INDESTRUCTIBILITY OF IDEALS AND MAD FAMILIES

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Abstract. In this survey paper we collect several known results on destroying tall ideals on countable sets and maximal almost disjoint families with forcing. In most cases we provide streamlined proofs of the presented results. The paper contains results of many authors as well as a preview of results of a forthcoming paper of Brendle, Guzmán, Hrušák, and Raghavan.

Introduction and Notation

The goal of this survey is to provide an overview of the state of the art in the area of destructibility and indestructibility of ideals on ω via various forcing notions. The background narrative is the question which maximal almost disjoint (MAD) families can be destroyed with mild forcing notions and the existence of various exotic MAD families.

The survey appeared in its previous incarnation as part of the PhD thesis [23] of the second author. The survey also contains several results from an upcoming paper of Brendle, Guzmán, Hrušák and Raghavan [7]. We hope that the main contribution of this text will be a streamlined presentation of related results which are scattered through the literature.

Our notation and terminology is fairly standard. We start by giving an overview of basic notions used in this paper. We will deal mostly with ideals of two types; ideals $\mathcal{I}$ on a countable set and σ-ideals $\mathscr{J}$ on the Baire space ω^ω or the Cantor space 2^ω. Although the ideals of the first kind will be formally living on various countable sets such as the rationals $\mathbb{Q}$, ω^2, $2^{<\omega}$, $\omega^{<\omega}$ etc., we will casually pretend that the underlying set is ω and we will establish our terminology and state theorems this way. We will always assume that any given ideal contains the ideal of all finite sets denoted by **fin**.

We say that an ideal $\mathcal{I}$ on ω is *tall* if for every $A \in [\omega]^\omega$ there is $I \in \mathcal{I}$ such that $A \cap I$ is infinite. Although we explicitly defined the notion of tallness only for ideals, we will use it for other families of subsets of ω as well. Such tall families are also synonymously called *hitting*. We will generally assume that every ideal on ω is tall. In fact, destroying or preserving tallness is the topic this survey is about.

We adopt the standard notation, $\mathcal{I}^*$ is the filter dual to $\mathcal{I}$ and $\mathcal{I}^+$ is the collection of $\mathcal{I}$-positive sets; $\mathcal{P}(\omega) \setminus \mathcal{I}$. We will generally adopt the custom to use the same terminology for both properties of ideals and properties of filters, i.e. we typically consider the statements "$\mathcal{I}$ is Φ" and "$\mathcal{I}^*$ is Φ" as synonymous. For a set $\mathcal{H} \subseteq \mathcal{P}(\omega)$ we denote $\mathcal{H}^\perp = \{ x \in [\omega]^\omega \mid |x \cap H| < \omega \text{ for each } H \in \mathcal{H} \}$. For a tall ideal (or just a hitting family in general) $\mathcal{I}$ define the cardinal

$$\operatorname{cov}^* \mathcal{I} = \min\{ |\mathcal{H}| \mid \mathcal{H} \subseteq \mathcal{I}, \mathcal{H} \text{ is hitting} \}.$$

We call this invariant the *covering number* of the ideal.

Ideals can be naturally seen as subsets of the Cantor space 2^ω and we can consider their topological properties. Let us recall a useful characterization of F_σ ideals due to Mazur. A *lower semicontinuous submeasure* is a function $\varphi : \mathcal{P}(\omega) \to [0, \infty]$ such that $\varphi(\emptyset) = 0$; if $A \subseteq B$, then $\varphi(A) \leq \varphi(B)$ (monotonicity); $\varphi(A \cup B) \leq \varphi(A) + \varphi(B)$ (subadditivity); and $\varphi(A) = \sup\{ \varphi(A \cap n) \mid n \in \omega \}$ for every $A \subseteq \omega$ (lower semicontinuity).

Proposition 1 (Mazur [44]). *Let $\mathcal{I}$ be an F_σ ideal on ω. There is a lower semicontinuous submeasure φ such that $\varphi(\{ n \}) = 1$ for every $n \in \omega$, and $\mathcal{I} = \operatorname{fin}(\varphi)$.*

We write $A \subseteq^* B$ if $A \setminus B \in \mathbf{fin}$. An ideal $\mathcal{I}$ on ω is a *P-ideal* if for every countable $\mathcal{C} \subseteq \mathcal{I}$ there exists $I \in \mathcal{I}$ such that $C \subseteq^* I$ for each $C \in \mathcal{C}$.

An important role in our considerations will be played by several definable ideals on countable sets. Let us give here the definitions of ideals we will need. All of these are easily seen to be tall. The ideal $\mathbf{fin} \times \mathbf{fin} \subset \mathcal{P}\left(\omega^2\right)$ consist of all subsets of $\omega \times \omega$ which have only finite intersection with all but finitely many columns. The ideal **nwd** consists of nowhere dense sets of the rationals $\mathbb{Q}$. The *density zero* ideal $\mathcal{Z}$ consist of all sets $A \subset \omega$ such $\lim_{n\to\infty} \frac{|A \cap n|}{n} = 0$. The ideal $\mathcal{ED}$ on ω^2 is a sub-ideal of the ideal $\mathbf{fin} \times \mathbf{fin}$; it is generated by the columns and by graphs of functions. I.e. a set is in $\mathcal{ED}$ if the size of its intersection with all but finitely columns is bounded by some number $n \in \omega$. The ideal $\mathcal{ED}$ is an F_σ ideal.

An *almost disjoint family* or just an *AD* family is a set of infinite subsets of ω such that the intersection of any two of its elements is finite. A *MAD* family is an almost disjoint family which is maximal among AD families with respect to the inclusion. As the finite MAD are of little interest to us, we will always implicitly assume that any given AD family is infinite. A straightforward application of the Kuratowski–Zorn lemma provides an easy construction of MAD families. However, the situation is much more interesting when we want to construct MAD families with additional properties.

For an AD family $\mathcal{A}$ we denote $\mathcal{I}(\mathcal{A})$ the ideal generated by $\mathcal{A}$. Notice that $\mathcal{A}$ is a MAD family iff $\mathcal{I}(\mathcal{A})$ is a tall ideal iff $\mathcal{A}$ is hitting. We will generally adopt the custom that any terminology, cardinal characteristic, etc. which we define for ideals will be naturally extended to AD families via the ideal $\mathcal{I}(\mathcal{A})$. I.e. "$\mathcal{A}$ is Φ" means "$\mathcal{I}(\mathcal{A})$ is Φ."

Notice that for a MAD family $\mathcal{A}$ we get $\operatorname{cov}^* \mathcal{A} = \operatorname{cov}^* \mathcal{I}(\mathcal{A}) = |\mathcal{A}|$. For an AD family $\mathcal{A}$ we define $\mathcal{I}^{++}(\mathcal{A})$ to be the collection of sets $X \subseteq \omega$ for which there are infinitely many $A \in \mathcal{A}$ such that $A \cap X$ is infinite. I.e. $\mathcal{I}^{++}(\mathcal{A}) \subseteq \mathcal{I}(\mathcal{B})^+$ for every AD family $\mathcal{B} \supseteq \mathcal{A}$, and $\mathcal{I}^{++}(\mathcal{A}) = \mathcal{I}(\mathcal{A})^+$ if an only if $\mathcal{A}$ is MAD.

For every ideal $\mathcal{I}$ on ω there is a natural σ-centered forcing which destroys $\mathcal{I}$ (as a hitting family), the *Mathias–Příkrý forcing* $\mathbf{M}(\mathcal{I})$, first considered in [43]. Although the poset is usually defined in the language of filters, we adopt here the dual language of ideals. The poset is defined as $\mathrm{M}(\mathcal{I}) = \left\{ (a, I) \colon a \in [\omega]^{<\omega}, I \in \mathcal{I} \right\}$, and $(a, I) \leq (b, J)$ if $b \sqsubseteq a$, $J \subseteq I$, and $(a \setminus b) \cap J = \emptyset$. The forcing adds a generic real $g = \bigcup \{ a \mid (a, I) \in G \}$, where G is an $\mathrm{M}(\mathcal{I})$ generic filter. The generic real g destroys the ideal $\mathcal{I}$; $g \in \mathcal{I}^{\perp}$.

Many standard forcings can be represented as quotient posets $\mathbf{P}_{\mathscr{I}}$. If $\mathscr{I}$ is a σ-ideal on the Baire space or the Cantor space, we denote $\mathbf{P}_{\mathscr{I}}$ the quotient poset of $\mathscr{I}$-positive Borel sets ordered by inclusion. These posets come with a comprehensive theory [59] and will play a central role in this paper.

The forcing $\mathbf{P}_{\mathscr{I}}$ adds a generic real which is defined as the unique element in the intersection of the generic filter. Note that this real does not belong to any Borel set in the ideal $\mathscr{I}$ coded in the ground model. Reals with this property are called $\mathscr{I}$-quasi-generic reals.

In particular, we will operate with the following σ-ideals and their associated forcings. For details see [59].

- $\mathscr{M}$ – the ideal of meager sets, $\mathbf{P}_{\mathscr{M}}$ is the Cohen forcing.
- $\mathscr{N}$ – the ideal of Lebesgue null sets, $\mathbf{P}_{\mathscr{N}}$ is the random forcing.
- $\mathscr{S}$ – the ideal of countable subsets of 2^{ω}. $\mathbf{P}_{\mathscr{S}}$ is the Sacks forcing.
- $\mathscr{K}_{\sigma}$ – the ideal generated by σ-compact subsets of ω^{ω}. $\mathbf{P}_{\mathscr{K}_{\sigma}}$ is the Miller forcing.
- $\mathscr{L}$ – the ideal on ω^{ω} generated by sets of form

$$A_g = \{ f \in \omega^{\omega} \mid (\exists^{\infty} n \in \omega)\, f(n) \in g(f \restriction n) \}$$

  where g ranges over all functions from $\omega^{<\omega}$ to ω. $\mathbf{P}_{\mathscr{L}}$ is the Laver forcing.

Theorem 2 (Hernández-Hernández–Hrušák [26, Theorem 3.7]). *For every tall analytic P-ideal $\mathcal{I}$ the following inequalities hold;*

$$\operatorname{add}\mathcal{N} \leq \operatorname{cov}^*\mathcal{I} \leq \operatorname{non}\mathcal{M}.$$

We will need the following definition, again see [59] for more information on the presented facts.

Definition 3 (Hrušák–Zapletal [34]). Suppose $\mathscr{I}$ is an σ-ideal such that the forcing $\mathbf{P}_{\mathscr{I}}$ is proper. The forcing $\mathbf{P}_{\mathscr{I}}$ has the *continuous reading of names* if for every $\mathscr{I}$-positive Borel set B and a Borel function $f : B \to 2^\omega$ there is an $\mathscr{I}$-positive Borel set $C \subseteq B$ such that the function $f \restriction C$ is continuous.

The continuous reading of names is a common and extremely useful property which is nevertheless somewhat 'slippery.' It is not really a property of the forcing but rather a property of the particular representation $\mathbf{P}_{\mathscr{I}}$.

Whenever $\mathbf{P}_{\mathscr{I}}$ is proper ω^ω-bounding, or if $\mathscr{I}$ is σ-generated by closed sets, then $\mathbf{P}_{\mathscr{I}}$ has the continuous reading of names. The following posets do have continuous reading of names: Cohen, random, Miller, Laver, Sacks. On the other hand, the poset for adding an eventually different real does not have the continuous reading of names. The Mathias–Příkrý poset $\mathbf{M}(\mathcal{I})$ does not have the continuous reading of names unless $\mathcal{I}$ is a P-ideal, see [34].

Proposition 4 (Zapletal [59]). *Let $\mathscr{I}$ be a σ-ideal on ω^ω such that $\mathbf{P}_{\mathscr{I}}$ has the continuous reading of names. For every $B \in \mathbf{P}_{\mathscr{I}}$ there is a G_δ set $D \in \mathbf{P}_{\mathscr{I}}$, $D \leq B$.*

Every ideal on the Baire or the Cantor space has a naturally associated trace ideal. For $a \subseteq \omega^{<\omega}$ define $\pi(a) = \{ x \in \omega^\omega \mid (\exists^\infty n \in \omega)\, x \restriction n \in a \}$, and similarly for $a \subseteq 2^{<\omega}$. For every a the set $\pi(a)$ is G_δ and every G_δ set is of this form.

Definition 5 (Brendle–Yatabe [12], Thümmel [58], Hrušák–Zapletal [34]). For an ideal $\mathscr{I}$ on ω^ω the *trace ideal* $\operatorname{tr}(\mathscr{I})$ of $\mathscr{I}$ is defined by $a \in \operatorname{tr}(\mathscr{I})$ iff $\pi(a) \in \mathscr{I}$. For ideals on 2^ω the definition is analogous.

The trace ideal $\operatorname{tr}(\mathscr{I})$ is always a tall ideal, as long as the ideal $\mathscr{I}$ contains all singletons. We will study the ideals $\operatorname{tr}(\mathcal{M})$, $\operatorname{tr}(\mathcal{N})$, $\operatorname{tr}(\mathcal{S})$, $\operatorname{tr}(\mathcal{K}_\sigma)$, and $\operatorname{tr}(\mathcal{L})$ in Section 1

Let us also establish some arboreal terminology. A *tree* T will generally be an initial subtree of the tree of finite sequences of integers $(\omega^{<\omega}, \subseteq)$ or $(2^{<\omega}, \subseteq)$, usually with no leaves. The space of maximal branches of

T is denoted $[T]$. For $s \in \omega^{<\omega}$ we denote $[s] = \{ f \in \omega^\omega \mid s \subseteq f \}$ and $[\![s]\!] = \{ t \in \omega^{<\omega} \mid s \subseteq t \}$. If $a, b \subseteq \omega^{<\omega}$ and for each $s \in a$ there is $t \in b$, $t \subseteq s$, we say that a *refines* b.

1. Destroying ideals

If $\mathcal{I}$ is a tall ideal and $\mathbf{P}$ is a partial order, we say that $\mathbf{P}$ *destroys* $\mathcal{I}$ if $\mathbf{P}$ forces that $\mathcal{I}$ is no longer tall in the respective generic extension. I.e. if $\mathbf{P}$ adds a new subset of ω that is almost disjoint with every element of $\mathcal{I}$. The following theorem is a collection of several results.

Theorem 6. *Let* $\mathbf{P}$ *be a partial order.*

(1) $\mathbf{P}$ *destroys* $\mathrm{tr}(\mathscr{S})$ *iff* $\mathbf{P}$ *adds new reals.*
(2) $\mathbf{P}$ *destroys* $\mathrm{tr}(\mathscr{K}_\sigma)$ *iff* $\mathbf{P}$ *adds unbounded reals.*
(3) $\mathbf{P}$ *destroys* $\mathrm{tr}(\mathscr{L})$ *iff* $\mathbf{P}$ *destroys* $\mathbf{fin} \times \mathbf{fin}$ *iff* $\mathbf{P}$ *adds dominating reals.*
(4) $\mathbf{P}$ *destroys* $\mathcal{ED}$ *iff* $\mathbf{P}$ *adds eventually different reals.*

We postpone the proof of the theorem to page 10. The theorem may suggest the following conjecture. If $\mathscr{I}$ is a σ-ideal on ω^ω and $\mathbf{P}$ is a forcing, then $\mathbf{P}$ adds $\mathcal{I}$-quasi-generic reals if and only if $\mathbf{P}$ destroys $\mathrm{tr}(\mathscr{I})$. The forward implications of the conjecture does of course hold, however, the other one may not, as demonstrated by the following example from [34]. The Mathias–Příkrý forcing of the $\mathrm{tr}(\mathscr{N})$ ideal $\mathbf{M}(\mathrm{tr}(\mathscr{N}))$ is a σ-centered forcing which destroys $\mathrm{tr}(\mathscr{N})$, but no σ-centered forcing can add an $\mathscr{N}$-quasi-generic real. We will later see that the problem here is that the forcing $\mathbf{M}(\mathrm{tr}(\mathscr{N}))$ does not have the continuous reading of names.

The Katětov order introduced in [35] is a powerful tool for classification of ideals and their destructibility. Let $\mathcal{I}$ be an ideal on a (countable) set X, $\mathcal{J}$ be an ideal on a (countable) set Y. We say that a function $f : Y \to X$ is a *Katětov morphism* if $f^{-1}[A] \in \mathcal{J}$ for every $A \in \mathcal{I}$. If there exists such Katětov morphism, we write $\mathcal{I} \leq_{\mathrm{K}} \mathcal{J}$ ($\mathcal{I}$ is *Katětov below* $\mathcal{J}$). It is easy to see that the Katětov order $\leq_{\mathrm{K}}$ is indeed a reflexive and transitive relation. If $\mathcal{I} \leq_{\mathrm{K}} \mathcal{J} \leq_{\mathrm{K}} \mathcal{I}$, we say that the ideals are Katětov equivalent; $\mathcal{I} \simeq_{\mathrm{K}} \mathcal{J}$. If $\mathcal{I} \leq_{\mathrm{K}} \mathcal{J} \not\leq_{\mathrm{K}} \mathcal{I}$, we write $\mathcal{I} <_{\mathrm{K}} \mathcal{J}$. There is a wide amount of literature concerning the Katětov order, the reader may consult e.g. [9, 19, 30, 31, 32, 45, 47, 52] for results related to the topic this survey.

Let us mention a few basic properties of the Katětov order.

- If $\mathcal{I} \subseteq \mathcal{J}$, then $\mathcal{I} \leq_{\mathrm{K}} \mathcal{J}$. In particular, the Fréchet ideal **fin** is Katětov below every ideal.
- $\mathcal{I} \simeq_{\mathrm{K}} \mathbf{fin}$ iff the ideal $\mathcal{I}$ is not tall.
- If $A \in \mathcal{I}^+$, then $\mathcal{I} \leq_{\mathrm{K}} \mathcal{I} \restriction A$.

If $\mathcal{I} \simeq_{\mathrm{K}} \mathcal{I}\restriction A$ for every $A \in \mathcal{I}^+$, we say that the ideal $\mathcal{I}$ is *Katětov uniform*. If the Katětov morphism in the definition of the order is moreover a fin-to-1 function, we talk about the *Katětov–Blass* order denoted $\leq_{\mathrm{KB}}$.

We extend these orders to AD families. For an AD family $\mathcal{A}$ we say that $\mathcal{A}$ is Katětov above/below $\mathcal{X}$ if $\mathcal{I}(\mathcal{A})$ is Katětov above/below $\mathcal{X}$. In particular, an AD family $\mathcal{A}$ is MAD iff $\mathbf{fin} <_{\mathrm{K}} \mathcal{A}$. The Katětov–Blass order is extended analogously.

Proposition 7 (Hrušák–García Ferreira [31]). *$\mathcal{A} \leq_{\mathrm{K}} \mathbf{fin} \times \mathbf{fin}$ for every infinite AD family $\mathcal{A}$.*

Proof. Choose pairwise different $\{A_n \mid n \in \omega\} \subset \mathcal{A}$ and let $f : \omega \times \omega \to \omega$ be any bijection such that $f[\{n\} \times \omega] \subseteq A_n$ for every $n \in \omega$. It is straightforward to check that f is the desired Katětov morphism. □

Lemma 8 (Hrušák–García Ferreira [31]). *Let $\mathcal{I}, \mathcal{J}$ be ideals such that $\mathcal{I} \leq_{\mathrm{K}} \mathcal{J}$.*

(1) $\operatorname{cov}^* \mathcal{J} \leq \operatorname{cov}^* \mathcal{I}$.
(2) *If a poset $\mathbf{P}$ destroys $\mathcal{J}$, then $\mathbf{P}$ destroys $\mathcal{I}$.*

Proof. Let f be the Katětov morphism. To prove the first clause, we can assume that $\operatorname{cov}^* \mathcal{I}$ is witnessed by a hitting family $\mathcal{H}$ such that $\bigcup \mathcal{H} = \omega$. Now $\left\{ f^{-1}[H] \mid H \in \mathcal{H} \right\} \subseteq \mathcal{J}$ is a hitting family.

Assume that $\mathbf{P}$ forces that $\dot{X}$ is a name for an infinite set almost disjoint with all elements of $\mathcal{J}$. Then $f[\dot{X}]$ is forced to be infinite and almost disjoint with every element of $\mathcal{I}$. □

Corollary 9 (Hrušák–García Ferreira [31]). *if $\mathcal{A} \leq_{\mathrm{K}} \mathcal{B}$ are MAD families, then $|\mathcal{B}| \leq |\mathcal{A}|$.*

The following theorem is the key to understanding the relation between many forcing notions and the destructibility of ideals. Particular instances of the theorem were proved by various authors [12, 28, 38]. The crucial assumption which failed in the $\mathbf{M}(\operatorname{tr}(\mathscr{N}))$ example was the continuous reading of names.

Theorem 10 (Hrušák–Zapletal [34]). *Let $\mathscr{I}$ be a σ-ideal on ω^ω such that $\mathbf{P}_{\mathscr{I}}$ is proper and has the continuous reading of names, and let $\mathcal{J}$ be an ideal on ω. The following are equivalent:*

(1) *There is a condition $B \in \mathbf{P}_{\mathscr{I}}$ which forces that $\mathcal{J}$ is not tall.*
(2) *There is $a \in \operatorname{tr}(\mathscr{I})^+$ such that $\mathcal{J} \leq_{\mathrm{K}} \operatorname{tr}(\mathscr{I})\restriction a$.*

Proof. Suppose there is a as in (2), $B = \pi(a) \in \mathbf{P}_{\mathscr{I}}$, and let $\dot{r}$ be a name for the generic real. Since $B \Vdash \dot{r} \in \pi(\check{a})$, the set $x = \{s \in a \mid s \subseteq r\}$ is forced by B to be infinite. We claim that B forces the set x to be almost disjoint with all elements of $\mathrm{tr}(\mathscr{I})\restriction a$. Let $C \le B$ and $d \in \mathrm{tr}(\mathscr{I})\restriction a$. The condition $C \setminus \pi(d)$ forces that $r \notin \pi(d)$ and thus $x \cap d$ is at most finite. The destruction of $\mathcal{J}$ now follows from the assumption on the Katětov order.

To prove the other implication, assume there is a condition $B \in \mathbf{P}_{\mathscr{I}}$ and a name for an infinite set $\dot{x} = \{\dot{x}_n \mid n \in \omega\}$ such that B forces that x is almost disjoint with elements of $\mathcal{J}$. Since $\mathbf{P}_{\mathscr{I}}$ has the continuous reading of names, we can use Proposition 4 to recursively define antichains $a_n \subset \omega^{<\omega}$ for $n \in \omega$ such that a_{n+1} refines a_n, for $a = \bigcup\{a_n \mid n \in \omega\}$ is $\pi(a) \subseteq B$ a condition in $\mathbf{P}_{\mathscr{I}}$, and for each $s \in a_n$ the set $[s] \cap \pi(a)$ is a condition which decides $\dot{x}_n$ to be some $g(s) \in \omega$. The function $g\colon a \to \omega$ is the desired Katětov morphism. Indeed, if $d \in (\mathrm{tr}(\mathscr{I})\restriction a)^+$, then $\pi(d) \in \mathbf{P}_{\mathscr{I}}$ forces that there for all $n \in \omega$ there is $s \in a_n$, $s \subset r$ for the generic real r. Now $g(s) \in x$ for each such s, $g[d] \cap x$ is forced to be infinite, and $g[d] \notin \mathcal{J}$. □

We see that to classify destructibility we need to understand the traces of ideals associated with forcings and their position in the Katětov order.

Lemma 11. *The ideal* $\mathrm{tr}(\mathscr{M})$ *is Katětov–Blass equivalent to* **nwd**.

Proof. Let $\preceq$ be an ordering of $\mathbb{Q}$ of type ω. For $s \in 2^{<\omega}$ recursively choose a clopen set of rationals U_s and $q(s) = \min_{\preceq} U_s$ such that

- $U_\emptyset = \mathbb{Q}$,
- $U_{s^\frown 0}, U_{s^\frown 1}$ is a partition of $U_s \setminus \{q(s)\}$,
- $\mathbb{Q} = \{q(s) \mid s \in 2^{<\omega}\}$, and
- $\{U_s \mid s \in 2^{<\omega}\}$ is a π-base of $\mathbb{Q}$, i.e. for every open set $O \subseteq \mathbb{Q}$ there is $s \in 2^{<\omega}$ such that $U_s \subseteq O$.

We claim that the bijection q is a Katětov morphism in both directions. Pick $N \in$ **nwd**, we need to prove that $\pi\left(g^{-1}[N]\right) \in \mathscr{M}$. Choose any $s \in 2^{<\omega}$, since N is nowhere dense and the sets U_t form a π-base, there is $t \supseteq s$ such that $U_t \cap N = \emptyset$. I.e. $[t] \subseteq [s]$ and $\pi\left(g^{-1}[N]\right) \cap [t] = \emptyset$.

To prove the other direction pick $a \in \mathrm{tr}(\mathscr{M})$ and choose any $s \in 2^{<\omega}$. We need to find $t \supseteq s$ such that $g[a] \cap U_t = \emptyset$. Since $[s] \cap \pi(a)$ is a meager G_δ set, there is $r \supseteq s$ such that $[r] \cap \pi(a) = \emptyset$. Thus there is $t \supseteq r$ such that $[\![t]\!] \cap a = \emptyset$, and consequently $g[a] \cap U_t = \emptyset$. □

To show that a given ideal $\mathcal{J}$ is indestructible with the forcing $\mathbf{P}_{\mathscr{I}}$ we need to argue that $\mathcal{J}$ is not Katětov below restrictions of $\mathrm{tr}(\mathscr{I})$. A viable strategy to prove this is to show that the covering number $\mathrm{cov}^* \mathcal{I}$ is consistently strictly smaller than the cov^* numbers of restrictions of $\mathrm{tr}(\mathscr{I})$, and use Lemma 8 and an absolutness argument. Therefore it is useful to determine the covering numbers of the trace ideals. The following proposition will help us with that.

Proposition 12 (Hrušák–Zapletal [34]). *Let $\mathscr{I}$ be an ideal on ω^ω generated by analytic sets. Then* $\operatorname{cov}\mathscr{I} \leq \operatorname{cov}^* \operatorname{tr}(\mathscr{I}) \leq \max(\operatorname{cov}\mathscr{I}, \mathfrak{d})$.

Proof. If $\mathcal{H} \subset \operatorname{tr}(\mathscr{I})$ is of size smaller than $\operatorname{cov}\mathscr{I}$, there is $x \in \omega^\omega$ such that $x \notin \pi(H)$ for each $H \in \mathcal{H}$, and thus $\{ x \restriction n \mid n \in \omega \} \cap H$ is finite for each $H \in \mathcal{H}$, the first inequality is proved.

Put $\kappa = \max(\operatorname{cov}\mathscr{I}, \mathfrak{d})$. Let $\mathbf{S}$ be the set of functions $f : \omega^{<\omega} \to [\omega^{<\omega}]^{<\omega}$ such that $f(s) \subseteq [\![s]\!]$ for each $s \in \omega^{<\omega}$. Since $\kappa \geq \mathfrak{d}$, there is a family of functions $\{ f_\alpha \in \mathbf{S} \mid \alpha \in \kappa \}$ such that for each $g \in \mathbf{S}$ there is $\alpha \in \kappa$ such that $g(s) \cup \{ s \} \subseteq f_\alpha(s)$ for each $s \in \omega^{<\omega}$. Since every every analytic set is a union of at most $\mathfrak{d}$ many compact sets and $\kappa \geq \operatorname{cov}\mathscr{I}$, there are is a set $\left\{ T_\beta \mid \beta \in \kappa \right\}$ of finitely branching subtrees of $\omega^{<\omega}$ such that $\bigcup \left\{ [T_\beta] \mid \beta \in \kappa \right\} = \omega^\omega$ and $[T_\beta] \in \mathscr{I}$ for each $\beta \in \kappa$.

Fix $\alpha, \beta \in \kappa$. We will recursively choose an increasing function $k\colon \omega \to \omega$ and define finite sets $a_{\alpha,\beta}(n) \subset \omega^{<\omega}$ such that

- $k(0) = 0$, $a_{\alpha,\beta}(0) = f_\alpha(\emptyset)$,
- $k(n+1) > \max\left\{ |s| \mid s \in a_{\alpha,\beta}(n) \right\}$, and
- $a_{\alpha,\beta}(n+1) = \bigcup \left\{ f_\alpha(s) \mid s \in T_\beta, |s| = k(n+1) \right\}$.

Once this is done, let $a_{\alpha,\beta} = \bigcup \left\{ a_{\alpha,\beta}(n) \mid n \in \omega \right\}$. Note that $\pi(a_{\alpha,\beta}) = [T_\beta] \in \mathscr{I}$.

Moreover, for every $t \in \omega^{<\omega}$ and $\alpha \in \kappa$ let $b_{\alpha,t} = \bigcup \{ f_\alpha(t^\frown n) \mid n \in \omega \}$. We have that $\pi(b_{\alpha,t}) = \emptyset$, i.e. $b_{\alpha,t} \in \operatorname{tr}(\mathscr{I})$. The set $\left\{ a_{\alpha,\beta} \mid \alpha, \beta \in \kappa \right\} \cup \left\{ b_{\alpha,t} \mid t \in \omega^{<\omega}, \alpha \in \kappa \right\} \subseteq \operatorname{tr}(\mathscr{I})$ is of size κ, it remains to show that it is hitting.

Let $X \subset \omega^{<\omega}$ be an infinite set. Put

$$L = \{ s \in \omega^{<\omega} \mid [\![s]\!] \cap X \text{ is infinite} \}.$$

Assume first that there exist a maximal $s \in L$. Then

$$M = \{ n \in \omega \mid [\![s^\frown n]\!] \cap X \neq \emptyset \}$$

is infinite, there is $\alpha \in \kappa$ such that $f_\alpha(s^\frown n) \cap X \neq \emptyset$ for each $n \in M$, and $b_{\alpha,s} \cap X$ is infinite. On the other hand assume there is an infinite branch $r \in [L] \cap [T_\beta]$ for some $\beta \in \kappa$. For each $n \in \omega$ the intersection $X \cap [\![r \restriction n]\!]$ is nonempty, and there is $\alpha \in \kappa$ such that $f_\alpha(r \restriction n) \cap X$ is nonempty. Now $a_{\alpha,\beta} \cap X$ is infinite. □

Notice that the proof gives us, in fact, the following corollary.

Corollary 13 (Hrušák–Zapletal [34]). *Suppose that a forcing* **P** *does not add unbounded reals, i.e. ground model reals are preserved as a dominating family. Then* **P** *does not destroy the ideal* $\mathrm{tr}(\mathcal{K}_\sigma)$.

From Proposition 12 we get $\mathrm{cov}\,\mathcal{M} \leq \mathrm{cov}^*\,\mathbf{nwd} \leq \mathfrak{d}$. This can be improved. The presented proof of the following theorem is based on the proof in [3].

Proposition 14 (Keremedis [37]).

$$\mathrm{cov}\,\mathcal{M} = \mathrm{cov}^*\,\mathbf{nwd} = \mathrm{cov}^*\,\mathrm{tr}(\mathcal{M})$$

Proof. The second equality follows from Lemma 11. To prove the first one we need to show that $\mathrm{cov}^*\,\mathrm{tr}(\mathcal{M}) \leq \mathrm{cov}\,\mathcal{M}$. Suppose that $\{ T_\alpha \mid \alpha \in \kappa \}$ is a family of trees in $2^{<\omega}$ such that $[T_\alpha] \in \mathcal{M}$ for each $\alpha \in \kappa$, and $\kappa < \mathrm{cov}^*\,\mathrm{tr}(\mathcal{M})$. Since $\mathrm{tr}(\mathcal{M})\restriction[\![s]\!]$ is isomorphic to $\mathrm{tr}(\mathcal{M})$ for each $s \in 2^{<\omega}$, there is infinite $Y \subseteq [\![s]\!]$ almost disjoint with each T_α, $\alpha \in \kappa$.

If there exists some Y like this such that $\pi(Y) \neq \emptyset$ we are done since $\pi(Y) \cap [T_\alpha] = \emptyset$, demonstrating $\kappa < \mathrm{cov}\,\mathcal{M}$. Assume for contradiction that there is no such infinite Y. Notice that this implies that each $Y \in \{ T_\alpha \mid \alpha \in \kappa \}^\perp$ contains an infinite antichain. For $n \in \omega$ we build recursively infinite antichains $A_n \subset 2^{<\omega}$ such that

(1) $A_n \in \{ T_\alpha \mid \alpha \in \kappa \}^\perp$,
(2) A_{n+1} refines A_n, and
(3) $[\![s]\!] \cap A_{n+1}$ is infinite for each $s \in A_n$.

The construction starts with arbitrary suitable A_0. If A_n is defined, choose an infinite antichain $Y_s \in \{ T_\alpha \mid \alpha \in \kappa \}^\perp$, $Y_s \subset [\![s]\!]$ for each $s \in A_n$. Notice that if $s \in A_n \setminus T_\alpha$, then $Y_s \cap T_\alpha = \emptyset$ and $A_{n+1} = \bigcup\{ Y_s \mid s \in A_n \}$ is as required.

Once the construction is done enumerate $A_n = \{ s_n(i) \mid i \in \omega \}$ and for each $\alpha \in \kappa$ choose $f_\alpha : \omega \to \omega$ such that T_α is disjoint with $\{ s_n(i) \mid i > f_\alpha(n) \}$ for each $n \in \omega$. Since $\kappa < \mathrm{cov}^*\,\mathrm{tr}(\mathcal{M}) \leq \mathfrak{d}$ there is $g : \omega \to \omega$ not dominated by any f_α. We can now choose an increasing chain $\{ t_n \mid n \in \omega \}$ such that $t_n \in \{ s_n(i) \mid i > g(n) \}$. The choice of $t_{n+1} \supset t_n$ is possible because of condition (3). Moreover if $g(n) \geq f_\alpha(n)$ for some $n \in \omega$ and $\alpha \in \kappa$, then $t_m \notin T_\alpha$ for every $m \geq n$. Thus $Y = \{ t_n \mid n \in \omega \} \in \{ T_\alpha \mid \alpha \in \kappa \}^\perp$ and $\pi(Y) \neq \emptyset$, a contradiction. □

Now we can conclude:

Proposition 15 (Hrušák–Zapletal [34]).

(1) $\mathrm{cov}^*\,\mathrm{tr}(\mathcal{S}) = \mathfrak{c}$.
(2) $\mathrm{cov}^*\,\mathrm{tr}(\mathcal{K}_\sigma) = \mathfrak{d}$.
(3) $\mathrm{cov}^*\,\mathrm{tr}(\mathcal{M}) = \mathrm{cov}\,\mathcal{M}$.

(4) $\operatorname{cov}^* \operatorname{tr}(\mathscr{L}) = \mathfrak{b}$.

Proof. If $\mathcal{H} \subseteq \operatorname{tr}(\mathscr{S})$ is of size less than $\mathfrak{c}$, there is $x \in 2^\omega \setminus \bigcup\{\, \pi(H) \mid H \in \mathcal{H}\,\}$ which witnesses that $\mathcal{H}$ is not hitting. For $\operatorname{tr}(\mathscr{K}_\sigma)$ the equality follows from $\operatorname{cov} \mathscr{K}_\sigma = \mathfrak{d}$ and Proposition 12. The $\operatorname{tr}(\mathscr{M})$ case is Proposition 14. The last item follows from Proposition 12 and $\mathbf{fin} \times \mathbf{fin} \leq_{\mathrm{K}} \operatorname{tr}(\mathscr{L})$. □

For many standard forcings the trace ideals are, in fact, Katětov homogeneous. This means that when determining destructibility via the Katětov order, we do not really need to consider the restrictions of the trace ideals.

Lemma 16. *Let $\mathscr{I}$ be an ideal on ω^ω and let $a \subseteq \omega^{<\omega}$ such that $\pi(a) = \omega^\omega$ Then* $\operatorname{tr}(\mathscr{I}) \restriction a \leq_{\mathrm{K}} \operatorname{tr}(\mathscr{I})$. *If $\mathscr{I}$ is an ideal on 2^ω, we get* $\operatorname{tr}(\mathscr{I}) \restriction a \leq_{\mathrm{KB}} \operatorname{tr}(\mathscr{I})$

Proof. We may assume that $\emptyset \in a$. Define a function $f : \omega^{<\omega} \to a$ by declaring that $f(t)$ is the largest element of $a \cap \{\, t \restriction n \mid n \in \omega\,\}$. Notice that since $\pi(a) = \omega^\omega$, the preimage of no point contains an infinite chain, and in the case of 2^ω the function f is **fin**-to-1 (use compactness). For $b \subseteq a$ we get $\pi\left(f^{-1}[b]\right) = \pi(b)$ and f is the desired morphisms. □

Definition 17. Let $\mathscr{I}, \mathscr{J}$ be ideals on ω^ω. We say that $\mathscr{I}$ is *continuously Katětov below* $\mathscr{J}$; $\mathscr{I} \leq_{\mathrm{CK}} \mathscr{J}$, if there is a continuous Katětov morphism witnessing $\mathscr{I} \leq_{\mathrm{K}} \mathscr{J}$.

Proposition 18 (Meza-Alcántara [45])**.** *Let $\mathscr{I}, \mathscr{J}$ be ideals on ω^ω. If $\mathscr{I} \leq_{\mathrm{CK}} \mathscr{J}$, then* $\operatorname{tr}(\mathscr{I}) \leq_{\mathrm{K}} \operatorname{tr}(\mathscr{J})$.

Proof. Let $F : \omega^\omega \to \omega^\omega$ be the continuous Katětov morphism witnessing $\mathscr{I} \leq_{\mathrm{CK}} \mathscr{J}$. Define $f : \omega^{<\omega} \to \omega^{<\omega}$ as $f(s) = \bigcap f\big[[s]\big] \restriction |s|$ for $s \in \omega^{<\omega}$. We claim f is a Katětov morphism witnessing $\operatorname{tr}(\mathscr{I}) \leq_{\mathrm{K}} \operatorname{tr}(\mathscr{J})$. Indeed, if $a \subseteq \omega^{<\omega}$ and $x \in \pi\left(f^{-1}[a]\right)$, then $F(x) \in \pi(a)$ due to the continuity of F, and consequently $\pi\left(f^{-1}[a]\right) \subseteq F^{-1}[\pi(a)]$. □

Proposition 19. *The ideals* $\operatorname{tr}(\mathscr{S})$ *and* **nwd** *are Katětov–Blass uniform, the ideals* $\operatorname{tr}(\mathscr{K}_\sigma)$ *and* $\operatorname{tr}(\mathscr{L})$ *are Katětov uniform.*

Proof. Every uncountable Borel subset of 2^ω contains a copy of the Cantor set, every $\mathscr{K}_\sigma$ positive Borel subset of ω^ω contains a copy of a superperfect tree [36], every $\mathscr{L}$ positive Borel subset of ω^ω contains a homeomorphic copy of a Laver tree [10], and the result for $\operatorname{tr}(\mathscr{S})$, $\operatorname{tr}(\mathscr{K}_\sigma)$ and $\operatorname{tr}(\mathscr{L})$ follows by the previous lemmas. Finally if $a \notin \mathbf{nwd}$, then it contains a copy of the rationals. □

Proof of Theorem 6. If $\mathbf{P}$ destroys any ideal, it has to add reals. If $r \in 2^\omega$ is a new real, the set $\{\, r \restriction n \mid n \in \omega\,\}$ destroys $\operatorname{tr}(\mathscr{S})$. If a new unbounded branch in ω^ω is added, the branch witnesses that $\mathrm{tr}(\mathscr{K}_\sigma)$ is destroyed. For the other implication see Corollary 13.

It is easy to see that if $\mathbf{P}$ adds a dominating real, then it destroys both $\mathbf{fin} \times \mathbf{fin}$ and $\mathrm{tr}(\mathscr{L})$. Since the Laver forcing adds a dominating real, it destroys $\mathbf{fin}\times\mathbf{fin}$ and by Theorem 10 and Proposition 19 $\mathbf{fin}\times\mathbf{fin} \leq_K \mathrm{tr}(\mathscr{L})$. Lemma 8 now states that if $\mathrm{tr}(\mathscr{L})$ is destroyed, $\mathbf{fin} \times \mathbf{fin}$ is also destroyed. And if $\mathbf{fin} \times \mathbf{fin}$ is destroyed, there is an infinite partial function $f : A \to \omega$ which is almost disjoint with each element of $(\mathbf{fin} \times \mathbf{fin}) \cap V$. In particular, $g <^* f$ for every $g : A \to \omega$, $g \in V$, and there is a new dominating real.

Finally a new almost disjoint real obviously destroys $\mathcal{ED}$, and if $\mathbf{P}$ destroys $\mathcal{ED}$, it makes $V \cap \omega^\omega$ meager, hence it adds an eventually different real (see e.g. [4, Lemma 2.4.8]). □

We now get the following characterizations.

Proposition 20 (Brendle–Yatabe [12])**.** *Let $\mathcal{I}$ be an ideal on ω. The following are equivalent.*

(1) *Sacks forcing destroys $\mathcal{I}$.*
(2) *Every forcing which adds a real destroys $\mathcal{I}$.*
(3) $\mathcal{I} \leq_K \mathrm{tr}(\mathscr{S})$.

Proposition 21 (Brendle–Yatabe [12])**.** *Let $\mathcal{I}$ be an ideal on ω. The following are equivalent.*

(1) *Miller forcing destroys $\mathcal{I}$.*
(2) *Every forcing which adds an unbounded real destroys $\mathcal{I}$.*
(3) $\mathcal{I} \leq_K \mathrm{tr}(\mathscr{K}_\sigma)$.

Proposition 22 (Hrušák [28], Kurilić [38])**.** *Let $\mathcal{I}$ be an ideal on ω. The following are equivalent.*

(1) *Cohen forcing destroys $\mathcal{I}$.*
(2) $\mathcal{I} \leq_K \mathbf{nwd}$.

Theorem 10 characterizes destructibility of ideals for forcings $\mathbf{P}_{\mathscr{I}}$ using the Katětov order. It turns out that for many standard ideals we can, in fact, use the Katětov–Blass order instead. Brendle and Yatabe [12] called this property of ideals a very weak fusion.

Definition 23. Let $\mathscr{I}$ be a σ-ideal on ω^ω such that $\mathbf{P}_{\mathscr{I}}$ is proper and has the continuous reading of names. We say that $\mathscr{I}$ has a *very weak fusion* if for every ideal $\mathcal{J}$ on ω the following conditions are equivalent.

(1a) There is a condition $B \in \mathbf{P}_{\mathscr{I}}$ which forces that $\mathcal{J}$ is not tall.
(1b) There is $a \in \mathrm{tr}(\mathscr{I})^+$ such that $\mathcal{J} \leq_K \mathrm{tr}(\mathscr{I}) \restriction a$.
(2) There is $a \in \mathrm{tr}(\mathscr{I})^+$ such that $\mathcal{J} \leq_{KB} \mathrm{tr}(\mathscr{I}) \restriction a$.

The equivalence of (1a) and (1b) is of course exactly Theorem 10, so the ideal has a very weak fusion if these conditions imply (2).

Proposition 24 (Brendle–Yatabe [12]). *The ideals $\mathscr{S}$, $\mathscr{M}$, $\mathscr{N}$, and $\mathscr{K}_\sigma$ do have a very weak fusion.*

Proof. Suppose that $\mathbf{P}_{\mathscr{I}}$ ($\mathscr{I}$ being one of our ideals) destroys an ideal $\mathcal{J}$ on ω. We are searching for $a \in \mathrm{tr}(\mathscr{I})^+$ and a **fin**-to-1 Katětov morphism.

Cohen forcing; $\mathscr{M}$. Let $s \in \omega^{<\omega}$ be a condition that forces that $\dot{X}$ is a name for a set in $\mathcal{J}^\perp$. We can recursively find set $a \subseteq [\![s]\!]$ which is dense in $[\![s]\!]$, and a 1-to-1 function $f : a \to \omega$ such that $t \Vdash f(t) \in \dot{X}$ for each $t \in a$. It is obvious that $a \in \mathrm{tr}(\mathscr{M})^+$ and it is easy to check that f is a Katětov morphism.

Sacks forcing; $\mathscr{S}$. Let $p \subseteq 2^{<\omega}$ be a Sacks tree that forces that $\dot{X}$ is a name for a set in $\mathcal{J}^\perp$. We will use a fusion-like construction. For all $s \in 2^{<\omega}$ recursively define Sacks trees p_s with stem $t_s \in 2^{<\omega}$, and $f(t_s) \in \omega$ such that

- $p_\emptyset \leq p$,
- $p_u < p_v$ for $v \subset u \in 2^{<\omega}$,
- $t_{s^\frown 0}$ and $t_{s^\frown 1}$ are incomparable in $2^{<\omega}$,
- $p_s \Vdash f(t_s) \in \dot{X}$,
- $f(t_u) \neq f(t_v)$ for $u \neq v \in 2^{<\omega}$.

Again $a = \{ t_s \mid s \in 2^{<\omega} \} \in \mathrm{tr}(\mathscr{S})^+$, $f : a \to \omega$ is 1-to-1, and it is easy to check that f is a Katětov morphism.

Miller forcing; $\mathscr{K}_\sigma$. The proof is similar to the case of the Sacks forcing, just use the appropriate fusion for the Miller forcing.

Random forcing; $\mathscr{N}$. Let $p \subseteq 2^{<\omega}$ be a tree such that $[p]$ has positive Lebesgue measure and forces that $\dot{X} = \{ \dot{x}_n \mid n \in \omega \}$ is a name for a set in $\mathcal{J}^\perp$. By the usual proof of the forcing being ω^ω-bounding we may assume that there are finite sets F_n and $h_n \colon F_n \to \omega$ for $n \in \omega$ such that

- F_n is a finite maximal antichain in p,
- F_{n+1} refines F_n,
- $p \cap [\![s]\!] \Vdash \dot{x}_n = h_n(s)$ for each $s \in F_n$.

Find a set $W \in [\omega]^\omega$ such that $h_n[F_n] < h_m[F_m]$ for each $n, m \in W$, $n < m$. Let $a = \bigcup\{ F_n \mid n \in W \}$ and $f = \bigcup\{ h_n \mid n \in W \}$; $\pi(a) = [p] \in \mathscr{N}^+$ and f is **fin**-to-1. The fact that f is a Katětov morphism is, same as for the other ideals, easy to check. □

There are many other important results on destroying ideals which will not be explicitly used in the present paper. Let us however mention at least a couple of these.

Theorem 25 (Laflamme [39]). *Every F_σ ideal can be destroyed without adding unbounded reals.*

Theorem 26 (Laflamme [39]). *Every F_σ ideal can be destroyed with a forcing which preserves P-ultrafilters.*

Theorem 27 (Zapletal [59]). *Every F_σ ideal can be destroyed without adding unbounded or splitting reals.*

Let us provide here a definition of a forcing which has the properties stated in the previous theorem, this definition comes from [26]. Let $\mathcal{I}$ be an F_σ ideal and let $\mathcal{I} = \mathrm{Fin}(\varphi)$ for some lower semicontinuous submeasure φ. The forcing $\mathbf{P}_\varphi$ consists of finitely branching trees $T \subset \omega^{<\omega}$ ordered by inclusion, which fulfill the property that for each $n \in \omega$ there are only finitely $t \in T$ such that $\varphi(\{ i \mid t^\frown i \in T \}) < n$.

These results imply that ideals which are Katětov below an F_σ ideal are relatively easy to destroy. Ideals which do not have an F_σ ideal $\leq_{\mathrm{K}}$-above them are called *Laflamme*. In fact, being Laflamme is equivalent to being not extendible to an F_σ ideal.

Lemma 28. *An ideal $\mathcal{I}$ on ω is Laflamme iff there is no F_σ ideal $\mathcal{J} \supseteq \mathcal{I}$.*

Proof. Suppose there exists an F_σ ideal $\mathcal{K} = \mathrm{Fin}(\varphi)$ given by a lower semicontinuous submeasure φ and a Katětov morphism f witnessing $\mathcal{I} \leq_{\mathrm{K}} \mathcal{K}$. For $n \in \omega$ let define a closed set $C_n = \left\{ X \subseteq \omega \mid \varphi\left(f^{-1}[X]\right) \leq n \right\}$. Now $\mathcal{J} = \bigcup\{ C_n \mid n \in \omega \}$ is an F_σ ideal containing $\mathcal{I}$. □

The case of the density zero ideal $\mathcal{Z}$ is quite more complicated.

Theorem 29 (Raghavan–Shelah [51]). *The density zero ideal $\mathcal{Z}$ can not be destroyed without adding unbounded reals, and* $\mathrm{cov}^* \mathcal{Z} \leq \mathfrak{d}$.

Theorem 30 (Raghavan [49]). *If a forcing destroys the density zero ideal $\mathcal{Z}$, then it adds a dominating real or a real which is not promptly split. In particular $\mathcal{Z}$ can not be destroyed by a Suslin c.c.c. forcing without adding dominating reals.*

For the definition of a promptly split real see [49].

Question 31 (Hernández-Hernández–Hrušák [26]). Is $\mathrm{cov}^* \mathcal{Z} \leq \mathfrak{b}$ a theorem of ZFC?

2. Destroying MAD families

Our main focus is investigating destructibility of MAD families. We say that a MAD family $\mathcal{A}$ is destroyed in a generic extension if it is no longer maximal, that is the ideal is $\mathcal{I}(\mathcal{A})$ is destroyed. The results summarized so far give us the following.

Proposition 32. *Let $\mathcal{A}$ be a MAD family.*

(1) [28] *If $|\mathcal{A}| < \mathfrak{c}$, then $\mathcal{A}$ is Sacks indestructible.*
(2) [12] *If $|\mathcal{A}| < \mathfrak{d}$, then $\mathcal{A}$ is Miller indestructible.*
(3) [28, 38] *If $|\mathcal{A}| < \operatorname{cov} \mathscr{M}$, then $\mathcal{A}$ is Cohen indestructible.*

Proof. We have $\operatorname{cov}^* \mathcal{I}(\mathcal{A}) = |A|$, and the proposition follows from Propositions 20, 21, and 22 combined with (1) of Lemma 8. □

Since it possible to construct a MAD family inside any given tall ideal, there are always MAD families which are destroyed whenever a real is added. For example, every MAD contained in $\operatorname{tr}(\mathscr{S})$ is of this sort. The existence of indestructible MAD families is more interesting.

Problem (Steprāns [57]). Is there a Cohen indestructible MAD family?

Problem (Hrušák [29]). Is there a Sacks indestructible MAD family?

These families are known to exist under several additional assumptions, but it is currently unknown whether they exist in ZFC.

The following proposition follows from the fact that every AD family is Katětov below $\mathbf{fin} \times \mathbf{fin}$,

Proposition 33. *If a forcing P adds a dominating real, then P destroys every MAD family.*

Definition 34. We call a MAD family $\mathcal{A}$ *tight* if for every

$$\left\{ X_n \in \mathcal{I}(\mathcal{A})^+ \mid n \in \omega \right\}$$

there is $B \in \mathcal{I}(\mathcal{A})$ such that $B \cap X_n$ is infinite for each $n \in \omega$.

Tight MAD families were first considered by Malykhin who called them of ω-MAD families [42]. Notice that in the definition of tightness we can equivalently demand only that all the intersections $B \cap X_n$ are nonempty instead of infinite.

Tightness implies Cohen indestructibility, the other implication is only partially correct.

Proposition 35 (Malykhin [42], Hrušák–García Ferreira [31], Kurilić [38]). *Let $\mathcal{A}$ be a MAD family.*

(1) *If $\mathcal{A}$ is tight, then $\mathcal{A}$ is Cohen indestructible.*
(2) *If $\mathcal{A}$ is Cohen indestructible, then there is $X \in \mathcal{I}(\mathcal{A})^+$ such that $\mathcal{A} \restriction X$ is tight.*

Proof. Let $\mathcal{A}$ be a tight MAD family, we will show that $\mathcal{I}(\mathcal{A}) \not\leq_K$ **nwd**. Let $f : \mathbb{Q} \to \omega$ be a candidate for a Katětov morphism, $\{ U_n \mid n \in \omega \}$ be a base of open sets of $\mathbb{Q}$. If $f[U_n] \in \mathcal{I}(\mathcal{A})$ for some $n \in \omega$, the f is not a Katětov morphism so suppose this is not the case. Since $\mathcal{A}$ is tight, there is $B \in \mathcal{I}(\mathcal{A})$ such that $B \cap f[U_n]$ is infinite for each $n \in \omega$. Thus $f^{-1}[B]$ is a dense set, $f^{-1}[B] \notin$ **nwd**.

To prove the second statement, assume that $\mathcal{A}$ has no tight restrictions, we show that the Cohen forcing destroys $\mathcal{A}$. For all $s \in \omega^{<\omega}$ recursively define sets $X_S \in \mathcal{I}(\mathcal{A})^+$ such that $\{X_{s^\frown n} \mid n \in \omega\}$ witnesses that $\mathcal{A} \restriction X_s$ is not tight. The construction can start with $X_\emptyset = \omega$. Let $c \in \omega^\omega$ be a Cohen real, let X be an infinite pseudo-intersection of $\left\{ X_{c \restriction n} \mid n \in \omega \right\}$. The set X is forced to be almost disjoint with all elements of $\mathcal{A}$; for every $A \in \mathcal{A}$ the set $\{ s \in \omega^{<\omega} \mid |A \cap X_s| < \omega \}$ is dense. □

Consequently tight MAD families exist if and only if Cohen indestructible MAD families exist. On the other hand, Cohen indestructibility does not in general imply tightness. A Cohen indestructible MAD which is not tight can be constructed e.g. under CH.

Lemma 36 (Hrušák–García Ferreira [31], Kurilić [38]). *Let $\mathcal{A}$ be an AD family of size less than $\mathfrak{b}$. For every*

$$\{X_n \mid n \in \omega\} \subseteq \mathcal{I}(\mathcal{A})^+$$

there exists $B \in \mathcal{A}^\perp$ such that $|B \cap X_n| = \omega$ for each $n \in \omega$.

Proof. Since $\mathfrak{b} \leq \mathfrak{a}$, the family $\mathcal{A}$ has no MAD restrictions, and we may assume that $\{X_n \mid n \in \omega\} \subseteq \mathcal{A}^\perp$ and the set consists of pairwise disjoint elements. For $A \in \mathcal{A}$ let $f_A \colon \omega \to \omega$ be a function such that $A \cap X_n \subseteq f_A(n)$. Since $|\mathcal{A}| < \mathfrak{b}$, there is a function $g : \omega \to \omega$ dominating every f_A for every $A \in \mathcal{A}$. Now $B = \bigcup\{X_n \setminus g(n) \mid n \in \omega\}$ is the desired set. □

We say that MAD families in a given class *exist generically* if for every AD family $\mathcal{A}$ of size less than $\mathfrak{c}$ there is a MAD family $\mathcal{B} \supseteq \mathcal{A}$ in the given class.

Proposition 37 (Kurilić [38], Hrušák–García Ferreira [31]). *If $\mathfrak{b} = \mathfrak{c}$, then tight MAD families exist generically.*

Proof. Given an AD family $\mathcal{A}$ of size less than $\mathfrak{c}$, just keep extending $\mathcal{A}$ by adding elements obtained by an application of Lemma 36 while keeping track of all possible witnesses for non-tightness. □

Proposition 38 (Guzmán–Hrušák–Martínez-Ranero–Ramos-García [22]). *There is an AD family of size $\mathfrak{b}$ which cannot be extended to a tight MAD family.*

Proof. Fix a bijection $\mathrm{b}\colon \omega^{<\omega} \to \omega$ and a set $X \subset \omega^\omega$ of size $\mathfrak{b}$ which is not contained in any σ-compact set. For $f \in \omega^\omega$ define

$$A_f = \left\{ \bigcup \{ \mathrm{b}(f \restriction n) \} \times f(n) \mid n \in \omega \right\}.$$

Let $\mathcal{A}_0 = \{ A_x \mid x \in X \}$ and notice that $\mathcal{A}_0$ is an AD family on $\omega \times \omega$ of size $\mathfrak{b}$. Notice also that for every $g \in \omega^\omega$ the set $N(g) = \left\{ f \in \omega^\omega \mid \left| g \cap A_f \right| < \omega \right\}$ is σ-compact. For each $n \in \omega$ choose a countable AD family $\mathcal{C}_n$ on $C_n = \{ n \} \times \omega$ and let $\mathcal{A} = \mathcal{A}_0 \bigcup \{ \mathcal{C}_n \mid n \in \omega \}$. Now $\mathcal{A}$ is an AD family of size $\mathfrak{b}$ and $C_n \in \mathcal{I}^{++}(\mathcal{A})$ for each $n \in \omega$. Whenever a set B intersects C_n for every $n \in \omega$, we can find a function $g \subseteq B$, $g \in \omega^\omega$, and $x \in X \setminus N(g)$. I.e. $B \cap A_x$ is infinite. However $A_x \cap C_n$ is only finite for each $n \in \omega$, and the set $\{ C_n \mid n \in \omega \}$ witnesses that no MAD extension of $\mathcal{A}$ can be tight. □

Corollary 39 (Guzmán–Hrušák–Martínez-Ranero–Ramos-García [22])**.** *Tight MAD families exist if and only if* $\mathfrak{b} = \mathfrak{c}$.

We will look now into the generic existence of other MAD families.

Definition 40. Let $\mathcal{J}$ be a tall ideal on ω. We define $\mathfrak{a}(\mathcal{J})$ to be minimal size of an AD family $\mathcal{A}$ such that $\mathcal{A} \cup \mathcal{A}^\perp \subseteq \mathcal{J}$.

The next Lemma is just a straightforward application of the definition.

Lemma 41. *Let $\mathcal{J}$ be a tall ideal on ω, $\mathcal{A}$ be an infinite AD family of size less than $\mathfrak{a}(\mathcal{J})$, and $f : \omega \to \omega$ be a* **fin***-to-1 function. There exists an AD family $\mathcal{B} \supseteq \mathcal{A}$, $|\mathcal{A}| = |\mathcal{B}|$ and $B \in \mathcal{I}(\mathcal{B})$ such that $f^{-1}[B] \in \mathcal{J}^+$.*

Proof. If the choice $\mathcal{A} = \mathcal{B}$ does not work, $\mathcal{C} = \left\{ f^{-1}[A] \mid A \in \mathcal{A} \right\}$ is an AD family contained in $\mathcal{J}$ of size less than $\mathfrak{a}(\mathcal{J})$. There is $B \in \mathcal{C}^\perp \cap \mathcal{I}^+$ and we can put $\mathcal{B} = \mathcal{A} \cup \{ f[B] \}$. □

Proposition 42 (Guzmán–Hrušák–Martínez-Ranero–Ramos-García [22])**.** *Let $\mathscr{I}$ be a σ-ideal on ω^ω that has a very weak fusion and* $\mathrm{tr}(\mathscr{I})$ *is Katětov–Blass uniform. Then $\mathbf{P}_{\mathscr{I}}$ indestructible MAD families exist generically if and only if* $\mathfrak{a}(\mathrm{tr}(\mathcal{I})) = \mathfrak{c}$.

Proof. Every MAD family extending a witness of $\mathfrak{a}(\mathrm{tr}(\mathcal{I}))$ is contained in $\mathrm{tr}(\mathscr{I})$ and must be destroyed by $\mathbf{P}_{\mathscr{I}}$. On the oder hand, if an AD family is smaller than $\mathfrak{a}(\mathrm{tr}(\mathcal{I})) = \mathfrak{c}$, then we can extend it to a MAD family $\mathcal{A}$ by applying Lemma 41, making sure there is no Katětov–Blass morphism witnessing $\mathcal{A} \leq_{\mathrm{KB}} \mathrm{tr}(\mathscr{I})$. □

In particular we are getting $\mathfrak{a}(\mathrm{tr}(\mathscr{M})) = \mathfrak{b}$. A variation of the covering number will be useful for studying these invariants.

Definition 43. Let $\mathcal{J}$ be a tall ideal on ω. We define $\operatorname{cov}^+(\mathcal{J})$ to be smallest size of a family $\mathcal{H} \subseteq \mathcal{J}$ such that for each $X \in \mathcal{J}^+$ there is $H \in \mathcal{H}$ such that $X \cap H$ is infinite.

Clearly $\operatorname{cov}^+(\mathcal{I}) \leq \operatorname{cov}^*(\mathcal{I})$ and $\operatorname{cov}^+(\mathcal{I}) \leq \mathfrak{a}(\mathcal{I})$. We say that a set $A \subset 2^{<\omega}$ is *off-branch* if $\pi(A) = \emptyset$. Every off-branch set is in $\operatorname{tr}(\mathscr{S})$.

Lemma 44. *The minimal size of a family $\mathcal{B}$ of off-branch sets such that for every $A \in \operatorname{tr}(\mathscr{S})^+$ there is $B \in \mathcal{B}$ such that $|B \cap A| = \omega$, is $\operatorname{cov}^+ \operatorname{tr}(\mathscr{S})$.*

Proof. Since every set in $\operatorname{tr}(\mathscr{S})^+$ has an infinite off branch subset, there is nothing to prove if $\operatorname{cov}^+ \operatorname{tr}(\mathscr{S}) = \mathfrak{c}$. Assume this is not the case and $\mathcal{H}$ is the witness for $\operatorname{cov}^+ \operatorname{tr}(\mathscr{S})$. There is a set of prefect trees $\{ T_\alpha \subseteq 2^{<\omega} \mid \alpha \in \mathfrak{c} \}$ such that $\{ [T_\alpha] \mid \alpha \in \mathfrak{c} \}$ is a partition of 2^ω. Since $\left|\bigcup \{ \pi(H) \mid H \in \mathcal{H} \}\right| < \mathfrak{c}$, there is α such $[T_\alpha] \cap \pi(H) = \emptyset$ for each $H \in \mathcal{H}$. The set of splitting nodes of T_α is isomorphic to $2^{<\omega}$, the sets $T_\alpha \cap H$ are off-branch and as required in the Lemma (in an isomorphic copy of $2^{<\omega}$). □

Proposition 45 (Guzmán–Hrušák–Martínez-Ranero–Ramos-García [22])**.**

$$\operatorname{cov} \mathscr{M} \leq \operatorname{cov}^+ \operatorname{tr}(\mathscr{S})$$

Proof. Let $\mathcal{A}$ of size less than $\operatorname{cov} \mathscr{M}$ be a set of off-branch sets in $2^{<\omega}$, we will find $B \in \operatorname{tr}(\mathscr{S})^+ \cap \mathcal{A}^\perp$. Represent the Cohen poset as the end-extension order on the set of finite initial subtrees of $2^{<\omega}$. There are countably many dense set which force that the generic object is a Sacks tree. For $A \in \mathcal{A}$ let D_A be the set of all finite tress T such that $[\![s]\!] \cap A = \emptyset$ for every s which is a maximal node of T. Since each such A is off-branch, every D_A is a dense set in the Cohen forcing. We defined $< \operatorname{cov} \mathscr{M}$ many dense subsets of the Cohen poset, so there exists a filter intersecting them all. Our set B is the Sacks tree which is the union of this filter. □

Corollary 46 (Brendle–Yatabe [12])**.** *If* $\operatorname{cov} \mathscr{M} = \mathfrak{c}$*, then Sacks indestructible MAD families exist generically.*

Proposition 47 (Guzmán–Hrušák–Martínez-Ranero–Ramos-García [22])**.** *If* $\mathfrak{a} \leq \mathfrak{a}(\operatorname{tr}(\mathscr{S}))$*, then there is a Sacks indestructible MAD family.*

Proof. If $\mathfrak{a} < \mathfrak{c}$, use Proposition 32. If $\mathfrak{c} = \mathfrak{a} = \mathfrak{a}(\operatorname{tr}(\mathscr{S}))$, use Corollary 46. □

Question 48 (Guzmán–Hrušák–Martínez-Ranero–Ramos-García [22])**.** Is $\mathfrak{a} \leq \mathfrak{a}(\operatorname{tr}(\mathscr{S}))$ a ZFC theorem?

A positive answer for this question would provide a solution to the problem of Hrušák.

3. Shelah–Steprāns ideals

We turn our attention now towards the following major open problems.

Problem (Roitman, see e.g. [46]). Does $\mathfrak{d} = \omega_1$ imply $\mathfrak{a} = \omega_1$?

Problem (Brendle–Raghavan [8]). Does $\mathfrak{b} = \mathfrak{s} = \omega_1$ imply $\mathfrak{a} = \omega_1$?

Since both the cardinal invariants $\mathfrak{b}$, $\mathfrak{s}$ are provably smaller or equal to $\mathfrak{d}$, see e.g. [5], the two problems are not independent. The general consistency of $\mathfrak{d} < \mathfrak{a}$ has been established by Shelah [54] and later $\omega_1 < \mathfrak{s} < \mathfrak{b} < \mathfrak{a}$ was shown to be consistent by Fischer and Mejía [18]. However, these results were not achieved by a standard linear forcing iteration and the method does not seem to help to resolve the stated problems. The problems appear to be interconnected with the following question. Given a MAD family $\mathcal{A}$, is there a proper forcing destroying $\mathcal{A}$ while preserving dominating families/unbounded families/splitting families?

A natural idea would be to prove that every MAD family is contained in an F_σ ideal and use Theorem 26 to destroy it. This however does not work, Laflamme proved that under CH there is MAD family which cannot be extended to an F_σ ideal, see [39]. Let us also mention some results on destructibility of MAD families.

Theorem 49 (Hrušák–García Ferreira [31]). (CH) *For any proper ω^ω-bounding forcing P of size $\mathfrak{c}$ there exists a P-indestructible, Cohen-destructible MAD family.*

Shelah proved the following important result [53], see also [11].

Proposition 50 (Shelah [53]). *Every MAD family can be destroyed by a proper forcing that does not add dominating reals.*

And recently Guzmán and Kalajdzievski proved in [24] the following.

Proposition 51 (Guzmán–Kalajdzievski [24]). *Every MAD family can be destroyed by a proper forcing that does not add dominating reals and preserves P-ultrafilters.*

By the results of Brendle and Raghavan [11] the result of Shelah is tightly connected with the Mathias–Příkrý poset $\mathbf{M}(\mathcal{I})$, and the link between the properties of the ideal $\mathcal{I}$ and the forcing $\mathbf{M}(\mathcal{I})$. The following definition turned out to be the key for understanding this phenomenon.

Definition 52. Let $\mathcal{I}$ be an ideal on ω. A set $X \subseteq [\omega]^{<\omega}$ is called $\mathcal{I}$-*universal* if for each $I \in \mathcal{I}$ there is $s \in X$, $s \cap I = \emptyset$. The collection of all $\mathcal{I}$-universal sets is denoted $(\mathcal{I}^{<\omega})^+$.

This terminology is adopted from Laflamme [40] who used the term universal in connection with the dual filter, sets were called $\mathcal{F}$-universal for a filter $\mathcal{F}$. We will also use a modification of this notion.

Definition 53. Let $\mathcal{I}$ be an ideal on ω. A sequence $\langle X_n \subseteq [\omega]^{<\omega} \mid n \in \omega \rangle$ is called an *$\mathcal{I}$-universal γ-sequence* if for each $I \in \mathcal{I}$ for all but finitely many $n \in \omega$ there is $s \in X_n$, $s \cap I = \emptyset$.

The question of adding dominating reals by the Mathias–Příkrý was first investigated by Canjar [13]. Assuming $\mathfrak{d} = \mathfrak{c}$, he constructed a maximal ideal $\mathcal{I}$ such that the poset $\mathbf{M}(\mathcal{I})$ does not add dominating reals. Ideals with this property were later called Canjar ideals [21], for further results see e.g [6, 14, 17, 20, 25, 33]. We will state an equivalent combinatorial property as a definition.

Definition 54. An ideal $\mathcal{I}$ on ω is called *Canjar* (or *Menger*) if for every sequence $\langle X_n \subseteq [\omega]^{<\omega} \mid n \in \omega \rangle$ of $\mathcal{I}$-universal sets there exist $Y_n \in [X_n]^{<\omega}$ such that $\bigcup\{ Y_n \mid n \in \omega \}$ is an $\mathcal{I}$-universal set.

Definition 55. An ideal $\mathcal{I}$ on ω is called *Hurewicz* if for every sequence $\langle X_n \subseteq [\omega]^{<\omega} \mid n \in \omega \rangle$ of $\mathcal{I}$-universal sets there exist $Y_n \in [X_n]^{<\omega}$ such that $\bigcup\{ Y_n \mid n \in a \}$ is $\mathcal{I}$-universal for every $a \in [\omega]^{\omega}$.

Every Hurewicz ideal is Canjar. It is easy to see that F_σ ideals do have these properties. On the other hand, there are no other analytic Canjar ideals; Arkhangel'skiĭ proved [1] that all Menger sets of reals are already F_σ.

Theorem 56 (Hrušák–Minami [33], Chodounský–Repovš–Zdomskyy [14]). *Let $\mathcal{I}$ be an ideal on ω. The following are equivalent*

(1) *The ideal $\mathcal{I}$ is Canjar.*
(2) *The ideal $\mathcal{I}$ is a Menger subspace of the Cantor space 2^ω.*
(3) *The forcing $\mathbf{M}(\mathcal{I})$ does not add dominating reals.*

Theorem 57 (Chodounský–Repovš–Zdomskyy [14]). *Let $\mathcal{I}$ be an ideal on ω. The following are equivalent*

(1) *The ideal $\mathcal{I}$ is Hurewicz.*
(2) *The ideal $\mathcal{I}$ is a Hurewicz subspace of the Cantor space 2^ω.*
(3) *The forcing $\mathbf{M}(\mathcal{I})$ preserves all unbounded families of the ground model, i.e. it is almost ω^ω-bounding.*

In fact, for Hurewicz ideals the Mathias–Příkrý forcing even preserves certain splitting families, this fact was first observed by Zdomskyy[1]. We

[1]Personal communication.

say that $S \subset \omega$ *block splits* a set of pairwise disjoint sets $\{ Z_n \mid n \in \omega \}$ if $|\{ n \in \omega \mid Z_n \subset S \}| = |\{ n \in \omega \mid Z_n \cap S = \emptyset \}| = \omega$. We say that $\mathcal{S} \subseteq \mathcal{P}(\omega)$ is a *block splitting* family if for every pairwise disjoint set Z of finite sets there is $S \in \mathcal{S}$ which splits Z. Every block splitting family is automatically splitting and $\mathcal{P}(\omega)$ is a block splitting family.

Proposition 58 (Zdomskyy). *If $\mathcal{I}$ is a Hurewicz ideal on ω, then the Mathias–Příkrý forcing forcing $\mathbf{M}(\mathcal{I})$ preserves block splitting families.*

Proof. Let $\mathcal{S}$ be a block splitting family and let $\dot{P} = \{ \dot{p}_n \mid n \in \omega \}$ be a set of names for pairwise disjoint finite sets. We may assume that each $\dot{p}_n$ is forced to be disjoint with n. For $n \in \omega$ let X_n be set of all $t \in [\omega \setminus n]^{<\omega}$ such that for each $s \subseteq n$ there exists $F_n^s(t) \in [\omega]^{<\omega}$ and $I_n^s(t) \in \mathcal{I}$ such $\left(s \cup t, I_n^s(t)\right) \Vdash p_n = F_n^s(t)$.

Claim. *For each $n \in \omega$ the set X_n is $\mathcal{I}$-universal.*

This can proved using a folklore argument, for $I \in \mathcal{I}$ in 2^n many steps define an $\sqsubseteq$-increasing sequence $\left\{ t_i \in [\omega \setminus I]^{<\omega} \mid i \in 2^n \right\}$ and a $\subseteq$-increasing sequence $\left\{ I_n^i(t_i) \supseteq I \mid i \in 2^n, I_n^i(t_i) \cap t_{i-1} = \emptyset \right\}$ such that the conditions $\left(s \cup t_i, I_n^i(t_i)\right)$ decide $\dot{p}_n$ to be some $F_n^i(t_i)$. ■

Since $\mathcal{I}$ is Hurewicz we may find $\left\{ Y_n \in [X_n]^{<\omega} \right\}$ as in the definition of the Hurewicz property. Let $Z_n = \bigcup \left\{ F_n^s(t) \mid s \subseteq n, t \in Y_n \right\}$ for $n \in \omega$. Since each Z_n is finite and disjoint with n, there exists $b \in \omega^\omega$ such that the sets Z_n are pairwise disjoint when n ranges over b. Let $S \in \mathcal{S}$ be a set block splitting $\{ Z_n \mid n \in b \}$, we show that it is forced that S block splits $\dot{P}$. Pick any $k \in \omega$ and a condition $(s, I) \in \mathbf{M}(\mathcal{I})$, we will find $n \geq k$ and a stronger condition which forces that $\dot{p}_n \subseteq S$. We may assume that $s \subseteq k$. Define $a = \{ n \in b \setminus k \mid Z_n \subset S \}$, this set is infinite. Since $\bigcup \{ Y_n \mid n \in a \}$ is an $\mathcal{I}$-universal set, there is $n \in a$ and $t \in Y_n$ such that $t \cap I = \emptyset$. Thus the condition (s, I) is compatible with $\left(s \cup t, I_n^s(t)\right)$ and can be extended to force that $\dot{p}_n = F_n^s(t) \subseteq Z_n \subset S$. The argument showing that S is forced to be disjoint with $\dot{p}_n$ for infinitely many $n \in \omega$ is exactly the same. □

With destroying MAD families in mind Aurichi and Zdomskyy proved the following theorems.

Theorem 59 (Aurichi–Zdomskyy [2]). (CH) *or* (MA) *There exists a Cohen-indestructible MAD family which is not Canjar.*

And also the converse.

Theorem 60 (Zdomskyy [60]). *It is consistent that every tight MAD family is Hurewicz. In particular, this does hold in the Laver model.*

The rest of this survey is a preview of the forthcoming paper [7], discuss here the results and we include proofs of a few simple statements for completeness. For the other proofs see the the cited paper. The notion of Shelah–Steprāns MAD families was introduced by Raghavan [50], it is related to the notion of a *strongly separable* MAD family which was introduced by Shelah and Steprāns in [55]. We first study the Shelah–Steprāns property for general ideals.

Definition 61. An ideal $\mathcal{I}$ on ω is called *Shelah–Steprāns* if for every $\mathcal{I}$-universal set X there exists an infinite $Y \in [X]^\omega$ such that $\bigcup Y \in \mathcal{I}$.

An equivalent reformulation of the definition gives us that the ideal $\mathcal{I}$ is Shelah–Steprāns if for every $X \subseteq [\omega]^{<\omega} \setminus \emptyset$ there is either $I \in \mathcal{I}$ which intersects all elements of X or I contains infinitely many elements of X.

Lemma 62 (Brendle–Guzmán–Hrušák–Raghavan [7]). *Let $\mathcal{I}, \mathcal{J}$ be ideals on ω such that $\mathcal{I} \leq_K \mathcal{J}$. If $\mathcal{I}$ Shelah–Steprāns, then $\mathcal{J}$ is also Shelah–Steprāns.*

Proof. Let f be the Katětov morphisms between $\mathcal{I}$ and $\mathcal{J}$, and let X be an $\mathcal{J}$-universal set. First notice that $X' = \{ f[s] \mid s \in X \}$ is an $\mathcal{I}$-universal set; for each $I \in \mathcal{I}$ there is $s \in X$ disjoint with $f^{-1}[I] \in \mathcal{J}$, i.e. $f[s] \in X'$ is disjoint with I. Since $\mathcal{I}$ is Shelah–Steprāns, there is $Y' \in [X']^\omega$ such that $A = \bigcup Y' \in \mathcal{I}$. Now $f^{-1}[A] \in \mathcal{J}$ is a set which contains every $s \in X$ for which $f[s] \in Y'$. □

Proposition 63 (Brendle–Guzmán–Hrušák–Raghavan [7]). *Every non-meager ideal on ω is Shelah–Steprāns.*

Proof. Let $\mathcal{I}$ be a non-meager ideal and let $X \subseteq [\omega]^{<\omega} \setminus \emptyset$ be an $\mathcal{I}$-universal set. Since $\mathbf{fin} \subseteq \mathcal{I}$ there is an infinite $Y \in [X]^\omega$ which consists of pairwise disjoint sets. The set $M = \{A \subseteq \omega \mid y \subset A$ for infinitely many $y \in Y\}$ is dense G_δ, i.e. there exists $I \in \mathcal{I} \cap M$ and $\mathcal{I}$ is Shelah–Steprāns. □

There are also meager Shelah–Steprāns ideals, the prototypical example is the ideal $\mathbf{fin} \times \mathbf{fin}$. In fact, this example is critical for the Borel ideals.

Proposition 64 (Brendle–Guzmán–Hrušák–Raghavan [7]). *A Borel ideal $\mathcal{I}$ is Shelah–Steprāns iff $\mathbf{fin} \times \mathbf{fin} \leq_K \mathcal{I}$.*

The Shelah–Steprāns property can be also expressed in topological terms; asking for existence of sets separating the ideal and its dual.

Proposition 65 (Brendle–Guzmán–Hrušák–Raghavan [7]). *An ideal $\mathcal{I}$ is Shelah–Steprāns if and only if there is no F_σ set $F \subset \mathcal{P}(\omega)$ such that $\mathcal{I} \subseteq F$ and $\mathcal{I}^* \cap F = \emptyset$.*

And we can use [56, Corollary 1.5] to deduce the following.

Proposition 66 (Brendle–Guzmán–Hrušák–Raghavan [7]). *If $\mathcal{I}$ is Shelah–Steprāns ideal, then $\mathcal{I}$ cannot be extended to an $F_{\sigma\delta}$ ideal.*

In particular every Shelah–Steprāns ideal is Laflamme. An interesting property of Shelah–Steprāns ideals is that these ideals are very much indestructible.

Proposition 67 (Brendle–Guzmán–Hrušák–Raghavan [7]). *Shelah–Steprāns ideals are Cohen indestructible and random indestructible.*

In fact, the proposition does hold for many general forcing notions. Assuming a suitable determinacy condition, definable proper almost ω^ω-bounding forcings which have the continuous reading of names do not destroy Shelah–Steprāns ideals. For details see [7] (as with all the other results concerning the Shelah–Steprāns property).

We will say that an AD family $\mathcal{A}$ is Shelah–Steprāns if the ideal it generates $\mathcal{I}(\mathcal{A})$ is Shelah–Steprāns. Notice that every Shelah–Steprāns AD family is automatically MAD; if $B \in \mathcal{A}^\perp$, then the singletons of B form an $\mathcal{I}(\mathcal{A})$-universal set. We already know that such MAD families are Cohen indestructible, in fact, Shelah–Steprāns MAD families are even tight [7]. A cardinality assumption again implies generic existence of Shelah–Steprāns MAD families.

Proposition 68 (Minami–Sakai[47]). *If $\mathfrak{p} = \mathfrak{c}$, then Shelah–Steprāns MAD families exist generically.*

On the other hand, unlike other types of MAD families with various combinatorial properties, Shelah–Steprāns MAD families are consistently known not to exist.

Theorem 69 (Raghavan [50]). *It is consistent with* ZFC *that there are no Shelah–Steprāns MAD families (and $\mathfrak{b} = \mathfrak{c}$).*

Shelah–Steprāns MAD families are indestructible by many definable forcings. Every MAD family $\mathcal{A}$ is of course destroyed by the Mathias–Příkrý forcing $\mathbf{M}(\mathcal{I}(\mathcal{A}))$. Surprisingly, for Shelah–Steprāns MAD families this forcing does not add dominating or unsplit reals.

Proposition 70 (Brendle–Guzmán–Hrušák–Raghavan [7]). *Every Shelah–Steprāns MAD family $\mathcal{A}$ is Hurewicz. In particular the Mathias–Příkrý forcing $\mathbf{M}(\mathcal{I}(\mathcal{A}))$ is c.c.c., does not add unsplit nor dominating reals (is even almost ω^ω-bounding), and destroys $\mathcal{A}$.*

We will define one more combinatorial property of ideals and MAD families. It is one of the strongest properties of MAD families considered in the literature.

Definition 71 (Brendle–Guzmán–Hrušák–Raghavan [7])**.** An ideal $\mathcal{I}$ on ω is called *raving* if for every $\mathcal{I}$-universal γ-sequence $\{X_n \mid n \in \omega\}$ (see Definition 53) exists $I \in \mathcal{I}$ such that $[I]^{<\omega} \cap X_n \neq \emptyset$ for all but finitely many $n \in \omega$.

It is easy to see that every raving ideal is Shelah–Steprāns. We will say that a MAD family $\mathcal{A}$ is raving if the ideal $\mathcal{I}(\mathcal{A})$ is raving.

As usual, the property is inherited upwards in the Katětov order.

Lemma 72 (Brendle–Guzmán–Hrušák–Raghavan [7])**.** *Let $\mathcal{I}, \mathcal{J}$ be ideals on ω such that $\mathcal{I} \leq_{\mathrm{K}} \mathcal{J}$. If the ideal $\mathcal{I}$ is raving, then $\mathcal{J}$ is raving as well.*

Proof. The proof is identical to the proof of Lemma 62. □

Shelah–Steprāns and raving MAD families can be constructed using the so called parametrized diamond principles $\diamondsuit(\mathfrak{b})$ and $\diamondsuit(\mathfrak{d})$ introduced in [48, 27]. These principles can be interpreted as a strengthening of statement that the corresponding cardinal invariant is equal to ω_1 by adding a certain 'guessing' component. These diamond principles do typically hold in standard models whenever the corresponding cardinal invariant is ω_1, see [48].

The following theorem is in particular a strengthening of the fact that $\diamondsuit(\mathfrak{b})$ already implies $\mathfrak{a} = \omega_1$. The problem of Roitman in certain sense questions whether the guessing component of the parametrized diamond is necessary for this implication.

Theorem 73 (Brendle–Guzmán–Hrušák–Raghavan [7])**.**

$\diamondsuit(\mathfrak{b})$ *There is a Shelah–Steprāns MAD family.*

$\diamondsuit(\mathfrak{d})$ *There is raving MAD family.*

The canonical approach to get a MAD family with strong combinatorial properties is to add one with forcing using countable approximations. Let **P**-MAD be a poset consisting of countable AD families on ω ordered by reverse inclusion. It is easy to see that this is σ-closed forcing adding a MAD family.

Proposition 74 (Brendle–Guzmán–Hrušák–Raghavan [7])**.** *The generic MAD family added by* **P**-*MAD is raving.*

Proof. Let $\mathcal{B} = \{B_n \mid n \in \omega\}$ be an condition in **P**-MAD forcing that $X = \{X_i \mid i \in \omega\}$ is a universal γ-sequence. (We may assume that each X_i is nonempty.) For $i \in \omega$ denote $E_i = i \cup \bigcup\{B_n \mid n < i\}$. Find an interval partition $\{P_n \mid n \in \omega\}$ of ω such that if $i \in P_n$, then X_i contains an element s_i disjoint with E_n. Notice that $A = \bigcup\{s_i \mid i \in \omega\} \in \mathcal{B}^\perp$ and $\mathcal{B} \cup \{A\}$ is a condition in **P**-MAD which forces that A is in the generic MAD and A contains an element of X_i for every $i \in \omega$. □

Genericity of certain objects over the morel $L(\mathbb{R})$ is sometimes possible to express in combinatorial terms. The study of this phenomenon was initiated by Todorcevic (see [16]) and further studied e.g. in [41, 15]. However, the case of the **P**-MAD forcing is not yet fully understood.

Problem (Brendle–Guzmán–Hrušák–Raghavan [7]). Is there a combinatorial characterization of **P**-MAD generic family $\mathcal{A}$ or the ideal $\mathcal{I}(\mathcal{A})$ over $L(\mathbb{R})$?

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