PREPRINT manuscript No.
(will be inserted by the editor)

Complex polynomials and tessellations in the Riemann sphere

Leidy J. González–Cely
Departamento de Ciencias Básicas,
Universidad Autónoma de Bucaramanga
lgonzalez673@unab.edu.co
ORCID 0000-0001-7198-8384

Jesús Muciño–Raymundo
Centro de Ciencias Matemáticas,
Universidad Nacional Autónoma de México
muciray@matmor.unam.mx
ORCID 0000-0002-4431-6088

October 16, 2024

Abstract Let P be a complex polynomial of degree $n \geq 2$ in the Riemann sphere, and let γ be an oriented Jordan path running through its critical values. A classical algorithm, with roots in the pioneering work of H. A. Schwarz and F. Klein, states that the inverse image of γ under P determines a finite tessellation with tiles that are topological k–polygons with two alternate colors in the Riemann sphere. Following a question by W. P. Thurston, we study under what conditions a finite graph (or equivalently a finite tessellation of the Riemann sphere) originates from a generic polynomial P and an oriented Jordan path γ as above.

Mathematical Subject Classification (2020) 57M15; 57M12; 30F10

Keywords Complex polynomials, Riemann sphere, graphs, tessellations

1 Introduction

Tessellations are implicitly related to rational functions, very roughly speaking:

a complex rational function R of degree $n \geq 2$ determines a tessellation of the Riemann sphere by $2n$ tiles, which are topological k–polygons with alternate colors.

More than a century ago, the former tessellations of this kind originated in the pioneering work of H. A. Schwarz [16] and F. Klein [5], [8]. We review their geometrical algorithm. Let $R : \widehat{\mathbb{C}}_z \longrightarrow \widehat{\mathbb{C}}_w$ be a complex rational function, and let $\gamma \subset \widehat{\mathbb{C}}_w$ be an oriented Jordan path running through the critical values $\mathcal{V}_R$ of R. We recognize that γ is a graph with vertices $\mathcal{V}_R$. The pullback graph $R^*\gamma$ is well defined with vertices of even valences at least 4 at the critical points of R, and vertices of valence 2 at the cocritical points of R. Recall that a cocritical point of R is non critical such that it assumes a critical value under R. The tessellation associated with R is the decomposition $\widehat{\mathbb{C}}_z \backslash R^*\gamma$, where every open connected component of the decomposition is a tile of the tessellation. In fact, the tiles have alternate colors, say blue and gray. The cocritical points play a crucial role, by allowing us to recognize the tiles of the tessellation as topological k–polygons, here k is the number of critical values of R. Our problem can be seen as a converse of the above algorithm, therefore from graphs and tessellations to polynomials. It can be stated as follows.

Characterize the graphs Γ, with tessellations $\widehat{\mathbb{C}}_z \backslash \Gamma$ that arise from polynomials $P : \widehat{\mathbb{C}}_z \longrightarrow \widehat{\mathbb{C}}_w$ and oriented Jordan paths γ running through the critical values $\mathcal{V}_P$ of P.

To explain the ideas, Figures 1, 4 and 7 illustrate the simplest examples of the difference between an initial graph Γ and its related $P^*\gamma$. Around 2010, W. P. Thurston conducted a discussion group about "*the shape of rational maps*". This group found specific conditions under which a graph Γ can be recognized as $R^*\gamma$ for a generic rational function; see S. Koch *et al.* [9]. Obviously, polynomials are non generic rational functions, because the infinity is a point of a higher ramification order. We recall that a *generic polynomial* P of degree $n \geq 2$ has $n-1$ distinct finite critical values. Our main result is as follows.

Theorem 1 *Let $\Gamma \subset \widehat{\mathbb{C}}_z$ be a finite, oriented, connected graph with tessellation $\widehat{\mathbb{C}}_z \backslash \Gamma$. The following assertions are equivalent.*

1) Under edge subdivision, Γ can be transformed into $P^*\gamma$ for some generic polynomial P of degree $n \geq 2$ and an oriented Jordan path $\gamma \subset \widehat{\mathbb{C}}_w$ running through the critical values $\mathcal{V}_P$.

2) The graph Γ has $n-1 \geq 1$ vertices of valence 4 and a vertex at ∞ of valence $2n$ such that
i) there exists an alternating colouring of $\widehat{\mathbb{C}}_z \backslash \Gamma$ with n blue and n gray tiles, and
ii) each blue tile has ∞ as a boundary vertex.

Assertion (1) states that the tessellation $\widehat{\mathbb{C}}_z \backslash \Gamma$ arises from a polynomial function. As a novel aspect, our proof of (2) $\Rightarrow$ (1) is constructive. Given a graph Γ as in (2), the proof provides an algorithm for finding a suitable edge subdivision of Γ, which transforms the graph Γ in one of the shape $P^*\gamma$. Clearly, the result is topological in the following sense. If we allow a small enough continuous ϵ–perturbation of a generic polynomial P_0 and its corresponding Jordan path γ_0, then the pairs $\{(P_\epsilon, \gamma_\epsilon)\}$ topologically determine the same Γ in the Riemann sphere.

The structure of the article is as follows. In Section 2, we review the Schwarz–Klein classical algorithm describing the tessellations of complex rational functions, see Theorem 2, and the result of W. P. Thurston *et al.* In Section 3, the easy implication (1)$\Rightarrow$(2) is provided. In Example 3, we illustrate how consistent labellings and edge subdivision operations allow us to go from graphs Γ to rational functions. The core of our work is in Section 4, where we achieve the proof of (2)$\Rightarrow$(1) through a consistent labelling and edge subdivision operations. A comment about the constructive nature of the proof is in Section 5.

Let us recall a necessarily incomplete list of highlights in the subject. H. A. Schwarz [16] considered tessellations by triangles in his study of hypergeometric differential equations, see [7] §10.3 and [19] Ch. 5 for modern descriptions. In order to understand complex analytic functions, the works of F. Klein reveal his mastery regarding graphs and tessellations, as seen in [5] and [8]. A. Speiser [17] and the brothers F. and R. Nevanlinna [14] Ch. 6, §5, [13] Ch. XI, §2, established relations between transcendental complex analytic functions and tessellations. More recently, G. V. Belyĭ [3] and A. Grothendieck [6] considered the celebrated correspondence between tessellations by triangles in Riemann surfaces, rational functions with three critical values and dessins d'enfants.

2 Graphs and complex rational functions

Tessellations and graphs appear in many instances in the study of complex analytic functions and Riemann surfaces, with very intricate meanings and notations. We provide accurate ad hoc concepts.

Definition 1 A *tessellation* of the Riemann sphere $\widehat{\mathbb{C}}$ is a union

$$\mathscr{T} = \underbrace{T_1 \cup \ldots \cup T_n}_{\text{blue tiles}} \cup \underbrace{T'_1 \cup \ldots \cup T'_n}_{\text{gray tiles}} \subset \widehat{\mathbb{C}}, \quad n \geq 2, \tag{1}$$

where the $2n$ *tiles* $\{T_\alpha, T'_\alpha\}_{\alpha=1}^n$ are open Jordan domains, such that:
i) The union of their closures $\cup_{\alpha=1}^n \overline{T_\alpha} \cup_{\alpha=1}^n \overline{T'_\alpha}$ is $\widehat{\mathbb{C}}$.
ii) If the intersection of the closures of any two tiles is non-empty, then it consists of a finite number of points and/or a finite number of edges (simple paths).
iii) If two tiles are adjacent along an edge, then they have alternate colors, blue and gray.

In all this work, T_α denotes a blue tile and T'_α is gray. There are n blue tiles and n gray tiles, which is called the *global balance condition* in [9]. By using the points and edges in (ii), a tessellation $\mathscr{T}$ determines an underlying graph Γ as follows.

Definition 2 A `t`*–graph* Γ is a finite oriented connected graph in $\widehat{\mathbb{C}}$, with vertices $V(\Gamma)$ of even valence equal or greater than 4 and edges $E(\Gamma)$, such that:
i) $\mathscr{T}(\Gamma) \doteq \widehat{\mathbb{C}} \setminus \Gamma$ is a tessellation, as in (1).
ii) Each blue tile T_α is in the left side of the oriented edges of its boundary $\partial\overline{T_\alpha}$.

Considering the above definitions in mind, a tessellation $\mathscr{T}$ and a `t`–graph Γ are essentially equivalent objects, where the alternated coloring in Definition 1 corresponds to the oriented edges in Definition 2. Hence, the name `t`–graph must be understood as an abbreviation of "tessellation graph". The tessellations from complex rational functions require a more accurate notion, as follows.

Definition 3 An `R`*–map* $\widehat{\Gamma}$ is a finite, oriented, connected graph in $\widehat{\mathbb{C}}$, with vertices $V(\widehat{\Gamma})$ of even valence equal or greater than 2 and edges $E(\widehat{\Gamma})$, such that:
i) If we forget the vertices of valence 2 of $\widehat{\Gamma}$, then we obtain Γ such that:
$$\mathscr{T}(\widehat{\Gamma}) \doteq \mathscr{T}(\Gamma) = T_1 \cup \ldots \cup T_n \cup T'_1 \cup \ldots \cup T'_n$$
is a tessellation as in (1).
ii) Each boundary $\partial\overline{T_\alpha}$ (resp. $\partial\overline{T'_\alpha}$) of a tile has $k \geq 2$ edges of $\widehat{\Gamma}$.
iii) Local balance: for each subgraph $\Gamma_0 \subset \Gamma$ homeomorphic to $\mathbb{S}^1$, with the induced orientation of Γ, there are strictly more blue tiles than gray tiles on the left side of Γ_0.

To be explicit, consider a vertex 0 of valence 2 in $\widehat{\Gamma}$ and its two edges, thus $(-1, 0) \cup \{0\} \cup (0, 1)$. The operation of forgetting the vertex 0 replaces the above by the unique edge $(-1, 1)$. Note that $\widehat{\Gamma}$ satisfies the global and local balance conditions in the main result of [9].

Remark 1 An `R`–map $\widehat{\Gamma}$ has two numerical attributes:
• its *degree* $n \geq 2$, and
• its k*–gonality* since the tiles of its tessellation $\mathscr{T}(\widehat{\Gamma})$ are topological k–polygons.

Example 1 (`R`*–maps* $\widehat{\Gamma}$ *defined by* $R^*\gamma$*)* 1. Let $R(z) = (z^4 - 1)/2z^2$ be a rational function with critical values $\mathcal{V}_R = \{i, -i, \infty\}$ and critical points $\mathcal{C}_R = \{0, (1+i)/\sqrt{2}, (1-i)/\sqrt{2}, (-1+i)/\sqrt{2}, (-1-i)/\sqrt{2}, \infty\}$. The choice of $\gamma = i\mathbb{R} \cup \{\infty\}$ determines an octahedron as a tessellation associated to $\widehat{\Gamma} = R^*\gamma$, see Figure 1.a. The set of cocritical points is empty, hence $\widehat{\Gamma} = \Gamma$.
2. Let $P(z) = z^3 - 3z$ be a polynomial with critical values $\mathcal{V}_P = \{-2, 2, \infty\}$, and we choose $\gamma = \mathbb{R} \cup \{\infty\}$. According to Figure 1.b, the tiles of $\mathscr{T}(P^*\gamma)$ are topological triangles and the `R`–map $\widehat{\Gamma} = P^*\gamma$ has vertices ζ_1, ζ_2 of valence 2 (corresponding to the cocritical points of P). Figure 1.c illustrates the associated `t`–graph Γ with tiles that are topological digons and triangles; hence $\widehat{\Gamma} \neq \Gamma$.
3. Let $Q(z) = (1 + 3z - 3z^2 + 3z^3)/(3 - 3z + 3z^2 + z^3)$, since $\mathcal{V}_R = \{1, i, -i\}$, the choice $\gamma = \{\,|w| = 1\}$ is suitable. Figure 1.d shows the `R`–map $\widehat{\Gamma} = Q^*\gamma$. Clearly, there exist $M_1, M_2 \in PSL(2, \mathbb{C})$ such that the right–left equivalence $P = M_1 \circ Q \circ M_2$ holds, for the polynomial $P(z)$ in (2).

The following result illuminates the theory of rational functions and tessellations, and we provide a suitable version.

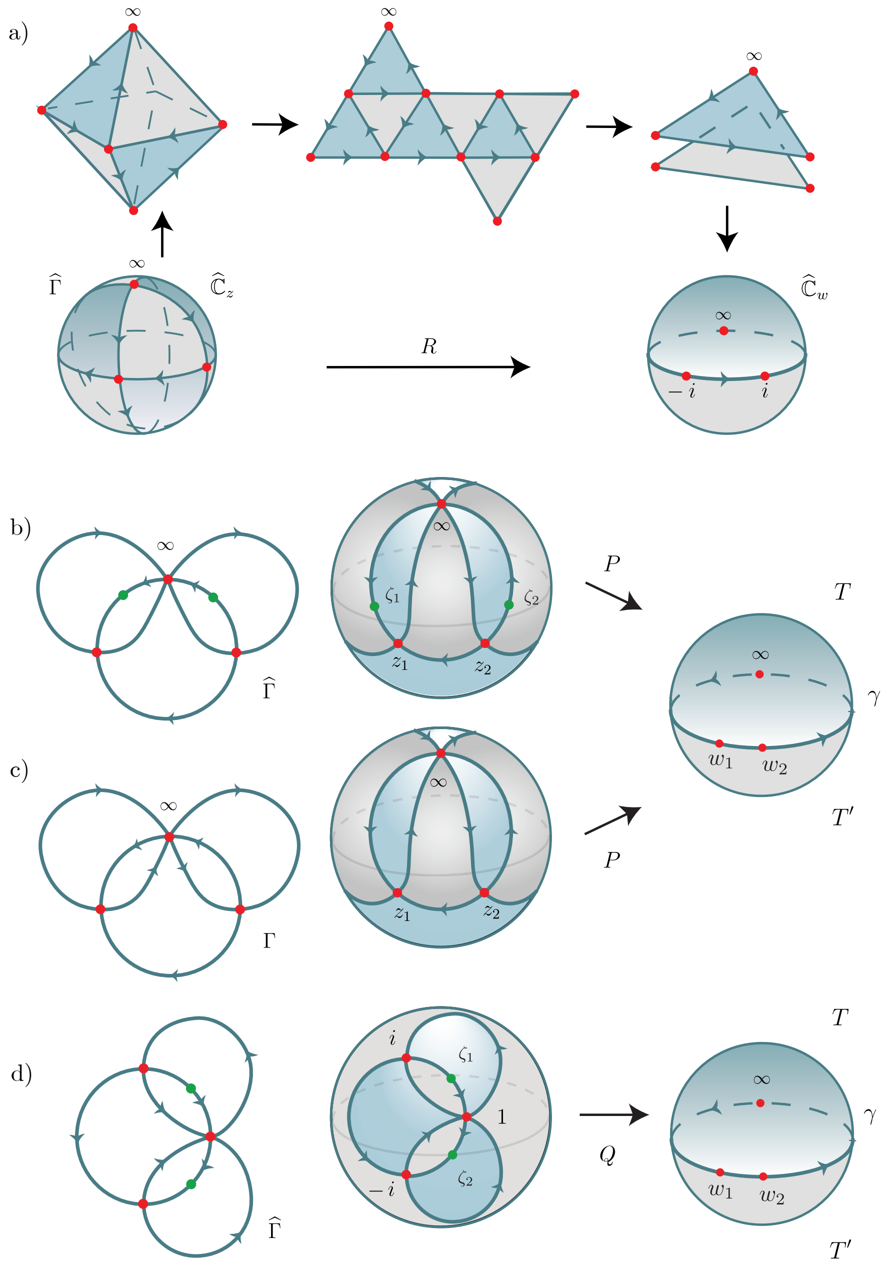

Fig. 1 Tessellation from rational functions, critical points are red, cocritical points are green. a) An R–map $\widehat{\Gamma} = R^*\gamma$ from a rational function R, its tessellation is an octahedron. b) An R–map $\widehat{\Gamma} = P^*\gamma$ from a cubic polynomial P. c) The underlying t–graph Γ of P is obtained from $\widehat{\Gamma}$ by forgetting its vertices of valence 2. d) An R–map $\widehat{\Gamma} = Q^*\gamma$ from a rational function Q, which is right–left Möbius equivalent to the above P.

Theorem 2 (B. Riemann, H. A. Schwarz, F. Klein, W. P. Thurston)

1) Let $R : \widehat{\mathbb{C}}_z \longrightarrow \widehat{\mathbb{C}}_w$ be a rational function of degree $n \geq 2$, and let $\gamma \subset \widehat{\mathbb{C}}_w$ be an oriented Jordan path running through the critical values $\mathcal{V}_R$. The pair (R, γ) determines an R–map $\widehat{\Gamma} = R^*\gamma$ with vertices

$$V(\widehat{\Gamma}) = \{\underbrace{\textit{critical points of } R}_{\textit{even valence } \geq 4}\} \cup \{\underbrace{\textit{cocritical points of } R}_{\textit{valence } =2}\} \qquad (2)$$

and a tessellation $\mathscr{T}(\widehat{\Gamma})$.

2) An R*–map* $\widehat{\Gamma}$ *in* $\widehat{\mathbb{C}}_z$ *with vertices of valence* 2 *or* 4*, having* $2n-2 \geq 2$ *vertices of valence* 4*, and* $2n$ *tiles, determines a (nonunique) a rational function* $R : \widehat{\mathbb{C}}_z \longrightarrow \widehat{\mathbb{C}}_w$.

Note that, assertion (2) works for generic rational functions, see Theorem 3, while (1) assumes any rational function of degree at least 2. We attribute the Theorem 2 to H. A. Schwarz [16] and F. Klein [8], [5] since it is implicitly described in these works. The present result can be seen as a combinatorial version of the Riemann's existence Theorem, compare with [2], [18] Ch. 6, [10] p. 74, and [4] p. 85. Moreover, in [13] Ch. XI §2 the Nevanlinna brothers expand the technique in Theorem 2 to the study of transcendental functions, which leads to tessellations with an infinite number of tiles; see [15], [12], [1], [11] for contemporary applications. After Theorem 2, the name R–map[1] must be understood as a coarse abbreviation of "rational function".

Furthermore, a comparison with the following concept is illustrative. A *branched cover* $\mathcal{R} : \widehat{\mathbb{C}}_z \longrightarrow \widehat{\mathbb{C}}_w$ is a continuous map such that there exists a finite set $\{w_j\}_{j=1}^k \subset \widehat{\mathbb{C}}_w$ satisfying the following:
i) the set $\mathcal{R}^{-1}\big(\{w_j\}_{j=1}^k\big)$ is a finite set, and
ii) the map $\mathcal{R} : \widehat{\mathbb{C}}_z \backslash \mathcal{R}^{-1}\big(\{w_j\}_{j=1}^k\big) \longrightarrow \widehat{\mathbb{C}}_w \backslash \{w_j\}_{j=1}^k$ is a topological covering.
The degree of $\mathcal{R}$ is the number of preimages $\{\mathcal{R}^{-1}(w_0)\}$, for $w_0 \in \widehat{\mathbb{C}}_w \backslash \{w_j\}_{j=1}^k$.

Corollary 1 *An* R*–map* $\widehat{\Gamma}$ *determines a branched cover* $\mathcal{R} : \widehat{\mathbb{C}}_z \longrightarrow \widehat{\mathbb{C}}_w$ *of degree* $n \geq 2$*. Conversely, a branched cover* $\mathcal{R}$ *determine a (nonunique)* R*–map with degree* n*.* □

Clearly, the non uniqueness of assertion (2) in Theorem 2 is up to topological equivalence. Two branched covers $\mathcal{R}_1$, $\mathcal{R}_2$ are right–left topologically equivalent when $\mathcal{R}_2 = \phi_2 \circ \mathcal{R}_1 \circ \phi_1$, for ϕ_1, ϕ_2 orientation preserving homeomorphisms of the Riemann sphere.

Proof of Theorem 2. Let us consider assertion (1). Let $R(z)$ be a rational function of degree $n \geq 2$ with k critical values

$$\mathcal{V}_R = \{w_1, \ldots, w_j, \ldots, w_k\} \subset \widehat{\mathbb{C}}_w, \quad \text{here } 2 \leq k \leq 2n-2.$$

We assume that $\gamma \subset \widehat{\mathbb{C}}_w$ is an oriented Jordan path running through $\mathcal{V}_R$. In graph theory language, γ is an oriented embedded graph with vertices $V(\gamma) = \mathcal{V}_R$ of valence 2 and oriented edges $E(\gamma) = \{\overline{w_j\, w_{j+1}}\}_{j=1}^k \cup \overline{w_k\, w_1}$. The associated tessellation is

$$\mathscr{T}(\gamma) = \widehat{\mathbb{C}}_w \backslash \gamma = T \cup T',$$

where T and T' are open k–sided topological polygons with vertices $\mathcal{V}_R$ and boundary γ. The blue polygon T is in the left side of γ. In the category of graphs, the pullback $\widehat{\Gamma} = R^*\gamma$ of γ under R is well defined. Thus, $\widehat{\Gamma}$ is an R–graph with

$$\text{vertices } \ V(\widehat{\Gamma}) = \bigcup_j R^{-1}(w_j) \ \text{ and } \ \text{edges } \ E(\widehat{\Gamma}) = \bigcup_j R^{-1}\big(\{\overline{w_j\, w_{j+1}}\}_{j=1}^k \cup \overline{w_k\, w_1}\big).$$

It should be mentioned that the vertices of $\widehat{\Gamma}$ include the following:
• the critical points $\mathcal{C}_R = \{z_j\}$ of R as vertices of even valence ≥ 4, and
• the cocritical points $\mathcal{C}c_R = \{\zeta_\kappa\}$ of R as vertices of valence 2, according with Equation (2).
Moreover,

$$\mathscr{T}(\widehat{\Gamma}) = \underbrace{T_1 \cup \ldots \cup T_n}_{R^{-1}(T) \text{ blue tiles}} \cup \underbrace{T_1' \cup \ldots \cup T_n'}_{R^{-1}(T') \text{ gray tiles}}, \tag{3}$$

is a tessellation with tiles that are topological k–polygons with alternate colors. The proof of assertion (1) is done.

For assertion (2), we assume the existence of an R–map $\widehat{\Gamma}$ as a topological graph in the sphere $\mathbb{S}^2$.

As a first step, we construct a C^1 diffeomorphism $\mathcal{R} : \mathbb{S}^2 \backslash V(\widehat{\Gamma}) \longrightarrow \widehat{\mathbb{C}}_w$. Consider the circle $\gamma = \mathbb{R} \cup \{\infty\} \subset \widehat{\mathbb{C}}_w$ furnished with vertices $\{w_1, \ldots, w_k = \infty\}$ and the respective k segments as edges. Assume that the spheric length of the edges in $\mathbb{S}^2$ is $2\pi/k$. Thus, γ is a graph with associated tessellation

[1] Despite the abundance of previous references, we have not found a more suitable name.

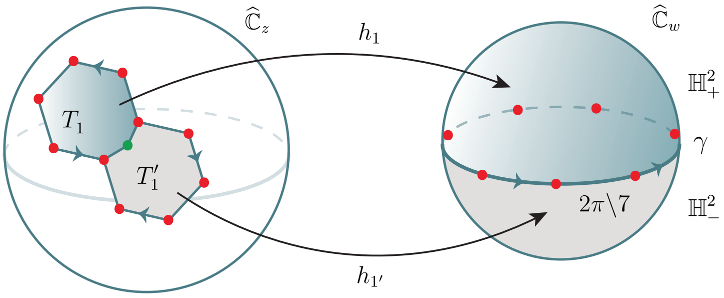

Fig. 2 Two topological k–polygons T_1, T_1' are mapped to the half planes $\mathbb{H}^2_+$, $\mathbb{H}^2_-$, here we sketch the case $k = 7$. The critical points are red, a cocritical point is green.

$$\widehat{\mathbb{C}}_w \backslash \gamma = \mathbb{H}^2_+ \cup \mathbb{H}^2_-,$$

here the upper half plane $\mathbb{H}^2_+ = \{\mathfrak{Re}\,(w) > 0\}$ is a blue tile, see Figure 2. Let $\{z_j\}$ the vertices of valence 4 of $\widehat{\mathit{\Gamma}}$ and their associated $\mathtt{t}$–map $\mathit{\Gamma}$, as in Definition 3.i. Without loss of generality, assume that each oriented edge $\overline{z_j z_{j+1}} \subset \overline{T_1}$ of $\mathit{\Gamma}$ is a C^1 embedded trajectory in $\mathbb{S}^2$. For each edge, we consider a bijective, continuous parametrization $L_j(t) : [0, 2\pi/k] \longrightarrow \overline{z_j z_{j+1}} \subset \mathbb{S}^2$. The parametrization induces a distance on the edge, with length $2\pi k$. There exists a C^1 diffeomorphism $h_1 : \overline{T_1} \longrightarrow \overline{\mathbb{H}^2_+} \subset \widehat{\mathbb{C}}_w$. In addition, we can assume that h_1 is an orientation preserving isometry from the oriented boundary of T_1 (with the metric induced by $\{L_j(t)\}$) to $\partial\overline{\mathbb{H}^2_+} = \gamma$ (with the spheric metric). See Figure 2.
Assuming that the gray tile $T_1' \subset \mathbb{S}^2$ is adjacent with T_1, there exists a C^1 diffeomorphism $h_{1'} : \overline{T_1'} \longrightarrow \overline{\mathbb{H}^2_-} \subset \widehat{\mathbb{C}}_w$, with analogous metric properties as the above h_1. Note that, the maps h_1 and $h_{1'}$ coincide in the intersection $\partial\overline{T_1} \cap \partial\overline{T_1'} \subset \widehat{\mathit{\Gamma}}$ and up to slight modification (if it is necessary), they define a C^1 diffeomorphism from $\overline{T_1} \cup \overline{T_1'}$ minus the vertices of $\widehat{\mathit{\Gamma}}$ to $\widehat{\mathbb{C}}_w \backslash \{w_1, \ldots, w_k\}$.

By using the hypothesis in the vertices of valence 4 and 2, there exists a *consistent labelling of the vertices of* $\widehat{\mathit{\Gamma}}$, thus

$$\mathcal{L}_c : V(\widehat{\mathit{\Gamma}}) \longrightarrow \{1, \ldots, k\}, \quad z_j \longmapsto \mathcal{L}_c(z_j), \tag{4}$$

where

- $\mathcal{L}_c$ is a bijection in the vertices of valence 4, and
- in the boundary of each blue tile T_α, the cyclic order of the labels $\mathcal{L}_c(z_\iota)$ of the vertices $\{z_\iota\} = \widehat{\mathit{\Gamma}} \cap \partial\overline{T_\alpha}$ coincides with the counterclockwise order of $\{z_\iota\}$ in $\partial\overline{T_\alpha}$. For the existence of $\mathcal{L}_c$, see [9] p. 225.

Using $\mathcal{L}_c$, we extend the above maps to cover all the tiles of the tessellation in $\mathbb{S}^2$, the complete local C^1 diffeomorphism is denoted as $\mathcal{R} : \mathbb{S}^2 \backslash V(\widehat{\mathit{\Gamma}}) \longrightarrow \widehat{\mathbb{C}}_w$. Clearly, the local behavior of $\mathcal{R}$ at each vertex z_j of $\widehat{\mathit{\Gamma}}$ is topologically equivalent to $\{z \longmapsto z^{\nu/2}\}$, where $\nu = 2, 4$ means the valence of the vertex z_j.

Secondly, we endow $\mathbb{S}^2$ with a complex structure, in a such way that the branched cover $\mathcal{R}$ is recognized as a complex rational function $R(z)$. Several proofs are available and probably the most elementary is as follows. Consider the complex structure J on the (real) tangent bundle $T\widehat{\mathbb{C}}_w$, defined as $J\frac{\partial}{\partial x} = \frac{\partial}{\partial y}$, $J\frac{\partial}{\partial y} = -\frac{\partial}{\partial x}$. Since $\mathcal{R}$ is a C^1 local diffeomorphism, the pullback of J is well defined. It follows that

$$\mathcal{R} : \big(\mathbb{S}^2 \backslash V(\widehat{\mathit{\Gamma}}),\, \mathcal{R}^* J\big) \longrightarrow \widehat{\mathbb{C}}_w$$

is a holomorphic nonsingular function. By applying the elementary Riemann's extension theorem to the conformal punctures at the vertices $V(\widehat{\mathit{\Gamma}})$, the function $\mathcal{R}$ holomorphically extends with critical points at $z_j \in V(\widehat{\mathit{\Gamma}})$. By the uniformization Theorem, $(\mathbb{S}^2, \mathcal{R}^* J)$ is the Riemann sphere $\widehat{\mathbb{C}}_z$ and $\mathcal{R}$ is a holomorphic map, hence a rational function $R(z)$ on $\widehat{\mathbb{C}}_z$ of degree n. □

In particular, not every $\mathtt{t}$–graph is a $\mathtt{R}$–map, see Figures 3 and 10 in [9] for examples in the rational non–polynomial case. Moreover, the existence of a rational functions $R(z)$ from a $\mathtt{t}$–map $\mathit{\Gamma}$ was established as follows.

Theorem 3 (W. P. Thurston et al. [9]) *A* $\mathtt{t}$*–graph* Γ *with* $2n-2$ *vertices of valence 4 (by forgetting its vertices of valence 2) is equal to* $\mathcal{R}^*\gamma$ *for a branched cover* $\mathcal{R}:\mathbb{S}^2 \longrightarrow \mathbb{S}^2$ *of degree* $n \geq 2$ *and suitable* γ*, if and only if*

i) the tiles of the tessellation $\mathscr{T}(\Gamma) = \widehat{\mathbb{C}}_z \backslash \Gamma$ *are Jordan domains,*

ii) Γ *is globally balanced, i.e.* $\mathscr{T}(\Gamma)$ *has alternated* n *blue tiles and* n *gray tiles,*

iii) Γ *is locally balanced, i.e. for each subgraph* $\Gamma_0 \subset \Gamma$ *homeomorphic to* $\mathbb{S}^1$*, with the induced orientation of* Γ*, there are strictly more blue tiles than gray tiles on the left side of* Γ_0*.* □

Theorem 3 works for pairs (R,γ), where the rational function R of degree n attains the maximal number $2n-2$ of critical points. According to our definitions and as a matter of record:

• A pair (R,γ) denotes a rational function R and an oriented Jordan path $\gamma \subset \widehat{\mathbb{C}}_w$ running through the critical values $\mathcal{V}_R$.

• $\widehat{\Gamma}$ is an $\mathtt{R}$–map, with vertices of even valence ≥ 2.

• Γ is a $\mathtt{t}$–graph, with vertices of even valence ≥ 4.

• $\mathscr{T}$ is a tessellation of the Riemann sphere $\widehat{\mathbb{C}}_z$.

We have the following diagram:

$$\begin{tikzcd}
& \widehat{\Gamma} \arrow[dd, shift right, "\textcircled{2}"'] \arrow[dr, leftrightarrow, "\textcircled{4}"] & \\
\mathscr{T} \arrow[dr, leftrightarrow, "\textcircled{1}"'] & & (R,\gamma). \\
& \Gamma \arrow[uu, shift right, "\textcircled{3}"'] \arrow[ur, "\textcircled{5}"'] &
\end{tikzcd}$$

By definition, map ① is an equivalence. Map ② obtains a $\mathtt{t}$–graph Γ from a $\mathtt{R}$–map $\widehat{\Gamma}$ by forgetting the vertices of valence 2 of $\widehat{\Gamma}$, Figure 1.b–c illustrates this operation. Under additional hypothesis for Γ in Theorem 3, the map ③ determines an $\mathtt{R}$–map $\widehat{\Gamma}$ which is obtained from a $\mathtt{t}$–graph Γ by edge subdivision, adding vertices of valence 2. The map ④ is Theorem 2, and ⑤ is the composition of ③ and ④. If $\widehat{\Gamma}$ and Γ are related by ② or ③, then the tessellations $\mathscr{T}(\widehat{\Gamma})$ and $\mathscr{T}(\Gamma)$ coincide.

If the degree n of generic polynomials increases, then the number of cocritical points grows as $(n-1)(n-2)$. Hence, starting with a $\mathtt{t}$–map Γ, the edge subdivision operation ③ requires to knowning the assignation of exactly $(n-1)(n-2)$ cocritical points in suitable edges of Γ. We illustrate this trouble with the next family of tessellations.

Example 2 (A family of quartic polynomials and their $\mathtt{R}$*–maps)* Consider the family of polynomials

$$P_e(z) = ez - \frac{z^2}{2} + \frac{z^4}{4}, \quad e \in \mathbb{R}. \tag{5}$$

For $e \notin \{0, \pm 0.385\}$, the corresponding polynomials $P_e(z)$ are generic. If $-0.385 \leq e \leq 0.385$, then we use $\gamma = \mathbb{R} \cup \{\infty\}$. In the other cases, the choice γ is cumbersome. For simplicity, we describe the affine aspect of the tessellations.

Case $e = 0$. The polynomial is even, $P_0(z) = P_0(-z)$. There are only two critical values, three critical points and two cocritical points. A sketch of $\widehat{\Gamma}_0$ is in Figure 3.a, and its tessellation is by topological triangles.

Case $0 < e < 0.385$. The polynomial $P_e(z)$ has three critical values, three critical points and six cocritical points. Figure 3.b illustrates $\widehat{\Gamma}_e$ and its tessellation is by topological quadrangles.

Case $e = 0.385$. The associated $P_e(z)$ has only two critical values, two critical points. Figure 3.c. illustrates $\widehat{\Gamma}_e$ and its tessellation by topological triangles.

Case $0.385 < e$. Figure 3.d illustrates the case $e = 2$, thus $\widehat{\Gamma}_2$, for other values of e the corresponding R–maps are topologically equivalent.
Cases $e < 0$. Up to a rotation by π of $\mathbb{C}_z$, the graph $\widehat{\Gamma}_e$ coincides with the corresponding cases $\widehat{\Gamma}_{-e}$ described above.

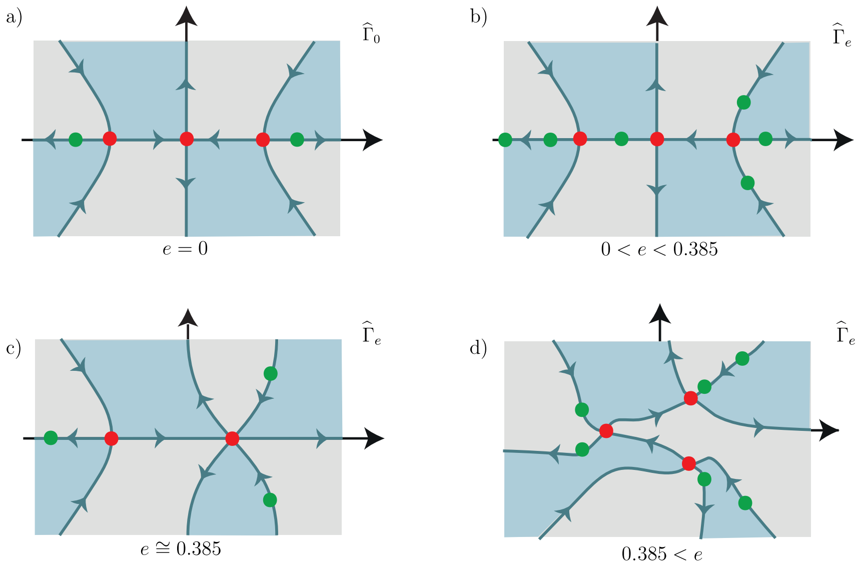

Fig. 3 R–maps $\widehat{\Gamma}_e$ and their tessellations in $\mathbb{C}_z$ of the family of polynomials $P_e(z) = ez - (1/2)z^2 + (1/4)z^4$, with real e. In particular for $0 < e < 0.385$, $\widehat{\Gamma}_0$ is different from $\widehat{\Gamma}_e$. Moreover, if we forget the cocritical points, they determine topologically the same t–graph Γ.

3 Theorem 1 proof (1) ⇒ (2) and consistent labellings

Let Γ be as in (1) of Theorem 1. By hypothesis, the edge subdivision operation adds vertices $\{\zeta_\kappa\}$ of valence 2 to Γ. The resulting graph is an R–map $\widehat{\Gamma}$, as in Definition 3. Recalling Theorem 2 assertion (2), there exists an oriented Jordan path $\gamma \subset \widehat{\mathbb{C}}_w$ running through the critical values of the corresponding polynomial $P(z)$, say $\{w_1, \ldots, w_j, \ldots, w_n = \infty\}$.
Thus each tile T_α of $\mathscr{T}(\widehat{\Gamma})$ is a n–sided topological polygon, and the path γ determines an order of the critical values. The proof (1) ⇒ (2) is done.

In order to prove the converse assertion, an implicit datum is the *consistent labelling of an* R*–map*. The following nontrivial example originated from a rational function is illustrative.

Example 3 (Consistent labelling and edge subdivision) Let Γ be a t–graph with tessellation $\mathscr{T}(\Gamma)$ as in Figure 4.a. Theorem 3 asserts that $\mathscr{T}(\Gamma)$ originates from a generic rational map of degree 3.
In order to recognize an associated R–graph, we suppose that 4 critical values are required. Assume that the critical values have labels $\{1, \ldots, 4\}$ such that the order $1 < \ldots < 4$ coincides with the (cyclic order) of the critical values $\{w_j\}$ in the Jordan path $\gamma \doteq \mathbb{R} \cup \{\infty\}$.
We perform two operations. Firstly, we label the vertices $\{z_j\}$ of Γ with $\{1, \ldots, 4\}$, thus we have a function

$$\mathcal{L} : V(\Gamma) \longrightarrow \{1, \ldots, 4\}, \quad z_j \longmapsto \mathcal{L}(z_j).$$

In such way that the cyclic order of the labels in all the oriented boundaries $\partial \overline{T_\alpha}$ of the blue tiles coincides with the order of $1 < \ldots < 4$ in the Jordan path $\gamma \doteq \mathbb{R} \cup \{\infty\}$, see Figure 4.b. The reader can verify that the choice of $\mathcal{L}$ is the hard part of the construction. As second operation, the labels allow performing the edge subdivision of the edges of Γ, as follows. An edge of the blue tile T_1 has vertices with labels 1 and 4,

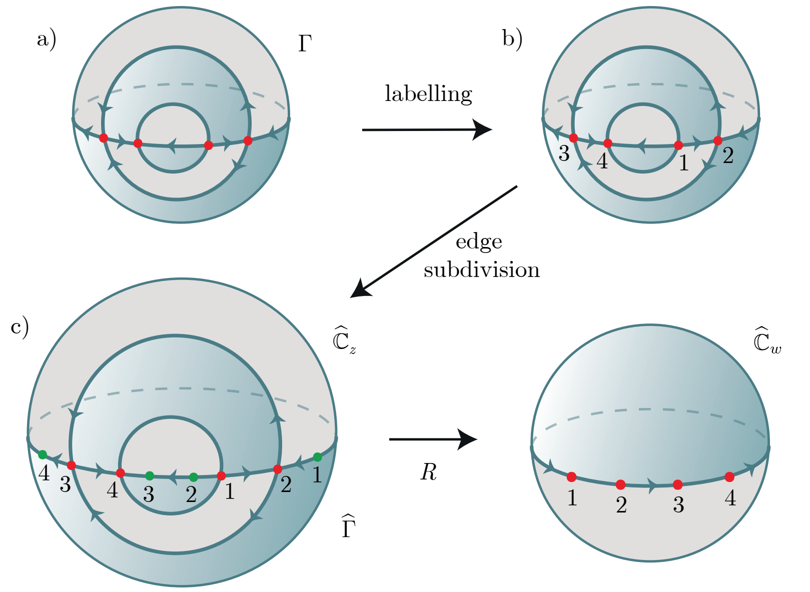

Fig. 4 a) Tessellation of a t–graph Γ. b) A consistent labelling of the vertices of Γ. c) The edge subdivision of Γ determines $\widehat{\Gamma}$, hence a rational function R.

by edge subdivision, we introduce two hidden vertices of valence 2, and label them with 2 and 3, see Figure 4.c. The edges in the boundary of T_2 have labels $1, \ldots, 4$, and no edge subdivision is required. We continue the operations in an analogous way. The complete edge subdivision operation then produces an R–map $\widehat{\Gamma}$ such that all the tiles are topological quadrangles. Note that the labelling $\mathcal{L}_c$ extends to the hidden vertices of valence 2 of $\widehat{\Gamma}$. According to Theorem 2, we can verify that the pair

$$\big(R(z) = (z-3)^2/(z^3-4),\, \gamma = \mathbb{R} \cup \{\infty\}\big)$$

determines the R–map $\widehat{\Gamma}$.

We return to the polynomial case and ask about labelling the vertices of Γ with analogous properties as in the above example.

As first step, we note that there exists a *topological order* $\prec$ in each boundary $\partial\overline{T_\alpha}$, which depends on the orientation of the edges of $\widehat{\Gamma}$ and the existence of the vertex $\infty \in \widehat{\mathbb{C}}_w$, with valence $2n$ in $\widehat{\Gamma}$. That is the main technical difference with the rational non–polynomial case.
In fact, let z_ι, z_j be finite critical or cocritical points of a polynomial $P(z)$. Assume that we run through the boundary $\partial\overline{T}_\alpha \backslash \{\infty\}$ of a blue tile T_α of $\mathscr{T}(\widehat{\Gamma})$, with counterclockwise direction, starting at $z_\iota \neq \infty$. If we encounter z_j (before than ∞), then we define

$$z_\iota \prec z_j \text{ in } \partial\overline{T_\alpha} \backslash \{\infty\}.$$

If $z_j \neq \infty$, then by definition $z_j \prec \infty$ in $\partial\overline{T_\alpha}$. Obviously, the subscripts ι, j have no meaning with respect to the order $\prec$.

Example 4 In Figure 1.b, looking at the corresponding blue tiles, we have $\zeta_1 \prec z_1$ and $z_2 \prec \zeta_2$.

Corollary 2 *Let $\widehat{\Gamma} = P^*\gamma$ be an* R*–map from a generic polynomial $P(z)$ of degree $n \geq 2$ and be γ a Jordan path, as in Theorem 1. Then $\widehat{\Gamma}$ has a labelling*

$$\mathcal{L} : V(\widehat{\Gamma}) = \{\textit{critical points of } P\} \cup \{\textit{cocritical points of } P\} \longrightarrow \{1, \ldots, n\}, \quad z_j \longmapsto \mathcal{L}(z_j), \tag{6}$$

which satisfies

$$\textit{if } \; z_\iota \prec z_j \; \textit{ in } \; \partial\overline{T_\alpha}, \quad \textit{then } \; \mathcal{L}(z_\iota) < \mathcal{L}(z_j) \; \textit{ in the order } \; \{1, \ldots, n\}. \tag{7}$$

□

Remark 2 1. As a consequence of equation (7), we note that necessarily $\mathcal{L}(\infty) = n$.
2. A generic polynomial P of degree n has n critical points in $\widehat{\mathbb{C}}_z$. Hence, the labelling $\mathcal{L}$ of $\widehat{\Gamma}$ is injective if and only if the cocritical point set of P is empty, thus the polynomial is z^2 (up to right–left affine equivalence).

4 Theorem 1 proof of (2) ⇒ (1)

Let Γ be a $\mathtt{t}$–graph as is Theorem 1, assertion (2), with $n \geq 2$ vertices

$$V(\Gamma) = \{\underbrace{z_1, \ldots, z_j, \ldots, z_{n-1}}_{\text{valence } 4}, \underbrace{z_n = \infty}_{\text{valence } 2n}\} \tag{8}$$

and $3n - 2$ oriented edges

$$E(\Gamma) = \{\overline{z_\iota z_j}\}, \quad \text{where } \iota,\ j \in 1, \ldots, n, \tag{9}$$

here z_ι, z_j correspond to the initial and final vertices. The associated tessellation is as in (1).

The main difficulty in ensuring the existence of a polynomial $P(z)$ from the $\mathtt{t}$–graph Γ lies in the fact that the tiles T_α of its tessellation $\mathscr{T}(\Gamma)$ are not a priori topological n–polygons, which is the n–gonality property in Remark 1.

4.1 Scheme of the proof

Step 1. Construct a consistent labelling $\mathcal{L}_c$ for Γ. This useful concept is analogous to Corollary 2, as follows.

Definition 4 Let z_ι, z_j be two vertices of a $\mathtt{t}$–map Γ in the boundary $\partial\overline{T_\alpha}$ of a blue tile. The *topological order* $\prec$ *in* $\partial\overline{T_\alpha}$ is as follows: if z_ι, z_j are both different of ∞ and we can run (with the counterclockwise orientation) through $\partial\overline{T_\alpha}\backslash\{\infty\}$ starting at z_ι and arriving at z_j, then

$$z_\iota \prec z_j.$$

Furthermore, if one of the vertices is ∞, then $z_j \prec \infty$.

Note that, the vertices z_ι, z_j are not necessarily adjacent, and the vertex ∞ is the maximal element of the order $\prec$ in $\partial\overline{T_\alpha}$.

Remark 3 1. Clearly, $\prec$ is a partial order for all the vertices of Γ.
2. By abusing of the notation, the symbol $\prec$ in Definition 4 is the same that the used in Corollary 2. If Γ is a $\mathtt{t}$–map obtained by an $\mathtt{R}$–map $\widehat{\Gamma}$ forgetting its vertices of valence 2, then the partial order of the vertices of Γ in Definition 4 coincides with the partial order of the vertices of $\widehat{\Gamma}$ in Corollary 2.

For simplicity, throughout the text we omit the adjective partial for the orders. The order $\prec$ is the key ingredient for finding a suitable $\widehat{\Gamma}$.

Definition 5 A *consistent labelling* $\mathcal{L}_c$ for Γ is a bijective function

$$\begin{aligned} \mathcal{L}_c : V(\Gamma) &\longrightarrow \{1, \ldots, n\} \\ z_j &\longmapsto \mathcal{L}_c(z_j) \\ \infty &\longmapsto n \end{aligned}$$

such that for all blue tile T_α of $\mathscr{T}(\Gamma)$,

$$\text{if } \ z_\iota \prec z_j \ \text{ in } \ \partial\overline{T_\alpha}, \ \text{ then } \ \mathcal{L}_c(z_\iota) < \mathcal{L}_c(z_j) \ \text{ in the order } \ 1 < 2 < \ldots < n.$$

This labelling $\mathcal{L}_c$ allows us to make a globally consistent assignment of the finite vertices of Γ to the vertices of some Jordan path $\gamma \subset \widehat{\mathbb{C}}_w$.

Step 2. By the labelling $\mathcal{L}_c$ and edge subdivision operations, we add hidden vertices to Γ.

The *hidden vertices* $\{\zeta_\kappa\}$ *of* Γ are vertices of $\widehat{\Gamma}$, which satisfy that the tiles T_α of the corresponding $\widehat{\Gamma} = \Gamma \cup \{\zeta_\kappa\}$ are topological n–polygons. We add the hidden vertices as in the two cases below. Let $\overline{z_\iota z_j}$ be an edge of Γ.

Case 1. Assume that $z_\iota \prec z_j$ in $\partial\overline{T_\alpha}$ and $\mathcal{L}_c(z_\iota) = k$, $\mathcal{L}_c(z_\iota) = k + \nu + 1$ with $\nu \geq 1$.
Note that $z_j = \infty$ is allowed, which by definition has label n. Then by edge subdivision, ν hidden vertices in $\overline{z_\iota z_j}$ are constructed. By using consistent labelling, the required hidden vertices are $\zeta_1, \ldots, \zeta_\nu$ and moreover they satisfy that

$$\begin{array}{rcccccccc} \text{with respect to order} & z_\iota & \prec & \zeta_1 & \prec \ldots \prec & \zeta_\nu & \prec & z_j \\ \text{with labels} & k & < & k+1 & < \ldots < & k+\nu & < & k+\nu+1, \end{array} \quad \text{in } \partial\overline{T_\alpha},$$

see Example 5. Furthermore, the second row shows that the consistent labelling $\mathcal{L}_c$ of Γ extends to the hidden vertices.

Case 2. Assume an oriented edge $\overline{\infty z_\iota}$ of Γ.
If the label $\mathcal{L}_c(z_\iota) = 1$, then no hidden vertices are required.
If the label $\mathcal{L}_c(z_\iota) = \nu + 1 \geq 2$, then by edge subdivision $\zeta_1, \ldots, \zeta_\nu$ hidden vertices are constructed. As above, we have

$$\begin{array}{rccccc} \text{with respect to order} & \zeta_1 & \prec \ldots \prec & \zeta_\nu & \prec & z_\iota \\ \text{with labels} & 1 & < \ldots < & \nu & < & \nu+1, \end{array} \quad \text{in } \partial\overline{T_\alpha}.$$

The second row shows that the consistent labelling of Γ extends to the hidden vertices.

Example 5 (Consistent labelling and edge subdivision) According to Figure 5, we consider $z_\iota \prec z_j$ in ∂T_α and assume a consistent labelling $\mathcal{L}_c$ for Γ, such that $\mathcal{L}_c(z_\iota) = 2$ and $\mathcal{L}_c(z_j) = 5$. The edge $\overline{z_\iota z_j}$ requires two hidden vertices ζ_3, ζ_4. In addition, the edge $\overline{\infty z_\iota}$ requires one hidden vertex ζ_1. Moreover, if we make the choice of γ with vertices

$$\{w_1 < \ldots < w_{n-1} < w_n = \infty\} \subset \gamma \doteq \mathbb{R} \cup \{\infty\} \subset \widehat{\mathbb{C}}_w,$$

then the definition of a branched cover $\mathcal{P}$ at the vertices and hidden vertices of Γ should satisfy that

$$\mathcal{P}(\infty) = w_n = \infty,\ \mathcal{P}(\zeta_1) = w_1,\ \mathcal{P}(z_\iota) = w_2,\ \mathcal{P}(\zeta_3) = w_3,\ \mathcal{P}(\zeta_4) = w_4,\ \mathcal{P}(z_j) = w_5.$$

Step 3. Construct an R–map $\widehat{\Gamma}$ associated with Γ.
We define a branched cover $\mathcal{P}$ in the vertices of Γ as the composition $\mathcal{P} = \mathcal{I} \circ \mathcal{L}_c$, thus

$$\begin{array}{ccccc} \{z_1, \ldots, z_{n-1}\} & \xrightarrow{\ \mathcal{L}_c\ } & \{1, \ldots, n-1, n\} & \xrightarrow{\ \mathcal{I}\ } & \{w_1, \ldots, w_{n-1}, w_n = \infty\} \\ z_j & \longrightarrow & \mathcal{L}_r(z_j) = m & \longrightarrow & w_m. \end{array} \tag{10}$$

Secondly, we will extend continuously $\mathcal{P}$ to the hidden vertices in $\widehat{\Gamma}$ and its tile boundaries $\partial\overline{T_\alpha}$. We will extend $\mathcal{P}$ to the interior of the tiles T_α. By Theorem 2, $\mathcal{P}$ is a branched cover and determines a complex polynomial $P(z)$. Clearly, the hidden vertices in $\widehat{\Gamma}$ should be the cocritical points of $P(z)$.

Example 6 (A consistent labelling) Consider the t–map Γ in Figure 1.c. The oriented Jordan path $\gamma = \mathbb{R} \cup \{\infty\}$ has vertices $\{w_1, w_2, \infty\}$ with $w_1 < w_2 < \infty$. A consistent labelling for the vertices z_1, z_2, ∞ of Γ is

$$\mathcal{L}_c(z_1) = 2,\ \mathcal{L}_c(z_2) = 1,\ \mathcal{L}_c(\infty) = 3.$$

By edge subdivision operations, we add two hidden vertices ζ_1, ζ_2 of valence 2, to obtain an R–map $\widehat{\Gamma}$ with vertices $V(\widehat{\Gamma}) = \{z_1, z_2, \infty, \zeta_1, \zeta_2\}$, as in Figure 1.b. For the boundaries $\partial\overline{T_\alpha}$ in the tessellation of $\widehat{\Gamma}$, we have that the order $\prec$ satisfies

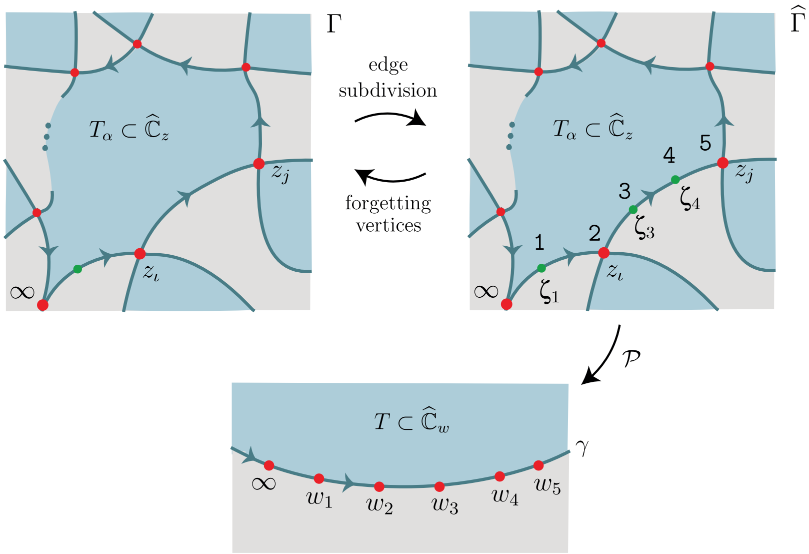

Fig. 5 Assuming the existence of a consistent labelling $\mathcal{L}_r(z_\iota) = 2$ $\mathcal{L}_r(z_j) = 5$ for Γ, the edge subdivision operation transforms Γ to $\widehat{\Gamma}$, by adding three vertices ζ_ς of valence 2. We sketch the associated continuous branched covering $\mathcal{P}$.

$$\zeta_1 \prec z_1 \prec \infty \text{ in } \partial T_1, \quad z_2 \prec z_1 \prec \infty \text{ in } \partial T_2, \quad z_2 \prec \zeta_2 \prec \infty \text{ in } \partial T_3.$$

The tiles of the tessellation of $\widehat{\Gamma}$ are topological triangles.

Example 7 (Nonuniqueness of consistent labellings) As additional difficulty, in general a $\mathtt{t}$–map Γ has many labellings; however, the consistent ones are few. Let Γ with vertices $V(\Gamma) = \{z_1, z_2, z_3, \infty\}$, an affine sketch of it is in the upper row of Figure 6. The orientation of Γ determines the blue tiles of $\mathscr{T}(\Gamma)$. There are six possible labellings

$$\mathcal{L}_\kappa : V(\Gamma) = \Big\{z_1, z_2, z_3, \infty\Big\} \longrightarrow \{1, 2, 3, 4\}, \quad z_j \longmapsto \mathcal{L}_\kappa(z_j), \ \ \infty \longmapsto 4,$$

as follows:

a) $\mathcal{L}_1(z_1) = 1 \ \ \mathcal{L}_1(z_2) = 2 \ \ \mathcal{L}_1(z_3) = 3\,;$

b) $\mathcal{L}_2(z_1) = 1 \ \ \mathcal{L}_2(z_2) = 3 \ \ \mathcal{L}_2(z_3) = 2\,;$

c) $\mathcal{L}_3(z_1) = 2 \ \ \mathcal{L}_3(z_2) = 1 \ \ \mathcal{L}_3(z_3) = 3\,;$

d) $\mathcal{L}_4(z_1) = 2 \ \ \mathcal{L}_4(z_2) = 3 \ \ \mathcal{L}_4(z_3) = 1\,;$

e) $\mathcal{L}_5(z_1) = 3 \ \ \mathcal{L}_5(z_2) = 1 \ \ \mathcal{L}_5(z_3) = 2\,;$

f) $\mathcal{L}_6(z_1) = 3 \ \ \mathcal{L}_6(z_2) = 2 \ \ \mathcal{L}_6(z_3) = 1.$

Which labellings $\mathcal{L}_\kappa$ are consistent for Γ? Consider $\mathcal{L}_1$ see Figure 6.a, according to the order $\prec$, we have that $z_3 \prec z_2$ on $\partial\overline{T_1}$ and $\mathcal{L}_1(z_3) = 3 > \mathcal{L}_1(z_2) = 2$. Therefore, the orders $\prec$ and $<$ do not match in $\partial\overline{T_1}$. By simple inspection, only the labellings $\mathcal{L}_2$ and $\mathcal{L}_4$ are consistent. Recalling Example 2, the labelling $\mathcal{L}_2$ (resp. $\mathcal{L}_4$) determines any of the polynomials

$$P_e(z) = ez - \tfrac{z^2}{2} + \tfrac{z^4}{4}, \quad \text{for } e \in (0,\, 0.385, 0) \quad \big(\text{resp. } e \in (-0.385,\, 0)\big).$$

4.2 Test trees

Let Γ be a $\mathtt{t}$–graph as in Theorem 1, assertion (2). In order to construct a consistent labelling $\mathcal{L}_c$ for Γ, an auxiliary graph is the following.

Definition 6 The *test tree* $\mathcal{T}$ *associated with* Γ has n vertices

$$V(\mathcal{T}) = \{\mathtt{v}_1, \ldots, \mathtt{v}_\alpha, \ldots, \mathtt{v}_n\} \subset \mathbb{C}_z, \quad \text{where } \mathtt{v}_\alpha \in T_\alpha,$$

and $n-1$ edges

$$E(\mathcal{T}) = \{\overline{\mathtt{v}_\alpha \mathtt{v}_\beta}\} \subset \mathbb{C}_z, \quad \alpha, \beta \in \{1, \ldots, n\}, \ \alpha \neq \beta,$$

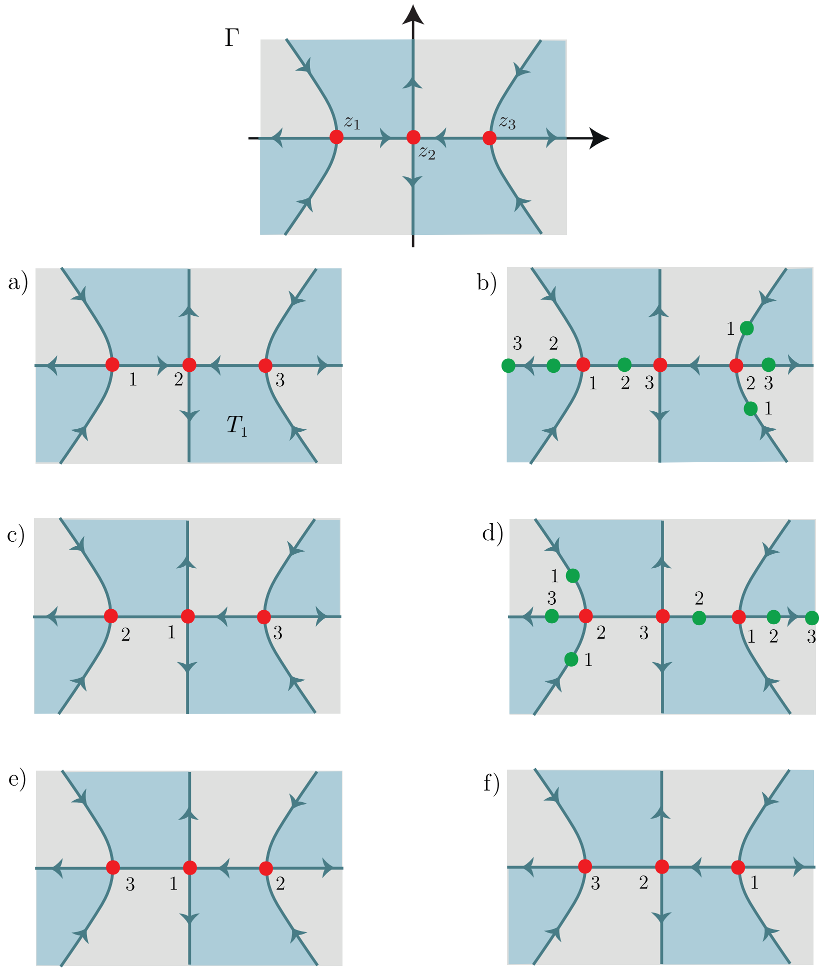

Fig. 6 From a combinatorial point of view, there are six labellings $\mathcal{L}_\kappa$ for Γ in the upper row, which are sketch in (a)–(f). The consistent labellings correspond to (b) and (d).

such that each $\overline{\mathtt{v}_\alpha \mathtt{v}_\beta}$ runs through one finite vertex $z_j \in \partial\overline{T_\alpha} \cap \partial\overline{T_\beta}$ of Γ.

Since the vertices of Γ in $\mathbb{C}_z$ have valence 4, (we recall the genericity hypothesis in Theorem 1), the test tree $\mathcal{T}$ is a well defined embedded graph.
In fact, if some cycle appears in $\mathcal{T}$, it encloses a gray tile T'_α, which is a contradiction, note that the vertex ∞ of Γ is in the closure of every tile of the tessellation $\mathscr{T}(\Gamma)$. The edges of $\mathcal{T}$ do not have orientation, that is $\overline{\mathtt{v}_\alpha \mathtt{v}_\beta} = \overline{\mathtt{v}_\beta \mathtt{v}_\alpha}$.

Remark 4 There is a bijection between the vertices $V(\Gamma)$ of Γ in $\mathbb{C}_z$ and the edges $E(\mathcal{T})$ of its test tree $\mathcal{T}$. Hence, a labelling $\mathcal{L}$ for Γ determines a labelling

$$\mathcal{L} : E(\mathcal{T}) \longrightarrow \{1, \ldots, n-1\}$$

for the edges its test tree $\mathcal{T}$ and vice versa. Note that the label n is not required for any labelling $\mathcal{L}$ of $E(\mathcal{T})$. Moreover, by abusing of language, we say that $\mathcal{T}$ *has a consistent labelling* when the corresponding labelling of Γ is consistent, see Definition 5.

Example 8 (On the scheme of the proof) Consider Γ as in Figure 7.a, the sketch of its test tree $\mathcal{T}$ is on the right, and the superposition of $\Gamma \cup \mathcal{T}$ is the center drawing. The edges of $\mathcal{T}$ are in bijective correspondence with the affine vertices of Γ. A consistent labelling for Γ is

$$\mathcal{L}_c(z_1) = 2,\ \mathcal{L}_c(z_2) = 3,\ \mathcal{L}_c(z_3) = 1,$$

which is equivalent to a consistent labelling for $\mathcal{T}$

$$\mathcal{L}_c(\overline{\mathtt{v}_1\mathtt{v}_2}) = 2,\ \mathcal{L}_c(\overline{\mathtt{v}_2\mathtt{v}_3}) = 3,\ \mathcal{L}_c(\overline{\mathtt{v}_3\mathtt{v}_4}) = 1.$$

In summary, Figure 7 illustrates the complete scheme for the proof (2)⇒(1) of Theorem 1.

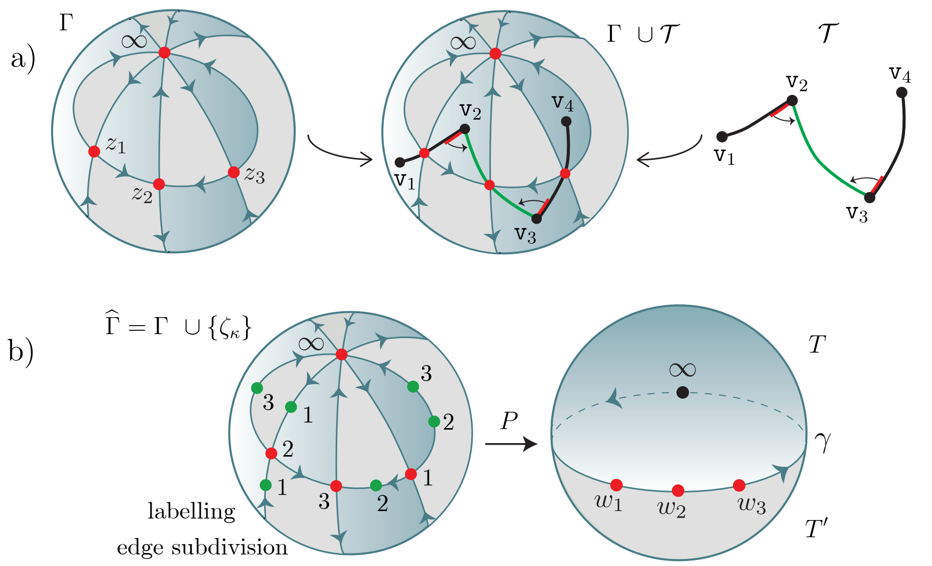

Fig. 7 a) A $\mathtt{t}$–map Γ and its tessellation, the tiles are topological digons and triangles, on the right the test tree $\mathcal{T}$. b) A consistent labelling for Γ and $\mathcal{T}$ determines the associated $\mathtt{R}$–map $\widehat{\Gamma} = \Gamma \cup \{\zeta_k\}$ such that its tiles are topological triangles. The existence of an associated generic polynomial P follows.

In order to construct a consistent labelling $\mathcal{L}_c$ for the test tree $\mathcal{T}$, we furnished its edges with additional information. Recall two notions from graph theory. Let $\overline{\mathtt{v}_\alpha\mathtt{v}_\beta}$ be an edge of $\mathcal{T}$;

it is a *leaf* when one of its extreme vertices has valence 1,

it is a *bridge* when its extreme vertices have a valence equal to or greater than 2.

Let $\mathtt{v}_\alpha \in V(\mathcal{T})$ be a vertex of $\mathcal{T}$ of valence at least 2, the ordered subtree of $\mathcal{T}$ formed by the incident edges to $\mathtt{v}_\alpha$ is the *star of* $\mathtt{v}_\alpha$, denoted

$$star(\mathtt{v}_\alpha) = \{\overline{\mathtt{v}_\alpha\mathtt{v}_\beta}, \ldots, \overline{\mathtt{v}_\alpha\mathtt{v}_\sigma}\}.$$

Definition 7 Consider, $\overline{\mathtt{v}_\alpha\mathtt{v}_\beta}$ an edge of $\mathcal{T}$, it assumes one of the following properties:

1) It is an *initial* leaf when in addition $\overline{\mathtt{v}_\alpha\mathtt{v}_\beta}^I$ of $star(\mathtt{v}_\alpha)$ is the edge that intersects the lower vertex $z_\iota \in \partial\overline{T_\alpha} \cap \{z_1, \ldots, z_{n-1}\}$ of Γ, according to the order $\prec$ on $\partial\overline{T_\alpha}$, see Figure 8.

2) It is a *non initial leaf.*

3) It is a *non initial bridge.*

4) It is an *initial bridge.*

5) It is a *double initial bridge.*

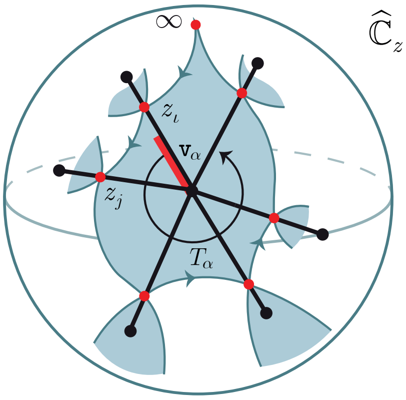

Fig. 8 The geometrical meaning of the initial edge of a test tree $\mathcal{T}$; it is the first edge associated with z_ι, when we travel along the boundary $\partial\overline{T}_\alpha$ starting from ∞ in the counterclockwise orientation.

We explain the five cases in the definition. For the sake of clarity, from now on we use a pictorial notation for the test trees $\mathcal{T}$. For the edges, the leaves are black and the bridges are green. For initial edges, we add a small red bar and a circular arrow, following the geometrical rule in Figure 8. According to this pictorial notation, there are five kinds of edges which can appear in a test tree $\mathcal{T}$, Figure 9 illustrates them. Every *star*($\mathtt{v}_\alpha$) of a test tree $\mathcal{T}$ has exactly one initial edge. In addition, an edge $\overline{\mathtt{v}_\alpha \mathtt{v}_\beta}$ can be initial for both *star*($\mathtt{v}_\alpha$) and *star*($\mathtt{v}_\beta$).

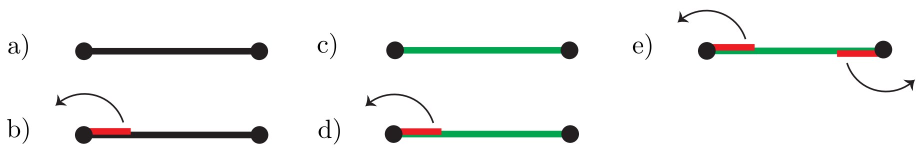

Fig. 9 The edges of a test tree $\mathcal{T}$ belong to one of five kinds: a) a non initial leaf, b) an initial leaf, c) a non initial bridge, d) an initial bridge, e) a double initial bridge. The oriented circular arrows have counterclockwise orientation.

Example 9 Figure 10 illustrates two kinds of edges of test trees $\mathcal{T}$ and their tessellations. The example of a non initial bridge appears in the test tree $\mathcal{T}$ of Figure 7. The assignment of small red bars and circular arrows follows the geometrical rule in Figure 9.

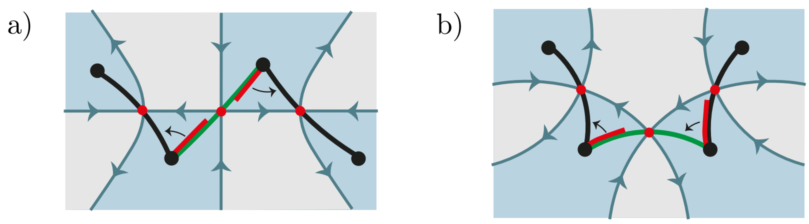

Fig. 10 Examples of test trees $\mathcal{T}$ with edges of kinds; non initial leaf, initial leaf, initial bridge, doble initial bridge.

4.3 Construction of consistent labellings

Proposition 1 *Every* $\mathtt{t}$*–map* Γ *admits at least one consistent labelling* $\mathcal{L}_c$.

Proof. Let $\mathcal{T}$ be the test tree associated with the $\mathtt{t}$–map Γ. If $\mathcal{T}$ has 2 vertices, then a consistent labelling $\mathcal{L}_c$ exists by simple inspection, we deal with generic polynomials of degree 3. Recall Examples 1, 2 and 7.

We consider the case where $\mathcal{T}$ has $n \geq 4$ vertices and $n-1$ edges (we deal with polynomials of degree $n \geq 4$). By induction hypothesis on the number of vertices of $\mathcal{T}$; we assume that $\mathcal{T}$ originates from a tree denoted $\mathcal{T}_{n-1}$ by adding one vertice $\mathtt{v}_n$ and one edge $\overline{\mathtt{v}_\alpha \mathtt{v}_n}$; thus

$$\mathcal{T}_{n-1} \doteq \mathcal{T} \backslash (\mathtt{v}_n \cup \overline{\mathtt{v}_\alpha \mathtt{v}_n}) \tag{11}$$

is a connected tree. Again, by induction hypothesis $\mathcal{T}_{n-1}$ has a consistent labelling, denoted

$$\mathcal{L}_{n-1} : E(\mathcal{T}_{n-1}) \longrightarrow \{1, \ldots, n-2\},$$

we recall Definition 5 and Remark 4. In Section 5, the simplest case $n = 4$ and $\mathcal{T}_{n-1} = \mathcal{T}_3$ with 2 edges is described. We want to perform a consistent labelling for $\mathcal{T}$.

Case 1. We assume that the valence of $\mathtt{v}_\alpha$ in $\mathcal{T}_{n-1}$ is at least 2 and the new edge $\overline{\mathtt{v}_\alpha \mathtt{v}_n}$ is a non initial leaf in $\mathcal{T}$ of the star with center in $\mathtt{v}_\alpha$; see Figure 11.

i) We consider the star of $\mathtt{v}_\alpha$ in the tree $\mathcal{T}$. The edge $\overline{\mathtt{v}_\beta \mathtt{v}_\alpha}$ is the predecessor edge of $\overline{\mathtt{v}_\alpha \mathtt{v}_n}$. Assume that the label under $\mathcal{L}_{n-1}$ of $\overline{\mathtt{v}_\beta \mathtt{v}_\alpha}$ is k.

ii) Multiply by 2 all the labels of $\mathcal{L}_{n-1}$, in particular the label of $\overline{\mathtt{v}_\beta \mathtt{v}_\alpha}$ is $2k$; see the second row in Diagram 12.

iii) We assign a provisional label $2k+1$ for the edge $\overline{\mathtt{v}_\alpha \mathtt{v}_n}$.

iv) Recalling Remark 4, we note that the numerical order $<$ in the duplicated labels coincides with the topological order $\prec$ from Γ as in equation (7).

v) We eliminate the gaps between the labels in (iii) by a shift (down arrows) as follows

$$\begin{array}{rcccccccc}
\mathcal{L}_{n-1} \text{ of } \mathcal{T}_{n-1} \text{ in (11):} & 1, & \ldots, & k, & & k+1, & k+2, & \ldots, & n-2 \\
& \downarrow & & \downarrow & & \downarrow & \downarrow & & \downarrow \\
& 2, & \ldots, & 2k, & 2k+1, & 2(k+1), & 2(k+2), & \ldots, & 2(n-2) \\
& \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow \\
\mathcal{L}_c \text{ of } \mathcal{T}: & 1, & \ldots, & k, & k+1, & k+2, & k+3, & \ldots, & n-1.
\end{array} \tag{12}$$

We are done, and the resulting labels determine a consistent labelling $\mathcal{L}_c$ for the edges of $\mathcal{T}$. By recalling Remark 4, $\mathcal{L}_c$ is a consistent labelling for Γ.

Case 2. We assume that the valence of $\mathtt{v}_\alpha$ in $\mathcal{T}_{n-1}$ is at least 2 and the new edge $\overline{\mathtt{v}_\alpha \mathtt{v}_n}$ is an initial leaf in $\mathcal{T}$ of the star with center in $\mathtt{v}_\alpha$; see Figure 12.

i) In $\mathcal{T}_{n-1}$, we consider the first edge $\overline{\mathtt{v}_\alpha \mathtt{v}_\delta}$ in the star of $\mathtt{v}_\alpha$ and assume that the label of $\overline{\mathtt{v}_\alpha \mathtt{v}_\delta}$ under $\mathcal{L}_{n-1}$ is k.

ii) Multiply by 2 all the labels of $\mathcal{T}_{n-1}$, in particular now the label of $\overline{\mathtt{v}_\alpha \mathtt{v}_\delta}$ is $2k$.

iii) We define the provisional label $2k-1$ for the new edge $\overline{\mathtt{v}_\alpha \mathtt{v}_n}$.

iv) Recalling Remark 4, we note that the numerical order $<$ in the duplicated labels coincide with the topological order $\prec$ from Γ as in equation (7).

v) We eliminate the gaps between the labels in (iii) by a shift, as in equation (12),

$$\begin{array}{rcccccccc}
\mathcal{L}_{n-1} \text{ of } \mathcal{T}_{n-1} \text{ in (11):} & 1, & \ldots, & k, & & k+1, & k+2, & \ldots, & n-2 \\
& \downarrow & & \downarrow & & \downarrow & \downarrow & & \downarrow \\
& 2, & \ldots, & 2k, & 2k+1, & 2(k+1), & 2(k+2), & \ldots, & 2(n-2) \\
& \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow \\
\mathcal{L}_c \text{ of } \mathcal{T}: & 1, & \ldots, & k, & k+1, & k+2, & k+3, & \ldots, & n-1.
\end{array}$$

This performs a consistent labelling $\mathcal{L}_c$ for the edges of $\mathcal{T}$. By recalling, Remark 4, we have a consistent labelling $\mathcal{L}_c$ for Γ.

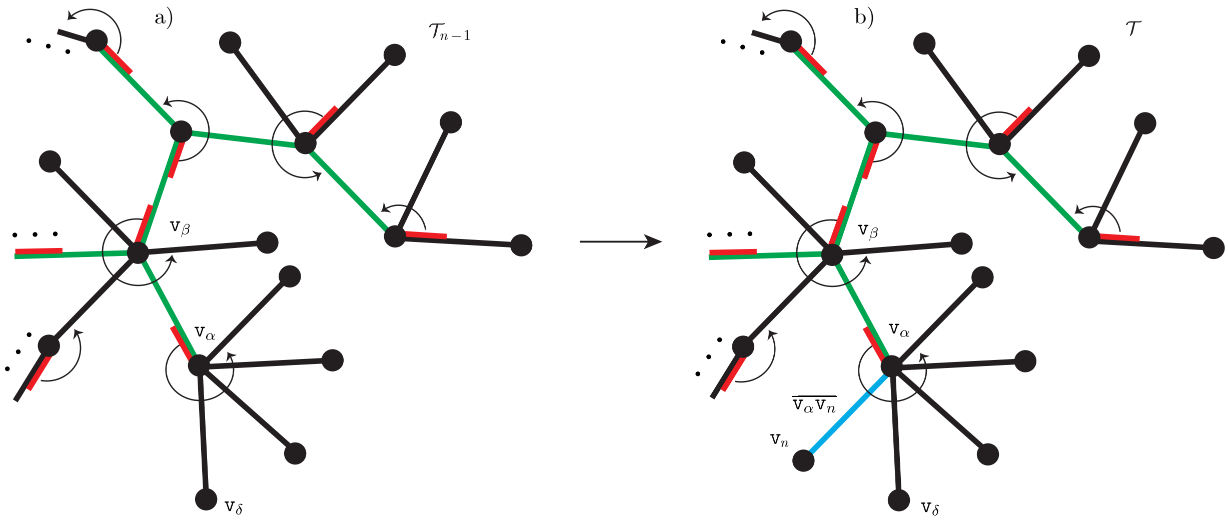

Fig. 11 a) The valence of $\mathtt{v}_\alpha$ in the tree $\mathcal{T}_{n-1}$ is at least 2 and and $\mathtt{v}_\alpha$ is the initial extreme of its bridge. b) In order to obtain $\mathcal{T}$; we add a new blue edge $\overline{\mathtt{v}_\alpha\mathtt{v}_n}$ to the star of $\mathtt{v}_\alpha$ in $\mathcal{T}_{n-1}$. The new edge $\overline{\mathtt{v}_\alpha\mathtt{v}_n}$ is a non initial leaf of $\mathcal{T}$.

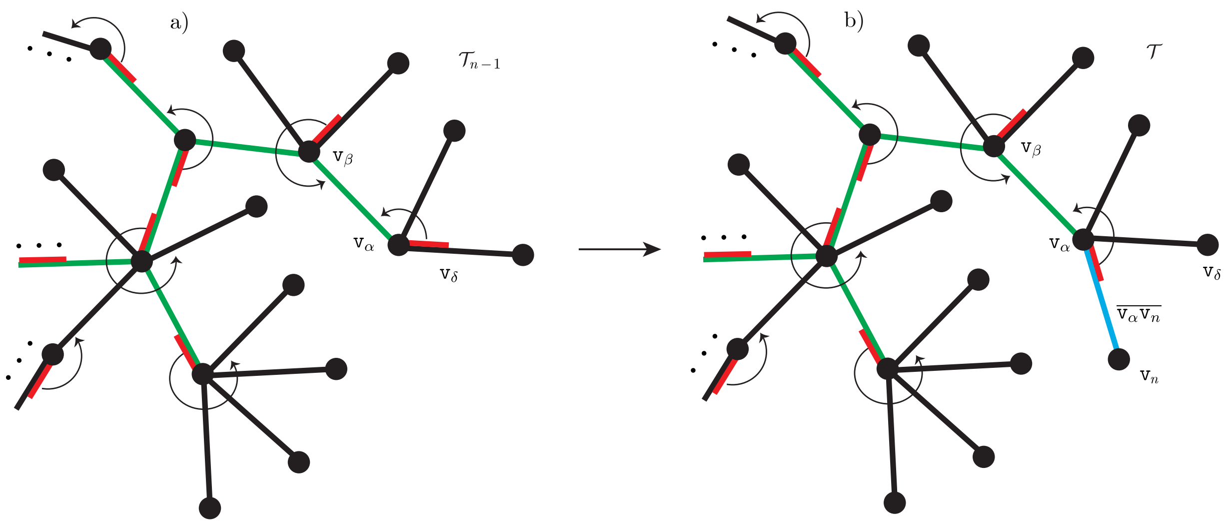

Fig. 12 a) The valence of $\mathtt{v}_\alpha$ in $\mathcal{T}_{n-1}$ is at least 2 and $\mathtt{v}_\alpha$ is the non initial extreme of its bridge. b) In order to obtain $\mathcal{T}$; we add a blue edge $\overline{\mathtt{v}_\alpha\mathtt{v}_n}$ to the star of $\mathtt{v}_\alpha$ in $\mathcal{T}_{n-1}$. The new edge $\overline{\mathtt{v}_\alpha v_n}$ is an initial leaf of $\mathcal{T}$.

Case 3. We assume that the valence of $\mathtt{v}_\alpha$ in $\mathcal{T}_{n-1}$ is 1 and the new edge $\overline{\mathtt{v}_\alpha\mathtt{v}_n}$ is an extreme vertex of $\mathcal{T}$.

The edge $\overline{\mathtt{v}_\beta\mathtt{v}_\alpha}$ is a leaf of $\mathcal{T}_{n-1}$. After the addition of the new edge $\overline{\mathtt{v}_\alpha\mathtt{v}_n}$, we have that $\overline{\mathtt{v}_\beta\mathtt{v}_\alpha}$ is a bridge of $\mathcal{T}$, see Figure 13. Hence, $\mathtt{v}_\alpha$ has valence 2 in $\mathcal{T}$. Figure 13 shows that there are two possibilities:
• the new edge $\overline{\mathtt{v}_\alpha\mathtt{v}_n}$ is a (non initial) leaf, or
• it is an initial leaf.
In any case, this leads us to one of the previous Cases 1 or 2.

The proof of Proposition 1 is done.

□

Figure 13 shows that a tree $\mathcal{T}$ (without the information of initial edges) does not determine a single $\mathtt{t}$–map Γ, a single tessellation (up to right–left topological equivalence).

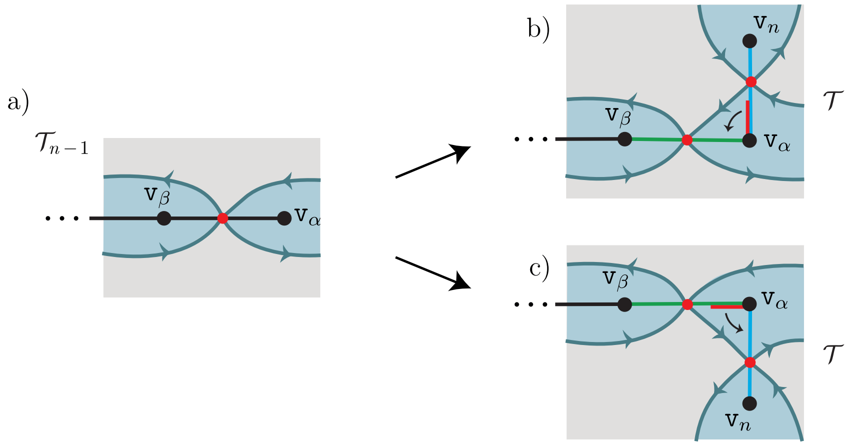

Fig. 13 a) The valence of $\mathtt{v}_\alpha$ in $\mathcal{T}_{n-1}$ is 1, it is the extreme vertex of a leaf. b)–c) In order to obtain $\mathcal{T}$; we add a blue edge $\overline{\mathtt{v}_\alpha \mathtt{v}_n}$. There are two possible tessellations (b) and (c) associated to $\mathcal{T}$. The choice of the tessellation determines which is the new initial edge of $\mathcal{T}$ at $\mathtt{v}_\alpha$.

5 A constructive algorithm

Let us consider an $\mathtt{R}$–map $\widehat{\Gamma_3}$ of a generic polynomial of degree 3, and let $\mathcal{T}_3$ be its test tree, provided with its consistent labelling $\mathcal{L}_2$, as in Figure 14.a. For degree 3, there is only one class of generic polynomials up to right–left topological equivalence.

In order to construct all the $\mathtt{R}$–maps of generic polynomials of degree 4, we obtain new trees $\mathcal{T}$ by adding a vertex $\mathtt{v}_4$ to $\mathcal{T}_3$. The respective blue edges $\overline{\mathtt{v}_\alpha \mathtt{v}_4}$ are illustrated in clockwise order from Figure 14.b–h. Since the original $\widehat{\Gamma}_3$ has seven edges, a priori there are seven different ways to add the new edge. In Figure 14, we describe the new seven test trees $\mathcal{T}$ provided with their consistent labellings $\mathcal{L}_c$:

b) The new edge $\overline{\mathtt{v}_\alpha \mathtt{v}_4}$ is initial, and we apply Case 2 in proof of Proposition 1. Note that the new consistent labelling $\mathcal{L}_3$ is different from the original $\mathcal{L}_2$; the original edges with labels 1 and 2, now have labels 2 and 3, respectively. The resulting polynomial appeared in Figure 3.d, note that the assignation of blue tiles depends in the choice of an orientation of γ.
c) The new edge $\overline{\mathtt{v}_\alpha \mathtt{v}_4}$ is non initial, and we apply Case 1. Up to orientation preserving homeomorphism of $\widehat{\mathbb{C}}_z$, the resulting polynomial is $P(z) = \frac{z}{4} + \frac{z^2}{2} - \frac{z^4}{4}$, with $\Gamma = P^{-1}(\mathbb{R} \cup \{0\})$.
d) The new edge $\overline{\mathtt{v}_\alpha \mathtt{v}_4}$ is initial, we apply Case 2. The resulting polynomial appeared in Figure 3.d.
e) Up to orientation preserving homeomorphisms of $\widehat{\mathbb{C}}_z$, it coincides with (b).
f) Up to orientation preserving homeomorphisms of $\widehat{\mathbb{C}}_z$, it coincides with (d).
g) The new edge $\overline{\mathtt{v}_\alpha \mathtt{v}_4}$ is initial, we apply Case 1. Up to orientation preserving homeomorphism of $\widehat{\mathbb{C}}_z$, the resulting polynomial is $P(z) = \frac{z}{4} - \frac{z^2}{2} + \frac{z^4}{4}$, with $\Gamma = P^{-1}(\mathbb{R} \cup \{0\})$, as in Figure 6.b.
h) Up to orientation preserving homeomorphisms of $\widehat{\mathbb{C}}_z$, this $\mathcal{T}$ coincides with (b).

Summing up, given any $\mathtt{t}$–map Γ with $n-1$ edges and its tessellation $\mathscr{T}(\Gamma)$ in $\widehat{\mathbb{C}}_z$, we can construct its test tree $\mathcal{T}$, by adding suitable $n-3$ edges to the tree $\mathcal{T}_3$, as in Figure 14.a. In each step, a consistent labelling $\mathcal{L}_c$ for the respective $\mathtt{t}$–map exists. This provides a constructive algorithm which recognizes the associated $\mathtt{R}$–map $\widehat{\Gamma}$ with Γ, as Theorem 1 asserts.

References

1. A. Alvarez–Parrilla, J. Muciño–Raymundo, *Dynamics of singular complex analytic vector fields with essential singularities II*, J. Singul. (2022), 1–78. `https://www.journalofsing.org/volume24/article1.html`
2. I. Bauer, F. Catanese, *Generic lemniscates of algebraic functions,* Math. Ann. 307, 3 (1997), 417–444. `https://doi.org/10.1007/s002080050042`

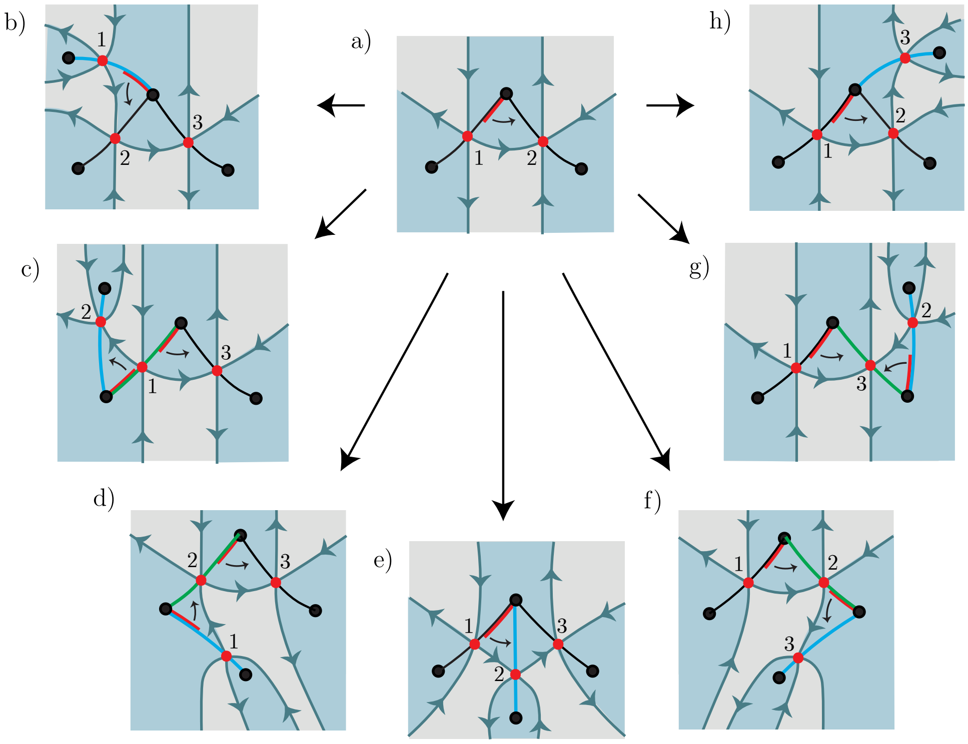

Fig. 14 a) We sketch the $\mathbb{R}$–map of a generic polynomial of degree 3, its test tree $\mathcal{T}_3$ with 2 edges, and its consistent labelling. In (b)–(h), an new blue edge $\overline{v_\alpha v_4}$ is added. We sketch the new trees $\mathcal{T}$ with their consistent labellings $\mathcal{L}_c$.

3. G. V. Belyĭ, *On Galois extensions of a maximal cyclotonic field*, Mathematics of the USSR–Izvestiya, 14, 2 (1980) 247–256. `https://doi.org/10.1070/IM1980v014n02ABEH001096`
4. R. Cavalieri, E. Miles, *Riemann Surfaces and Algebraic Curves*, London Math. Soc. Student Texts 87, Cambridge U. K., (2016). `https://doi.org/10.1017/CBO9781316569252`
5. E. Chislenko, Y. Tschinkel, *The Felix Klein Protocols*, Notices of the American Mathematical Society, 54, 8 (2007) 961–970. `https://www.ams.org/notices/200708/tx070800960p.pdf`
6. A. Grothendieck, 1984. *Esquisse d'un Programme*, 1984 manuscript, In The Grothendieck Theory of Dessins d'Enfants, L. Schneps et al. Eds., London Math. Soc., Vol. 20 Cambridge University Press (1997), I, 7-48. `https://doi.org/10.1017/CBO9780511569302`
7. E. Hille, *Ordinary Differential Equations in the Complex Domain*, Dover, N. Y., USA (1997).
8. F. Klein, *Lectures on the Icosahedron,* Dover Phoenix Editions, USA (2003).
9. S. Koch, T. Lei, *On balanced graphs, following W. Thurston*, in What's Next? The Mathematical Legacy of William P. Thurston, D. P. Thurston Ed., Princeton Univ. Press, (2020) 213–231.
10. S. K. Lando, A. K. Zvonkin, *Graphs on Surfaces and their Applications*, Encyclopaedia of Mathematical Sciences, 141, Low–Dimensional Topology II, Springer–Verlag, Berlin, (2004). `https://doi.org/10.1007/978-3-540-38361-1`
11. G. León–Gil, J. Muciño–Raymundo, *Integrability and adapted complex structures to smooth vector fields on the plane*, Lobachevskii J. Math. 43, 1 (2022) 110–126. `https://doi.org/10.1134/S1995080222040151`
12. D. Masoero, *Painleve I, coverings of the sphere and Belyĭ functions*, Constructive Approximation, 39, 1 (2014) 43–74. `https://doi.org/10.1007/s00365-013-9185-3`
13. R. Nevanlinna, *Analytic Functions*, Die Grundlehren der mathematischen Wissenschaften, Band 162, Springer–Verlag, New York, Berlin, (1970). `https://link.springer.com/book/10.1007/978-3-642-85590-0`
14. R. Nevanlinna, V. Paatero, *Introduction to Complex Analysis.* Addison-Wesley, Reading, Massachusetts USA, 1969.
15. R. Peretz *Maximal Domains for entire functions,* J. Anal. Math. 61 (1993) 1–28 `https://doi.org/10.1007/BF02788836`
16. H. A. Schwarz, *Ueber diejenigen Fälle, inwelchen die Gaussische hypergeometrische Reihe eine algebraische Function ihres vierten Elementes darstellt*, J. Reine Angewandte Mathematik, Volumen 75, (1873) 292–335. `https://eudml.org/doc/148203`
17. A. Speiser, *Über Riemannsche Flächen,* Comment. Math. Helv. 2, 1, 284–293 (1930). `https://eudml.org/doc/138546`
18. H. Völklein, *Groups as Galois Groups*, Cambridge Studies in Advanced Mathematics 53, Cambridge University Press, Cambridge, New York, (1996). `https://doi.org/10.1017/CBO9780511471117`

19. M. Yoshida, *Fuschsian Differential Equations*, Friedr. Vieweg & Sohn, Braunschweig, (1987). https://doi.org/10.1007/978-3-663-14115-0