Square-bracket operations clubs

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Abstract

This paper continues the investigation of the three square-bracket operations $[\cdot\cdot]$ from chapter 5 of [40]. We say that a square-bracket operation $[\cdot\cdot]$ has the *Ramsey club property* if for every club $C \subseteq \omega_1$, there is an uncountable subset $W \subseteq \omega_1$ such that $[\alpha\beta] \in C$ for every $\alpha, \beta \in W$. The second author proved that the Proper Forcing Axiom implies that all the square-bracket operations induced by Aronszajn trees have this property. We extend this result to the other two classes. We conclude that each of the statements "all square-bracket operations have the Ramsey club property" and "No square-bracket operation has the Ramsey club property" are consistent with ZFC. In other words, ZFC is unable to decide the status of the Ramsey club property for any square-bracket operation. Furthermore, we analyze the status of the Ramsey club property for square-bracket operations under Martin's Axiom and the Continuum Hypothesis.

1 Introduction

The famous Ramsey's theorem states that every graph on ω contains either a countably infinite complete subgraph or a countably infinite independent subgraph[1]. The straightforward generalization to ω_1 would assert that every graph on ω_1 contains either an uncountable complete subgraph or an uncountable independent subgraph. However, as noted by Sierpiński, this generalization is false. In fact, it in a spectacular way, as can be seen by the following theorem of the second author:

Theorem 1 (T. [40], [38]) *There is $c : [\omega_1]^2 \longrightarrow \omega_1$ such that for every $A \in [\omega_1]^{\omega_1}$ and $\gamma \in \omega_1$, there are $\alpha, \beta \in A$ such that $c(\alpha, \beta) = \gamma$.*

In other words, the pairs of ω_1 can be colored with uncountably many colors in such a way that every color appears in every uncountable set (for more results

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[1]Recall that a graph is complete (independent) if any two vertices are (not) connected.

on this type, see [30] or [35]). Given a function $F : [\omega_1]^2 \longrightarrow \omega_1$ and a subset $X \subseteq \omega_1$, define the graph $G(F, X)$ on ω_1 by connecting α and β with an edge in case $F(\alpha, \beta) \in X$. This family of graphs can be very interesting when F has strong combinatorial properties. In chapter 5 of the book [40], the *square-bracket operations* $[\cdot\cdot]_{\mathcal{C}}$, $[\cdot\cdot]_{T,a}$ and $[\cdot\cdot]_{\mathcal{R},e}$ are introduced, where $\mathcal{C}$ is a C-sequence, T is a special Aronszajn tree with an specializing mapping a, $\mathcal{R}$ is an ω_1-sequence of reals and e is a sequence of mappings. The definitions and main properties of these operations will be reviewed in the Preliminaries Section. In this paper, the term "square-bracket operation" will always refer to one of these three functions. The following is another result of the second author:

Theorem 2 (T. [40]) *Let F be a square-bracket operation and $S, Z \subseteq \omega_1$. If the symmetric difference $S \triangle Z$ is stationary, then the graphs $G(F, S)$ and $G(F, Z)$ are orthogonal to each other (they do not contain uncountable isomorphic subgraphs).*

With this result, the second author was able to solve the *basis problem for uncountable graphs:*[2] 2^{ω_1} is the minimum size of a family $\mathcal{G}$ of uncountable graphs such that every uncountable graph contains an isomorphic copy of some member of $\mathcal{G}$. We believe this result justifies a deeper investigation into the graphs induced by the square-bracket operations, which is the central topic of the present paper. We begin with the following definition:

Definition 3 *Let $F : [\omega_1]^2 \longrightarrow \omega_1$ and $X \subseteq \omega_1$. We say that X is* F-Ramsey *if $G(F, X)$ contains either an uncountable complete subgraph or an uncountable independent subgraph.*

We will investigate which subsets of ω_1 are Ramsey for the square-bracket operations. The following is result is proved in [40].

Theorem 4 (T.) *Let $[\cdot\cdot]$ be a square-bracket operation and $W \in [\omega_1]^{\omega_1}$. The set $\{[\alpha\beta] \mid \alpha, \beta \in W\}$ contains a club.*

Note that this result implies Theorem 1. Another easy corollary is the following:

Corollary 5 *Let $[\cdot\cdot]$ be a square-bracket operation and $X \subseteq \omega_1$.*

1. *If $G([\cdot\cdot], X)$ contains an uncountable complete subgraph, then X contains a club.*

[2] This result should be contrasted with the following well-known consequences of PFA: Any two ω_1-dense subsets of reals are isomorphic (see [10] or [12]), there is a five element base for the uncountable linear orders (see [29]) and there are exactly five cofinal types of directed partial orders of size ω_1 (see [44]).

2. *If $G([\cdot\cdot], X)$ contains an uncountable independent subgraph, then X is non-stationary.*

Note that Theorem 4 implies that if $S \subseteq \omega_1$ is stationary (in particular, if it is a club), then $G([\cdot\cdot], S)$ does not contain an uncountable independent subgraph. This allows us to reformulate Definition 3 for clubs as follows:

Definition 6 *Let $[\cdot\cdot]$ be a square-bracket operation and $C \subseteq \omega_1$ a club. We say that C is* $[\cdot\cdot]$-Ramsey *if $G([\cdot\cdot], C)$ contains an uncountable complete subgraph.*

By the first point of Corollary 5, the problem of identifying the subsets $X \subseteq \omega_1$ for which $G([\cdot\cdot], X)$ contains an uncountable complete subgraph can be reduced to the case where X is a club. Moreover, since the question of which subsets yield a graph with an uncountable independent subgraph is dual to the question for complete subgraphs, we are led to the following problem:

Which clubs are $[\cdot\cdot]$-Ramsey?

The first result in this direction can be found in the book [40] (Lemma 5.4.4):

Theorem 7 (T. [40]) *The* Proper Forcing Axiom *(PFA) implies that if T is a special Aronszajn tree and $a : T \longrightarrow \omega$ is an specialization mapping, then every club is $[\cdot\cdot]_{T,a}$-Ramsey.*

This theorem answered a question raised by Woodin while he was working with his forcing $\mathbb{P}_{\mathsf{max}}$ (see [46]). The following questions motivated our research on this topic:

1. Is the conclusion of Theorem 7 independent of ZFC? What is its status under MA or CH?
2. Does Theorem 7 hold for the other square-bracket operations?

This work addresses these questions. We will show that Theorem 7 holds for the other square-bracket operations and that its conclusion is independent from ZFC (for all square-bracket operations). In fact, for some of the square-bracket operations, the conclusion is even independent from MA and its negation is consistent with CH. The most important open question remaining is whether CH is consistent with the statement that every club is $[\cdot\cdot]$-Ramsey.

The structure of the paper is as follows. Section 2 covers the necessary notation and definitions that will be used throughout the paper. The next five sections provide the required background on elementary submodels, Aronszajn

trees, walks on ordinals, forcing and Baumgartner's club forcing. The reader may skip these preliminary sections and return to them as needed. The definitions of the square-bracket operations from the book [40] are reviewed in Section 8. In Section 9 we prove that for every club $C \subseteq \omega_1$, there exists a square-bracket operation $[\cdot\cdot]$ (of any type) for which C is $[\cdot\cdot]$-Ramsey. On the other hand, in Section 10 we prove that the generic club added by Baumgartner's club forcing is not $[\cdot\cdot]$-Ramsey for any square-bracket operation $[\cdot\cdot]$ from the ground model. Consequently, ZFC can not prove the conclusion of Theorem 7. Furthermore, we prove that MA is consistent with the existence of a square-bracket operation induced by a C-sequence for which not all clubs are Ramsey. In Section 11 we extend these results for all square-bracket operations and not just the ones from the ground model. In Sections 13 and 14 we extend Theorem 7 for the remaining square-bracket operations. The final section presents a collection of open questions.

2 Notation

We say that $(T, <_T)$ is a *tree* if it is a partial order with a least element and for every $s \in T$, the set $s^{\downarrow} = \{t \in T \mid t <_T s\}$ is a well-order. Given an ordinal α, by T_α we denote the α-level of T, which is the set of all $s \in T$ for which $s^{\downarrow}$ is isomorphic to α. The *height of* T is the least α for which T_α is empty. For $s \in T$, the *height of* s is the (unique) ordinal α such that $s \in T_\alpha$. The height of s will be denoted as $|s|$ (although we are using the same symbol that is used to denote cardinality, there should not be any room for confusion). If $\gamma \leq |s|$, denote by $s \restriction \gamma$ the only element of T_γ for which $s \restriction \gamma \leq_T s$ (so $s \restriction |s| = s$). For $C \subseteq \omega_1$, denote $T \restriction C = \{s \in T \mid |s| \in C\}$. We say that T is a *Hausdorff tree* if for every limit ordinal α and $s, t \in T_\alpha$, if $s^{\downarrow} = t^{\downarrow}$, then $s = t$ (equivalently, the tree topology of T is Hausdorff). We say that $s, t \in T$ are *incompatible* (denoted by $s \perp t$) if $s \not\leq_T t$ and $t \not\leq_T s$. Denote by $\triangle(s, t)$ the largest α for which $s \restriction \alpha = t \restriction \alpha$. A branch through a tree is a maximal linearly ordered subset.

Although our main focus is on trees of height ω_1, we will also encounter subtrees of $2^{<\omega}$. Given $s, t \in 2^{<\omega}$, the concatenation of s and t is denoted by $s^\frown t$. We will occasionally denote $s^\frown n$ instead of $s^\frown (n)$ (for some $n \in \omega$). The *cone of* s is defined as $\langle s \rangle = \{f \in 2^\omega \mid s \subseteq f\}$. Let $S \subseteq 2^{<\omega}$ be a tree (the tree ordering is $\subseteq$). We say that $s \in S$ is a *splitting node* if both $s^\frown 0$ and $s^\frown 1$ belong to S. If s is a splitting node of S and it extends exactly n-many splitting nodes, we will say s is an *n-splitting node of S*. The set of all n-splitting nodes of S is denoted by $\mathsf{split}_n(S)$. A *Sacks tree (or perfect tree)* is a tree in which any node can be extended to a splitting node. The *set of branches of* S is defined as $[S] = \{f \in 2^\omega \mid \forall n \in \omega\, (f \restriction n \in S)\}$.

By $\mathsf{LIM}(\omega_1)$ we denote the set of all countable limit ordinals. Given $S \subseteq \omega_1$, its set of *accumulation points* (denoted by S') is the set of all $\alpha \in \omega_1$ such that

$S \cap \alpha$ is unbounded in α. By $[\alpha, \beta]$ we denote the closed interval $\{\xi \mid \alpha \leq \xi \leq \beta\}$. This should not be confused with the square-bracket operations, which are always denoted by $[\alpha\beta]$ (with no comma). The intervals (α, β) and $(\alpha, \beta]$ have the expected meaning. The expression $\langle \delta_n \rangle \longrightarrow \delta$ means that the sequence $\langle \delta_n \rangle$ converges to δ (with the usual order topology of ω_1). The class of ordinals is denoted by OR.

The quantifier $\exists^{\omega_1}$ means "there are uncountably many" while $\forall^{\infty}$ means "for almost all" (for all except finitely many). For A a set of the ordinal numbers, we will denote by $\mathsf{OT}(A)$ its order type. If f is a function, by $\mathsf{dom}(f)$ we denote its domain and $\mathsf{im}(f)$ is its image.

3 Preliminaries on countable elementary submodels

Elementary submodels play a fundamental role while studying the combinatorics of ω_1. We now review the basic facts about them that are needed for our work and refer the reader to the survey [16] to learn more. For a cardinal κ, define $\mathsf{H}(\kappa)$ as the set consisting of all sets whose transitive closure has size less than κ. Arguments involving elementary submodels, often start by choosing a large enough regular κ that is needed for the argument in question. To avoid constant repetition, we will omit the reference to κ when there is no need to explicitly specify it. In this way, we will simply write "let M be a countable elementary submodel" instead of "let M be a countable elementary submodel of some $\mathsf{H}(\kappa)$, for a large enough regular κ". It is often the case that the reference to κ is not needed. However, this is not the case when working with models as side conditions (see [39] and [41]), as we will do in sections 13 and 14. In this two sections, we will be more carefully naming the cardinal that is needed.

Let M be a countable elementary submodel. The *height of* M is defined as $\delta_M = M \cap \omega_1$. It is well-known that δ_M is a countable ordinal. We will need the following result regarding the height of a model:

Proposition 8 *Let M be a countable elementary submodel and $C, S \in M$ be two subsets of ω_1.*

1. *If C is a club, then $\delta_M \in C$.*
2. *If $\delta_M \in S$, then S is stationary.*

Another very useful result is the following:

Lemma 9 *Let M be a countable elementary submodel and $A \in M$. If $A \nsubseteq M$, then A is uncountable.*

Recall the following notion:

Definition 10 *Let δ be an ordinal. We say that δ is* indecomposable *if it is closed under (ordinal) addition. Equivalently, $\gamma + \delta = \delta$ for every $\gamma < \delta$. By $\mathsf{IND}(\omega_1)$ we denote the set of all countable indecomposable ordinals.*

The height of a model is an indecomposable ordinal (it is also closed under ordinal multiplication and exponentiation). It is not hard to see that $\mathsf{IND}(\omega_1)$ is a club.

Definition 11 *Let $\gamma \leq \omega_1$. We say that $\{M_\alpha \mid \alpha < \gamma\}$ is a* continuos chain of elementary submodels *if the following conditions hold:*

1. *There is an uncountable regular κ such that each M_α is an elementary submodel of $\mathsf{H}(\kappa)$.*
2. *If $\alpha < \beta < \gamma$, then $M_\alpha \in M_\beta$.*
3. *If $\alpha < \gamma$ is a limit ordinal, then $M_\alpha = \bigcup_{\xi<\alpha} M_\xi$.*

4 Preliminaries on Aronszajn trees

An Aronszajn tree is a tree of height ω_1 such that all of its levels are countable and has no uncountable branches. A large part of the combinatorics of ω_1 deals with Aronszajn trees. The reader may consult the survey [43] to learn a lot on Aronszajn trees (see also [13] and [24]). An Aronszajn tree T *is special* if there is $a : T \longrightarrow \omega$ such that for every $n \in \omega$, the set $a^{-1}(\{n\})$ is an antichain. We call a *an specialization mapping for* T.

Let $T^1, ..., T^n$ be trees of height ω_1. The *tree product* $S = T^0 \otimes ... \otimes T^n$ consists of all sequences $\overline{s} = (s_0, ..., s_n) \in T^0 \times ... \times T^n$ such that $|s_0| = ... = |s_n|$. For $\overline{s}, \overline{t} \in S$, define $\overline{s} \leq_S \overline{t}$ if $s_i \leq_{T^i} t_i$ for every $i \leq n$. It is easy to see verify that $(S, \leq_S)$ is a tree of height ω_1 and S_δ is precisely the set of all $\overline{s} = (s_0, ..., s_n) \in S$ with $|s_0| = ... = |s_n| = \delta$. If $\overline{s} = (s_0, ..., s_n) \in S$ and $\delta < |\overline{s}|$, we have that $\overline{s} \restriction \delta = (s_0 \restriction \delta, ..., s_n \restriction \delta) \in S_\delta$ (we take this opportunity to remind the reader that in this context, $|\overline{s}|$ denotes the height of s in the tree S, not its cardinality as a sequence). A set $A = \{\overline{s}_\alpha \mid \alpha \in \omega_1\} \subseteq S$ is *height increasing* if $|\overline{s}_\alpha| < |\overline{s}_\beta|$ for all $\alpha < \beta$.

Definition 12 *Let $T^1, ..., T^n$ be trees of height ω_1, $S = T^0 \otimes ... \otimes T^n$, $A = \{\overline{s}_\alpha \mid \alpha \in \omega_1\} \subseteq S$, $k \in \omega$ and $\delta \in \omega_1$. We say that A* is k-distributed at level *δ if for every $\overline{u}_0, ..., \overline{u}_k \in S_\delta$, there are uncountably many $\alpha \in \omega_1$ such that $\overline{s}_\alpha \restriction \delta \cap \overline{u}_i = \emptyset$ for all $i \leq k$.*

The following lemma from [43] (Lemma 5.9) is particularly useful when working with Aronszajn trees.

Proposition 13 *Let $T^1, ..., T^n$ be trees of height ω_1, $S = T^0 \otimes ... \otimes T^n$ and $A = \{\overline{s}_\alpha \mid \alpha \in \omega_1\} \subseteq S$ that is height increasing. For every $k \in \omega$, there is $\delta \in \omega_1$ such that A is k-distributed at level δ.*

A consequence of this result is the following proposition (an alternative proof can be found in Claim 3.2 of [47]).

Proposition 14 *Let T be an Aronszajn tree and A an uncountable subset of T. There are two incomparable nodes $s, t \in T$ such that the sets $\{u \in T \mid s <_T u\}$ and $\{u \in T \mid t <_T u\}$ are uncountable.*

An "elementary submodel" version of the preceding proposition is also true.

Proposition 15 *Let T be an Aronszajn tree, $A \subseteq T$, M a countable elementary submodel with $T, A \in M$ and $t \in A \setminus M$. There are $\xi \in M$ and $u \in T \restriction \xi$ such that:*

1. *$t \perp u$.*
2. *There are uncountably many $s \in A$ such that $u \leq_T s$.*

5 Preliminaries on forcing

We assume the reader is already familiar with the method of forcing as presented in [23]. If $\mathbb{P}$ is a forcing and M is a countable elementary submodel with $\mathbb{P} \in M$, we say that $p \in \mathbb{P}$ is an $(M, \mathbb{P})$-*generic condition* if for every $D \subseteq \mathbb{P}$ open dense set with $D \in M$, the set $D \cap M$ is predense below p. Moreover, p is an $(M, \mathbb{P})$-*totally generic condition* if $p \in D$ for every $D \in M$ that is an open dense subset of $\mathbb{P}$. It is well-known that if p is $(M, \mathbb{P})$-generic and $G \subseteq \mathbb{P}$ is a generic filter with $p \in G$, then $M[G]$ is a generic extension of M and $M[G] \cap V = M$. The following equivalence of generic conditions is very useful (see [41]).

Lemma 16 *Let $\mathbb{P}$ be a forcing, $p \in \mathbb{P}$ and M an elementary submodel with $\mathbb{P} \in M$. The following are equivalent:*

1. *p is an $(M, \mathbb{P})$-generic condition.*
2. *For every $A \subseteq \mathbb{P}$ with $A \in M$ and $q \leq p$, if $q \in A$, then there is $r \in A \cap M$ that is compatible with q.*

A forcing $\mathbb{P}$ is *(totally) proper* if for every countable elementary submodel M with $\mathbb{P} \in M$ and $p \in M \cap \mathbb{P}$, there is $q \leq p$ that is an $(M, \mathbb{P})$-*(totally) generic condition.* It can be proved a forcing is totally proper if and only if it is proper and does not add new reals (see [17]).

We say that $\mathbb{P}$ has the κ-*chain condition* ($\mathbb{P}$ is κ-c.c.) if $\mathbb{P}$ does not have antichains of size κ. We write c.c.c. instead of ω_1-c.c. The following is well-known (see [27] or [41]).

Proposition 17 *Let $\mathbb{P}$ be c.c.c. and M a countable elementary submodel with $\mathbb{P} \in M$. Every $p \in \mathbb{P}$ is $(M, \mathbb{P})$-generic.*

In regard with two step iterations, we have the following result (see [1]).

Proposition 18 *Let $\mathbb{P}$ be a forcing notion, $\dot{\mathbb{Q}}$ a $\mathbb{P}$-name for a partial order and M a countable elementary submodel with $\mathbb{P}, \dot{\mathbb{Q}} \in M$. For $(p, \dot{q}) \in \mathbb{P} * \dot{\mathbb{Q}}$, the following are equivalent:*

1. *$(p, \dot{q})$ is $(M, \mathbb{P} * \dot{\mathbb{Q}})$-generic.*
2. *p is $(M, \mathbb{P})$-generic and $p \Vdash$ "$\dot{q}$ is $(M[\dot{G}], \dot{\mathbb{Q}}[\dot{G}])$-generic" (where $\dot{G}$ is the canonical $\mathbb{P}$-name for the generic filter).*

Recall the following notion:

Definition 19 *Let $\mathbb{P}$ be a suborder of $\mathbb{Q}$. We say that* $\mathbb{P}$ is a regular suborder of $\mathbb{Q}$ *if the following conditions hold:*

1. *If $p \perp_{\mathbb{P}} r$, then $p \perp_{\mathbb{Q}} r$.*
2. *For every $q \in \mathbb{Q}$, there is $p \in \mathbb{P}$ such that for every $r \in \mathbb{P}$, if $r \leq p$, then r is compatible with q. In this situation, we say that p is a reduction of q to $\mathbb{P}$.*

If $\mathbb{P}$ is a regular suborder of $\mathbb{Q}$, then forcing with $\mathbb{Q}$ adds a $(V, \mathbb{P})$-generic filter. We will say that $\mathbb{P}$ and $\mathbb{Q}$ are *forcing equivalent* if they give rise to the same generic extensions.

Definition 20 *Let $\mathbb{P}$ and $\mathbb{Q}$ be partial orders. We say that $f : \mathbb{P} \longrightarrow \mathbb{Q}$* is a dense embedding *if for every $p, r \in \mathbb{P}$, the following conditions hold:*

1. *If $p \leq r$, then $f(p) \leq f(r)$.*
2. *If $p \perp r$, then $f(p) \perp f(r)$.*

3. The image of f is dense in $\mathbb{Q}$.

It is well-known that if there is a dense embedding between two partial orders, then they are forcing equivalent. See [24] for more information on regular suborders and dense embeddings.

The *Martin number* $\mathfrak{m}$ is the smallest cardinal κ for which there exists a c.c.c. forcing $\mathbb{P}$ and a family $\mathcal{D}$ of κ dense subsets of $\mathbb{P}$ such that there is no filter $G \subseteq \mathbb{P}$ that intersects every member of $\mathcal{D}$. Martin's Axiom (MA) is the statement that $\mathfrak{m} = \mathfrak{c}$ (where $\mathfrak{c}$ denotes the size of the continuum). The following is a very well-known result (see Theorem V.4.1 of [24]).

Theorem 21 *Let $\lambda > \omega$ be a regular cardinal such that $2^{<\lambda} = \lambda$. There is $\mathbb{P}$ a c.c.c. forcing of size λ such that $\mathbb{P} \Vdash$ "$\mathfrak{m} = \mathfrak{c} = \lambda$".*

6 Preliminaries on walks on ordinals

The method of *walks on ordinals* (introduced by the second author) is a powerful technique for constructing structures on ω_1. Virtually all interesting combinatorial objects on ω_1, including Aronszajn trees, Hausdorff gaps, Countryman lines, homogenous Eberlein compacta, colorings on ω_1 with strong combinatorial properties, L-spaces, Lindelöf groups with non-Lindelöf square, ultrafilters, Fréchet groups... and much more can be constructed using this method. This section will review the necessary result on walks on ordinals that are needed for our work. The reader may learn more of this method in the books [40] and [14]. Further applications can be found in [37], [42], [26], [30], [31], [20], [32], [33], [15], [28], [35], [36], [34] or [25] among many others.

We say that $\mathcal{C} = \{C_\alpha \mid \alpha \in \omega_1\}$ is a C-*sequence* if the following conditions hold for every $\alpha \in \omega_1$:

1. $C_\alpha \subseteq \alpha$.
2. $C_{\alpha+1} = \{\alpha\}$.
3. If α is a limit ordinal, then $C_\alpha = \{C_\alpha(n) \mid n \in \omega\}$ is a cofinal subset of α of order type ω.

Fix $\mathcal{C}$ a C-sequence. When we write $C_\alpha = \{C_\alpha(n) \mid n \in \omega\}$, we always implicitly assume that $C_\alpha(n) < C_\alpha(n+1)$ for every $n \in \omega$. For $\alpha < \beta$ two countable ordinals, the *step from β to α* is defined as $\mathsf{stp}(\alpha, \beta) = \mathsf{min}(C_\beta \setminus \alpha)$. The *upper trace of the walk from β to α* is define as $\mathsf{Tr}(\alpha, \beta) = \{\beta_0, ..., \beta_n\}$ where $\beta_0 = \beta$, $\beta_n = \alpha$ and $\beta_{i+1} = \mathsf{stp}(\alpha, \beta_i)$ ($\mathsf{Tr}(\alpha, \beta)$ is a finite set since there are no infinite decreasing sequences of ordinals). Note that The *full code of the walk* is defined as $\rho_0(\alpha, \beta) = \langle |C_{\beta_0} \cap \alpha|, |C_{\beta_1} \cap \alpha|, ..., |C_{\beta_{n-1}} \cap \alpha| \rangle$, which is an

element of $\omega^{<\omega}$. Define $\lambda(\alpha,\beta) = \mathsf{max}\{\bigcup C_{\beta_i} \cap \alpha \mid i < n\}$ and note that it is an ordinal smaller than α. Both the upper trace and the full code can be defined recursively:

$$\mathsf{Tr}(\alpha,\beta) = \{\beta\} \cup \mathsf{Tr}(\alpha,\mathsf{min}(C_\beta \setminus \alpha))$$

$$\rho_0(\alpha,\beta) = \langle |C_\beta \cap \alpha| \rangle^\frown \rho_0(\alpha,\mathsf{min}(C_\beta \setminus \alpha))$$

A simple, yet crucial result is the following:

Proposition 22 *Let $\alpha < \gamma < \beta$ be countable ordinals. The following conditions are equivalent:*

1. $\gamma \in \mathit{Tr}(\alpha,\beta)$.
2. $\mathit{Tr}(\alpha,\beta) = \mathit{Tr}(\gamma,\beta) \cup \mathit{Tr}(\alpha,\gamma)$.
3. $\lambda(\gamma,\beta) < \alpha$.
4. $\rho_0(\alpha,\beta) = \rho_0(\gamma,\beta)^\frown \rho_0(\alpha,\gamma)$.

Therefore, if $\lambda(\gamma,\beta) < \alpha$, then the walk from β to α decomposes as the walk from β to γ followed by the walk from γ to α. For β a countable ordinal, define $\rho_{0\beta} : \beta \longrightarrow \omega^{<\omega}$ by $\rho_{0\beta}(\alpha) = \rho_0(\alpha,\beta)$. The *Aronszajn tree induced by* ρ_0 is then defined as $T(\rho_0) = \{\rho_{0\beta} \restriction \alpha \mid \alpha \leq \beta < \omega_1\}$. In [40] it is proved that $T(\rho_0)$ is a Hausdorff, special Aronszajn tree that does not branch at limit levels. For convenience, we write $\triangle_{\mathcal{C}}(\alpha,\beta)$ instead of $\triangle_{T(\rho_0)}(\rho_{0\alpha},\rho_{0\beta})$. The following lemma is well-known, we prove it for completeness.

Lemma 23 *The set $\{\rho_{0\alpha} \mid \alpha \in \mathit{LIM}(\omega_1)\}$ is an antichain of $T(\rho_0)$.*

Proof. If α is a limit ordinal and $\xi < \alpha$, then $\rho_{0\alpha}(\xi)$ is a sequence of length 1 if and only if $\xi \in C_\alpha$. The conclusion of the lemma easily follows. ■

The following lemma is well-known (for example, it follows from Fact 2 in [30]). We provide a proof here for the convenience of the reader.

Lemma 24 *Let $\delta < \omega_1$. If $\xi < \delta$, then $\xi \leq \lambda(\alpha,\delta)$ for almost all $\alpha \in (\xi,\delta)$. In particular, if $\langle \delta_n \rangle \longrightarrow \delta$, then $\langle \lambda(\delta_n,\delta) \rangle \longrightarrow \delta$.*

Proof. Suppose there is an infinite set $A \subseteq (\xi,\delta)$ such that $\lambda(\alpha,\delta) < \xi$ for all $\alpha \in A$. By Proposition 22, it follows that $A \subseteq \mathsf{Tr}(\xi,\delta)$. Hence, $\mathsf{Tr}(\xi,\delta)$ is infinite, which is a contradiction. ■

7 Preliminaries on Baumgartner's club forcing

Baumgartner's club forcing is the standard method for adding a club subset to ω_1 using finite conditions. It was introduced and studied by Baumgartner in [12]. This is a very interesting forcing notion with several applications and is a natural example of a proper forcing that is not ω-proper. The finiteness of its conditions facilitates various amalgamation arguments, which in turn yield rich combinatorial properties for the generic club. This section reviews the fundamentals of this forcing notion. For further reading, the reader is referred to [21], [12], [2], [48], [49], [4], [7]. By $\mathsf{Int}(\omega_1)$ we denote the set $\{(\alpha,\beta] \mid \alpha<\beta<\omega_1\}$.

Definition 25 Baumgartner's club forcing *(denoted by $\mathbb{BC}$) consists of all pairs $p=(a,I)$ with the following properties:*

1. $a \in [\omega_1]^{<\omega}$.
2. $I \subseteq \mathsf{Int}(\omega_1)$ *is finite.*
3. $a \cap \bigcup I = \emptyset$.

Given $p=(a_p,I_p)$ and $q=(a_q,I_q)$ in $\mathbb{BC}$, define $p \leq q$ if $a_q \subseteq a_p$ and $\bigcup I_q \subseteq \bigcup I_p$. This is the presentation of Baumgartner's club forcing from [48] (there are other several presentations, all yielding forcing equivalent notions). For convenience, we will say that α *is above* p if $a_p \subseteq \alpha$ and $\bigcup I_p \subseteq \alpha$.

If $G \subseteq \mathbb{BC}$ is a generic filter, the *Baumgartner's generic club* is defined as $C_{gen}=\bigcup\{a_p \mid p\in G\}$. It is easy to verify that C_{gen} is forced to be a club.

Since $\mathbb{BC}$ is a partial order of size ω_1, then it satisfies the ω_2-chain condition. Furthermore, it is a proper forcing. In fact, identifying the generic conditions over a given countable elementary submodel is a very easy:

Proposition 26 ([12]) *Let M be a countable elementary submodel and $p=(a_p,I_p)\in\mathbb{BC}$. The following conditions are equivalent:*

1. *p is an $(M,\mathbb{BC})$-generic condition.*
2. *$\delta_M \in a_p$ (where $p=(a_p,I_p)$).*

With this last result and Propositions 17 and 18, we get the following:

Corollary 27 *Let $\dot{\mathbb{Q}}$ be a $\mathbb{BC}$-name for a c.c.c. partial order and M a countable elementary submodel with $\dot{\mathbb{Q}}\in M$. For $(p,\dot{x})\in\mathbb{BC}*\dot{\mathbb{Q}}$, the following conditions are equivalent:*

1. *$(p,\dot{x})$ is $(M,\mathbb{BC}*\dot{\mathbb{Q}})$-generic.*

2. $\delta_M \in a_p$.

The properness and the ω_2-chain condition of $\mathbb{BC}$ imply that no cardinals are collapsed after performing a forcing extension. Let $p, r \in \mathbb{BC}$. Note that if p and r are compatible, then $p \wedge r = (a_r \cup a_p, I_p \cup I_q)$ is the largest common extension of both p and r. For $p, q \in \mathbb{BC}$, define $q \sqsubseteq p$ if there is $\delta \in a_p$ such that $a_q = a_p \cap \delta$ and $I_q = \{(\alpha, \beta] \in I_p \mid \beta < \delta\}$. Note that if $q \sqsubseteq p$, then $p \leq q$. If M is a countable elementary submodel and p is an $(M, \mathbb{BC})$-generic condition, define $p_M = (a_p \cap M, I_p \cap M)$. We have the following:

Proposition 28 *Let M be a countable elementary submodel of some $H(\kappa)$ and p an $(M, \mathbb{BC})$-generic condition. The following holds:*

1. $p_M = (a_p \cap M, I_p \cap M) \sqsubseteq p$. *In particular,* $p \leq p_M$.
2. $p_M \in M$.
3. *If $r \in M$ and $r \leq p_M$, then r and p are compatible.*

This last proposition implies that $\mathbb{BC}$ is *strongly proper* (see [41]). How can we add many Baumgartner clubs? Since $\mathbb{BC}$ is not c.c.c., a finite support product (which in this case, is equivalent to a finite support iteration) would collapse ω_1. What about the countable support product? In general, if $\mathbb{P}$ is a forcing notion and X is a set of ordinals, by $\prod^{CS}_{\alpha \in X} \mathbb{P}$ we denote the *countable support product of* $\mathbb{P}$, which consists of all countable partial functions from X to $\mathbb{P}$. Although the countable support product is often useful for some applications (for example, to add many Sacks or Silver reals), it is of no use in this context. In [4] it is shown that the countable support product also collapses ω_1 (this is an instance of a more general phenomenon: Hrušák proved in [19] that any countable support product of infinitely many proper forcing of size at most $\mathfrak{c}$ that add unbounded reals will collapse ω_1). Of course, we could use a countable support iteration. However, this approach not only restricts us to models where the continuum is at most ω_2, but also causes us to lose the "finite nature" that is such an appealing feature of Baumgartner's original forcing. Fortunately for us, in [4] Asperó found a method for adding many Baumgartner clubs without collapsing cardinals. This method behaves much like a product (one can think of it as lying between the finite and countable support products), while still preserving some of the "finite essence" of the Baumgartner club forcing. We will now review this method, all the result are due to Asperó and can be found in [4].

Definition 29 $\mathbb{BC}_0$ *denotes the set of all $p = (a, I) \in \mathbb{BC}$ such that $a \subseteq IND(\omega_1)$ and $I = \emptyset$. Elements of $\mathbb{BC}_0$ are often called* pure conditions.

It is easy to see that any finite set of conditions in $\mathbb{BC}_0$ has a lower bound in $\mathbb{BC}_0$. For X a set of ordinals, by $\mathbb{BC}^X$ we denote the countable support product $\prod^{CS}_{\alpha\in X} \mathbb{BC}$.

Definition 30 *Let $X \subseteq \mathsf{OR}$, $p \in \mathbb{BC}^X$ and $\gamma \in \omega_1$.*

1. *The* support of p *is* $\mathsf{sup}(p) = \{\alpha \in X \mid p(\alpha) \neq 1_{\mathbb{BC}}\}$.
2. *The* pure support of p *is* $\mathsf{sup}_{\mathsf{pr}}(p) = \{\alpha \in \mathsf{sup}(p) \mid p(\alpha) \in \mathbb{BC}_0\}$.
3. *The* impure support of p *is* $\mathsf{sup}_{\mathsf{imp}}(p) = \mathsf{sup}(p) \setminus \mathsf{sup}_{\mathsf{pr}}(p)$.
4. *The* full image of p *is* $\mathsf{Fim}(p) = \bigcup\{a_{p(\alpha)} \mid \alpha \in \mathsf{sup}(p)\}$.
5. *The* γ-support of p *is* $\mathsf{sup}_\gamma(p) = \{\alpha \in \mathsf{sup}(p) \mid \gamma \in a_{p(\alpha)}\}$.

If γ is an indecomposable ordinal, we denote by γ^+ the first indecomposable ordinal above γ. Although this notation is also used for successor cardinals, no confusion should arise. We can now define Asperó's forcing for adding many Baumgartner clubs.

Definition 31 *Let $X \subseteq \mathsf{OR}$. The Asperó product $\mathbb{BC}^{(X)}$ is the suborder of $\mathbb{BC}^X$ consisting of all $p \in \mathbb{BC}^X$ with the following properties:*

1. *Both $\mathsf{sup}_{\mathsf{imp}}(p)$ and $\mathsf{Fim}(p)$ are finite.*
2. *If δ is an indecomposable ordinal, then $\mathsf{OT}(\mathsf{sup}_\delta(p)) < \delta^+$.*

The order on $\mathbb{BC}^{(X)}$ is inherited from $\mathbb{BC}^X$. Although our definition of the Asperó product differs slightly from the one in [4], these differences are inconsequential.

Let $p, q \in \mathbb{BC}^{(X)}$. Note that p and q are compatible in $\mathbb{BC}^X$ if and only if they are compatible in $\mathbb{BC}^{(X)}$. It is straightforward to check that if $Y \subseteq X$ and $p \in \mathbb{BC}^{(X)}$, then the restriction $p \restriction Y$ is in $\mathbb{BC}^{(Y)}$. We also have the following:

Lemma 32 *Let $Y \subseteq X$. The forcing $\mathbb{BC}^{(Y)}$ is a regular suborder of $\mathbb{BC}^{(X)}$. Moreover, if $p \in \mathbb{BC}^{(X)}$, then $p \leq p \restriction y$ and $p \restriction y$ is a reduction of p to $\mathbb{BC}^{(Y)}$.*

It is clear that $\mathbb{BC}^{(\{\gamma\})}$ is isomorphic to $\mathbb{BC}$, hence $\mathbb{BC}^{(X)}$ adds many Baumgartner clubs.

Lemma 33 *Let $X, Y \subseteq \mathsf{OR}$.*

1. *If* $X \cap Y = \emptyset$, *then* $\mathbb{BC}^{(X \cup Y)}$ *is isomorphic to* $\mathbb{BC}^{(X)} \times \mathbb{BC}^{(Y)}$. *In particular, they are forcing equivalent.*

2. *If* $\mathsf{OT}(X) = \mathsf{OT}(Y)$, *then* $\mathbb{BC}^{(X)}$ *and* $\mathbb{BC}^{(Y)}$ *are isomorphic.*

Regarding properness, the following result is the analogue of Proposition 26:

Theorem 34 (Asperó, [4]) *Let* $X \subseteq \mathsf{OR}$, M *a countable elementary submodel with* $X \in M$ *and* $p \in \mathbb{BC}^{(X)}$. *The following are equivalent:*

1. p *is* $(M, \mathbb{BC}^{(X)})$*-generic.*

2. $M \cap X \subseteq \mathsf{sup}_{\delta_M}(p)$.

Although this last result is not stated explicitly in [4], it follows directly from the proof of Lemma 3.7 of that paper.

Corollary 35 (Asperó, [4]) *Let* $X \subseteq \mathsf{OR}$. *The Asperó product* $\mathbb{BC}^{(X)}$ *is proper and has the* ω_2*-*c.c. *In particular, it does not collapse cardinals.*

Regarding the size of the continuum after forcing with $\mathbb{BC}^{(X)}$, we quote the following result:

Proposition 36 *Let* $\kappa \geq \omega_2$ *be a cardinal such that* $\kappa^{\omega_1} = \kappa$. *If* $G \subseteq \mathbb{BC}^{(\kappa)}$ *is a generic filter, then* $V[G]$ *is a model of* $\kappa = 2^{\omega} = 2^{\omega_1}$.

For similar constructions to $\mathbb{BC}^{(X)}$, we refer the reader to [5] and [6].

8 The square-bracket operations

This section reviews the definitions of the three square-bracket operations from chapter 5 of the book [40], which are the protagonists of this work. A key property of these operations is that every uncountable subset of ω_1 realizes club many values. We will study the Ramsey clubs associated with each operation.

The first square-bracket operation is defined using walks on ordinals. Let $\mathcal{C}$ be a C-sequence. Define $[\cdot\cdot]_{\mathcal{C}} : [\omega_1]^2 \longrightarrow \omega_1$ such that for every $\alpha < \beta$:

$$\begin{array}{|lcl|} \hline [\alpha\beta]_{\mathcal{C}} & = & \mathsf{min}\{\mathsf{Tr}(\triangle_{\mathcal{C}}(\alpha,\beta),\beta) \setminus \alpha\} \\ & = & \mathsf{min}\{\mathsf{Tr}(\alpha,\beta) \cap \mathsf{Tr}(\triangle_{\mathcal{C}}(\alpha,\beta),\beta)\} \\ \hline \end{array}$$

In this way, when walking from β to $\triangle_{\mathcal{C}}(\alpha,\beta)$, the smallest point encountered that is greater than or equal to α is precisely $[\alpha\beta]_{\mathcal{C}}$. Equivalently, it is the minimal point common to both the walk from β to α and the walk from β to $\triangle_{\mathcal{C}}(\alpha,\beta)$ (see [40]). It is worth pointing out that the definition in [40] uses $\triangle_{\mathcal{C}}(\alpha,\beta)-1$ instead of $\triangle_{\mathcal{C}}(\alpha,\beta)$. This distinction is irrelevant to our arguments.

The second square-bracket operation is defined with a sequence of (distinct) reals $\mathcal{R}=\{r_\alpha \mid \alpha\in\omega_1\}\subseteq 2^\omega$ and a sequence $e=\langle e_\beta \mid \beta\in\omega_1\rangle$ where $e_\beta : \beta+1\longrightarrow\omega$ is an injective function such that $e_\beta(\beta)=0$. For convenience we will call a sequence with this properties a *nice sequence*. For α and β countable ordinals, denote $\triangle_{\mathcal{R}}(\alpha,\beta)=\triangle(r_\alpha,r_\beta)$. Define $[\cdot\cdot]_{\mathcal{R},e}:[\omega_1]^2\longrightarrow\omega_1$ such that for every $\alpha<\beta$:

$$\boxed{[\alpha\beta]_{\mathcal{R},e}=\mathsf{min}\{\xi\in[\alpha,\beta]\mid e_\beta(\xi)\leq\triangle_{\mathcal{R}}(\alpha,\beta)\}}$$

The computation of $[\alpha\beta]_{\mathcal{R},e}$ proceeds by finding the smallest m for which $r_\alpha(m)\neq r_\beta(m)$ and then taking the least $\xi\in[\alpha,\beta]$ such that $e_\beta(\xi)\leq m$.

The third square-bracket operation is defined with a Hausdorff, special Aronszajn tree T and an specialization mapping $a:T\longrightarrow\omega$. This time, the operation is defined on T instead of ω_1. Given $s,t\in T$, define $s\wedge t$ as the largest lower bound for both s and t. Note that $s\wedge t$ is well-defined since T is a Hausdorff tree. For $s,t\in T$, let $[s\ ,t]=\{z\in T\mid s\leq_T z\leq_T t\}$.

Define $[\cdot\cdot]_{T,a}:[T]^2\longrightarrow T$ such that for every $s,t\in T$ with $|s|\leq|t|$:

$$\boxed{[st]_{T,a}=\mathsf{min}\{z\in[t\restriction|s|\ ,t]\mid z=t\vee a(z)\leq a(s\wedge t)\ \}}$$

To compute $[st]_{T,a}$, first set $m=a(s\wedge t)$. Then, find the first node $z\leq_T t$ with $|z|\geq|s|$ and $a(z)\leq m$. If no node such z exists, let $[st]_{T,a}=t$. Since the operation is not defined on ω_1, in this context the vertex set of the graph $G([\cdot\cdot],X)$ (for $X\subseteq\omega_1$) is the tree T. Two nodes s and t are connected by an edge if the height of $[st]_{T,a}$ is in X.

The reader is encouraged to review the proofs that the square-bracket operations defined above all have the property that every uncountable subset of ω_1 realizes club-many distinct values (Lemmas 5.1.4, 5.4.1, and 5.5.1 of [40]), as ideas from these proofs will be used frequently.

9 Finding Ramsey clubs

Let $[\cdot\cdot]$ be a square-bracket operation and $C\subseteq\omega_1$ a club. Recall that C is $[\cdot\cdot]$-*Ramsey* if there exists an uncountable $W\subseteq\omega_1$ such that $\{[\alpha\beta]\mid\alpha,\beta\in W\}\subseteq C$

(in other words, W induces a complete subgraph of $G([\cdot\cdot], C)$). In this section, we prove that for every club C, there exists a choice of parameters for the square-bracket operations $[\cdot\cdot]$ such that C is $[\cdot\cdot]$-Ramsey.

Theorem 37 *Let $C \subseteq \omega_1$ be a club, $\mathcal{R} = \{r_\alpha \mid \alpha \in \omega_1\} \subseteq 2^\omega$ a sequence of reals and T a Hausdorff special Aronszajn tree. There exist a C-sequence $\mathcal{C}$, a nice sequence e and a specializing mapping $a : T \longrightarrow \omega$ such that C is $[\cdot\cdot]$-Ramsey, where $[\cdot\cdot]$ is any of the operations $[\cdot\cdot]_{\mathcal{C}}$, $[\cdot\cdot]_{\mathcal{R},e}$ or $[\cdot\cdot]_{T,a}$.*

Proof. We may assume $C \subseteq \mathsf{LIM}(\omega_1)$ and recall that C' denotes the accumulation points of C. We first construct the C-sequence $\mathcal{C}$. Find a sequence $\mathcal{C} = \{C_\alpha \mid \alpha \in \omega_1\}$ such that for every $\alpha \in \omega_1$:

1. $C_{\alpha+1} = \{\alpha\}$.
2. If $\alpha \in C'$, then $C_\alpha = \{C_\alpha(n) \mid n \in \omega\} \subseteq C$ is a cofinal subset of α of order type ω.
3. If α is a limit ordinal, but $\alpha \notin C'$, then $C_\alpha = \{C_\alpha(n) \mid n \in \omega\}$ is a cofinal subset of α of order type ω and $C_\alpha(0) = \bigcup(C \cap \alpha)$.

We emphasize that for α limit: In case $\alpha \in C'$, then C_α is contained in C, while if $\alpha \notin C'$, then at most the first point of C_α is in C. Moreover, if $C \cap \alpha \neq \emptyset$, then $C_\alpha(0)$ is in C. We claim that if $\alpha, \beta \in C'$ (with $\alpha < \beta$), then $\mathsf{Tr}(\alpha, \beta) \subseteq C$. We proceed by contradiction, let $\mathsf{Tr}(\alpha, \beta) = \{\beta_0, ..., \beta_{n+1}\}$ (where $\beta_0 = \beta$, $\beta_{n+1} = \alpha$ and $\beta_{i+1} = \mathsf{stp}(\alpha, \beta_i)$). Let i be the first such that $\beta_{i+1} \notin C$. Since $\beta_i \in C$, it follows that β_i is a limit ordinal. Furthermore, since $\beta_{i+1} \in C_{\beta_i}$, it follows that C_{β_i} is not a subset of C, hence $\beta_i \notin C'$. We know that $C \cap \beta_i \neq \emptyset$ (because α is in the intersection), so $C_{\beta_i}(0) = \bigcup(C \cap \beta_i)$ and then $\alpha \leq C_{\beta_i}(0)$. We get that $\beta_{i+1} = C_{\beta_i}(0)$, which is in C, so we get a contradiction. It follows that $\{[\alpha\beta]_{\mathcal{C}} \mid \alpha, \beta \in C'\} \subseteq C$, so C is $[\cdot\cdot]_{\mathcal{C}}$-Ramsey.

We now find the nice sequence e. Find $e = \langle e_\beta \mid \beta \in \omega_1 \rangle$ as follows: If $\beta \notin C$, then $e_\beta : \beta + 1 \longrightarrow \omega$ can be any injective function such that $e_\beta(\beta) = 0$. In case $\beta \in C$, we choose an injective function e_β with the following property:

For every $\gamma \in \beta$, if $\gamma \notin C$ and $\delta = \bigcup(C \cap \gamma)$, then $e_\beta(\delta) < e_\beta(\gamma)$

We claim that if $\alpha, \beta \in C$ (with $\alpha < \beta$), then $[\alpha\beta]_{\mathcal{R},e} \in C$. Assume this is not the case. Let $\gamma = [\alpha\beta]_{\mathcal{R},e}$ and suppose it is not in C. It follows that $\alpha < \gamma < \beta$. Since $\alpha \in C$, it follows that $\delta = \bigcup(C \cap \gamma)$ is in C as well. Note that $\alpha \leq \delta < \gamma$ and $e_\beta(\delta) < e_\beta(\gamma) \leq \triangle_{\mathcal{R}}(\alpha, \beta)$. In this way, $e_\beta(\delta) < \triangle_{\mathcal{R}}(\alpha, \beta)$, but this is a contradiction, since γ is the smallest element of the set $\{\xi \in [\alpha, \beta] \mid e_\beta(\xi) \leq \triangle_{\mathcal{R}}(\alpha, \beta)\}$. We conclude that $\{[\alpha\beta]_{\mathcal{R},e} \mid \alpha, \beta \in C\} \subseteq C$, so C is $[\cdot\cdot]_{\mathcal{R},e}$-Ramsey.

It remains to find the specializing mapping. Let $f : T \longrightarrow P$ be an specialization mapping, where P is the set of prime numbers. Define $a : T \longrightarrow \omega$ such that for every $s \in T$, the following holds:

1. If $|s| \in C$, then $a(s) = f(s)$.
2. If $|s| \notin C$ and $\delta = \bigcup (C \cap |s|)$, then $a(s) = f(s \restriction \delta)^{f(s)}$.

Note that $a(s)$ is a prime number if and only if $|s| \in C$. We claim that a is a specialization mapping. Proceed by contradiction, assume there are $s, t \in T$ such that $t <_T s$ and $a(s) = a(t)$. If $a(t)$ is prime, it follows that $a(t) = f(t)$ and $a(s) = f(s)$, which contradicts that f is an specialization mapping. We now assume $a(t)$ is not a prime number. Let $\delta = \bigcup (C \cap |s|)$. If $\delta \leq |t|$, then $t \restriction \delta = s \restriction \delta$ and $a(t) = f(t \restriction \delta)^{f(t)}$, while $a(s) = f(s \restriction \delta)^{f(s)}$. It follows that $f(t) = f(s)$, which again contradicts that f is a specialization mapping. The only remaining case is when $|t| < \delta$. Let $\gamma = \bigcup (C \cap |s|)$, which is smaller that δ. It follows that $a(s) = f(s \restriction \delta)^{f(s)}$ and $a(t) = f(t \restriction \gamma)^{f(t)}$. Since $t \restriction \gamma <_T s \restriction \delta$, it follows that $f(s \restriction \delta)$ and $f(t \restriction \gamma)$ are distinct prime numbers, so it is impossible that $a(s) = a(t)$.

We now show that if $s, t \in T \restriction C$ and $|s| < |t|$, then $|[st]_{T,a}| \in T$. Let $z = [st]_{T,a}$ and assume that $|z| \notin C$. It follows that $|s| < |z|$. Let $\delta = \bigcup (C \cap |z|)$, which is in C and $|s| \leq \delta$. We know that $a(z) = f(z \restriction \delta)^{f(z)}$. Since $\delta \in C$, we have that $a(z \restriction \delta) = f(z \restriction \delta)$, so $a(z \restriction \delta) < a(z) < a(s \wedge t)$. This contradicts the minimality of z. In this way, if U is an uncountable subset of $T \restriction C$ such that any two different nodes have different height, then $\{|[st]_{T,a}| \mid s, t \in U\} \subseteq C$, so C is $[\cdot\cdot]_{T,a}$-Ramsey. ■

Let $\mathcal{C} = \{C_\alpha \mid \alpha \in \omega_1\}$ be a C-sequence. We say that $\mathcal{C}$ is a *successor* C-*sequence* if for every α limit, every element of C_α is a successor ordinal. Such sequences are sometimes useful (see [40]). The C-sequence found in Theorem 37 is not a successor C-sequence. However, one can be obtained if desired.

Proposition 38 *Let $C \subseteq \omega_1$ be a club. There exists a successor C-sequence $\mathcal{D}$ such that C is $[\cdot\cdot]_{\mathcal{D}}$-Ramsey.*

Proof. We may assume $C \subseteq \mathsf{LIM}(\omega_1)$. Let $\mathcal{C}$ be the C-sequence obtained in the proof of Theorem 37. Define a new C-sequence $\mathcal{D} = \{D_\alpha \mid \alpha \in \omega_1\}$ where $D_\alpha(n) = C_\alpha(n) + 1$ for every α limit and $n \in \omega$. We claim that if $\alpha, \beta \in C'$ (with $\alpha < \beta$), then $[\alpha\beta] = [\alpha\beta]_{\mathcal{D}} \in C$. If $[\alpha\beta]$ is either α or β, there is nothing to do, so assume that $\alpha < [\alpha\beta] < \beta$. In particular, $\alpha \notin \mathsf{Tr}(\triangle, \beta)$, where $\triangle = \triangle_{\mathcal{D}}(\alpha, \beta)$. Let $\mathsf{Tr}(\triangle, \beta) = \{\beta_0, ..., \beta_i, \beta_{i+1}, ..., \beta_{n+1}\}$ where $\beta_0 = \beta$, $\beta_{i+1} = [\alpha\beta]$ and $\beta_{n+1} = \triangle$. Note that $\beta_{i+2} < \alpha < \beta_{i+1} = [\alpha\beta]$. If $[\alpha\beta]$ was a successor ordinal, we would have that $[\alpha\beta] - 1 < \alpha < [\alpha\beta]$, which is clearly impossible, so $[\alpha\beta]$ is a limit ordinal. Hence $\beta_i = [\alpha\beta] + 1$. Let $j < i$ be the largest such that β_j is a limit ordinal.

Claim 39 $\beta_{j+2} \in C$.

First consider the case where $\beta_j \in C'$. In this way, there is $\gamma \in C$ such that $\beta_{j+1} = \gamma+1$, so $\beta_{j+2} = \gamma$. In case $\beta_j \notin C'$, we know that $D_{\beta_j}(0) = \bigcup (C \cap \beta_j)+1$, which is in C and larger than α. It follows that $\beta_{j+1} = \bigcup (C \cap \beta_j) + 1$ and $\beta_{j+2} = \bigcup (C \cap \beta_j)$. This finishes the proof of the claim.

We now claim that $j = i - 1$. If this was not the case, we would have that $j < i - 1$ and then:

$$\begin{array}{rcl} j < i-1 & \implies & j+2 < i+1 \\ & \implies & j+2 \leq i \\ & \implies & j+2 < i \end{array}$$

Where the last implication holds because β_i is a successor and β_{j+2} is limit. But this is a contradicts the maximality of j. In this way, we have that $[\alpha\beta] = \beta_{i+1} = \beta_{j+2} \in C$. It follows that $\{[\alpha\beta] \mid \alpha, \beta \in C'\} \subseteq C$, so C is $[\cdot\cdot]_{\mathcal{D}}$-Ramsey. ■

10 Destroying square-bracket operations

We will say that a square-bracket operation $[\cdot\cdot]$ has the *Ramsey club property* if every club subset of ω_1 is $[\cdot\cdot]$-Ramsey. In this section we will prove that a Baumgartner club can not be $[\cdot\cdot]$-Ramsey for any square-bracket operation from the ground model. It follows that it is consistent some square-bracket operations do not have the Ramsey club property.

Theorem 40 $\mathbb{BC}$ *forces that the generic club is not* $[\cdot\cdot]$*-Ramsey, for any square-bracket operation* $[\cdot\cdot]$ *in* V.

Proof. Let $p_0 \in \mathbb{BC}$ and $\dot{B}$ such that $p_0 \Vdash$ "$\dot{B} \in [\omega_1]^{\omega_1}$". We want to extend p_0 to a condition that forces that there are $\alpha, \beta \in \dot{B}$ such that $[\alpha\beta]$ is not in $\dot{C}_{gen}$. Let $\langle M_\alpha \mid \alpha \leq \omega \rangle$ be a continuous chain of elementary submodels such that $p_0, \dot{B} \in M_0$, as well as the square-bracket operation we want to take care of. For convenience, we introduce the following notation:

1. $M = M_\omega$.
2. $\delta = \delta_M$.
3. $\delta_n = \delta_{M_n}$ for $n \in \omega$.
4. $p_0 = (a_0, I_0)$.

Note that $\delta = \bigcup_{n \in \omega} \delta_n$. Let $p_1 = (a_0 \cup \{\delta\}, I_0)$. We know that $p_1 \leq p_0$ and is an $(M, \mathbb{BC})$-generic condition (see Proposition 26). Since $\dot{B}$ is forced to be

uncountable, we can now find $p \leq p_1$ and $\beta > \delta$ such that $p \Vdash$ "$\beta \in \dot{B}$". Write $p = (a_p, I_p)$. By Proposition 28, we know that $p_M = (a_p \cap M, I_p \cap M)$ is in M and is weaker than p. Denote $p_M = (a_M, I_M)$.

The proof now splits into cases depending on the type of square-bracket operation under consideration. We begin with the case of the operation induced by a C-sequence, say $\mathcal{C}$. For notational convenience in this part of the proof, we will write $[\gamma\xi]$ for $[\gamma\xi]_{\mathcal{C}}$ and $\triangle(\gamma, \xi)$ for $\triangle_{\mathcal{C}}(\gamma, \xi)$ (for any $\gamma, \xi \in \omega_1$). We assume that $\mathcal{C} \in M_0$.

Let $g : T(\rho_0) \longrightarrow \omega$ be a specialization mapping with $g \in M_0$. Find $n \in \omega$ such that $\lambda(\delta, \beta)$ and p_M are in M_n, then choose $\xi < \delta_n$ such that $\lambda(\delta, \beta)$, $\lambda(\delta_n, \delta) < \xi$ and ξ is above p_M. Denote $s = \rho_{0\beta} \restriction \xi$ and let $i \in \omega$ such that $g(\rho_{0\beta} \restriction \delta_n) = i$. Define A as the set of all $t \in T(\rho_0)$ that satisfy the following properties:

1. $s \subseteq t$.
2. $g(t) = i$.
3. For every $\eta < \omega_1$, there is $q \in \mathbb{BC}$ for which:
   (a) $p_M \sqsubseteq q$.
   (b) There is $\alpha > \eta$ such that $q \Vdash$ "$\alpha \in \dot{B}$" and $t \subseteq \rho_{0\alpha}$.

It follows that $A \in M_n$, since all the parameters used to define in A belong to M_n. Moreover, note that $\rho_{0\beta} \restriction \delta_n \in A$, which implies that $A \neq \emptyset$. By elementarity., we can find $t \in A \cap M_n$. It follows that $s \subseteq t, \rho_{0\beta} \restriction \delta_n$ and that t and $\rho_{0\beta} \restriction \delta_n$ are incompatible in $T(\rho_0)$, since $g(t) = g(\rho_{0\beta} \restriction \delta_n)$. Let $\triangle = \triangle(\rho_{0\beta}, t)$. We look at $\mathsf{Tr}(\triangle, \delta_n)$. Since $t \in A$, we can find $q = (a_q, I_q) \in \mathbb{BC}$ and α such that:

1. $\alpha, q \in M_n$.
2. $\mathsf{Tr}(\triangle, \delta_n) \setminus \{\delta_n\} \subseteq \alpha$.
3. $p_M \sqsubseteq q$.
4. $q \Vdash$ "$\alpha \in \dot{B}$" and $t \subseteq \rho_{0\alpha}$.

It follows that $\triangle = \triangle(\alpha, \beta)$. Pick $\varepsilon < \delta_n$ that is above q. Define $r = (a_p \cup a_q, I_p \cup I_q \cup \{(\varepsilon, \delta_n]\})$. It follows that r extends p and q, it is $(M, \mathbb{BC})$-generic but not $(M_n, \mathbb{BC})$-generic. Note that $r \Vdash$ "$\alpha, \beta \in \dot{B}$" and $r \Vdash$ "$\delta_n \notin \dot{C}_{gen}$". We will now compute $[\alpha\beta]$. Recall that $\lambda(\delta, \beta), \lambda(\delta_n, \delta) < \triangle$. By Proposition 22 we get that:

$$\mathsf{Tr}(\triangle,\beta) \quad = \quad \mathsf{Tr}(\delta,\beta)\cup\mathsf{Tr}(\delta_n,\delta)\cup\mathsf{Tr}(\triangle,\delta_n)$$

Since $\alpha<\delta_n$ and $\mathsf{Tr}(\triangle,\delta_n)\setminus\{\delta_n\}\subseteq\alpha$, it follows that $[\alpha\beta]=\delta_n$, which is forced by r to be outside $\dot{C}_{gen}$.

We now consider the case of the operation induced by a sequence of reals $\mathcal{R}=\{r_\alpha\mid\alpha\in\omega_1\}\subseteq 2^\omega$ and $e=\langle e_\beta\mid\beta\in\omega_1\rangle$ a nice sequence. For notational convenience in this part of the proof, we will write $[\gamma\xi]$ for $[\gamma\xi]_{\mathcal{R},e}$ and $\triangle(\gamma,\xi)$ for $\triangle_{\mathcal{R}}(\gamma,\xi)$ (for any $\gamma,\xi\in\omega_1$). We are assuming that $\mathcal{R},e\in M_0$.

Find $n\in\omega$ such that $p_M\in M_n$. Define $m=e_\beta(\delta_n)$, $s=r_\beta\restriction m$ and $F=\{\xi<\beta\mid e_\beta(\xi)\leq m\}\cap M_n$. Since F is a finite set, we have that $F\in M_n$. Choose $\eta<\delta_n$ such that $F\subseteq\eta$ and η is above p_M. Let A be the set of all $t\in 2^{<\omega}$ such that there is $q\in\mathbb{BC}$ with the following properties:

1. $s\subseteq t$.
2. $p_M\sqsubseteq q$.
3. $q\Vdash$"$\exists^{\omega_1}\alpha\in\dot{B}\,(r_\alpha\in\langle t\rangle)$".

Note that if $k>m$, then $r_\beta\restriction k\in A$, as witnessed by p. Since $A\in M_n$ while r_β is not, we can find $t\in A$ for which $r_\beta\notin\langle t\rangle$. Denote $\triangle=\triangle(t,r_\beta)\geq m$. By elementarity., we can find $q_0\in\mathbb{BC}\cap M_n$ such that $p_M\sqsubseteq q_0$ and $q_0\Vdash$"$\exists^{\omega_1}\alpha\in\dot{B}\,(r_\alpha\in\langle t\rangle)$". Let $H=\{\xi<\beta\mid e_\beta(\xi)\leq\triangle\}\cap M_n$. We can now find $q=(a_q,I_q)\in M_n$ extending q_0 and $\alpha\in M_n$ such that $q\Vdash$"$\alpha\in\dot{B}$", $r_\alpha\in\langle t\rangle$ and $H\subseteq\alpha$. Note that $\triangle=\triangle(\alpha,\beta)$. Choose $\varepsilon\in M_n$ that is above q and define $r=(a_p\cup a_q,I_p\cup I_q\cup\{(\varepsilon,\delta_n]\})$. It follows that r extends p and q, it is $(M,\mathbb{BC})$-generic but not $(M_n,\mathbb{BC})$-generic. It follows that $r\Vdash$"$\alpha,\beta\in\dot{B}$" and $r\Vdash$"$\delta_n\notin\dot{C}_{gen}$". Moreover, it is easy to see that $[\alpha\beta]=\delta_n$, which is forced by r to be outside $\dot{C}_{gen}$.

The last case is when the operation is induced by a Hausdorff, special Aronszajn tree T and $a:T\longrightarrow\omega$ a specialization mapping. For notational convenience, we now write $[st]$ for $[st]_{T,r}$. We can assume that $T,a\in M_0$. One more change is needed, now $\dot{B}$ is forced to be an uncountable subset of T and $\beta>\delta$ is the height of some $t\in T$ such that $p\Vdash$"$t\in\dot{B}$".

Choose $n\in\omega$ such that $p_M\in M_n$. We can now define $m=a(t\restriction\delta_n)$ and $F=\{\eta<\beta\mid a(t\restriction\eta)\leq m\}\cap M_n$. Since F is a finite set, we have that $F\in M_n$. Choose $\xi<\delta_n$ such that $F\subseteq\xi$ and ξ is above p_M. Let A be the set of all $s\in T$ such that there is $q\in\mathbb{BC}$ with the following properties:

1. $t\restriction\xi\leq_T s$.
2. $p_M\sqsubseteq q$.

3. $q \Vdash$ "$s \in \dot{B}$".

It follows that $A \in M_n$ and $t \in A$. Apply Proposition 15 and find $\gamma \in M$ and $w \in T_\gamma$ such that $w \perp t$ and there are uncountably many $s \in A$ such that $w \leq_T s$. We look at $w \wedge t$. Since $\xi < |w \wedge t|$, it follows that $a(w \wedge t) > m = a(t \restriction \delta_n)$. Define $G = \{\eta < \beta \mid a(t \restriction \eta) \leq a(w \wedge t)\} \cap M_n$. Once again, $G \in M_n$ since it is a finite subset of M_n. We can now find $s \in A \cap M_n$ and $q \in \mathbb{BC} \cap M_n$ with the following properties:

1. $w \leq_T s$.
2. $|s| > \eta$ for every $\eta \in G$.
3. $p_M \sqsubseteq q$.
4. $q \Vdash$ "$s \in \dot{B}$".

Note that $s \wedge t = w \wedge t$. Choose $\varepsilon \in M_n$ that is above q and define $r = (a_p \cup a_q, I_p \cup I_q \cup \{(\varepsilon, \delta_n]\})$. It follows that r extends p and q, it is $(M, \mathbb{BC})$-generic but not $(M_n, \mathbb{BC})$-generic. Note that $r \Vdash$ "$s, t \in \dot{B}$" and $r \Vdash$ "$\delta_n \notin \dot{C}_{gen}$". Moreover, we have that $[st] = t \restriction \delta_n$, whose height is forced by r to be outside $\dot{C}_{gen}$. ■

We now want to prove the consistency of Martin Axiom (MA) with the existence of square-bracket operations that fail to have the Ramsey club property. To establish this result, we aim to show that the Baumgartner club causes this property to fail so badly (for square-bracket operations from the ground model) that it can not be fixed by a c.c.c. forcing. Unfortunately, we were only able to prove this result for the square-bracket operations induced by C-sequences.

Proposition 41 *Let $\dot{\mathbb{Q}}$ be a $\mathbb{BC}$-name for a c.c.c. forcing and $\mathcal{C}$ a C-sequence. $\mathbb{BC} * \dot{\mathbb{Q}}$ forces that the generic club is not $[\cdot\cdot]_{\mathcal{C}}$-Ramsey.*

Proof. We follow an approach similar to that of Theorem 40. However, this proof involves further complications because of the extra difficulties that arise when working with an iteration. For notational convenience, we will write $[\gamma\xi]$ for $[\gamma\xi]_{\mathcal{C}}$ and $\triangle(\gamma, \xi)$ for $\triangle_{\mathcal{C}}(\gamma, \xi)$ (for any $\gamma, \xi \in \omega_1$).

Let $(p_0, \dot{x}) \in \mathbb{BC} * \dot{\mathbb{Q}}$ and $\dot{B}$ such that $(p_0, \dot{x}) \Vdash$ "$\dot{B} \in [\omega_1]^{\omega_1}$". We want to extend $(p_0, \dot{x})$ to a condition forcing that there are $\alpha, \beta \in \dot{B}$ such that $[\alpha\beta]$ is not in $\dot{C}_{gen}$. Let $\langle M_\alpha \mid \alpha \leq \omega^2 \rangle$ be a continuous chain of elementary submodels such that $\dot{\mathbb{Q}}, (p_0, \dot{x}), \dot{B}, \mathcal{C} \in M_0$. For convenience, we introduce the following notation:

1. $M = M_{\omega^2}$.

2. $\delta = \delta_M$.

3. $\delta_\alpha = \delta_{M_\alpha}$ for $\alpha < \omega^2$.

4. $p_0 = (a_0, I_0)$.

Let $p_1 = (a_0 \cup \{\delta\}, I_0)$. By Corollary 27, we know that $(p_1, \dot{x}) \leq (p_0, \dot{x})$ is an $(M, \mathbb{BC} * \dot{\mathbb{Q}})$-generic condition. Since $\dot{B}$ is forced to be uncountable, we can now find $(p, \dot{y}) \leq (p_1, \dot{x})$ and $\beta > \delta$ such that $(p, \dot{y}) \Vdash$ "$\beta \in \dot{B}$". Write $p = (a_p, I_p)$. By Proposition 28, we know that $p_M = (a_p \cap M, I_p \cap M)$ is in M and is weaker than p. Denote $p_M = (a_M, I_M)$.

Let $g : T(\rho_0) \longrightarrow \omega$ be a specialization mapping with $g \in M_0$. Find $\gamma < \omega^2$ a limit ordinal such that $\lambda(\delta, \beta), p_M \in M_\gamma$. Choose $\eta < \delta_\gamma$ that is above p_M and $\lambda(\delta, \beta), \lambda(\delta_\gamma, \delta) < \eta$. Since γ is limit, we can find $\varepsilon < \gamma$ such that $\eta < \delta_\varepsilon$. Note that $\delta_\gamma \in \mathsf{Tr}(\delta_\varepsilon, \delta)$. Once again, since γ is limit, we can find $\alpha \in (\varepsilon, \gamma)$ such that $\mathsf{Tr}(\delta_\varepsilon, \delta_\gamma) \setminus \{\delta_\gamma\} \subseteq \delta_\alpha$. Define $q = (a_p \cup \{\delta_\varepsilon, \delta_\alpha\}, I_p \cup \{(\delta_\alpha, \delta_\gamma]\})$. It follows that q is a $\mathbb{BC}$-generic condition for M, M_ε and M_α, but it is not for M_γ. Let $G \subseteq \mathbb{BC}$ be a generic filter with $q \in G$.

We now go to $V[G]$. Note that $M[G], M_\varepsilon[G]$ and $M_\alpha[G]$ are forcing extensions of M, M_ε and M_α respectively. Denote $y = \dot{y}[G]$, $\mathbb{Q} = \dot{\mathbb{Q}}[G]$ and $i \in \omega$ such that $g(\rho_{0\beta} \restriction \delta_\varepsilon) = i$. Define A as the set of $w \in \mathbb{Q}$ for which there is $t \in T(\rho_0)$ such that:

1. $\rho_{0\beta} \restriction \lambda(\delta_\varepsilon, \delta_\gamma)$, $\rho_{0\beta} \restriction \eta \subseteq t$.

2. $g(t) = i$.

3. $w \Vdash$ "$\exists^{\omega_1} \mu \in \dot{B}(t \subseteq \rho_{0\mu})$".

If $w \in A$ and t is as above, we say that t is a witness for $w \in A$. Note that $A \in M_\varepsilon[G]$. We claim that $y \in A$ and $\rho_{0\beta} \restriction \delta_\varepsilon$ is a witness of it. We just need to check that $y \Vdash$ "$\exists^{\omega_1} \mu \in \dot{B}(t \subseteq \rho_{0\mu})$". In order to check this, let $H \subseteq \mathbb{Q}$ be a $(V[G], \mathbb{Q})$ generic filter with $y \in H$. We have that $M[G][H]$ is a generic extension of $M[G]$, which is a generic extension of M, so $\beta \notin M[G][H]$. It follows by Lemma 9 that $\{\mu \in \dot{B}[G \times H] \mid t \subseteq \rho_{0\mu}\}$ is uncountable. Since H was any generic filter that has y in it, it follows that $y \Vdash$ "$\exists^{\omega_1} \mu \in \dot{B}(t \subseteq \rho_{0\mu})$", just as we wanted.

We now know that $y \in A$. Since $\mathbb{Q}$ is c.c.c. and $A \in M_\varepsilon[G]$, it follows by Proposition 17 and Lemma 16 that there is $w \in A \cap M_\varepsilon[G]$ that is compatible with y. Pick $t \in T(\rho_0) \cap M_\varepsilon[G]$ a witness for $w \in A$. Since $g(t)$ and $g(\rho_{0\beta} \restriction \delta_\varepsilon)$ are the same, it follows that t and $\rho_{0\beta}$ are incomparable in $T(\rho_0)$. Choose $u \in \mathbb{Q}$ a common extension of w and y.

With the recently new knowledge acquired, we return to V. Let $\dot{u}$ and $\dot{w}$ be $\mathbb{BC}$-names for u and w. We can now find $r \leq q$ such that:

1. $r \Vdash$ "$\dot{w} \in \dot{A}$" and knows a $t \in M_\varepsilon$ that is a witness.
2. $r \Vdash$ "$\dot{u} \leq \dot{w}, \dot{y}$".

It follows that $(r, \dot{u}) \leq (p, \dot{y})$ (so $(r, \dot{u}) \Vdash$ "$\beta \in \dot{B}$") and that $(r, \dot{u}) \Vdash$ "$\exists^{\omega_1} \mu \in \dot{B}\,(t \subseteq \rho_{0\mu})$". Denote $\triangle = \triangle\,(\rho_{0\beta}, t)$ (recall that $\rho_{0\beta} \perp t$). Since $\lambda\,(\delta, \beta)\,, \lambda\,(\delta_\gamma, \delta)\,, \lambda\,(\delta_\varepsilon, \delta_\gamma) < \triangle$, we have that:

$$\mathsf{Tr}(\triangle, \beta) \quad = \quad \mathsf{Tr}(\delta, \beta) \cup \mathsf{Tr}(\delta_\gamma, \delta) \cup \mathsf{Tr}(\delta_\varepsilon, \delta_\gamma) \cup \mathsf{Tr}(\triangle, \delta_\varepsilon)$$

It is also worth recalling that $\mathsf{Tr}(\delta_\varepsilon, \delta_\gamma) \setminus \{\delta_\gamma\} \subseteq \delta_\alpha$, so $\mathsf{Tr}(\triangle, \beta) \cap \delta_\gamma \subseteq \delta_\alpha$. In this way, $\delta_\gamma = \mathsf{min}\{\mathsf{Tr}(\triangle, \beta) \setminus \delta_\alpha\}$. By Corollary 27, we know that $(r, \dot{u})$ is an $(M_\alpha, \mathbb{BC} * \dot{\mathbb{Q}})$-generic condition. Since $(r, \dot{u}) \Vdash$ "$\exists^{\omega_1} \mu \in \dot{B}\,(t \subseteq \rho_{0\mu})$", we can find $(r_0, \dot{u}_0)$ and μ such that:

1. $(r_0, \dot{u}_0) \leq (r, \dot{u})$.
2. $\mathsf{Tr}(\triangle, \beta) \cap \delta_\gamma \subseteq \mu$ and $\mu < \delta_\alpha$.
3. $t \subseteq \rho_{0\mu}$.
4. $(r_0, \dot{u}_0) \Vdash$ "$\mu \in \dot{B}$".

It follows that $\triangle = \triangle\,(\mu, \beta)$ and $[\mu\beta] = \delta_\gamma$, which r_0 forces that is not in $\dot{C}_{gen}$. ■

The difficulty in extending this result to the other square-bracket operations lies in the choice of α. While a natural choice exists for operations induced by C-sequences, it is unclear what the corresponding choice should be in the other cases.

By first adding a Baumgartner club and then forcing Martin's Axiom (MA) with a $\mathsf{c.c.c.}$ forcing, we obtain the following consistency result:

Theorem 42 *The statement "There exists a C-sequence whose square-bracket operation fails to have the Ramsey club property" is consistent with MA and an arbitrarily large continuum.*

It can be proved that the failure of the $[\cdot\cdot]_{\mathcal{C}}$-Ramsey club property for the Baumgartner generic club can not be fixed by a forcing of the form $\mathbb{COL}_\in\,(\kappa) * \dot{\mathbb{P}}$, where $\mathbb{COL}_\in\,(\kappa)$ denotes the $\in$-collapse of κ (see [41]) and $\dot{\mathbb{P}}$ is a name for an ω-proper forcing. This result will appear in a forth-comming paper.

11 Taking care of all square-bracket operations

In the previous section we proved the consistency of the existence of square-bracket operations that fail to have the Ramsey club property. In fact, the Baumgartner club witnesses the failure of this property for all operations from the ground model. We now aim to construct models in which no square-bracket operation has the Ramsey club property. Note that by Theorem 37, a single club cannot witness this failure for all such operations simultaneously. To build a model where every square-bracket operation fails to have the Ramsey club property, we will add many Baumgartner clubs using Asperó's product.

Theorem 43 *The statement "No square-bracket operation has the Ramsey club property" is consistent with an arbitrarily large continuum.*

Proof. Let $\kappa \geq \omega_2$ be a cardinal such that $\kappa^{\omega_1} = \kappa$. We claim that forcing with $\mathbb{BC}^{(\kappa)}$ produces a model as desired. Pick $G \subseteq \mathbb{BC}^{(X)}$ a generic filter. We already know that $V[G] \models \mathfrak{c} = \kappa$ (see Proposition 36). Let $[\cdot\cdot]$ be a square-bracket operation in $V[G]$. Since $[\cdot\cdot]$ is an object of size ω_1 and $\mathbb{BC}^{(\kappa)}$ has the ω_2-c.c., we can find $\alpha \in \kappa$ such that $[\cdot\cdot]$ only depends on $\mathbb{BC}^{(X)}$, where $X = \kappa \setminus \{\alpha\}$. Since $\mathbb{BC}^{(\kappa)}$ is forcing equivalent to $\mathbb{BC}^{(X)} \times \mathbb{BC}$, the result follows by Theorem 40. ■

We would like to emphasize following result, which follows by the previous theorem, Lemma 5.4.4. of [40] and the results established in the subsequent sections:

Corollary 44 *Both of the following statements are consistent (but not at the same time) with ZFC:*

1. *All square-bracket operations have the Ramsey club property.*
2. *No square-bracket operation has the Ramsey club property.*

In other words, ZFC is unable to decide the status of the Ramsey club property for any square-bracket operation.

Our aim is now to produce a model of Martin's Axiom in which no square-bracket operation (induced by a C-sequence) has the Ramsey club property. The desired model is obtained by first adding many Baumgartner clubs and then forcing MA in the usual way. To achieve this, we will require several preliminary technical lemmas.

Let X be a set of ordinals, $\beta \in X$ and $\dot{B}$ a $\mathbb{BC}^{(X)}$-name. We say that $\dot{B}$ *does not depend on* β if $\dot{B}$ is (equivalent) to a $\mathbb{BC}^{(X\setminus\{\beta\})}$-name. We have the following:

Proposition 45 *Let $\kappa \geq \omega_1$ be a regular cardinal, $\dot{\mathbb{P}}$ a $\mathbb{BC}^{(\kappa^+)}$-name for a partial order of size κ and $\dot{f}$ a $\mathbb{BC}^{(\kappa^+)}$-name for an operation on ω_1. There is $\beta \in \kappa^+$ such that:*

1. *$\dot{\mathbb{P}}$ and $\dot{f}$ do not depend on β.*
2. *$\mathbb{BC}^{(\kappa^+)} * \dot{\mathbb{P}}$ and $(\mathbb{BC}^{(\kappa^+ \setminus \{\beta\})} * \dot{\mathbb{P}}) \times \mathbb{BC}$ are forcing equivalent.*

Proof. We may assume there is a set P in the ground model such that $\dot{\mathbb{P}} = (P, \leq_{\dot{\mathbb{P}}})$. For every $x, y \in P$ and $\alpha, \beta \in \omega_1$ define the sets $D(x, y, \alpha, \beta)$ be the set of all $p \in \mathbb{BC}^{(\kappa^+)}$ such that either $p \Vdash$ "$x \leq_{\dot{\mathbb{P}}} y$" or $p \Vdash$ "$x \not\leq_{\dot{\mathbb{P}}} y$" and there is $\gamma \in \omega_1$ such that $p \Vdash$ "$\dot{f}(\alpha, \beta) = \gamma$". It is clear that each $D(x, y, \alpha, \beta)$ is an open dense subset of $\mathbb{BC}^{(\kappa^+)}$. Pick $A(x, y, \alpha, \beta) \subseteq D(x, y, \alpha, \beta)$ a maximal antichain and let A be the union of all these antichains. Define $B = \bigcup \{\mathsf{sup}(p) \mid p \in A\}$. Since $\mathbb{BC}^{(\kappa^+)}$ has the ω_2-c.c., it follows that $|B| \leq \kappa$, so we can find $\beta \in \kappa^+ \setminus B$. We claim β is as desired. It is clear that $\dot{\mathbb{P}}$ and $\dot{f}$ do not depend on β.

For ease of writing, denote $X = \kappa^+ \setminus \{\beta\}$. Define the function $F : (\mathbb{BC}^{(X)} * \dot{\mathbb{P}}) \times \mathbb{BC} \longrightarrow \mathbb{BC}^{(\kappa^+)} * \dot{\mathbb{P}}$ given by $F(p, \dot{x}, r) = (p^\frown r, \dot{x})$, where $p^\frown r \in \mathbb{BC}^{(\kappa^+)}$ is such that $\mathsf{sup}(p^\frown r) = \mathsf{sup}(p) \cup \{\beta\}$, $p \subseteq p^\frown r$ and $p^\frown r(\beta) = r$. It is easy to see that F is a dense embedding. ■

We will need the following result (see Theorem 2.1 of [11] and the remark below it).

Proposition 46 *Let κ be a regular cardinal, $\mathbb{P}$ a partial order and $\dot{\mathbb{Q}}$ a $\mathbb{P}$-name for a forcing.*

1. *If $\mathbb{P}$ is κ-c.c. and it forces that $\dot{\mathbb{Q}}$ is also κ-c.c., then $\mathbb{P} * \dot{\mathbb{Q}}$ is κ-c.c.*
2. *If $\mathbb{P} * \dot{\mathbb{Q}}$ is κ-c.c., then $\mathbb{P}$ is κ-c.c. and forces that $\dot{\mathbb{Q}}$ is κ-c.c.*

In particular, we get that if $\dot{\mathbb{P}}$ is a $\mathbb{BC}^{(X)}$-name for a c.c.c. partial order, then $\mathbb{BC}^{(X)} * \dot{\mathbb{P}}$ is ω_2-c.c. The following lemma follows by Proposition 36 and standard arguments.

Lemma 47 (GCH) *Let $\kappa > \omega_1$ be a regular cardinal and $X \subseteq OR$ of size κ. The following holds:*

1. *$\mathbb{BC}^{(X)}$ has size κ.*
2. *$\mathbb{BC}^{(X)}$ forces that "$\mathfrak{c} = 2^{<\kappa} = \kappa \wedge 2^\kappa = \kappa^+$".*

We can now prove the main result of the section:

Theorem 48 *The statement "No square-bracket operation induced by a C-sequence has the Ramsey club property" is consistent with MA and an arbitrarily large continuum.*

Proof. Assume GCH in the ground model, pick $\kappa \geq \omega_1$ a regular cardinal and let $G \subseteq \mathbb{BC}^{(\kappa^+)}$ be a generic filter. We go to $V[G]$. In here, we know that $\mathfrak{c} = \kappa^+$ and $2^{<\kappa^+} = 2^\kappa = \kappa^+$. By the standard proof of Theorem 21, there is a a finite support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha \leq \kappa^+ \rangle$ of c.c.c. forcings such that $\mathbb{P}_{\kappa^+}$ forces that "$\mathfrak{m} = \mathfrak{c} = \kappa^+$" and if $\alpha < \kappa^+$, then $|\mathbb{P}_\alpha| \leq \kappa$. For convenience, denote $\mathbb{P} = \mathbb{P}_{\kappa^+}$.

We return to V and claim that $\mathbb{Q} = \mathbb{BC}^{(\kappa^+)} * \dot{\mathbb{P}}$ is the partial order we are looking for. We already know that $\mathbb{Q}$ forces that "$\mathfrak{m} = \mathfrak{c} = \kappa^+$". It remains to prove that it also forces that no square-bracket operation has the Ramsey club property . Let $\dot{f}$ be a $\mathbb{Q}$-name for a square-bracket operation. Since $\mathbb{Q}$ is ω_2-c.c.c. and $\kappa^+ > \omega_1$ is regular, we can find $\alpha < \kappa^+$ such that $\dot{f}$ is equivalent to a $\mathbb{BC}^{(\kappa^+)} * \dot{\mathbb{P}}_\alpha$-name. Let $\dot{\mathbb{R}}$ be a $\mathbb{BC}^{(\kappa^+)} * \dot{\mathbb{P}}_\alpha$-name for the quotient $\mathbb{P}_{\kappa^+}/\mathbb{P}_\alpha$. In this way, $\mathbb{Q} \equiv (\mathbb{BC}^{(\kappa^+)} * \dot{\mathbb{P}}_\alpha) * \dot{\mathbb{R}}$ (where $\equiv$ denotes being forcing equivalent). Note that $\dot{\mathbb{R}}$ is forced to be c.c.c. by Proposition 46. Since $\mathbb{P}_\alpha$ is forced to have size at most κ, by Proposition 45, we can find $\beta \in \kappa^+$ such that the following conditions hold:

1. $\dot{\mathbb{P}}_\alpha$ and $\dot{f}$ do not depend on β.
2. $\mathbb{BC}^{(\kappa^+)} * \dot{\mathbb{P}}_\alpha \equiv (\mathbb{BC}^{(\kappa^+ \setminus \{\beta\})} * \dot{\mathbb{P}}_\alpha) \times \mathbb{BC}$.

In this way, we get the following:

$$\begin{aligned} \mathbb{Q} &= \mathbb{BC}^{(\kappa^+)} * \dot{\mathbb{P}} \\ &\equiv (\mathbb{BC}^{(\kappa^+)} * \dot{\mathbb{P}}_\alpha) * \dot{\mathbb{R}} \\ &\equiv ((\mathbb{BC}^{(\kappa^+ \setminus \{\beta\})} * \dot{\mathbb{P}}_\alpha) \times \mathbb{BC}) * \dot{\mathbb{R}} \end{aligned}$$

It follows that $\dot{f}$ will live in the extension by $\mathbb{BC}^{(\kappa^+ \setminus \{\beta\})} * \dot{\mathbb{P}}_\alpha$, which is then followed by $\mathbb{BC}$ and then a c.c.c. forcing. We can apply Proposition 41 and conclude that $\dot{f}$ will not have the Ramsey club property in the final extension. ■

Note that if Proposition 41 could be proved for the other square-bracket operations, then the argument of Theorem 48 would also apply for them.

12 Square-bracket operations and CH

Having settled the (non) relationship between Martin's Axiom and the Ramsey club property for the square-bracket operations, we now turn to the Continuum Hypothesis. It is easy to see that if CH holds in the ground model, then adding a single Baumgartner club preserves it. Thus, Theorem 40 implies that CH is

consistent with the existence of square-bracket operations that do not have the Ramsey club property. The goal of this section is to prove a stronger result: CH (and even $\diamondsuit$) is consistent with the failure of the Ramsey club property for every square-bracket operation. We cannot proceed as in previous sections, since adding many Baumgartner clubs would force the failure of CH. Instead, we will use a forcing notion for adding a club with countable conditions that was introduced by Baumgartner, Harrington, and Kleinberg in [9] (see also [18]).

Definition 49 *Let $\mathbb{SC}$ be the the set of all closed and countable subsets of ω_1. For $p, q \in \mathbb{SC}$, define $q \leq p$ if p is an initial segment of q.*

$\mathbb{SC}$ is the usual forcing for shooting a club through ω_1 with countable conditions. Clearly $\mathbb{SC}$ is a σ-closed tree. If $G \subseteq \mathbb{SC}$ is a generic filter, the *generic club* is defined as $C_{gen} = \bigcup G$. It is not hard to verify that C_{gen} is forced to be a club on ω_1. The following lemma is easy and well-known.

Lemma 50 *Let M be a countable elementary submodel.*

1. *If $p \in \mathbb{SC}$ is $(M, \mathbb{SC})$-generic, then $\delta_M \in p$.*
2. *Let $p \in \mathbb{SC} \cap M$. There is $q \leq p$ an $(M, \mathbb{SC})$-generic condition such that $\delta_M = \mathsf{max}(q)$.*

The paper [17] provides an example of a totally proper forcing $\mathbb{P}$ for which there exist conditions that are $(M, \mathbb{P})$-generic but not $(M, \mathbb{P})$-totally generic. Fortunately, this phenomenon does not occur with our forcing $\mathbb{SC}$.

Lemma 51 *Let M be a countable elementary submodel and $p \in \mathbb{SC}$. If p is $(M, \mathbb{SC})$-generic, then it is $(M, \mathbb{SC})$-totally generic.*

Proof. Let $D \in M$ be an open dense subset of $\mathbb{SC}$. By the genericity of p, there is $q \in D \cap M$ that is compatible with p. Since $\mathbb{SC}$ is a tree, it must be the case that either $p \leq q$ or $q \leq p$. Since $\mathsf{max}(q) < \delta_M \leq \mathsf{max}(p)$, then $q \leq p$. It follows that $p \in D$ because D is an open subset of $\mathbb{SC}$. ■

We also have the following:

Lemma 52 *Let $\{M_n \mid n \in \omega\}$ be a continuous increasing chain of countable elementary submodels and $M = \bigcup_{n \in \omega} M_n$. If $p \in \mathbb{SC}$ is $(M_n, \mathbb{SC})$-generic for every $n \in \omega$, then p is $(M, \mathbb{SC})$-generic.*

Proof. We will prove that p is $(M, \mathbb{SC})$-totally generic. Let $D \in M$ be an open dense subset of $\mathbb{SC}$. Choose $n \in \omega$ such that $D \in M_n$. Since p is $(M_n, \mathbb{SC})$-totally generic, it follows that $p \in D$. ■

To establish the main results of this section, we must first introduce new definitions and prove several auxiliary lemmas. We start with some preliminary results concerning uncountable subsets of the reals.

Definition 53 *Let X be a subset of 2^ω. The* kernel tree of X *(denoted by $S(X)$) consists of all $s \in 2^{<\omega}$ for which the set $\langle s \rangle \cap X$ is uncountable.*

We have the following simple observation.

Lemma 54 *If $X \subseteq 2^\omega$ is uncountable, then $S(X)$ is a Sacks tree.*

The following result is also easy, so we leave it to the reader.

Lemma 55 *Let $X \subseteq 2^\omega$ be uncountable, M a countable elementary submodel such that $X \in M$. If $r \in X \setminus M$, then $r \in [S(X)]$.*

We now introduce the definitions and results concerning Aronszajn trees that are necessary for the main theorem of this section.

Definition 56 *Let $(T, \leq_T)$ be an Aronszajn tree, $s \in T$, $p \in \mathbb{SC}$ and $\dot{B}$ an $\mathbb{SC}$-name for an uncountable subset of T. We say that:*

1. *p $\dot{B}$-*kills *s if there is $t <_T s$ such that $p \Vdash$ "$\forall z \in \dot{B}\,(t \not\leq_T z)$".*
2. *p $\dot{B}$-*freezes *s if for every $t <_T s$, we have that $p \Vdash$ "$\exists^{\omega_1} z \in \dot{B}\,(t \leq_T z)$".*
3. *p $\dot{B}$-*decides *s if p either $\dot{B}$-kills s or $\dot{B}$-freezes s.*

It is crucial to note that in both the killing and freezing definitions, the node t is required to be strictly below s. We have the following lemma.

Lemma 57 *Let $(T, \leq_T)$ be an Aronszajn tree, $\dot{B}$ an $\mathbb{SC}$-name for an uncountable subset of T, M a countable elementary submodel with $\dot{B}, T \in M$, $p \in \mathbb{SC}$ an $(M, \mathbb{SC})$-generic condition and $\delta = \delta_M$. The following holds:*

1. *p $\dot{B}$-decides every $s \in T_\delta$.*
2. *There is $s \in T_\delta$ such that p $\dot{B}$-freezes s.*

Proof. We start with the first point. Let $s \in T_\delta$. Since p is $(M, \mathbb{SC})$-totally generic, for every $t <_T s$, we have that one of the following statements holds:

1. $p \Vdash$ "$\exists^{\omega_1} z \in \dot{B}\,(t \leq_T z)$".
2. p forces that the set $\{z \in \dot{B} \mid t \leq_T z\}$ is at most countable.

If the first point holds for every $t <_T s$, then we have that p $\dot{B}$-freezes s and we are done. Assume there is $t <_T s$ such that forces that the set $\{z \in \dot{B} \mid t \leq_T z\}$ is at most countable. Appealing once again to total genericity, we know that there is $\alpha < \delta$ such that $p \Vdash$ "$\{|z| \mid z \in \dot{B} \wedge t \leq_T z\} \subseteq \alpha$". It follows that $p \Vdash$ "$\forall z \in \dot{B}\,(s \restriction \alpha \not\leq_T z)$, hence p $\dot{B}$-kills s.

We will now prove that p $\dot{B}$-freezes a node at the level δ. Since $\dot{B}$ is forced to be uncountable, we can find $z \in T$ with $|z| > \delta$ and $q \leq p$ such that $q \Vdash$ "$z \in \dot{B}$". Let $s = z \restriction \delta$. It follows that p can not $\dot{B}$ kill s, so p B-freezes it. ■

Let $e = \langle e_\beta \mid \beta \in \omega_1 \rangle$ be a nice sequence. Define the tree $T(e) = \{e_\beta \restriction \alpha \mid \alpha \leq \beta < \omega_1\}$. We say that a square-bracket operation $[\cdot\cdot]$ is an *Aronszajn square-bracket operation* if it is of one of the following types:

1. $[\cdot\cdot]_{\mathcal{C}}$ where $\mathcal{C}$ is a C-sequence.
2. $[\cdot\cdot]_{T,a}$ where T is a Hausdorff, special Aronszajn tree and $a : T \longrightarrow \omega$ is a specialization mapping.
3. $[\cdot\cdot]_{\mathcal{R},e}$ where $\mathcal{R} = \{r_\alpha \mid \alpha \in \omega_1\} \subseteq 2^\omega$ and $e = \langle e_\beta \mid \beta \in \omega_1 \rangle$ is a nice sequence for which $T(e)$ is an Aronszajn tree.

We can now prove the following:

Proposition 58 $\mathbb{SC}$ *forces that the generic club is not* $[\cdot\cdot]$*-Ramsey for any Aronszajn square-bracket operation* $[\cdot\cdot]$ *in* V.

Proof. Let $p \in \mathbb{SC}$ and $\dot{B}$ such that $p \Vdash$ "$\dot{B} \in [\omega_1]^{\omega_1}$". We want to extend p to a condition forcing that there are $\alpha, \beta \in \dot{B}$ such that $[\alpha\beta]$ is not in $\dot{C}_{gen}$. Let $\langle M_\alpha \mid \alpha \leq \omega^2 \rangle$ be a continuous chain of countable elementary submodels such that $\dot{B} \in M_0$, as well as the square-bracket operation we want to take care of. For convenience, we introduce the following notation:

1. $M = M_{\omega^2}$.
2. $\delta = \delta_M$.
3. $\delta_\alpha = \delta_{M_\alpha}$ for $\alpha < \omega^2$.

The proof now splits into cases depending on the type of square-bracket operation under consideration. We begin with the case of the operation induced by a C-sequence, say $\mathcal{C}$. We assume that $\mathcal{C} \in M_0$. For notational convenience in this part of the proof, we will write T for $T(\rho_0)$, $[\gamma\xi]$ for $[\gamma\xi]_{\mathcal{C}}$ and $\triangle(\gamma, \xi)$ for $\triangle_{\mathcal{C}}(\gamma, \xi)$ (for any $\gamma, \xi \in \omega_1$). To simplify notation, a subset $W \subseteq \omega_1$ will occasionally be identified with the corresponding set $\{\rho_{0\alpha} \mid \alpha \in W\}$.

Fix an enumeration $T_\delta = \{s_n \mid n \in \omega\}$ such that each element of T_δ is listed infinitely many times. We will now recursively construct a sequence $\langle \varepsilon_n, \gamma_n, \mu_n, q_n, z_n, t_n, \triangle_n \rangle_{n\in\omega}$ satisfying the following properties[3]:

1. $\omega n < \varepsilon_n < \mu_n < \gamma_n < \omega^2$.
2. $\gamma_n < \varepsilon_{n+1}$.
3. $q_{n+1} \leq q_n \leq p$ and $q_n \in M$.
4. q_n is both $(M_{\varepsilon_n}, \mathbb{SC})$-generic and $(M_{\mu_n}, \mathbb{SC})$-generic.
5. $\mathsf{max}(q_n) = \delta_{\gamma_n}+1$ and $\delta_{\gamma_n} \notin q_n$ (in particular, q_n is not $(M_{\gamma_n}, \mathbb{SC})$-generic).
6. $z_n \in T_{\delta_{\varepsilon_n}}$ and q_n $\dot{B}$-freezes it. Moreover, if q_n $\dot{B}$-freezes $s_n \restriction \delta_{\varepsilon_n}$, then $z_n = s_n \restriction \delta_{\varepsilon_n}$.
7. $t_n \in M_{\varepsilon_n}$.
8. t_n and z_n are incompatible nodes of T.
9. $\triangle_n = \triangle(z_n, t_n)$.
10. $\lambda(\delta_{\gamma_n}, \delta) < \triangle_n$
11. $\mathsf{Tr}(\triangle_n, \delta) = \mathsf{Tr}(\delta_{\gamma_n}, \delta) \cup \mathsf{Tr}(\triangle_n, \delta_{\gamma_n})$.
12. $\mathsf{Tr}(\triangle_n, \delta_{\gamma_n}) \setminus \{\delta_{\gamma_n}\} \subseteq \delta_{\mu_n}$
13. $q_n \Vdash$ "$\exists^{\omega_1}\alpha \in \dot{B}\,(t \subseteq \rho_{0\alpha})$".

The construction proceeds by recursion. For notational convenience, let $q_{-1} = p$, $\varepsilon_{-1} = \mu_{-1} = \gamma_{-1} = 0$. Suppose we have constructed the sequence up to stage $n-1$, we now define the items for stage n. Since ω^2 is a limit of limit ordinals, we can find $\gamma_n < \omega^2$ a limit ordinal such that γ_{n-1}, $\omega n < \gamma_n$ and $q_{n-1} \in M_{\gamma_n}$. Now, since γ_n is limit, it is possible to find $\varepsilon_n < \gamma_n$ such that $\gamma_{n-1}, \omega n < \varepsilon_n$ and $\lambda(\delta_{\gamma_n}, \delta)$, $q_{n-1} \in M_{\varepsilon_n}$. We can now apply Lemma 50 and find $p_n \in M_{\gamma_n}$ an $(M_{\varepsilon_n}, \mathbb{SC})$-generic condition extending q_{n-1} such that $\mathsf{max}(p_n) = \delta_{\varepsilon_n}$.

Since p_n is $(M_{\varepsilon_n}, \mathbb{SC})$-generic, Lemma 57 implies that p_n $\dot{B}$-decides $s_n \restriction \delta_{\varepsilon_n}$. If p_n $\dot{B}$-freezes $s_n \restriction \delta_{\varepsilon_n}$, we set $z_n = s_n \restriction \delta_{\varepsilon_n}$. If instead p_n $\dot{B}$-kills $s_n \restriction \delta_{\varepsilon_n}$, we let z_n be any node in $T_{\delta_{\varepsilon_n}}$ that is $\dot{B}$-frozen by p_n (the existence of such a node is guaranteed by Lemma 57). Find $\eta < \delta_{\varepsilon_n}$ such that $\lambda(\delta_{\gamma_n}, \delta)$, $\lambda(\delta_{\varepsilon_n}, \delta_{\gamma_n},) < \eta$. Since p_n $\dot{B}$-freezes z_n, it follows that $p_n \Vdash$ "$\exists^{\omega_1}\alpha \in \dot{B}\,(z_n \restriction \eta \subseteq \rho_{0\alpha})$". By Proposition 15 and since p_n is $(M_{\varepsilon_n}, \mathbb{SC})$-totally generic, it is possible to find $t_n \in T$ with the following properties:

[3]The reader may wish to skip this lengthy list initially, as the motivation for these properties will become clear in the next paragraphs.

1. $t_n \in M_{\varepsilon_n}$.

2. z_n and t_n are incompatible nodes of T.

3. $z_n \restriction \eta \subseteq t_n$.

4. $p_n \Vdash$ "$\exists^{\omega_1} \alpha \in \dot{B}\,(t \subseteq \rho_{0\alpha})$".

Let $\triangle_n = \triangle\,(z_n, t_n)$. Clearly $\eta \leq \triangle_n$, so in particular we have that $\lambda\,(\delta_{\gamma_n}, \delta)$, $\lambda\,(\delta_{\varepsilon_n}, \delta_{\gamma_n},) < \triangle_n$. We can now apply Proposition 22 and conclude that $\mathsf{Tr}(\triangle_n, \delta) = \mathsf{Tr}(\delta_{\gamma_n}, \delta) \cup \mathsf{Tr}(\triangle_n, \delta_{\gamma_n})$. Since γ_n is a limit ordinal, we can find $\mu_n < \gamma_n$ such that $\mathsf{Tr}(\triangle_n, \delta_{\gamma_n}) \setminus \{\delta_{\gamma_n}\} \subseteq \delta_{\mu_n}$, $\varepsilon_n < \mu_n$ and $p_n \in M_{\gamma_n}$. By Lemma 50, there is $\overline{p}_n \in M_{\gamma_n}$ an $(M_{\mu_n}, \mathbb{SC})$-generic condition extending p_n. Finally, let $q_n = \overline{p}_n \cup \{\delta_{\gamma_n} + 1\}$. This finishes the construction.

Define $q = \bigcup_{n \in \omega} q_n \cup \{\delta\}$. It is straightforward to verify that $q \in \mathbb{SC}$ and extends p. Moreover, q is $(M, \mathbb{SC})$-generic because it is $(M_{\varepsilon_n}, \mathbb{SC})$-generic for every $n \in \omega$ (see Lemma 52). Since $\dot{B}$ is forced to be uncountable, we can find $r \leq q$ and $\beta > \delta$ such that $r \Vdash$ "$\beta \in \dot{B}$".

Lemma 24 implies that the sequence $\langle \lambda(\delta_{\gamma_m}, \delta) \rangle_{m \in \omega}$ converges to δ. In particular, $\lambda(\delta_{\gamma_m}, \delta) > \lambda\,(\delta, \beta)$ holds for almost all $m \in \omega$. Moreover, since in our enumeration $T_\delta = \{s_n \mid n \in \omega\}$ every element appears infinitely many often, it is possible to find an $n \in \omega$ such that $\lambda\,(\delta, \beta) < \lambda(\delta_{\gamma_n}, \delta)$ and $s_n = \rho_{0\beta} \restriction \delta$. Given that q $\dot{B}$-freezes s_n, we conclude that q_n $\dot{B}$-freezes $s_n \restriction \delta_{\varepsilon_n}$, and therefore $z_n = s_n \restriction \delta_{\varepsilon_n}$. Since r is $(M_{\mu_n}, \mathbb{SC})$-totally generic, there is $\alpha < \delta_{\mu_n}$ such that $r \Vdash$ "$\alpha \in \dot{B}$", $t_n \subseteq \rho_{0\alpha}$ and $\mathsf{Tr}(\triangle_n, \delta_{\gamma_n}) \setminus \{\delta_{\gamma_n}\} = \mathsf{Tr}(\triangle_n, \delta_{\gamma_n}) \cap M_{\mu_n} \subseteq \alpha$. We will now compute $[\alpha\beta]$. First note that:

$$\begin{array}{rcl} \triangle\,(\alpha, \beta) & = & \triangle\,(t_n, s_n) \\ & = & \triangle\,(t_n, z_n) \\ & = & \triangle_n \end{array}$$

Recall that $\lambda\,(\delta, \beta) < \lambda(\delta_{\gamma_n}, \delta) < \triangle_n$. By Proposition 22:

$$\begin{array}{rcl} \mathsf{Tr}(\triangle_n, \beta) & = & \mathsf{Tr}(\delta, \beta) \cup \mathsf{Tr}(\triangle_n, \delta) \\ & = & \mathsf{Tr}(\delta, \beta) \cup \mathsf{Tr}(\delta_{\gamma_n}, \delta) \cup \mathsf{Tr}(\triangle_n, \delta_{\gamma_n}) \end{array}$$

Since $\mathsf{Tr}(\triangle_n, \delta_{\gamma_n}) \setminus \{\delta_{\gamma_n}\} \subseteq \alpha < \delta_{\gamma_n}$, it follows that $[\alpha\beta] = \mathsf{min}\{\mathsf{Tr}(\triangle_n, \beta) \setminus \alpha\} = \delta_{\gamma_n}$. Finally, recall that $r \Vdash$ "$\alpha, \beta \in \dot{B}$" and $\delta_{\gamma_n} \notin r$. This concludes the proof for the square-bracket operation induced by a C-sequence.

We now consider the case of the operation induced by a sequence of reals $\mathcal{R} = \{r_\alpha \mid \alpha \in \omega_1\} \subseteq 2^\omega$ and a nice sequence $e = \langle e_\beta \mid \beta \in \omega_1 \rangle$ for which $T =$

$T(e)$ is an Aronszajn tree. We are assuming that $\mathcal{R}, e \in M_0$. For notational convenience in this part of the proof, we will write $[\gamma\xi]$ for $[\gamma\xi]_{\mathcal{R},e}$ and $\triangle(\gamma,\xi)$ for $\triangle_{\mathcal{R}}(\gamma,\xi)$ (for any $\gamma,\xi \in \omega_1$). Fix an enumeration $T_\delta = \{s_n \mid n \in \omega\}$.

Let $\dot{X}$ be the $\mathbb{SC}$-name for the set $\{r_\alpha \mid \alpha \in \dot{B}\}$. By Lemma 54, its kernel tree is forced to be a Sacks tree. Furthermore, since $\mathbb{SC}$ is σ-closed, (by extending p if needed) we may assume that there is a Sacks tree S such that $p \Vdash$ "$S = S(\dot{X})$". Since both p and $\dot{B}$ are in M_0, it follows that $S \in M_0$. We will now recursively construct a sequence $\langle \gamma_n, \mu_n, q_n, m_n \rangle_{n\in\omega}$ satisfying the following properties:

1. $\omega n < \mu_n < \gamma_n < \omega^2$.
2. $\gamma_n < \mu_{n+1}$.
3. $q_{n+1} \leq q_n \leq p$ and $q_n \in M$.
4. q_n is $(M_{\mu_n}, \mathbb{SC})$-generic.
5. $\mathsf{max}(q_n) = \delta_{\gamma_n}+1$ and $\delta_{\gamma_n} \notin q_n$ (in particular, q_n is not $(M_{\gamma_n}, \mathbb{SC})$-generic).
6. $s_n(\delta_{\gamma_n}) < m_n$ and $\mathsf{split}_{s_n(\delta_{\gamma_n})}(S) \subseteq 2^{<m_n}$.
7. $\{\xi \in M_{\gamma_n} \mid s_n(\xi) \leq m_n\} \subseteq \delta_{\mu_n}$.

The construction proceeds by recursion. For notational convenience, let $q_{-1} = p, \mu_{-1} = \gamma_{-1} = 0$. Suppose we have constructed the sequence up to stage $n-1$, we now define the items for stage n. Since ω^2 is a limit of limit ordinals, we can find $\gamma_n < \omega^2$ a limit ordinal such that $\gamma_{n-1}, \omega n < \gamma_n$ and $q_{n-1} \in M_{\gamma_n}$. Choose m_n such that $\mathsf{split}_{s_n(\delta_{\gamma_n})}(S) \subseteq 2^{<m_n}$. Note that $s_n(\delta_{\gamma_n}) < m_n$. Since γ_n is limit, we can now find $\mu_n < \gamma_n$ such that $\gamma_{n-1}, \omega n < \mu_n$, $q_{n-1} \in M_{\mu_n}$ and $\{\xi \in M_{\gamma_n} \mid s_n(\xi) \leq m_n\} \subseteq \delta_{\mu_n}$. This last requirement is possible because s_n is an injective function. We can now apply Lemma 50 and find $p_n \in M_{\gamma_n}$ an $(M_{\mu_n}, \mathbb{SC})$-generic condition extending q_{n-1}. Define $q_n = p_n \cup \{\delta_{\gamma_n}+1\}$. This finishes the construction.

Define $q = \bigcup_{n\in\omega} q_n \cup \{\delta\}$. Since q is $(M_{\mu_n}, \mathbb{SC})$-generic for every $n \in \omega$, Lemma 52 implies that q is $(M, \mathbb{SC})$-generic. Choose $r \leq q$ and $\beta > \delta$ such that $r \Vdash$ "$\beta \in \dot{B}$". It follows by Lemma 55 that $r_\beta \in [S]$.

Let $n \in \omega$ be such that $s_n = e_\beta \restriction \delta$ and $t \in S$ be the only node satisfying $t \subseteq r_\beta$ and $t \in \mathsf{split}_{s_n(\delta_{\gamma_n})}(S)$. Define $\triangle = |t|$ and note that $s_n(\delta_{\gamma_n}) \leq \triangle < m_n$. Since t is a splitting node of S, the node $z = t^\frown(1 - r_\beta(\triangle))$ also belongs to S. Consequently, we have that $r \Vdash$ "$\exists^{\omega_1}\alpha \in \dot{B}\,(z \subseteq r_\alpha)$". We now use the fact that r is $(M_{\mu_n}, \mathbb{SC})$-totally generic to find $\alpha \in M_{\mu_n}$ such that $z \subseteq r_\alpha$, $r \Vdash$ "$\alpha \in \dot{B}$" and $\{\xi < \delta_{\mu_n} \mid e_\beta(\xi) \leq m_n\} \subseteq \alpha$.

We claim that $[\alpha\beta] = \delta_{\mu_n}$. Recall that $[\alpha\beta]$ is the least $\xi \in [\alpha, \beta]$ for which $e_\beta(\xi) \leq \triangle(\alpha, \beta) = \triangle$. On one hand, we already knew that $e_\beta(\delta_{\gamma_n}) = s_n(\delta_{\gamma_n}) \leq \triangle$. On the other hand, for $\xi < \delta_{\gamma_n}$ with $e_\beta(\xi) \leq \triangle$, we have the following:

$$\begin{array}{rcl} e_\beta(\xi) \leq \triangle & \longrightarrow & e_\beta(\xi) < m_n \\ & \longrightarrow & s_n(\xi) < m_n \\ & \longrightarrow & \xi < \delta_{\mu_n} \end{array}$$

Since $\xi < \delta_{\mu_n}$ and $e_\beta(\xi) \leq m_n$, it follows that $\xi < \alpha$. We conclude that $[\alpha\beta] = \delta_{\mu_n}$. Finally, recall that $r \Vdash$ "$\alpha, \beta \in \dot{B}$" and $\delta_{\gamma_n} \notin r$. This concludes the proof for this square-bracket operation.

The last case is when the operation is induced by a Hausdorff, special Aronszajn tree T and $a : T \longrightarrow \omega$ a specialization mapping. For notational convenience, we now write $[st]$ for $[st]_{T,r}$. We can assume that $T, a \in M_0$. One more change is needed, now $\dot{B}$ is forced to be an uncountable subset of T. Fix an enumeration $T_\delta = \{s_n \mid n \in \omega\}$. We will now recursively construct a sequence $\langle \varepsilon_n, \gamma_n, \mu_n, q_n, z_n, w_n, m_n, \triangle_n \rangle_{n \in \omega}$ satisfying the following properties:

1. $\omega n < \varepsilon_n < \mu_n < \gamma_n < \omega^2$.
2. $\gamma_n < \varepsilon_{n+1}$.
3. $q_{n+1} \leq q_n \leq p$ and $q_n \in M$.
4. q_n is both $(M_{\varepsilon_n}, \mathbb{SC})$-generic and $(M_{\mu_n}, \mathbb{SC})$-generic.
5. $\mathsf{max}(q_n) = \delta_{\gamma_n} + 1$ and $\delta_{\gamma_n} \notin q_n$ (in particular, q_n is not $(M_{\gamma_n}, \mathbb{SC})$-generic).
6. $m_n = a(s_n \restriction \delta_{\gamma_n})$.
7. $z_n \in T_{\delta_{\varepsilon_n}}$ and q_n $\dot{B}$-freezes it. Moreover, if q_n $\dot{B}$-freezes $s_n \restriction \delta_{\varepsilon_n}$, then $z_n = s_n \restriction \delta_{\varepsilon_n}$.
8. $w_n \in T \cap M_{\varepsilon_n}$ and is incompatible with z_n.
9. $w_n \Vdash$ "$\exists^{\omega_1} t \in \dot{B}\,(w_n \subseteq t)$".
10. $\triangle_n = \triangle(z_n, w_n)$.
11. $\{\xi < \delta_{\gamma_n} \mid a(s_n \restriction \xi) \leq m_n\} \subseteq \triangle_n$.
12. $\{\xi < \delta_{\gamma_n} \mid a(s_n \restriction \xi) \leq a(z_n \wedge w_n)\} \subseteq \delta_{\mu_n}$.

The construction proceeds by recursion. For notational convenience, let $q_{-1} = p, \varepsilon_{-1} = \mu_{-1} = \gamma_{-1} = 0$. Suppose we have constructed the sequence up to stage $n-1$, we now define the items for stage n. Since ω^2 is a limit of limit ordinals, we can find $\gamma_n < \omega^2$ a limit ordinal such that γ_{n-1}, $\omega n < \gamma_n$ and $q_{n-1} \in M_{\gamma_n}$. Define $m_n = a(s_n \restriction \delta_{\gamma_n})$ and find $\eta < \delta_{\gamma_n}$ such that $\{\xi < \delta_{\gamma_n} \mid a(s_n \restriction \xi) \leq m_n\} \subseteq \eta$. Since γ_n is a limit ordinal, we can find $\varepsilon_n < \gamma_n$

such that γ_{n-1}, $\omega n < \varepsilon_n$ and $\eta, q_{n-1} \in M_{\varepsilon_n}$. We can now apply Lemma 50 and find $p_n \in M_{\gamma_n}$ an $(M_{\varepsilon_n}, \mathbb{SC})$-generic condition extending q_{n-1} such that $\mathsf{max}(p_n) = \delta_{\varepsilon_n}$. Since p_n is $(M_{\varepsilon_n}, \mathbb{SC})$-generic, Lemma 57 implies that p_n $\dot{B}$-decides $s_n \restriction \delta_{\varepsilon_n}$. If p_n $\dot{B}$-freezes $s_n \restriction \delta_{\varepsilon_n}$, we set $z_n = s_n \restriction \delta_{\varepsilon_n}$. If instead p_n $\dot{B}$-kills $s_n \restriction \delta_{\varepsilon_n}$, we let z_n be any node in $T_{\delta_{\varepsilon_n}}$ that is $\dot{B}$-frozen by p_n (the existence of such a node is guaranteed by Lemma 57).

Recall that $p_n \Vdash$ "$\exists^{\omega_1} t \in \dot{B}\,(z_n \restriction \eta \subseteq t)$" and p_n is an $(M_{\varepsilon_n}, \mathbb{SC})$-totally generic condition. We can use Proposition 14 to find two incompatible nodes $w^0, w^1 \in T \cap M_{\varepsilon_n}$ such that $z_n \restriction \eta \subseteq w^0, w^1$ and $p_n \Vdash$ "$\exists^{\omega_1} t \in \dot{B}\left(w^i \subseteq t\right)$" for each $i \in 2$. Pick any $w_n \in \left\{w^0, w^1\right\}$ that is incompatible with z_n. Define $\triangle_n = \triangle\,(z_n, w_n)$ and note that $\eta < \triangle_n$, so $\{\xi < \delta_{\gamma_n} \mid a\,(s_n \restriction \xi) \leq m_n\} \subseteq \triangle_n$. Since γ_n is a limit ordinal, we can find $\mu_n < \gamma_n$ such that $\varepsilon_n < \mu_n$, $p_n \in M_{\mu_n}$ and $\{\xi < \delta_{\gamma_n} \mid a\,(s_n \restriction \xi) \leq a\,(z_n \wedge w_n)\} \subseteq \delta_{\mu_n}$. By Lemma 50, there is $\overline{p}_n \in M_{\gamma_n}$ an $(M_{\mu_n}, \mathbb{SC})$-generic condition extending p_n. We now define $q_n = \overline{p}_n \cup \{\delta_{\gamma_n} + 1\}$ and finish the recursive construction.

Define $q = \bigcup_{n \in \omega} q_n \cup \{\delta\}$, which is an $(M, \mathbb{SC})$-generic condition by Lemma 52. Choose $r \leq q$ and $s \in T$ with $|s| > \delta$ such that $r \Vdash$ "$s \in \dot{B}$". Let $n \in \omega$ such that $s_n = s \restriction \delta$. Given that r $\dot{B}$-freezes s_n, we conclude that q_n $\dot{B}$-freezes $s_n \restriction \delta_{\varepsilon_n}$. Hence $z_n = s_n \restriction \delta_{\varepsilon_n}$. Since r is $(M_{\mu_n}, \mathbb{SC})$-totally generic, it follows that there is $t \in T \cap M_{\mu_n}$ such that $w_n <_T t$, $r \Vdash$ "$t \in \dot{B}$" and $\{\xi < \delta_{\gamma_n} \mid a\,(s_n \restriction \xi) \leq a\,(z_n \wedge w_n)\} \subseteq |t|$.

We claim now that $[ts] = s \restriction \delta_{\gamma_n}$. Recall that $m_n = a\,(s \restriction \delta_{\gamma_n})$ and note that $\triangle\,(t, s) = \triangle\,(z_n, w_n) = \triangle_n$, so $t \wedge s$ is equal to $z_n \wedge w_n$. Furthermore, $\{\xi < \delta_{\gamma_n} \mid a\,(s_n \restriction \xi) \leq m_n\} \subseteq \triangle_n$ and $m_n \leq a\,(t \wedge s)$. It then follows that $a\,(s \restriction \delta_{\gamma_n}) \leq a\,(t \wedge s)$. Moreover, if $\xi < \delta_{\gamma_n}$ and $a\,(s_n \restriction \xi) \leq \triangle_n$, then $\xi < |t|$. We conclude that $[ts] = s \restriction \delta_{\gamma_n}$. Finally, recall that $r \Vdash$ "$\alpha, \beta \in \dot{B}$" and $\delta_{\gamma_n} \notin r$. This concludes the proof. ■

We will now prove that a σ-closed forcing can not fix the failure of the Ramsey club property.

Lemma 59 *Let $[\cdot\cdot]$ be a square-bracket operation and $C \subseteq \omega_1$ a club that is not $[\cdot\cdot]$-Ramsey. If $\mathbb{P}$ is a σ-closed forcing, then $\mathbb{P}$ forces that C is not $[\cdot\cdot]$-Ramsey.*

Proof. Assume that there are $p \in \mathbb{P}$ and $\dot{B}$ a $\mathbb{P}$-name for an uncountable subset of ω_1 for which $p \Vdash$ "$\{[\alpha\beta] \mid \alpha, \beta \in \dot{B}\} \subseteq C$". Recursively, find $\{(p_\alpha, \gamma_\alpha) \mid \alpha \in \omega_1\}$ such that for every $\alpha < \beta < \omega_1$, the following conditions hold:

1. $p_\alpha \leq p$.
2. $p_\beta \leq p_\alpha$ and $\gamma_\alpha < \gamma_\beta$.
3. $p_\alpha \Vdash$ "$\gamma_\alpha \in \dot{B}$".

Let $E = \{\gamma_\alpha \mid \alpha \in \omega_1\}$. Since C is not $[\cdot\cdot]$-Ramsey, there are $\alpha < \beta$ such that $[\gamma_\alpha \gamma_\beta] \notin C$. It follows that p_β forces that $\gamma_\alpha, \gamma_\beta$ are in $\dot{B}$, but $[\gamma_\alpha \gamma_\beta]$ is not in C, which is a contradiction. ■

The main result of the section now easily follows:

Theorem 60 *The statement "No Aronszajn square-bracket operation has the Ramsey club property" is consistent with* $\diamondsuit$.

Proof. We start with a model of $\diamondsuit$ and GCH. Let $\mathbb{SC}_{\omega_2}$ be the countable support iteration of length ω_2 of the forcing $\mathbb{SC}$. This iteration is σ-closed, preserves $\diamondsuit$ (see Lemma V.5.8 and Exercise IV.7.46 of [24]) and is ω_2-$\mathsf{c.c.}$ (see Theorem 2.10 of [1]). It follows that every Aronszajn square-bracket operation will appear in an intermediate extension. By Proposition 58, any such operation will fail to have the Ramsey club property in an intermediate model. Finally, by Lemma 59, this failure is preserved throughout the remainder of the iteration. ■

13 PFA and square-bracket operations induced by a sequence of reals

So far we have studied the failure of the Ramsey club property. We now turn our attention to proving that it can consistently hold. As mentioned in the introduction, the second author (answering a question of Woodin) proved that the *Proper Forcing Axiom (PFA)* implies the Ramsey club property holds for the square-bracket operations induced by special Aronszajn trees (see Lemma 5.4.4 of [40]). We will show that the same is true for the other two types of square-bracket operations. This section focuses on the Ramsey club property for the square-bracket operations defined from a sequence of reals and a nice sequence. This proof is the simplest of the three. For the remainder of this section, fix a club $C \subseteq \mathsf{LIM}(\omega_1)$, a sequence $\mathcal{R} = \{r_\alpha \mid \alpha \in \omega_1\} \subseteq 2^\omega$ and a nice sequence $e = \langle e_\beta \mid \beta \in \omega_1 \rangle$. For notational convenience, in this section we will write $[\gamma\xi]$ for $[\gamma\xi]_{\mathcal{R},e}$ and $\triangle(\gamma,\xi)$ for $\triangle_{\mathcal{R}}(\gamma,\xi)$ (for any $\gamma, \xi \in \omega_1$). Moreover, for $\beta \in \omega_1$ and $n \in \omega$, we define the set $F_n(\beta) = \{\xi \leq \beta \mid e_\beta(\xi) \leq n\}$ (note that $\beta \in F_n(\beta)$). In this way, it is clear that if $\alpha < \beta$, then $[\alpha\beta] = \mathsf{min}\big(F_{\triangle(\alpha,\beta)}(\beta) \setminus \alpha\big)$.

We introduce more notation. We say that a set $E \subseteq [\omega_1]^m$ *is unbounded* if for every $\xi < \omega_1$, there is $b \in E$ with $\xi < \mathsf{min}(b)$. For $\overline{s} = \langle s_i \rangle_{i<m} \in (2^{<\omega})^m$, define $\langle \overline{s} \rangle$ as the set of all $(f_i)_{i<m} \in (2^\omega)^m$ for which $f_i \in \langle s_i \rangle$ for every $i < m$. Given $Y \in [\omega_1]^{<\omega}$, denote $\mathcal{R}_Y = \langle r_\alpha \rangle_{\alpha \in Y}$. The following lemma is probably well-known, we provide a proof for the convenience of the reader.

Lemma 61 *Let* $m \in \omega$, $E \subseteq [\omega_1]^m$ *and* M *a countable elementary submodel such that* $E, \mathcal{R} \in M$. *Assume there is* $X = \{\alpha_i \mid i < m\} \in E$ *with* $X \cap M = \emptyset$.

For every $n \in \omega$, there exists $\overline{s} = \langle s_i \rangle_{i<m} \in (2^{<\omega})^m$ such that for all $i, j < m$, the following conditions hold:

1. $|s_i| = |s_j|$.
2. $s_i \restriction n = r_{\alpha_i} \restriction n$.
3. $r_\alpha \notin \langle s_i \rangle$ *for every* $\alpha \in X$.
4. *The set* $\{Y \in E \mid \mathcal{R}_Y \in \langle \overline{s} \rangle\}$ *is unbounded.*

Proof. For every $i < m$, denote $t_i = r_{\alpha_i} \restriction n$. Let D be the set of all $Y = \{\beta_i \mid i < m\} \in E$ such that $r_{\beta_i} \in \langle t_i \rangle$ for every $i < m$. Note that $D \in M$ and $X \in D$. Since $X \cap M = \emptyset$, it follows by elementarity. that D is unbounded. In this way, we can find $Y = \{\beta_i \mid i < m\} \in D$ such that $Y \cap M$ and $Y \cap X$ are both empty. Find $l \in \omega$ such that $\triangle(r_{\alpha_i}, r_{\beta_i})$, $\triangle\left(r_{\beta_i}, r_{\beta_j}\right)$, $\triangle\left(r_{\alpha_i}, r_{\alpha_j}\right) < l$ for all $i, j < m$ and define $s_i = r_{\beta_i} \mid l$. It is easy to see that $\overline{s} = \{s_i \mid i < m\}$ is as desired. ■

The following forcing notion was inspired by the one from Lemma 5.4.4 of [40].

Definition 62 *Define $\mathbb{P}(C)$ as the set of all $p = (\mathcal{M}, \mathcal{N})$ with the following properties:*

1. *$\mathcal{N}$ is a finite $\in$-chain of countable elementary submodels of $\mathsf{H}(\omega_2)$ such that $\mathcal{R}, e$ and C belong to all of them.*
2. *$\mathcal{M} \subseteq \mathcal{N}$.*
3. ***Completeness condition.*** *If $M, L \in \mathcal{M}$ are distinct, then $[\delta_M \delta_L] \in C$.*
4. ***Crossing models condition.*** *For every $M, L \in \mathcal{M}$ distinct and $N \in \mathcal{N}$, if $N \in M$, then $\mathsf{min}(F_{\triangle(\delta_M, \delta_L)}(\delta_M) \setminus N) \in C$.*

We highlight several key points regarding this notion. Let $p = (\mathcal{M}, \mathcal{N})$ be an element of $\mathbb{P}(C)$.

1. If $N \in \mathcal{N}$, then $\delta_N \in C$ (see Proposition 8).
2. In the crossing models condition, the case $N = L$ is a consequence of the completeness condition. Therefore, when verifying this condition for a potential element of $\mathbb{P}(C)$, we may always assume $N \neq L$.
3. In the crossing models condition, no relationship is assumed between M, N and L other than $N \in M$ and $M \neq L$. In particular, it is possible that $N \in L \in M$, $L \in N \in M$ or $N \in M \in L$.

Let $p = (\mathcal{M}_p, \mathcal{N}_p)$ and $q = (\mathcal{M}_q, \mathcal{N}_q)$ in $\mathbb{P}(C)$. Define $p \leq q$ if $\mathcal{M}_q \subseteq \mathcal{M}_p$, $\mathcal{N}_q \subseteq \mathcal{N}_p$ and $\mathcal{M}_q = \mathcal{M}_p \cap \mathcal{N}_q$. For $p, q \in \mathbb{P}(C)$, define $q \sqsubseteq p$ if there is $N \in \mathcal{N}_p$ such that $\mathcal{M}_q = \mathcal{M}_p \cap N$ and $\mathcal{N}_q = \mathcal{N}_p \cap N$. Clearly if $q \sqsubseteq p$, then $p \leq q$.

We now introduce several "invariants" for conditions in $\mathbb{P}(C)$ which will be useful for analyzing the combinatorial properties of the forcing.

Definition 63 *Let $p = (\mathcal{M}, \mathcal{N}) \in \mathbb{P}(C)$. Define the following items:*

1. $\mathsf{ht}(p) = \{\delta_N \mid N \in \mathcal{N}\}$ *and* $\mathsf{ht}_{\mathcal{M}}(p) = \{\delta_M \mid M \in \mathcal{M}\}$.
2. d_p *is the smallest natural number that is larger than all of the following:*
   (a) $\triangle(\delta_M, \delta_N)$ *for* $M, N \in \mathcal{N}$ *distinct.*
   (b) $e_{\delta_M}(\delta_N)$ *for* $M, N \in \mathcal{N}$ *with* $N \in M$.
3. $T_p = \langle r_{\delta_N} \restriction d_p \rangle_{N \in \mathcal{N}_p}$.
4. $S(p) = \bigcup \{F_{d_p}(\delta_N) \mid N \in \mathcal{N}\}$.
5. $H(p) = (S(p), <, \mathsf{ht}(p), \mathsf{ht}_{\mathcal{M}}(p), d_p, e, T_p, \triangle)$.

It follows that $S(p)$ is a finite subset of ω_1 that contains $\mathsf{ht}(p)$. Let $p, q \in \mathbb{P}(C)$ and $f : S(p) \longrightarrow S(q)$ a bijection. We say that f *is an isomorphism from* $H(p)$ *to* $H(q)$ if for every $\alpha, \beta \in S(p)$, the following holds:

1. $\alpha < \beta$ if and only if $f(\alpha) < f(\beta)$.
2. $\alpha \in \mathsf{ht}(p)$ if and only if $f(\alpha) \in \mathsf{ht}(q)$.
3. $\alpha \in \mathsf{ht}_{\mathcal{M}}(p)$ if and only if $f(\alpha) \in \mathsf{ht}_{\mathcal{M}}(q)$.
4. $d_p = d_q$.
5. $e_\beta(\alpha) = e_{f(\beta)}(f(\alpha))$.
6. If $\alpha \in \mathsf{ht}(p)$, then $r_\alpha \restriction d_p = r_{f(\alpha)} \restriction d_q$.
7. $\triangle(\alpha, \beta) = \triangle(f(\alpha), f(\beta))$.

We say that $H(p)$ and $H(q)$ are isomorphic if there is an isomorphism between them. Since such an isomorphism is an isomorphism between the finite linear orders $(S(p), <)$ and $(S(q), <)$, it is necessarily unique. It is easy to see that there are only countably many isomorphism types.

Definition 64 *We say that M is a* big model *if the following conditions hold:*

1. *There is $\kappa > \omega_2$ a large enough regular cardinal such that M is a countable elementary submodel of $\mathsf{H}(\kappa)$.*

2. *$C, \mathcal{R}, e, \mathbb{P}(C) \in M$.*

For M a big model, denote $\overline{M} = M \cap \mathsf{H}(\omega_2)$, which is an elementary submodel of $\mathsf{H}(\omega_2)$. We now prove the following key proposition, which will be used to establish the main result of this section.

Proposition 65 *Let M be a big model and $p = (\mathcal{M}_p, \mathcal{N}_p) \in \mathbb{P}(C)$. If $\overline{M} \in \mathcal{N}_p$, then p is $(M, \mathbb{P}(C))$-generic.*

Proof. Let $D \in M$ be an open dense set of $\mathbb{P}(C)$. By extending p if necessary, we may assume that $p \in D$. We need to prove that p is compatible with an element of $D \cap M$. Denote $\delta = \delta_M$, which is an element of C. Define $X = \mathsf{ht}(p) \setminus M$ and enumerate $X = \{\delta_i \mid i < m\}$ in increasing order (hence $\delta_0 = \delta$).

Let $p_M = (\mathcal{M}_p \cap M, \mathcal{N}_p \cap M)$. It is easy to see that $p_M \in \mathbb{P}(C) \cap M$ and $p_M \sqsubseteq p$. Find $\eta_0 < \delta$ such that $S(p) \cap M \subseteq \eta_0$. We now define E as the set of all $Y \in [\omega_1]^m$ for which there is $r \in D$ with the following properties:

1. $p_M \sqsubseteq r$.

2. $\mathsf{ht}(r) = \mathsf{ht}(p_M) \cup Y$.

3. $S(r) \cap \eta_0 = S(p) \cap \eta_0$ (which is the same as $S(p) \cap M$).

4. $H(p)$ and $H(r)$ are isomorphic and the isomorphism is the identity on $S(p) \cap M$.

We claim that $E \in M$. Although the third and fourth clause involve M and p, they only depend on the isomorphism type of p (which is in M) and the finite set $S(p) \cap M \in M$. Consequently, both clauses can be reformulated without reference to objects outside M. For $Y \in E$ and r as above, we will say that r is a witness for $Y \in E$.

Apply Lemma 61 and find $\overline{s} = \langle s_i \rangle_{i<m}$ with the following properties:

1. $|s_i| = |s_j|$ for all $i, j < m$.

2. $r_{\delta_i} \restriction d_p = s_i \restriction d_p$ for $i < m$.

3. $r_\alpha \notin \langle s_i \rangle$ for all $\alpha \in X$ and $i < m$.

4. The set $\{Y \in E \mid \mathcal{R}_Y \in \langle \overline{s} \rangle\}$ is unbounded.

Denote $l = |s_0|$ (which is the same as $|s_i|$ for every $i < m$). Find $\eta \in (\eta_0, \delta)$ such that $F_l(\beta) \cap M \subseteq \eta$ for every $\beta \in X$. We can now find $Y \in E \cap M$ with $\eta < \mathsf{min}(Y)$ and $\mathcal{R}_Y \in \langle \overline{s} \rangle$. By elementarity., there exists $r \in D \cap M$ that that is a witness of $Y \in E$. Let $h : S(p) \longrightarrow S(r)$ be the unique isomorphism from $H(p)$ to $H(r)$. Recall that h is the identity on M.

We claim that p and r are compatible. It suffices to show that $q = (\mathcal{M}_p \cup \mathcal{M}_r, \mathcal{N}_p \cup \mathcal{N}_r)$ belongs to $\mathbb{P}(C)$. Clearly $\mathcal{N}_p \cup \mathcal{N}_r$ is a finite $\in$-chain and contains $\mathcal{M}_p \cup \mathcal{M}_r$. It remains to verify that q satisfies both the completeness condition and the crossing models condition.

To verify that q satisfies the completeness condition, it suffices to show that for any $\alpha, \beta \in \mathsf{ht}_{\mathcal{M}}(p) \setminus M$, we have that $[h(\alpha)\, \beta] \in C$. Denote $\triangle = \triangle(h(\alpha), \beta)$. Recall that $h(\alpha) \in M$ while $\beta \notin M$, so $h(\alpha) < \beta$. We proceed by cases, depending on whether α and β are equal.

Case: $\alpha \neq \beta$

It follows that $\triangle = \triangle(h(\alpha), \beta) = \triangle(\alpha, \beta)$ and is smaller than d_p, so $F_{\triangle}(\beta) \cap M \subseteq \eta_0 \subseteq h(\alpha)$. We have the following:

$$\begin{array}{rcl} [h(\alpha)\, \beta] & = & \mathsf{min}(F_{\triangle}(\beta) \setminus h(\alpha)) \\ & = & \mathsf{min}(F_{\triangle}(\beta) \setminus M) \\ & = & \mathsf{min}\big(F_{\triangle(\alpha,\beta)}(\beta) \setminus M\big) \end{array}$$

If $\delta < \beta$, then $\mathsf{min}\big(F_{\triangle(\alpha,\beta)}(\beta) \setminus M\big) \in C$ by the crossing models condition of p. On the other hand, if $\delta = \beta$, then $\mathsf{min}\big(F_{\triangle(\alpha,\beta)}(\beta) \setminus M\big) = \beta$, which belongs to C.

Case: $\alpha = \beta$

Here we have that $e_{\beta}(\delta) < d_p \leq \triangle < l$. It follows that $F_{\triangle}(\beta) \cap M \subseteq F_l(\beta) \cap M \subseteq \eta \subseteq h(\beta)$. We have the following:

$$\begin{array}{rcl} [h(\alpha)\, \beta] & = & \mathsf{min}(F_{\triangle}(\beta) \setminus h(\beta)) \\ & = & \mathsf{min}(F_{\triangle}(\beta) \setminus M) \\ & = & \delta \in C. \end{array}$$

This completes the proof that q satisfies the completeness condition. We now verify the crossing models condition. It suffices to show that for any distinct $\alpha, \beta \in \mathsf{ht}_{\mathcal{M}}(p) \cup \mathsf{ht}_{\mathcal{M}}(r)$ and any $N \in \mathcal{N}_p \cup \mathcal{N}_r$ with $\delta_N < \beta$ and $\alpha \neq \delta_N$, we have that $\mathsf{min}(F_{\triangle}(\beta) \setminus N) \in C$, where $\triangle = \triangle(\alpha, \beta)$. The proof proceeds by cases, depending on the membership of N.

Case: $N \in \mathcal{N}_p \cap \mathcal{N}_r$

Note that if both $\alpha, \beta \in \mathsf{ht}(p)$ or both $\alpha, \beta \in \mathsf{ht}(r)$, then the result follows immediately since p and r are conditions in $\mathbb{P}(C)$. The proof now proceeds by subcases, depending on the relationship between $\triangle$ and d_p.

Subcase: $N \in \mathcal{N}_p \cap \mathcal{N}_r$ and $\triangle < d_p$.

First assume that $\alpha \in \mathsf{ht}(p) \setminus \mathsf{ht}(r)$ (so $\alpha \notin M$) and $\beta \in \mathsf{ht}(r) \setminus \mathsf{ht}(p)$. It follows from $\triangle < d_p$ that $h(\alpha) \neq \beta$ and $\triangle = \triangle(h(\alpha), \beta)$. Since $N \in \mathcal{N}_r$, the crossing models property of r implies that $\mathsf{min}(F_{\triangle}(\beta) \setminus N) \in C$. We now assume that $\beta \in \mathsf{ht}(p) \setminus \mathsf{ht}(r)$ (so $\beta \notin M$) and $\alpha \in \mathsf{ht}(r) \setminus \mathsf{ht}(p)$. Once again, since $\triangle < d_p$, we have that $h(\beta) \neq \alpha$ and $\triangle = \triangle(h(\beta), \alpha)$. Since $N \in \mathcal{N}_p$, the crossing models property of p implies that $\mathsf{min}(F_{\triangle}(\beta) \setminus N) \in C$.

Subcase: $N \in \mathcal{N}_p \cap \mathcal{N}_r$ and $d_p \leq \triangle$.

We have that $r_\alpha \restriction d_p = r_\beta \restriction d_p$. Note that $e_\beta(\delta_N) < d_p \leq \triangle$. It follows that $\mathsf{min}(F_{\triangle}(\beta) \setminus N) = \delta_N \in C$.

Case: $N \in \mathcal{N}_r \setminus \mathcal{N}_p$

Note that if $\alpha, \beta \in \mathsf{ht}(r)$, then the result follows immediately since r is a condition in $\mathbb{P}(C)$. The proof now proceeds by subcases, depending on whether β is in $\mathsf{ht}(r)$.

Subcase: $N \in \mathcal{N}_r \setminus \mathcal{N}_p$ and $\beta \notin \mathsf{ht}(r)$.

Since $\beta \notin \mathsf{ht}(r)$, it follows that $\beta \notin M$. Moreover, $F_{\triangle}(\beta) \cap M \subseteq \eta < \delta_N$. In this way, $\mathsf{min}(F_{\triangle}(\beta) \setminus N) = \mathsf{min}(F_{\triangle}(\beta) \setminus M)$. If $\beta = \delta$, then $\mathsf{min}(F_{\triangle}(\beta) \setminus M) = \delta$, which is in C. Assume that $\beta \neq \delta$ (so $\delta < \beta$). If $\triangle \geq d_p$, since $e_\beta(\delta) < d_p$, it follows that $\mathsf{min}(F_{\triangle}(\beta) \setminus M) = \delta$. We now assume that $\triangle < d_p$. Let $\gamma \in \mathsf{ht}_{\mathcal{M}}(p)$ such that $\triangle(\gamma, \beta) = \triangle$. We conclude that $\mathsf{min}(F_{\triangle}(\beta) \setminus M) \in C$ by the crossing models property of p.

Subcase: $N \in \mathcal{N}_r \setminus \mathcal{N}_p$ and $\beta \in \mathsf{ht}(r)$.

Note that $\beta \in M$. If α is also in M, then $\alpha, \beta \in \mathsf{ht}(r)$ and the result follows since r is a condition. Assume that $\alpha \notin M$. If $h(\alpha) \neq \beta$, then $\triangle = \triangle(h(\alpha), \beta)$ and is smaller than d_p. We have that $\mathsf{min}(F_{\triangle}(\beta) \setminus N) = \mathsf{min}(F_{\triangle(h(\alpha),\beta)}(\beta) \setminus N)$ and it is in C because of the crossing models property of r. We now assume that $h(\alpha) = \beta$. It follows that $e_\beta(\delta_N) < d_p \leq \triangle$. In this way, $\mathsf{min}(F_{\triangle}(\beta) \setminus N) = \delta_N$, which is in C.

Case: $N \in \mathcal{N}_p \setminus \mathcal{N}_r$

It must be the case that $N \notin M$. Since $\delta_N < \beta$, then $\beta \notin M$, so $\beta \in \mathsf{ht}(p)$. If α is also in $\mathsf{ht}(p)$, the result follows immediately since $p \in \mathbb{P}(C)$. Assume that $\alpha \in \mathsf{ht}(r) \setminus \mathsf{ht}(p)$. If $h(\beta) \neq \alpha$, then $\triangle < d_p$ and $\triangle = \triangle\left(\beta, h^{-1}(\alpha)\right)$. It follows that $\mathsf{min}(F_{\triangle}(\beta) \setminus N) = \mathsf{min}(F_{\triangle(\beta,h^{-1}(\alpha))}(\beta) \setminus N)$ and it is in C because of the crossing models property of p. Assume that $h(\beta) = \alpha$, hence $d_p \leq \triangle$, so $e_\beta(\delta_N) < d_p \leq \triangle$. We conclude that $\mathsf{min}(F_{\triangle}(\beta) \setminus N) = \delta$, which is in C. We finally finished the proof. ■

We conclude the following:

Corollary 66 *$\mathbb{P}(C)$ is a proper forcing.*

Proof. Let M be a big model and $q \in M \cap \mathbb{P}(C)$. Define $p = (\mathcal{M}_q, \mathcal{N}_q \cup \{\overline{M}\})$. It follows that $p \leq q$ and it is $(M, \mathbb{P}(C))$-generic by Proposition 65. ■

The main result of this section now easily follows. This is the same argument as the one from Lemma 5.4.4 of [40].

Theorem 67 *PFA implies that every square-bracket operation of the form $[\cdot\cdot]_{\mathcal{R},e}$ has the Ramsey club property.*

Proof. Assume PFA and let C, $\mathcal{R}$ and e as before, with $\mathbb{P}(C)$ the corresponding forcing notion. Given $G \subseteq \mathbb{P}(C)$ a filter, define the set $W(G) = \bigcup\{\delta_M \mid \exists p \in G\,(M \in \mathcal{M}_p)\}$. It follows from the definition of the forcing that $\{[\alpha\beta] \mid \alpha, \beta \in W(G)\} \subseteq C$. To complete the proof, it suffices to show that there is $p \in \mathbb{P}(C)$ that forces that $W(\dot{G})$ is uncountable (where $\dot{G}$ is the canonical name of the generic filter).

Let M be a big model. Define $p = (\{\overline{M}\}, \{\overline{M}\})$. Since p is $(M, \mathbb{P}(C))$-generic and $p \Vdash$"$\delta_M \in W(\dot{G})$", it follows from Lemma 9 that p forces that $W(\dot{G})$ is uncountable. ■

14 PFA and square-bracket operations induced by C-sequences

It remains to prove that PFA implies the Ramsey club property for the square-bracket operations arising from C-sequences. The proof for this case appears to be the hardest of the three. For the remainder of this section, fix a club $C \subseteq \mathsf{LIM}(\omega_1)$ and $\mathcal{C} = \{C_\alpha \mid \alpha \in \omega_1\}$ a C-sequence. For notational convenience, in this section we will write $[\gamma\xi]$ for $[\gamma\xi]_{\mathcal{C}}$ and $\triangle(\gamma, \xi)$ for $\triangle_{\mathcal{C}}(\gamma, \xi)$ (for any $\gamma, \xi \in \omega_1$). We need to prove some preliminary results.

Definition 68 *Let $X \subseteq \omega_1$. We define the following:*

1. $Tr[X] = \bigcup\{Tr(\alpha, \beta) \mid \alpha, \beta \in X\}$.
2. $\triangle(X) = \{\triangle(\alpha, \beta) \mid \alpha, \beta \in X\}$.
3. $\lambda(X) = \{\lambda(\alpha, \beta) \mid \alpha, \beta \in X\}$.
4. *$S(X)$ is the smallest set that contains X and is closed under the previous operations.*

We now have the following:

Proposition 69 *If $X \subseteq \omega_1$ is finite, then $S(X)$ is also finite.*

Proof. Let $X \subseteq \omega_1$ be a finite set. Define recursively a sequence of sets by $X_0 = X$ and $X_{n+1} = X_n \cup \mathsf{Tr}[X_n] \cup \triangle(X_n) \cup \lambda(X_n)$. It is straightforward to verify that $S(X) = \bigcup_{n\in\omega} X_n$ and that $X_n \subseteq X_m$ whenever $n \leq m$. It follows that $|S(X)| \leq \omega$ (this can also be deduced from the observation that $S(X) \subseteq \mathsf{max}(X) + 1$). Assume that $S(X)$ is infinite. Let $S'(X)$ denote the set of accumulation points of $S(X)$. Since $S(X)$ is countably infinite, we conclude that $S'(X)$ has a maximum, which we will γ. Note that γ is a limit ordinal.

Denote $Y = S(X) \setminus \gamma$. Since γ is the largest accumulation point of $S(X)$, the set Y must be finite. Consequently, there exists $n \in \omega$ such that $Y \subseteq X_n$. Now, choose $\gamma_m \in C_\gamma$ such that:

1. $X_{n+1} \cap \gamma \subseteq \gamma_m$.
2. $\lambda(\gamma, \beta) < \gamma_m$ for all $\beta \in Y$ with $\gamma < \beta$.

We claim that $S(X) \cap (\gamma_m, \gamma)$ is empty. This will be a contradiction since γ is an accumulation point of $S(X)$. We will prove by induction that $X_l \cap (\gamma_m, \gamma) = \emptyset$. This is trivial in case $l \leq n+1$. Assume now that $n+1 \leq l$ and $X_l \cap (\gamma_m, \gamma) = \emptyset$, we will show that the same is true for $l+1$. Since λ and $\triangle$ are regressive functions and $X_l \setminus \gamma = Y$, it follows that $\lambda[X_l]$ and $\triangle(X_l)$ are disjoint with (γ_m, γ). It remains to prove that if $\beta \in Y$ and $\alpha \in X_l \cap \gamma$, then $\mathsf{Tr}(\alpha, \beta) \cap (\gamma_m, \gamma) = \emptyset$. We proceed by cases depending wether γ is in $\mathsf{Tr}(\alpha, \beta)$ or not.

First assume that $\gamma \in \mathsf{Tr}(\alpha, \beta)$. Since $\alpha \leq \gamma_m$, the walk from β to α, must pass through γ and then jump to $\mathsf{min}(C_\gamma \setminus \alpha) \leq \gamma_m$. It follows that $\mathsf{Tr}(\alpha, \beta) \cap (\gamma_m, \gamma) = \emptyset$. Now assume that $\gamma \notin \mathsf{Tr}(\alpha, \beta)$. Let $\mathsf{Tr}(\alpha, \beta) = \{\beta_0, ..., \beta_n\}$ where $\beta_0 = \beta$, $\beta_n = \alpha$ and $\beta_{i+1} = \mathsf{stp}(\alpha, \beta_i)$. Let i such that $\beta_i > \gamma$ but $\gamma < \beta_{i+1}$. Proposition 22 implies that $\beta_{i+1} \leq \lambda(\gamma, \beta)$. By our choice of γ_m, we have that $\lambda(\gamma, \beta) < \gamma_m$, hence $\beta_{i+1} < \gamma_m$. We conclude again that $\mathsf{Tr}(\alpha, \beta) \cap (\gamma_m, \gamma) = \emptyset$. ■

Let T be an Aronszajn tree, $A \subseteq T^{n+1}$, $\gamma \in \omega_1$ and $F = (t_0, ..., t_n) \in (T_\gamma)^{n+1}$ (we do not require that $t_i \neq t_j$ for $i \neq j$). The game $G_F(T, A)$ is defined as follows:

I	ξ_0		ξ_1		...	ξ_n	
II		s_0		s_1			s_n

The game lasts $n+1$ rounds. In round i, Player I plays $\xi_i \in \omega_1$ and Player II responds with $s_i \in T$, subject to the following rules:

1. $\xi_0 < \xi_1 < ... < \xi_n$.
2. $\xi_i < |s_i|$ for every $i \leq n$.

3. $t_i = s_i \restriction \gamma$ for every $i \leq n$.

Player II *wins the game* if the resulting sequence $(s_0, ..., s_n)$ is in A. Player I is a man and Player II is a woman. Since this is a clopen game, it is determined by the Gale-Stewart Theorem (see Theorem 20.1 of [22]).

Proposition 70 *Let T be an Aronszajn tree, $A \subseteq T^{n+1}$ and $\{M_0, ..., M_n\}$ an $\in$-chain of countable elementary submodels such that $T, A \in M_0$. Assume there is $(s_0, .., s_n) \in A$ such that $\delta_{M_0} \leq |s_0| < \delta_{M_1} \leq |s_1| \leq ... \leq \delta_{M_n} \leq |s_n|$. For every $\gamma \in \omega_1$, there is $F \in (T_\gamma)^{n+1}$ such that Player* II *has a winning strategy in $\mathrm{G}_F(T, A)$.*

Proof. By elementarity., it suffices to prove the conclusion of the proposition for $\gamma < \delta_{M_0}$. Let $F = (s_0 \restriction \gamma, ..., s_n \restriction \gamma)$. We claim that Player II has a winning strategy in the game $G_F(T, A)$. Assume this is false, so Player I has a winning strategy in the game. By elementarity., there is $\sigma \in M_0$ a winning strategy for this game. Consider the run of the game in where Player I follows σ and Player II plays s_i at round i. It is easy to see that this is a valid run of the game in which Player II wins, contradicting the assumption that σ is a winning strategy for Player I. ■

With the proposition, we can prove the following corollary:

Corollary 71 *Let T be an Aronszajn tree, $A \subseteq T^{n+1}$ and $\{M_0, ..., M_n\}$ an $\in$-chain of countable elementary submodels such that $T, A \in M_0$. Assume there is $(s_0, .., s_n) \in A$ such that $\delta_{M_0} \leq |s_0| < \delta_{M_1} \leq |s_1| \leq ... \leq \delta_{M_n} \leq |s_n|$. There are γ and F with the following properties:*

1. *$\gamma < \delta_{M_0}$.*
2. *$F = (t_0, ..., t_n) \in T^{n+1}$ and $t_i \neq s_j \restriction \gamma$ for every $i, j \leq n$.*
3. *Player* II *has a winning strategy in $\mathrm{G}_F(T, A)$.*

Proof. By the previous proposition, we know there is $B = \{F_\alpha \mid \alpha \in \omega_1\}$ such that for every $\alpha \in \omega_1$, we have that $F_\alpha \in (T_\alpha)^{n+1}$ and Player II has a winning strategy in $G_{F_\alpha}(T, A)$. By elementarity. we may even assume that $B \in M_0$. Proposition 13 implies that there is $\gamma_0 \in M_0$ such that B is $(n+1)$-distributed in at level γ_0. We can now find $\gamma \in M_0$ with $\gamma > \gamma_0$ such that the γ_0 projections of F_γ and $(s_0, .., s_n)$ are disjoint. It follows that γ and F are as desired. ■

We will now define the forcing notion required to prove the main result of this section. It is similar to the one of the previous section and is again based on the forcing from Lemma 5.4.4 of [40], but this version is the most complex of the three.

Definition 72 *Define $\mathbb{P}(C)$ as the set of all $p = (\mathcal{M}, \mathcal{N})$ with the following properties:*

1. *$\mathcal{N}$ is a finite $\in$-chain of countable elementary submodels of $\mathsf{H}(\omega_2)$ and both C and $\mathcal{C}$ belong to all of them.*

2. *$\mathcal{M} \subseteq \mathcal{N}$.*

3. ***Completeness condition.*** *If $M, L \in \mathcal{M}$ are distinct, then $[\delta_M \delta_L] \in C$.*

4. ***First crossing models condition.*** *If $M \in \mathcal{M}$ and $N, L \in \mathcal{N}$ are such that $N \in L \in M$, then $\mathsf{min}(\mathsf{Tr}(\delta_N, \delta_M) \setminus L) \in C$.*

5. ***Second crossing models condition.*** *Let $M, L \in \mathcal{M}$ distinct and $N \in \mathcal{N}$ with $N \in M$. If $\triangle = \triangle(\delta_M, \delta_L) \in N$, then $\mathsf{min}(\mathsf{Tr}(\triangle, \delta_M) \setminus N) \in C$.*

Similar to the forcing from the previous section, in the second crossing models condition, the case $N = L$ is a consequence of the completeness condition. Therefore, when verifying this condition for a potential element of $\mathbb{P}(C)$, we may always assume $N \neq L$. Furthermore, in this same condition, no relationship is assumed between M, N and L other than $N \in M$ and $M \neq L$. In particular, it is possible that $N \in L \in M$, $L \in N \in M$ or $N \in M \in L$.

Let $p = (\mathcal{M}_p, \mathcal{N}_p)$ and $q = (\mathcal{M}_q, \mathcal{N}_q)$ in $\mathbb{P}(C)$. Define $p \leq q$ if $\mathcal{M}_q \subseteq \mathcal{M}_p$, $\mathcal{N}_q \subseteq \mathcal{N}_p$ and $\mathcal{M}_q = \mathcal{M}_p \cap \mathcal{N}_q$. For $p, q \in \mathbb{P}(C)$, define $q \sqsubseteq p$ if there is $N \in \mathcal{N}_p$ such that $\mathcal{M}_q = \mathcal{M}_p \cap N$ and $\mathcal{N}_q = \mathcal{N}_p \cap N$. Clearly if $q \sqsubseteq p$, then $p \leq q$.

We now introduce the invariants for the elements of $\mathbb{P}(C)$.

Definition 73 *Let $p = (\mathcal{M}, \mathcal{N}) \in \mathbb{P}(C)$. Define the following items:*

1. *$\mathsf{ht}(p) = \{\delta_N \mid N \in \mathcal{N}\}$ and $\mathsf{ht}_{\mathcal{M}}(p) = \{\delta_M \mid M \in \mathcal{M}\}$*

2. *$S(p) = S(\mathsf{ht}(p))$ (see Definition 68).*

3. *$H(p) = (S(p), <, \mathsf{ht}_{\mathcal{M}}(p), \mathsf{ht}(p), \mathsf{Tr}, \triangle, \lambda, C)$.*

Let $p, q \in \mathbb{P}(C)$ and $h : S(p) \longrightarrow S(q)$ a bijection. We say that h *is an isomorphism from $H(p)$ to $H(q)$* if for every $\alpha, \beta \in S(p)$, the following holds:

1. $\alpha < \beta$ if and only if $h(\alpha) < h(\beta)$.

2. $\alpha \in \mathsf{ht}(p)$ if and only if $h(\alpha) \in \mathsf{ht}(q)$.

3. $\alpha \in \mathsf{ht}_{\mathcal{M}}(p)$ if and only if $h(\alpha) \in \mathsf{ht}_{\mathcal{M}}(q)$.

4. $h[\mathsf{Tr}(\alpha, \beta)] = \mathsf{Tr}(h(\alpha), h(\beta))$.

5. $h(\triangle(\alpha, \beta)) = \triangle(h(\alpha), h(\beta))$.

6. $h(\lambda(\alpha, \beta)) = \lambda(h(\alpha), h(\beta))$.

7. $\alpha \in C$ if and only if $h(\alpha) \in C$.

We say that $H(p)$ and $H(q)$ are isomorphic if there is an isomorphism between them. As in the previous section, there is at most one isomorphism from $H(p)$ to $H(q)$ and there are only countably many isomorphism types.

Definition 74 *We say that M is a* big model *if the following conditions hold:*

1. *There is $\kappa > \omega_2$ a large enough regular cardinal such that M is a countable elementary submodel of $\mathsf{H}(\kappa)$.*
2. *$C, \mathcal{C}, \mathbb{P}(C) \in M$.*

For M a big model, denote $\overline{M} = M \cap \mathsf{H}(\omega_2)$, which is an elementary submodel of $\mathsf{H}(\omega_2)$. We now prove the following

Proposition 75 *Let M be a big model and $p = (\mathcal{M}_p, \mathcal{N}_p) \in \mathbb{P}(C)$. If $\overline{M} \in \mathcal{N}_p$, then p is $(M, \mathbb{P}(C))$-generic.*

Proof. For convenience, we will denote $T = T(\rho_0)$. To simplify notation, a subset $W \subseteq \omega_1$ will occasionally be identified with the corresponding set $\{\rho_{0\alpha} \mid \alpha \in W\}$.

Let $D \in M$ be an open dense subset of $\mathbb{P}(C)$. By extending p if necessary, we may assume that $p \in D$. We need to prove that p is compatible with an element of $D \cap M$. Denote $\delta = \delta_M$, which is an element of C. Define $X = \mathsf{ht}(p) \setminus M$ and let m be the size of X. Note that δ is the smallest element of X. Let $p_M = (\mathcal{M}_p \cap M, \mathcal{N}_p \cap M)$. It is easy to see that $p_M \in \mathbb{P}(C) \cap M$ and $p_M \sqsubseteq p$.

Find $\eta_0 < \delta$ such that $S(p) \cap M \subseteq \eta_0$. We now define E as the set of all $Y \in [\omega_1]^m$ for which there is $r \in D$ with the following properties:

1. $p_M \sqsubseteq r$.
2. $\mathsf{ht}(r) = \mathsf{ht}(p_M) \cup Y$.
3. $S(r) \cap \eta_0 = S(p) \cap \eta_0$ (which is the same as $S(p) \cap M$).
4. $H(p)$ and $H(r)$ are isomorphic and if $h : S(p) \longrightarrow S(r)$ denote the (unique) isomorphism, then:
   (a) h is the identity mapping on $S(p) \cap M$.
   (b) $\rho_{0\alpha} \restriction \eta_0 = \rho_{0h(\alpha)} \restriction \eta_0$ for all $\alpha \in X$.

Note that $X \in E$. Moreover, E is in M. This is because the definition of E can can be reformulated without reference to objects outside M. For $Y \in E$ and r as above, we will say that r is a witness of $Y \in E$.

Claim 76 *There are η_1 and $F = (t_0, ..., t_{m-1})$ with the following properties:*

1. *$\eta_1 \in M$ and $\eta_0 < \eta_1 < \delta$.*
2. *$F = (t_0, ..., t_{m-1}) \in (T_{\eta_1})^m$ and $t_i \neq \rho_{0\alpha} \restriction \eta_1$ for every $\alpha \in X$ and $i < m$.*
3. *Player II has a winning strategy in the game $\mathrm{G}_F(T, E)$.*

Indeed, the existence of F and η_1 follows by Corollary 71. We can now find η with the following properties:

1. $\eta_1 < \eta < \delta$.
2. If $\alpha, \beta \in X$ and $t \in F$, then $\mathsf{Tr}(\triangle(t, \rho_{0\alpha}), \beta) \cap \delta \subseteq \eta$.

We now have the following:

Claim 77 *There is $Y \in E \cap M$ with the following properties:*

1. *$\eta < \mathit{min}(Y)$.*
2. *Every element of Y extends an element of F.*
3. *If $\gamma \in X$ and $\alpha, \beta \in Y$ are such that $\alpha < \beta$, then $\mathsf{Tr}(\alpha, \gamma) \cap M \subseteq \beta$.*

Let $\sigma \in M$ be a winning strategy for Player II in the game $G_F(T, E)$. Consider the following run of the game (in which Player II is following her strategy[4] σ).

I	η		ξ_0		...
II		$\rho_{0\varepsilon_0}$		$\rho_{0\varepsilon_1}$	

In which Player I plays as follows:

1. His first move is η.
2. After Player II played her move (say $\rho_{0\varepsilon_0}$), Player I chooses $\xi_0 \in M$ that is larger than η, ε_0 and $\bigcup_{\gamma \in X} \mathsf{Tr}(\varepsilon_0, \gamma) \cap M$.

[4]Note that in order for σ to be a winning strategy, it is necessary that all the moves of Player II are of the form $\rho_{0\varepsilon}$ for $\varepsilon < \omega_1$ (otherwise she inmediately loses the game).

3. After Player II played her move (say $\rho_{0\varepsilon_1}$), Player I chooses $\xi_1 \in M$ that is larger than ε_1 and $\bigcup_{\gamma\in X} \mathsf{Tr}(\varepsilon_1,\gamma)\cap M$.

$\vdots\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots$

Since σ is a winning strategy for Player II, it follows that $Y=(\varepsilon_0,...,\varepsilon_{m-1})$ is in E and clearly has the desired properties. This finishes the proof of the claim.

Fix Y as in the claim. By elementarity., there exists $r\in D\cap M$ that is a witnesses of $Y\in E$. Let $h: S(p)\longrightarrow S(r)$ be the unique isomorphism from $H(p)$ to $H(r)$. Recall that h is the identity on M. We claim that p and r are compatible. It suffices to show that $q=(\mathcal{M}_p\cup\mathcal{M}_r,\mathcal{N}_p\cup\mathcal{N}_r)$ belongs to $\mathbb{P}(C)$. Clearly $\mathcal{N}_p\cup\mathcal{N}_r$ is a finite $\in$-chain and contains $\mathcal{M}_p\cup\mathcal{M}_r$.

To verify that q satisfies the completeness condition, it suffices to show that for any $\alpha,\beta\in \mathsf{ht}_{\mathcal{M}}(p)\backslash M$, we have that $[h(\alpha)\,\beta]\in C$. Denote $\triangle=\triangle(h(\alpha),\beta)$. Recall that $h(\alpha)\in M$ while $\beta\notin M$, so $h(\alpha)<\beta$. It follows that $[h(\alpha)\,\beta]=\mathsf{min}(\mathsf{Tr}(\triangle,\beta)\setminus h(\alpha))$. We proceed by cases, depending on whether $\rho_{0\alpha}\restriction\delta$ and $\rho_{0\beta}\restriction\delta$ are equal.

Case: $\rho_{0\alpha}\restriction\delta\neq\rho_{0\beta}\restriction\delta$ (equivalently, $\triangle(\alpha,\beta)<\delta$)

In this case, we have that $\triangle=\triangle(\alpha,\beta)$ and $\mathsf{Tr}(\triangle,\beta)\cap M\subseteq\eta_0\subseteq h(\alpha)$. It follows that $\mathsf{Tr}(\triangle,\beta)\setminus h(\alpha)=\mathsf{Tr}(\triangle,\beta)\setminus\delta$ and therefore
$[h(\alpha)\,\beta]=\mathsf{min}(\mathsf{Tr}(\triangle,\beta)\setminus\delta)$. The proof now proceeds by subcases, depending on whether β is equal to δ.

Subcase: $\rho_{0\alpha}\restriction\delta\neq\rho_{0\beta}\restriction\delta$ and $\beta=\delta$.

We have that $\mathsf{Tr}(\triangle,\beta)\setminus\delta=\{\delta\}$, so $[h(\alpha)\,\beta]=\delta\in C$.

Subcase: $\rho_{0\alpha}\restriction\delta\neq\rho_{0\beta}\restriction\delta$ and $\beta\neq\delta$.

We know that $[h(\alpha)\,\beta]=\mathsf{min}(\mathsf{Tr}(\triangle,\beta)\setminus\delta)$, which is in C by the second crossing models condition of p.

Case: $\rho_{0\alpha}\restriction\delta=\rho_{0\beta}\restriction\delta$ (equivalently, $\delta\leq\triangle(\alpha,\beta)$)

The proof now proceeds by subcases, depending on the relationship of α and β with δ.

Subcase: $\rho_{0\alpha}\restriction\delta=\rho_{0\beta}\restriction\delta$ and $\alpha=\delta$ or $\beta=\delta$.

Lemma 23 implies that if $\rho_{0\alpha}\restriction\delta=\rho_{0\beta}\restriction\delta$ and $\alpha=\delta$ or $\beta=\delta$, then $\alpha=\beta=\delta$. We have that $\mathsf{Tr}(\triangle(h(\delta),\delta),\delta)\cap\delta\subseteq\eta\subseteq h(\delta)$, so $[h(\delta)\,\delta]=\delta\in C$.

Subcase: $\rho_{0\alpha}\restriction\delta=\rho_{0\beta}\restriction\delta$ and $\alpha,\beta\neq\delta$.

Recall that $\lambda(\delta,\beta) < \eta_0 < \triangle$. Proposition 22 implies that $\delta \in \mathsf{Tr}(\triangle,\beta)$. Since $\mathsf{Tr}(\triangle,\beta) \cap M \subseteq \eta \subseteq h(\alpha)$, it follows that $[h(\alpha)\,\beta] = \delta \in C$.

This completes the proof that q satisfies the completeness condition. We now verify the first crossing models condition. It suffices to show that for every $\alpha < \beta < \gamma$ with $\gamma \in \mathsf{ht}_{\mathcal{M}}(p) \setminus M$ and $\alpha,\beta \in \mathsf{ht}(p) \cup \mathsf{ht}(r)$, we have that $\mathsf{min}(\mathsf{Tr}(\alpha,\gamma) \setminus \beta) \in C$. The proof proceeds by cases.

Case: $\alpha \in \mathsf{ht}(p)$

If β is also in $\mathsf{ht}(p)$, then there is nothing to do since $p \in \mathbb{P}(C)$. Assume that $\beta \in \mathsf{ht}(r) \setminus \mathsf{ht}(p)$. Note that $\mathsf{Tr}(\alpha,\gamma) \cap M \subseteq \eta_0 \subseteq \beta$. It follows that $\mathsf{min}(\mathsf{Tr}(\alpha,\gamma) \setminus \beta) = \mathsf{min}(\mathsf{Tr}(\alpha,\gamma) \setminus \delta)$, which is in C by the first crossing models condition of p.

Case: $\alpha \in \mathsf{ht}(r) \setminus \mathsf{ht}(p)$ and $\beta \in \mathsf{ht}(p)$

Note that $\delta \leq \beta$. Since $\lambda(\delta,\gamma) < \eta_0 < \alpha$, Proposition 22 implies that $\mathsf{Tr}(\delta,\gamma)$ is an initial segment of $\mathsf{Tr}(\alpha,\gamma)$. It follows that $\mathsf{min}(\mathsf{Tr}(\alpha,\gamma) \setminus \beta) = \mathsf{min}(\mathsf{Tr}(\delta,\gamma) \setminus \delta)$, which is in C by the first crossing models condition of p.

Case: $\alpha \in \mathsf{ht}(r) \setminus \mathsf{ht}(p)$ and $\beta \in \mathsf{ht}(r)$

Since $\lambda(\delta,\gamma) < \eta_0 < \alpha$, Proposition 22 implies that $\delta \in \mathsf{Tr}(\alpha,\gamma)$. Furthermore, since $\mathsf{Tr}(\alpha,\gamma) \cap M \subseteq \beta$ (see Claim 77), it follows that $\mathsf{min}(\mathsf{Tr}(\alpha,\gamma) \setminus \beta) = \delta \in C$.

This completes the proof that q satisfies the first crossing models condition. We now verify the second crossing models condition. It suffices to show that for every $N \in \mathcal{N}_p \cup \mathcal{N}_r$ and $\alpha,\beta \in \mathsf{ht}_{\mathcal{M}}(p) \cup \mathsf{ht}_{\mathcal{M}}(r)$ are such that $\triangle = \triangle(\alpha,\beta) \in N$ and $\delta_N < \beta$, $\alpha \neq \delta_N$, we have that $\mathsf{min}(\mathsf{Tr}(\triangle,\beta) \setminus N) \in C$. As the reader probably guessed, the proof proceeds by cases.

Case: $N \in \mathcal{N}_p \cap \mathcal{N}_r$ (equivalently, $\mathcal{N} \in \mathcal{N}_p \cap M$)

If either $\alpha,\beta \in \mathsf{ht}(p)$ or $\alpha,\beta \in \mathsf{ht}(r)$, then there is nothing to do since both p and r are conditions, so we assume this is not the case.

Subcase: $N \in \mathcal{N}_p \cap \mathcal{N}_r$, $\alpha \in \mathsf{ht}(p) \setminus \mathsf{ht}(r)$ and $\beta \in \mathsf{ht}(r) \setminus \mathsf{ht}(p)$.

Note that $\alpha \notin M$. Moreover, since $\triangle \in N$, it follows that $\triangle < \eta_0$, which implies that $\triangle = \triangle(h(\alpha),\beta)$. In this way, $\mathsf{min}(\mathsf{Tr}(\triangle,\beta) \setminus N) = \mathsf{min}(\mathsf{Tr}(\triangle(h(\alpha),\beta),\beta) \setminus N)$, which is in C by the second crossing models condition of r.

Subcase: $N \in \mathcal{N}_p \cap \mathcal{N}_r$, $\alpha \in \mathsf{ht}(r) \setminus \mathsf{ht}(p)$ and $\beta \in \mathsf{ht}(p) \setminus \mathsf{ht}(r)$.

Note that $\beta \notin M$. Since $\triangle < \eta_0$, it follows that $\triangle = \triangle(h^{-1}(\alpha),\beta)$. In this way,

$\mathsf{min}(\mathsf{Tr}(\triangle,\beta)\setminus N) = \mathsf{min}(\mathsf{Tr}(\triangle\left(h^{-1}(\alpha),\beta\right),\beta)\setminus N)$, which is in C by the second crossing models condition of p.

Case: $N \in \mathcal{N}_r \setminus \mathcal{N}_p$

If both α, β belong to $\mathsf{ht}(r)$, then there is nothing to do, so we assume this is not the case.

Subcase: $N \in \mathcal{N}_r \setminus \mathcal{N}_p$, $\alpha \in \mathsf{ht}(r)$ and $\beta \notin \mathsf{ht}(r)$.

Note that $\beta \notin M$. Let $\overline{\alpha} = h^{-1}(\alpha)$. First consider the case where $\rho_{0\beta} \restriction \delta \neq \rho_{0\overline{\alpha}} \restriction \delta$. It follows that $\triangle = \triangle(\overline{\alpha},\beta)$ and $\mathsf{Tr}(\triangle,\beta) \cap M \subseteq \eta_0 \subseteq \eta$. Moreover, $\mathsf{min}(\mathsf{Tr}(\triangle,\beta) \setminus N) = \mathsf{min}(\mathsf{Tr}(\triangle(\overline{\alpha},\beta),\beta) \setminus M)$. If $\beta = \delta$, then this minimum is δ, which is in C. If $\delta < \beta$, the conclusion follows by the second crossing models of p. We now assume that $\rho_{0\beta} \restriction \delta = \rho_{0\overline{\alpha}} \restriction \delta$. We have that $\lambda(\delta,\beta) < \eta_0 < \triangle$. Proposition 22 implies that $\delta \in \mathsf{Tr}(\triangle,\beta)$. Moreover,
$\mathsf{Tr}(\triangle,\beta) \cap M \subseteq \eta \subseteq N$. It follows that $\mathsf{min}(\mathsf{Tr}(\triangle,\beta) \setminus N) = \delta \in C$.

Subcase: $N \in \mathcal{N}_r \setminus \mathcal{N}_p$, $\alpha \notin \mathsf{ht}(r)$ and $\beta \in \mathsf{ht}(r)$.

Note that $\alpha \notin M$. Let $\overline{\beta} = h^{-1}(\beta)$. First consider the case where $\rho_{0\alpha} \restriction \delta \neq \rho_{0\overline{\beta}} \restriction \delta$. It follows that $\triangle = \triangle(h(\alpha),\beta)$. Moreover, $\mathsf{min}(\mathsf{Tr}(\triangle,\beta) \setminus N) = \mathsf{min}(\mathsf{Tr}(\triangle(h(\alpha),\beta),\beta) \setminus N)$, which is in C by the second crossing models condition of r. We now assume that $\rho_{0\alpha} \restriction \delta = \rho_{0\overline{\beta}} \restriction \delta$. Since $\lambda(\delta,\overline{\beta}) \in M$ and h is an isomorphism that is the identity in M, we have the following:

$$\begin{array}{rcl} \lambda(\delta,\overline{\beta}) & = & h(\lambda(\delta,\overline{\beta})) \\ & = & \lambda(h(\delta),h(\overline{\beta})) \\ & = & \lambda(h(\delta),\beta) \end{array}$$

Since $\lambda(\delta,\overline{\beta}) < \eta_0 < \triangle$, it follows that $\lambda(h(\delta),\beta) < \triangle$. Proposition 22 implies that $\mathsf{Tr}(h(\delta),\beta)$ is an initial segment of $\mathsf{Tr}(\triangle,\beta)$. Since $h(\delta) \leq \delta_N$, we get that $\mathsf{min}(\mathsf{Tr}(\triangle,\beta) \setminus N) = \mathsf{min}(\mathsf{Tr}(h(\delta),\beta) \setminus N)$, which is in C by the first crossing models condition of p.

Subcase: $N \in \mathcal{N}_r \setminus \mathcal{N}_p$ and $\alpha, \beta \notin \mathsf{ht}(r)$.

Since $\triangle \in N$, it follows that $\triangle \in M$, which implies that $\triangle < \eta_0$. Moreover, $\mathsf{Tr}(\triangle,\beta) \cap M \subseteq \eta_0 \in N$. It follows that $\mathsf{min}(\mathsf{Tr}(\triangle,\beta) \setminus N) = \mathsf{min}(\mathsf{Tr}(\triangle,\beta) \setminus M)$, which is in C by the second crossing models condition of p.

Case: $N \in \mathcal{N}_p \setminus \mathcal{N}_r$

Since $\delta_N < \beta$, it follows that $\beta \in \mathsf{ht}(p) \setminus \mathsf{ht}(r)$. If α is also in $\mathsf{ht}(p)$, then there is nothing to do, so assume that $\alpha \in \mathsf{ht}(r) \setminus \mathsf{ht}(p)$. Let $\overline{\alpha} = h^{-1}(\alpha)$. If $\rho_{0\overline{\alpha}} \restriction \delta \neq \rho_{0\beta} \restriction \delta$, then $\triangle = \triangle(\overline{\alpha},\beta)$, so $\mathsf{min}(\mathsf{Tr}(\triangle,\beta) \setminus N)$ is equal to $\mathsf{min}(\mathsf{Tr}(\triangle(\overline{\alpha},\beta),\beta) \setminus N)$ and it is in C by the second crossing models condition of p. Assume now that $\rho_{0\overline{\alpha}} \restriction \delta = \rho_{0\beta} \restriction \delta$. It follows that $\lambda(\delta_N,\beta) < \triangle$. Proposition 22 implies that $\delta_N \in \mathsf{Tr}(\triangle,\beta)$, so $\mathsf{min}(\mathsf{Tr}(\triangle,\beta) \setminus N) = \delta_N \in C$. At last we finished the proof. ■

Arguing as in the previous section, we obtain the following result:

Theorem 78 *PFA implies that every square-bracket operation of the form* $[\cdot\cdot]_{\mathcal{C}}$ *has the Ramsey club property.*

With this result and Theorems 67 and 7, we conclude:

Corollary 79 *PFA implies that every square-bracket operation has the Ramsey club property.*

15 Open questions

We believe there remains much to be discovered on the topics explored in this paper. In this final section, we present several open questions that we have been unable to solve, hoping they may guide future research on the topic.

We proved that MA is consistent with the failure of the Ramsey club property for the square-bracket operations induced by C-sequences. We want to know whether this is also true for the other types of square-bracket operations.

Problem 80 *Is the statement "No square-bracket operation has the Ramsey club property" consistent with MA?*

As noted earlier, a positive resolution of the problem above would follow if we could prove Proposition 41 for the other square-bracket operations. We conjecture this is the case, but were unable to prove it.

In Proposition 58 for the case of the square-bracket operation induced by a set of reals and a nice sequence, we required the additional hypothesis that the tree generated by the nice sequence is Aronszajn. It is unclear if this assumption is necessary.

Problem 81 *Is the statement "No square-bracket operation induced by a set of reals and a nice sequence has the Ramsey club property" consistent with* $\diamondsuit$?

We conjecture that Proposition 58 holds for all square-bracket operations. A proof of this would solve the previous question.

Probably the most interesting problem on this topic is the following:

Problem 82 *Is CH consistent with the statement "every square-bracket operation has the Ramsey club property"? What about other guessing principles like* $\diamondsuit, \diamondsuit^*, \diamondsuit^+$ *or* $\diamondsuit^\#$*?*

It is possible that the method developed by Asperó and Mota in [8] provide a path to a positive answer for the first part of question. It would also be interesting to know if the *P-ideal Dichotomy* (PID) introduced by the second author in [45] (see also [3] and [41]) has any influence on this topic.

Problem 83 *Does PID imply that every square-bracket operation has the Ramsey club property?*

We would like to know if there are models where there are two square-bracket operations that behave different in respect to the Ramsey club property.

Problem 84 *Is it consistent that there exists two C-sequences $\mathcal{C}$ and $\mathcal{D}$ such that $[\cdot\cdot]_{\mathcal{C}}$ has the Ramsey club property but $[\cdot\cdot]_{\mathcal{D}}$ does not? (A similar question applies to the other classes of square-bracket operations).*

It would also be interesting to compare the behavior between different classes of square-bracket operations. For example:

Problem 85 *Is it consistent that every square-bracket operation induced by a C-sequence has the Ramsey club property, while no square-bracket operation induced by a sequence of reals and a nice sequence has such property?*

Naturally, we are interested in all variants of the previous question.

The influence of the underlying combinatorial structure of the square-bracket operation is not completely understood. For example, we could ask the following:

Problem 86 *Let $\mathcal{R} = \{r_\alpha \mid \alpha \in \omega_1\} \subseteq 2^\omega$ and $e = \langle e_\beta \mid \beta \in \omega_1\rangle$ a nice sequence. How does the combinatorial properties of $\mathcal{R}$ reflect on the properties of $[\cdot\cdot]_{\mathcal{R},e}$? For example, is there anything interesting to be said in case $\mathcal{R}$ is a Luzin or a Sierpiński set?*

Fix a square-bracket operation $[\cdot\cdot]$. For a subset $A \subseteq \omega_1$, define $C(A) = \{[\alpha\beta] \mid \alpha, \beta \in A\}$. It is natural to consider the following problem:

Problem 87 *What can we say about the following statement: "For every $A, B \in [\omega_1]^{\omega_1}$ there is $D \in [\omega_1]^{\omega_1}$ such that $C(D) \subseteq C(A) \cap C(B)$".*

Of course, such statement follows from PFA, but we do not know much more than that. Since we use the Proper Forcing Axiom to prove that all square-bracket operations have the Ramsey club property, it is natural to ask the following questions:

Problem 88 *Is the statement "all square-bracket operations have the Ramsey club property" consistent with an arbitrarily large continuum?*

Problem 89 *Is it possible to force that all square-bracket operations have the Ramsey club property without any Large Cardinal hypothesis?*

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