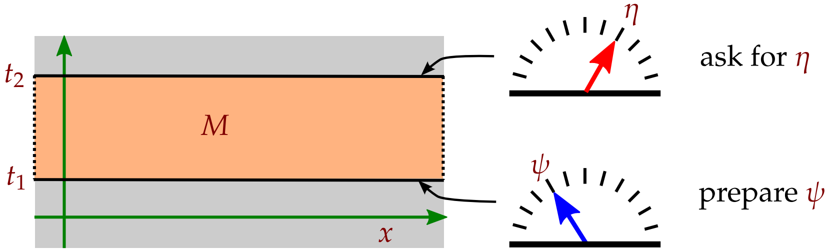
Lesson

Crossing symmetry is indispensable for state spaces associated to more general boundaries to make sense.

Probabilities: Born rule (I)

Consider a simple measurement:

- At t_1 we **prepare** a state ψ.
- At t_2 we **ask** whether the system is in state η.

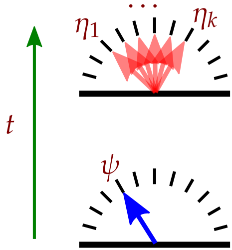


The **conditional probability** for this is $P(\eta|\psi) = |\langle\eta, U\psi\rangle|^2$

Probabilities in quantum theory are **conditional** and involve two ingredients: **preparation** and **question**.

Probabilities: Born rule (II)

Suppose we ask instead for a **range** of possible outcomes.



Ask for the outcome to be in the **subspace** $A \subseteq H$ spanned by the orthonormal vectors $\eta_1, \dots, \eta_k$. These can be seen as alternative and **exclusive** outcomes.

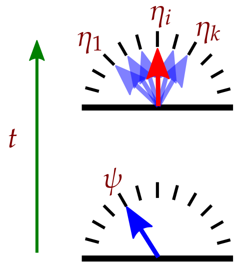
Prepare ψ.

The **conditional probability** for this is $P(A|\psi) = \sum_{i=1}^{k} |\langle \eta_k, U\psi \rangle|^2$

Probabilities depend generally on **subspaces**. The case of a single state arises as the special case of a one-dimensional subspace.

Probabilities: Born rule (III)

We may need to **condition on a range of outcomes**. Say, the apparatus registers only measurements in this range.



Ask for the outcome to be η_i given that the output lies in the **subspace** $S \subseteq H$ spanned by the orthonormal vectors $\eta_1, \ldots, \eta_k$.

Prepare ψ.

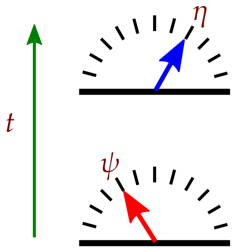
The **conditional probability** for this is

$$P(\eta_i | S, \psi) = \frac{|\langle \eta_i, U\psi \rangle|^2}{\sum_{i=1}^{k} |\langle \eta_k, U\psi \rangle|^2}$$

Probabilities arise generally as **quotients**.

Probabilities: Born rule (IV)

We may also need to **condition past events on future events**. Suppose a measurement is prepared with uniform probability over all states. (This is the **maximally mixed state**.) We register measurements with a fixed outcome only and ask for a specific initial state.



Register (or select) outcomes η only.

Prepare with uniform probability $\psi_1, \ldots, \psi_n$ (an ON-basis of H). Ask for the initial state ψ.

The **conditional probability** for this is $P(\psi|\eta) = |\langle \eta, U\psi \rangle|^2$

The **conditional structure** need not be related to the **causal structure**.

Probabilities

Generalizing the Born rule

Consider a spacetime region M. The associated amplitude ρ_M allows to extract probabilities for measurements in M.

Probabilities in quantum theory are generally conditional probabilities. They depend on two pieces of information. Here these are:

- $\mathcal{S} \subseteq \mathcal{H}_{\partial M}$ representing **preparation** or **knowledge**
- $\mathcal{A} \subseteq \mathcal{H}_{\partial M}$ representing **observation** or the **question**

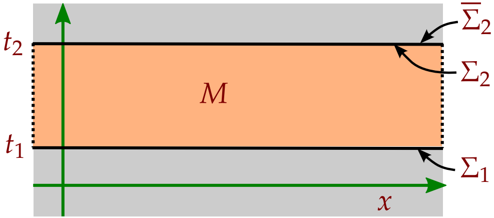
The probability that the physics in M is described by $\mathcal{A}$ given that it is described by $\mathcal{S}$ is: (here $\mathcal{A} \subseteq \mathcal{S}$) [RO 2005]

$$\boxed{P(\mathcal{A}|\mathcal{S}) = \frac{\sum_{i\in J} |\rho_M(\xi_i)|^2}{\sum_{i\in I} |\rho_M(\xi_i)|^2}}$$

Here $\{\xi_i\}_{i\in I}$ is an ON-basis of $\mathcal{S}$ and reduces on $J \subseteq I$ to an ON-basis of the subspace $\mathcal{A}$.

Recovering standard probabilities

Generalizing the Born rule



To compute the probability of measuring ψ_2 at t_2 given that we prepared ψ_1 at t_1 we set

$$\mathcal{S} = \psi_1 \otimes \mathcal{H}^*, \quad \mathcal{A} = \mathcal{H} \otimes \psi_2^*.$$

The resulting expression yields correctly

$$P(\mathcal{A}|\mathcal{S}) = \frac{|\rho_{[t_1,t_2]}(\psi_1 \otimes \psi_2^*)|^2}{1} = |\langle \psi_2, U_{[t_1,t_2]} \psi_1 \rangle|^2.$$

Observables are labeled

3 – The time-ordered product and composition of observables

- Standard observables of QFT are values of fields $\hat{\phi}(x)$ and their derivatives $\partial_0\hat{\phi}(x)$ at spacetime points x.
- These observables carry a label x specifying when (and where) they are applied.
- For consistency under changes of reference frame we need

$$[A(x), B(y)] = 0 \quad \text{if } x \text{ and } y \text{ are spacelike separated,}$$

that is, if there is a reference frame where x and y are instantaneous.

The time-ordered product

3 – The time-ordered product and composition of observables

- There is only one operationally meaningful composition of two observables, given by the commutative time-ordered product:

$$T\, A(x)B(y) := \begin{cases} A(x)B(y) & \text{if } x_0 > y_0 \\ B(y)A(x) & \text{if } x_0 < y_0 \end{cases}$$

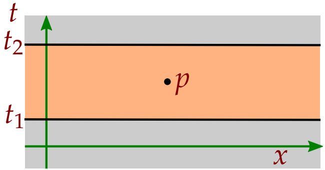
- In QFT all physically measurable quantities are constructed via the time-ordered product. The noncommutative operator product is never directly used.
- The operator product can be recovered from the time-ordered product. For equal times:

$$[A(t,\bar{x}), B(t,\bar{y})] = \lim_{\epsilon\to 0} TA(t+\epsilon,\bar{x})B(t-\epsilon,\bar{y}) - TB(t+\epsilon,\bar{y})A(t-\epsilon,\bar{x})$$

Observables in QFT are quantized through the path integral:

$$\langle\psi_2, \hat{A}\psi_1\rangle = \int_{K_{t_1}\times K_{t_2}} \mathcal{D}\varphi_1\mathcal{D}\varphi_2\ \psi_1(\varphi_1)\overline{\psi_2(\varphi_2)}\, Z^A_{[t_1,t_2]}(\varphi_1,\varphi_2)$$

$$Z^A_{[t_1,t_2]}(\varphi_1,\varphi_2) := \int_{K_{[t_1,t_2]},\phi|_{t_i}=\varphi_i} \mathcal{D}\phi\ A(\phi)e^{\mathrm{i}S(\phi)}$$

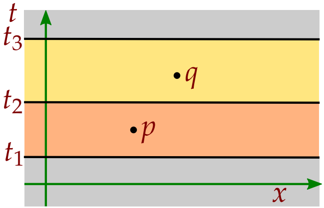


For example, for the classical observable $A = \phi(p)$, the quantization $\hat{A} = \hat{\phi}(p)$ is the usual field operator.

Lesson

Observables are naturally spacetime objects.

The observables of QFT inherit the composition property of the path integral. This is the origin of the time-ordered product.



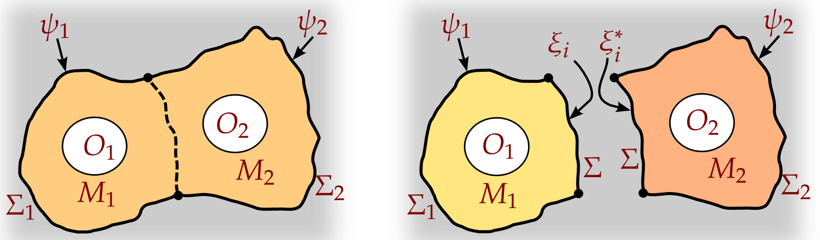
For example, if $A = \phi(p)\phi(q)$, then $\hat{A} = T\hat{\phi}(p)\hat{\phi}(q)$. This can also be obtained by spacetime composition of $\hat{\phi}(p)$ with $\hat{\phi}(q)$.

Lesson

Quantum observables are spacetime composable in the same way as amplitudes. Moreover, there is a correspondence between the classical product and quantum composition.

Observables in the GBF

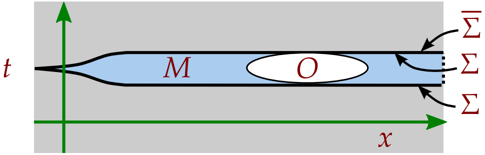
Observables are associated to regions M and encoded through **observable maps** $\rho_M^O : \mathcal{H}_{\partial M} \to \mathbb{C}$, similar to the amplitude maps.



Observables can be composed in the same way as amplitudes via gluing of the underlying regions. The same formula as for amplitudes applies. We denote their composition as

$$\rho_{M_1}^{O_1} \diamond \rho_{M_2}^{O_2} : \mathcal{H}_{\partial(M_1 \cup M_2)} \to \mathbb{C}$$

Recovering standard observables



- region: $M = [t,t] \times \mathbb{R}^3$
- boundary: $\partial M = \Sigma \cup \overline{\Sigma}$
- state space:
  $\mathcal{H}_{\partial M} = \mathcal{H}_\Sigma \otimes \mathcal{H}_{\overline{\Sigma}} = \mathcal{H}_\Sigma \otimes \mathcal{H}_\Sigma^*$

Recall $\mathcal{H}_\Sigma \cong \mathcal{H}$. In this geometry of an infinitesimally thin **slice** there is a correspondence between observable maps $\rho^O_{[t,t]} : \mathcal{H}_\Sigma \otimes \mathcal{H}^*_\Sigma \to \mathbb{C}$ and standard observables $\hat{O} : \mathcal{H} \to \mathcal{H}$ via matrix elements:

$$\rho^O_{[t,t]}(\psi_1 \otimes \psi_2^*) = \langle \psi_2, \hat{O}\psi_1 \rangle \quad \forall \psi_1, \psi_2 \in \mathcal{H}.$$

Observables and expectation values

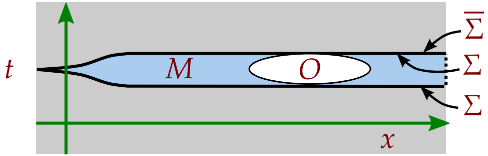
Consider a spacetime region M carrying an observable O. The associated observable map ρ_M^O allows to extract expectation values for measurements in M.

The **expectation value** of the observable O conditional on the system being prepared in the subspace $\mathcal{S} \subseteq \mathcal{H}_{\partial M}$ can be represented as follows: [RO 2010]

$$\boxed{\langle O \rangle_{\mathcal{S}} = \frac{\sum_{i\in I} \overline{\rho_M(\xi_i)}\, \rho_M^O(\xi_i)}{\sum_{i\in I} |\rho_M(\xi_i)|^2}}$$

Here $\{\xi_i\}_{i\in I}$ is an ON-basis of $\mathcal{S}$.

Recovering standard expectation values



To compute the expectation value of observable O at time t given by

$$\rho^O_{[t,t]}(\psi_1 \otimes \psi_2^*) = \langle \psi_2, \hat{O}\psi_1 \rangle$$

in the state ψ we set

$$\mathcal{S} = \psi \otimes \mathcal{H}^*.$$

The standard expectation value is then correctly recovered as

$$\langle O \rangle_{\mathcal{S}} = \frac{\rho^O_{[t,t]}(\psi \otimes \psi^*)}{1} = \langle \psi, \hat{O}\psi \rangle.$$

Composition correspondence

Suppose we are given classical observables O_1 and O_2 localized in adjacent spacetime regions M_1 and M_2 respectively. In the classical theory there is a natural composition of these observables given by the ordinary product $O_1 \cdot O_2$ in the joint spacetime region $M_1 \cup M_2$.

We then say that a quantization prescription $O_1 \mapsto \rho_{M_1}^{O_1}, O_2 \mapsto \rho_{M_2}^{O_2}$ satisfies the **composition correspondence property** if,

$$\boxed{\rho_{M_1 \cup M_2}^{O_1 \cdot O_2} = \rho_{M_1}^{O_1} \diamond \rho_{M_2}^{O_2}}$$

As already mentioned, quantum field theory satisfies this!

A remark on fermions

The formalism in the form presented so far only applies to bosonic theories. In the presence of fermionic degrees of freedom certain modifications apply [RO 2012]:

- All structures are equipped with a $\mathbb{Z}_2$-grading that distinguishes even and odd fermion number.
- Hilbert spaces are replaced by **Krein spaces**. These are indefinite inner product spaces decomposing into a positive definite and negative definite part.

$$\mathcal{H}_\Sigma = \mathcal{H}_\Sigma^+ \oplus \mathcal{H}_\Sigma^-$$

The reason that these Krein spaces are "invisible" in ordinary QFT has to do with the restriction to spacelike hypersurfaces and to a global choice of time orientation.

Learning from QFT (essay):

R. O., *Reverse engineering quantum field theory*, AIP Conf. Proc. **1508** (2012) 428–432. arXiv:1210.0944.

GBF foundations:

R. O., *General boundary quantum field theory: Foundations and probability interpretation*, Adv. Theor. Math. Phys. **12** (2008) 319-352. arXiv:hep-th/0509122.

GBF observables:

R. O., *Observables in the General Boundary Formulation*, Quantum Field Theory and Gravity, Springer, 2012, pp. 137–156. arXiv:1101.0367.
R. O., *Schrödinger-Feynman quantization and composition of observables in general boundary quantum field theory*, arXiv:1201.1877.

GBF fermions:

R. O., *Free fermi and bose fields in TQFT and GBF*. arXiv:1208.5038.