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Plane Polynomials and Hamiltonian Vector Fields Determined by Their Singular Points

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Abstract. Let $\Sigma(f)$ be the singular points of a polynomial $f \in \mathbb{K}[x, y]$ in the plane $\mathbb{K}^2$, where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$. Our goal is to study the singular point map $\mathfrak{S}_d$, it sends polynomials f of degree d to their singular points $\Sigma(f)$. Very roughly speaking, a polynomial f is essentially determined when any other g sharing the singular points of f satisfies that $f = \lambda g$; here both are polynomials of degree d, $\lambda \in \mathbb{K}^*$. In order to describe the degree d essentially determined polynomials, a computation of the required number of isolated singular points $\delta(d)$ is provided. A dichotomy appears for the values of $\delta(d)$; depending on a certain parity, the space of essentially determined polynomials is an open or closed Zariski set. We compute the map $\mathfrak{S}_3$, describing under what conditions a configuration of 4 points leads to a degree 3 essentially determined polynomial. Furthermore, we describe explicitly configurations supporting degree 3 non essential determined polynomials. The quotient space of essentially determined polynomials of degree 3 up to the action of the affine group $\mathit{Aff}(\mathbb{K}^2)$ determines a singular $\mathbb{K}$-analytic surface.

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1. Introduction

Very roughly speaking, the *singular point map* sends polynomials $f \in \mathbb{K}[x, y]$, of degree d, to their singular points

$$\mathfrak{S}_d : f \longmapsto \Sigma(f), \tag{1}$$

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where $\Sigma(f) \doteq I(f_x, f_y)$ is the affine algebraic variety (not necessarily reduced) generated by the ideal of partial derivatives of f, see Definition 3. Under what conditions is a degree d polynomial $f \in \mathbb{K}[x, y]$ essentially determined by its singular points $\Sigma(f) \subset \mathbb{K}^2$? Our approximation route uses a finite dimensional framework. Let $\mathbb{K}[x, y]^0_{\leq d}$ be the $\mathbb{K}$-vector space of polynomials having at most degree d (≥ 3) and a zero independent term, and let $\mathcal{P} = \{(x_\iota, y_\iota)\}$ be a configuration of n different points in the plane. The linear projective subspace of the polynomials with singular points at least in $\mathcal{P}$, denoted as

$$\mathcal{L}_d(\mathcal{P}) \doteq Proj\big(\{f \in \mathbb{K}[x, y]^0_{\leq d} \mid \mathcal{P} \subseteq \Sigma(f)\}\big), \tag{2}$$

is well defined. We say that a polynomial f is *essentially determined* by $\mathcal{P}$ when $\mathcal{L}_d(\mathcal{P})$ is a projective point $\{\lambda f \mid \lambda \in \mathbb{K}^*\}$, see Definition 4. All this leads us to the following.

Interpolation problem for singular points. Let $\mathcal{P} \subset \mathbb{K}^2$ be a configuration of n different points, we try to determine the projective subspace $\mathcal{L}_d(\mathcal{P})$ of polynomials of at most degree d with singular points at least in $\mathcal{P}$.

This problem has several novel features. The singular values $\{c_\iota\} \subset \mathbb{K}$ of f can appear in different level curves $\{f(x, y) - c_\iota = 0\}$; it is natural in Hamiltonian vector field theory and moduli spaces of polynomials, see Wightwick [18] and Fernández de Bobadilla [11]. This is the main difference from the widely considered problem of linear systems of curves in $\mathbb{CP}^2$, e.g., Miranda [15] and Ciliberto [8].

Very roughly speaking, for degree $d \geq 3$ the relevant data are the cardinality and position of the configuration $\mathcal{P}$, as a candidate to be a singular point configuration $\Sigma(f)$. For degree 3, the prescription of 4 singular points is suitable. For degree $d \geq 4$, however, the generic configuration $\mathcal{P}$ with $(d-1)^2$ points is too restrictive. Thus, the fiber $\mathfrak{S}_d^{-1}(\mathcal{P})$ will be generically empty. It follows that the position of the configurations $\mathcal{P}$ coming from polynomials is the hardest part to characterize. At this first stage, we consider mainly $\mathcal{P}$ as isolated points of multiplicity one, Remark 1 provides an explanation. Our first result describes the role of cardinality $\delta(d)$ of $\mathcal{P}$ in Eq. (2), see Proposition 1.

Dichotomy of the required number of singular points. If the dimension of $\mathbb{K}[x, y]^0_{\leq d}$ *is odd (resp. even), then the configurations* $\{\mathcal{P}\}$ *with* $\delta(d)$ *points and* $dim_{\mathbb{K}}(\mathcal{L}_d(\mathcal{P})) \geq 0$ *determine an open (resp. closed) Zariski set in the space of configurations with* $\delta(d)$ *points, denoted as* $Conf(\mathbb{K}^2, \delta(d))$.

We compute the singular point map $\mathfrak{S}_3$. Thus, a description for the 4 singular point configurations $\{\mathcal{P}\}$ with essentially determined polynomials is provided. Recall that the affine group $Aff(\mathbb{K}^2)$ acts on the space of polynomials, see Eq. (20). This action is rich enough and yet treatable for degree 3. Let

$$\mathscr{A} \doteq \{x_4 y_4 (x_4 + y_4 - 1)(x_4 + y_4)(x_4 - 1)(y_4 - 1) = 0\} \subset \mathbb{K}^2 = \{(x_4, y_4)\}$$

be an arrangement of six lines from two nested triangles, where of them is $\triangle = \{(0, 0), (1, 0), (0, 1)\}$. See Fig. 1a. We prove the following result.

Theorem 1. *Let f be a degree 3 polynomial having at least 4 singular points $\Sigma(f)$.*

(1) *f is essentially determined if and only if up to affine transformation the four singular points are*

$$\Sigma(f) = \{(0,0), (1,0), (0,1), (x_4, y_4)\} \quad \textit{and} \quad (x_4, y_4) \notin \mathscr{A}.$$

(2) *f is not essentially determined if and only if up to affine transformation the four singular points are*

$$\{(0,0), (1,0), (0,1), (x_4, y_4)\} \quad \textit{and} \quad (x_4, y_4) \in \mathscr{A}.$$

Moreover, in this case $\Sigma(f)$ can be four isolated points or two parallel lines.

In simple words, the 4-th point (x_4, y_4) generically determines the polynomial f. We compute the *fundamental domain* for this $\mathit{Aff}(\mathbb{K}^2)$-action and obtain a tessellation of $\mathbb{K}^2 = \{(x_4, y_4)\}$ with 24 tiles, as seen in Fig. 3. As expected, some interesting phenomena occur for configurations with nontrivial isotropy groups in $\mathit{Aff}(\mathbb{K}^2)$, Fig. 4 illustrates this. For degree $d \geq 3$, a particular family of configurations is the grid of $(d-1)^2$ points, from the intersection of two families of d parallel lines in $\mathbb{K}^2$, see Definition 8. They provide examples of nonessential determined polynomials with $(d-1)^2$ Morse singular points. A remaining open question is are these grids of $(d-1)^2$ points the unique mechanism in order to produce non essential determined Morse polynomials?

From the point of view of vector fields; under what conditions the singular points (i.e., zeros) of a Hamiltonian vector field determine it in a unique way? This is a very general and interesting issue in real and complex foliation theory, studied by Gómez-Mont and Kempf [13], Artes et al. [4], Campillo and Olivares [6] and Ramírez [17]. See Corollary 6. These related results are described in Sect. 7.

The content of this work is as follows. In Sects. 2 and 3, we study the problem of the dimension of linear systems for polynomials with singular points, using the degree as a parameter. In Sect. 4, we characterize polynomials essentially determined by their configurations of singular points; this proves Theorem 1. In Sect. 5, we focus on the degree 4 case. For each configuration of 6 points, we obtain a plane curve of degree 6 by parametrizing the essentially determined polynomials, see Proposition 2. Section 6 explores the behavior of pencils of Hamiltonian vector fields with common simple singularities.

2. Linear Systems $\mathcal{L}_d(\mathcal{P})$

Let $\mathbb{K}[x,y]^0_{\leq d}$ (resp. $\mathbb{K}[x,y]^0_{=d}$) be the $\mathbb{K}$-vector space of polynomials with at most degree $d \geq 3$ (resp. the set for degree $= d$) and a zero independent term. Consider

$$f(x,y) = \sum_{1 \leq \iota + j \leq d} a_{\iota j} x^{\iota} y^{j} \quad \in \mathbb{K}[x,y]^0_{\leq d}, \tag{3}$$

from which the $\mathbb{K}$-dimension of $\mathbb{K}[x,y]^0_{\leq d}$ is $\frac{1}{2}(d^2+3d)$ and its projectivization is

$$Proj\big(\mathbb{K}[x,y]^0_{\leq d}\big) = \big\{[f] \mid f \in \mathbb{K}[x,y]^0_{\leq d}\big\} = \mathbb{KP}^{\frac{1}{2}(d^2+3d-2)}, \tag{4}$$

where [ ] denotes a projective class. Recall that

$$Conf(\mathbb{K}^2, n) = \big\{\, \mathcal{P} = \{(x_1,y_1),\ldots,(x_n,y_n)\} \mid (x_\iota, y_\iota) \neq (x_j, y_j) \quad \text{for} \quad \iota \neq j\big\}/Sym(n) \tag{5}$$

is the *space of unordered configurations of* n *points* in $\mathbb{K}^2$, where the symmetric group $Sym(n)$ in n elements acts by exchanging the points. The configuration space $Conf(\mathbb{K}^2, n)$ is a $\mathbb{K}$-analytic manifold.

Definition 1. Given a configuration $\mathcal{P} \in Conf(\mathbb{K}^2, n)$, the *linear system of polynomials of at most degree* d *with singular points at least in* $\mathcal{P}$ is the projective subspace

$$\mathcal{L}_d(\mathcal{P}) = \big\{[f] \mid \mathcal{P} \subseteq \{f_x(x,y)=0\} \cap \{f_y(x,y)=0\}\big\} \subset Proj\big(\mathbb{K}[x,y]^0_{\leq d}\big). \tag{6}$$

In algebraic geometry language, $\{f_x(x,y)=0\}$ and $\{f_y(x,y)=0\}$ belong to the linear system of algebraic curves

$$\mathcal{L}_{d-1}\big(-\Sigma^n_{\alpha=1}(x_\iota,y_\iota)\big).$$

See [8,15]. In several places, however we consider f_x, f_y as functions and not just as algebraic curves.

The polynomials of at most degree d, the Hamiltonian polynomial vector fields and the polynomial vector fields, of at most degree $d-1$, are related by linear maps

$$\begin{array}{ccccc} \mathbb{K}[x,y]^0_{\leq d} & \stackrel{\cong}{\longleftrightarrow} & Ham(\mathbb{K}^2)_{\leq d-1} & \longrightarrow & \mathfrak{X}(\mathbb{K}^2)_{\leq d-1} \\ f & \longleftrightarrow & X_f = -f_y\frac{\partial}{\partial x} + f_x\frac{\partial}{\partial y} & \longrightarrow & X_f. \end{array}$$

In the space of Hamiltonian vector fields, $\mathcal{L}_d(\mathcal{P})$ determines a linear subspace

$$\{\lambda X_f \mid \mathcal{P} \subseteq \mathcal{Z}(\lambda X_f),\ \lambda \in \mathbb{K}^*\} \subset Ham(\mathbb{K}^2)_{\leq d-1}.$$

Set theoretically, the zeros $\mathcal{Z}(\lambda X_f)$ of the vector field X_f coincide with $\{f_x(x,y) = 0\} \cap \{f_y(x,y)=0\}$.

Definition 2. Let $f \in \mathbb{K}[x,y]$ be a nonconstant polynomial. Over $\mathbb{K}=\mathbb{C}$, the *Milnor number of* X_f *at a zero point* $(x_\iota,y_\iota) \in \mathcal{Z}(X)$ is

$$\mu_{(x_\iota,y_\iota)}(X) = \dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^2,(x_\iota,y_\iota)}}{< -f_y, f_x >},$$

where $\mathcal{O}_{\mathbb{C}^2,(x_\iota,y_\iota)}$ is the local ring of holomorphic functions at the point (x_ι,y_ι) and $< -f_y, f_x >$ is the ring generated by the partial derivatives.

Remark 1. 1. Over $\mathbb{K} = \mathbb{C}$, if (x_ι, y_ι) is an isolated singular point of f, then the notions of multiplicity for the intersection of the curves $\{f_x(x,y) = 0\} \cap \{f_y(x,y) = 0\}$ and the Milnor number for X_f coincide; see [14, p. 174].
2. A priori, we consider each point $(x_\iota, y_\iota) \in \mathcal{P}$ in (6) with multiplicity of intersection 1 for the algebraic curves $\{f_x(x,y) = 0\}$ and $\{f_y(x,y) = 0\}$.
3. By Bézout's theorem, the maximal number of isolated singularities of X_f on $\mathbb{C}^2$ is $(d-1)^2$. In this case, all the affine singularities are of multiplicity 1.
4. Moreover, the maximal number of isolated singularities of X_f extended to $\mathbb{CP}^2$ is

$$(d-1)^2 + d.$$

Here the upper bound d comes from the intersection of a generic projectivized level curve $\{f = c\}$ with the line at infinity; see [6,13] for the case of rational vector fields, which are not necessarily Hamiltonian.

Let $\mathbb{A}^2_{\mathbb{K}} = \operatorname{Spec} \mathbb{K}[x,y]$ be the affine scheme of the affine plane $\mathbb{K}^2$, see [10, pp. 48–49].

Definition 3. The *singular point map of degree d* is

$$\begin{aligned} \mathfrak{S}_d : \mathbb{K}[x,y]_{=d} &\longrightarrow \operatorname{Spec} \mathbb{K}[x,y] \\ f &\longmapsto \Sigma(f) = I(f_x, f_y), \end{aligned} \tag{7}$$

sending a polynomial of degree d to its singular points $\Sigma(f)$ as an affine algebraic variety (not necessarily reduced) generated by the ideal of partial derivatives of f.

In fact, $\Sigma(f)$ can be understood as a subscheme, with support at the points $\{f_x(x,y) = 0\} \cap \{f_y(x,y) = 0\}$, where the sheaf of ideals is defined by the germs of $I(f_x, f_y)$; compare with [6], [10, p. 100]. In a set theoretical language, $\Sigma(f)$ determines points and even algebraic curves. In the study of rational vector fields on $\mathbb{CP}^2$ however, the case of foliations with singularities along curves is removed, see [6,13].

Remark 2. The simplest case of the interpolation problem for singular points occurs when $\Sigma(f)$ is a finite set of points of multiplicity 1, i.e., $\{f_x(x,y) = 0\}$ and $\{f_y(x,y) = 0\}$ have transversal intersections. The $\Sigma(f)$ is a configuration in $Conf(\mathbb{K}^2, n)$, for $0 \le n \le (d-1)^2$.

Our former task is as follows: *Given a configuration $\mathcal{P}$, which is $dim_{\mathbb{K}}(\mathcal{L}_d(\mathcal{P}))$?*
To be clear, three relevant data must be considered the degree d of the polynomials $\{f\}$, the cardinality n and the position of the configuration $\mathcal{P}$. The following diagram explains:

cardinality n of $\mathcal{P}$ $\searrow$

position of $\mathcal{P}$ $\nearrow$

$$dim_{\mathbb{K}}(\mathcal{L}_d(\mathcal{P})) = \begin{cases} -1 & \mathcal{L}_d(\mathcal{P}) = \emptyset. \\ 0 & [f] = \mathcal{L}_d(\mathcal{P}) = \mathbb{KP}^0 \\ & f \text{ is essentially determined.} \\ \kappa \geq 1 & [f] \in \mathcal{L}_d(\mathcal{P}) = \mathbb{KP}^\kappa \\ & f \text{ is nonessential determined.} \end{cases} \tag{8}$$

The natural concepts are as follows.

Definition 4. Let $f \in \mathbb{K}[x, y]^0_{\leq d}$ be a polynomial and let $\mathcal{P}$ be a configuration of n points in $\mathbb{K}^2$.

(1) A polynomial f is *essentially determined by* $\mathcal{P}$ when $[f] = \mathcal{L}_d(\mathcal{P})$.
(2) A polynomial f is *nonessentially determined by* $\mathcal{P}$ when $[f] \in \mathcal{L}_d(\mathcal{P})$ and $dim_{\mathbb{K}}(\mathcal{L}_d(\mathcal{P})) \geq 1$.
(3) $\mathcal{P}$ is *a forbidden configuration* (for polynomials of at most degree d) when $\mathcal{L}_d(\mathcal{P}) = \emptyset$.
(4) The *set of degree d essentially determined polynomials* is

$$\mathcal{E}_d \doteq \bigcup_{\mathcal{P}} \mathcal{L}_d(\mathcal{P}) \subset Proj\big(\mathbb{K}[x, y]^0_{\leq d}\big), \tag{9}$$

where the union is over all configurations $\{\mathcal{P}\}$ such that $dim_{\mathbb{K}}(\mathcal{L}_d(\mathcal{P})) = 0$.

Remark 3. (1) The strict set theoretical inclusion $\mathcal{P} \subsetneq \Sigma(f)$ can be satisfied for essentially determined polynomials f. For example, in the case of a product of three lines, one possesses a multiplicity 1, say $f = L_1^2 L_2$.
(2) The set of degree 3 essentially determined polynomials $\mathcal{E}_3$ is a union of projective spaces; however, it is not a projective space, as Proposition 1 will show.
(3) As expected, many of the projective classes in $\mathcal{E}_d$ arise from Morse polynomials. The converse is not true, as seen in Corollary 7.

3. On the Number of Required Singular Points

A novel aspect of the interpolation problem for singular points is its cardinality; the configurations having a certain number $\delta(d)$ of points determine open or closed Zariski sets in $\mathbb{K}[x, y]^0_{\leq d}$. As a key point, the dimension $\frac{1}{2}(d^3 + 3d)$ of $\mathbb{K}[x, y]^0_{\leq d}$ can be even or odd. Starting with degree $d = 4$, the pattern of these dimensions is 4-periodic; even, even, odd odd, ... See the third column in Table 1.

TABLE 1. Dimensions and values for the interpolation problem

Degree d	$\delta(d)$ Eq. (10)	Number of columns in ϕ $\frac{1}{2}(d^2+3d)$	Number of rows in ϕ $2\delta(d)$	Zariski topology of $\{\mathcal{P}\} \subset$ $Conf(\mathbb{K}^2, \delta(d))$
3	4	9	8	Closed
4	7	14	14	Open
5	10	20	20	Open
6	13	27	26	Closed
7	17	35	34	Closed

Proposition 1 (A dichotomy of the number $\delta(d)$ of required singular points). *Let $\mathbb{K}[x,y]^0_{\leq d}$ be the set of polynomials having at most degree $d \geq 3$, and let*

$$\delta(d) \doteq \begin{cases} \frac{1}{4}(d^2+3d-2) & \text{when} \quad \frac{1}{2}(d^2+3d) \text{ is odd,} \\ \frac{1}{4}(d^2+3d) & \text{when} \quad \frac{1}{2}(d^2+3d) \text{ is even.} \end{cases} \tag{10}$$

1. *If the dimension of $\mathbb{K}[x,y]^0_{\leq d}$ is odd, then the configurations $\{\mathcal{P}\}$ with $\delta(d)$ points and $dim_{\mathbb{K}}(\mathcal{L}_d(\mathcal{P})) \geq 0$ determine an open Zariski set in $Conf(\mathbb{K}^2, \delta(d))$.*
2. *If the dimension of $\mathbb{K}[x,y]^0_{\leq d}$ is even, then the configurations $\{\mathcal{P}\}$ with $\delta(d)$ points and $dim_{\mathbb{K}}(\mathcal{L}_d(\mathcal{P})) \geq 0$ determine a closed Zariski set in $Conf(\mathbb{K}^2, \delta(d))$.*

Proof. Let $f(x,y) \in \mathbb{K}[x,y]^0_{\leq d}$ be a polynomial as in (3). Assume that $\mathcal{P} = \{(x_\iota, y_\iota) \mid \iota = 1, \ldots, n\}$ is set theoretically contained in $\Sigma(f)$. A priori, each point $(x_\iota, y_\iota) \in \mathcal{P}$ will drop the dimension of the vector space $\mathbb{K}[x,y]^0_{\leq d}$ by 2. In the linear framework, this leads to a linear system of $2n$ equations:

$$f_x(x_\iota, y_\iota) = f_y(x_\iota, y_\iota) = 0, \quad \iota = 1, \ldots, n, \tag{11}$$

with $\{a_{\iota j}\}$ as variables. Following Bézout's theorem for a moment, let us consider a configuration with $n = (d-1)^2$ points. We have a linear map

$$\begin{aligned} \phi : \mathbb{K}[x,y]^0_{\leq d} \cong \mathbb{K}^{\frac{1}{2}(d^2+3d)} &\longrightarrow \mathbb{K}^{2(d-1)^2} \\ f \longmapsto &\big(f_x(x_1,y_1), \ldots, f_x(x_{(d-1)^2}, y_{(d-1)^2}), \\ &f_y(x_1,y_1), \ldots, f_y(x_{(d-1)^2}, y_{(d-1)^2})\big). \end{aligned} \tag{12}$$

The interpolation matrix ϕ depends on $\mathcal{P}$, and for notational simplicity we omit this dependence. The matrix ϕ has $\frac{1}{2}(d^2+3d)$ columns, $2(d-1)^2$ rows and a very particular shape because of the partial derivatives involved in it, see Eqs. (17), (33) for explicit examples with $d = 3$, 4.

For degree $d = 3$ and a configuration $\mathcal{P}$ of 4 points; however, then the rank of the matrix ϕ associated with $\mathcal{P}$ is 8 if and only if $dim_{\mathbb{K}}(\mathcal{L}_3(\mathcal{P})) = 0$. If we consider degree $d \geq 4$, then the number of rows of ϕ is bigger than the number of columns. We must reduce the number n of required points in the configurations $\mathcal{P}$, this $n < (d-1)^2$. The number $\delta(d)$ in (10) determines two possibilities.

Case 1 in (10). For $\mathcal{P}$ with $\delta(d) = \frac{1}{4}(d^2+3d-2)$ points, the interpolation matrix ϕ has $\frac{1}{2}(d^2+3d)$ odd columns and $\frac{1}{2}(d^2+3d-2)$ even rows, for example for $(d+1) = 3, 6, 7$. Moreover,

$$(\text{number of columns of } \phi) - 1 = (\text{number of rows of } \phi).$$

The dimension of the kernel of ϕ is at least one, thus $dim_K(\mathcal{L}_d(\mathcal{P})) \geq 0$. There are $\frac{1}{2}(d^2+3d)$ minors A_j from the matrix $\phi(x_1, y_1, \ldots, x_{\delta(d)}, y_{\delta(d)})$. The complement of the algebraic equations

$$\{\Pi_j det(A_j(x_1, y_1, \ldots, x_{\delta(d)}, y_{\delta(d)})) = 0\} \subset Conf(\mathbb{K}, \delta(d))$$

describes the set of configurations having $dim_K(\mathcal{L}_d(\mathcal{P})) = 0$, corresponding to the essentially determined polynomials. These configurations of $\delta(d)$ points in $Conf_{\delta(d)}(\mathbb{K}^2)$ determine an open Zariski and dense set, which is the second part of assertion (1).

Case 2 in (10). The dimension of $\mathbb{K}[x,y]^0_{\leq n}$ is even and we assume $\frac{1}{4}(d^2+3d) \in \mathbb{N}$ points in $\mathcal{P}$. The interpolation matrix ϕ is square of even size, and there are $\frac{1}{2}(d^2+3d)$ columns and rows; for example when $d = 4, 5$.
If we assume $\mathcal{P}$ such that $\{det(\phi(x_1, y_1, \ldots, x_{\delta(d)}, y_{\delta(d)})) \neq 0\}$, then the only vector in the $\{a_{\iota j}\}$ variables solving the linear system (11) is zero. The set of desired polynomials is empty.

The configuration with nonempty polynomials

$$\left\{\mathcal{P} \mid det(\phi(x_1, y_1, \ldots, x_{\delta(d)}, y_{\delta(d)})) \neq 0\right\} \subset Conf(\mathbb{K}, \delta(d))$$

determines an algebraic set. □

Recalling (4), the *expected projective dimension of* $\mathcal{L}_d(\mathcal{P})$, which is the linear system of polynomials of at most degree d with singular points at least in $\mathcal{P} \in Conf(\mathbb{K}^2, n)$, is

$$max\left\{\frac{1}{2}(d^2+3d-2) - 2n,\ -1\right\}.$$

In Sect. 5, we provide an alternative for studying the even dimension case in Proposition 1.

4. Essentially Determined Polynomials of Degree 3

4.1. A Linear System

In order to apply elementary methods, we introduce a very simple configuration of 4 points, depending essentially on the fourth one (x_4, y_4). Secondly, we must find a polynomial $f(x_4, y_4, x, y)$ with a singular point set containing the above simple configuration. Let

$$\mathscr{A} \doteq \left\{xy(x+y-1)(x+y)(x-1)(y-1) = 0\right\} \qquad (13)$$

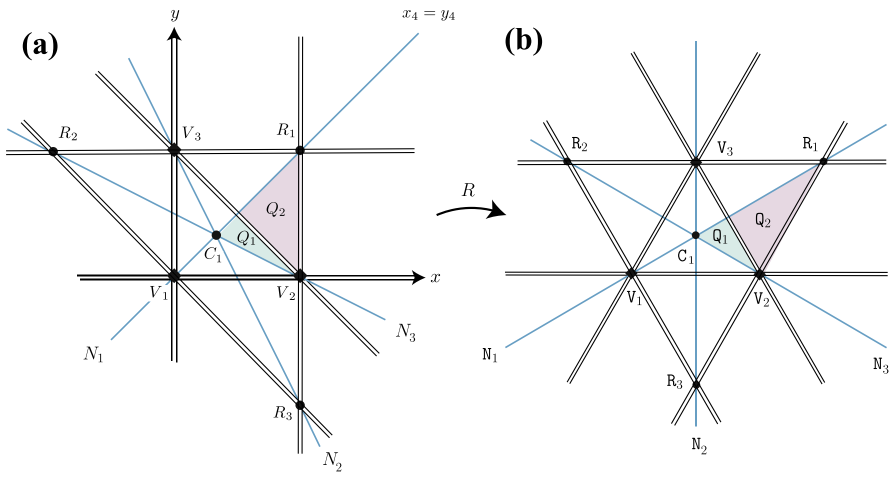

FIGURE 1. **a** The line arrangement $\mathscr{A}$ (of double lines) and the triangle $\triangle = \{V_1,\, V_2,\, V_3\}$. **b** The analogous objects under the linear map R, sending $\mathscr{A}$ to $\mathtt{A}$ and $\triangle$ to Δ

be an arrangement of six $\mathbb{K}$-lines; it is illustrated in Fig. 1a.

Lemma 1. *Let*

$$\begin{aligned}\mathcal{P} = \{&V_1 = (0,0),\, V_2 = (1,0),\, V_3 = (0,1),\, (x_4, y_4)\}\\ &\in Conf(\mathbb{K}^2, 4), \quad (x_4, y_4) \notin \mathscr{A},\end{aligned}$$

be a fourt point configuration. The polynomial

$$\begin{aligned} f(x_4, y_4, x, y) = {} & (y_4^2(y_4-1)(-1+2x_4+y_4)(2x^3-3x^2)\\ & + x_4^2(x_4-1)(-1+x_4+2y_4)(2y^3-3y^2)\\ & - 6x_4y_4(x_4-1)(y_4-1)(x^2y+xy^2-xy))a_6\\ & \in \mathbb{K}[x,y]_{=3}, \end{aligned} \tag{14}$$

for $a_6 \in \mathbb{K}^*$ is well defined and $\mathcal{P} = \Sigma\big(f(x_4, y_4, x, y)\big)$.

It will be convenient to write Eq. (14) as a map to the space of polynomials

$$f(x_4, y_4, \;,\;) : \mathbb{K}^2 \backslash \mathscr{A} \longrightarrow \mathbb{K}[x,y]_{=3}, \quad (x_4, y_4) \longmapsto f(x_4, y_4, x, y). \tag{15}$$

Proof. Let the following be a polynomial

$$\begin{aligned} f(x,y) = {} & a_1x^3 + a_2x^2y + a_3xy^2 + a_4y^3 + a_5x^2 + a_6xy + a_7y^2 + a_8x + a_9y\\ & \in \mathbb{K}[x,y]^0_{\leq 3}. \end{aligned} \tag{16}$$

For notational simplicity, only one subindex a_ι is considered. Let $\{(x_\iota, y_\iota) \mid \iota = 1, \dots, 4\}$ be an arbitrary configuration, and we require $(a_1, \dots, a_9)$ to be solutions of the linear system

$$\begin{pmatrix} & & & \vdots & & & & & \\ 3x_\iota^2 & 2x_\iota y_\iota & y_\iota^2 & 0 & 2x_\iota & y_\iota & 0 & 1 & 0 \\ 0 & x_\iota^2 & 2x_\iota y_\iota & 3y_\iota^2 & 0 & x_\iota & 2y_\iota & 0 & 1 \\ & & & \vdots & & & & & \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_9 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{17}$$

The interpolation matrix ϕ in (17) has 9 columns and 8 rows. The choice $\mathcal{P} = \{(0,0), (1,0), (0,1), (x_4, y_4)\}$ determines the linear system with only two equations

$$\begin{aligned} f_x(x,y) &= 3a_1x^2 + 2a_2xy + a_3y^2 + 2a_5x + a_6y + a_8 = 0, \\ f_y(x,y) &= a_2x^2 + 2a_3xy + 3a_4y^2 + a_6x + 2a_7y + a_9 = 0. \end{aligned}$$

Obviously, $(0,0) \in \mathcal{P}$ implies the vanishing of the linear part $f_x(0,0) = a_8 = 0 = a_9 = f_y(0,0)$. The linear conditions imposed by $(1,0)$ and $(0,1)$ are

$$\begin{cases} f_x(1,0) = 3a_1 + 2a_5 = 0 & a_1 = -\frac{2}{3}a_5, \\ f_y(1,0) = a_2 + a_6 = 0 & a_6 = -a_2, \end{cases}$$
$$\begin{cases} f_x(0,1) = a_3 + a_6 = 0 & a_6 = -a_3, \\ f_y(0,1) = 3a_4 + 2a_7 = 0 & a_4 = \frac{-2}{3}a_7. \end{cases}$$

The solution of this system

$$\begin{aligned} f(x_4, y_4, x, y) = a_6 \Bigg(& \frac{y_4(-1 + 2x_4 + y_4)}{3x_4(x_4 - 1)}x^3 - x^2y - xy^2 + \frac{x_4(-1 + x_4 + 2y_4)}{3y_4(y_4 - 1)}y^3 \\ & + \frac{y_4(1 - 2x_4 - y_4)}{2x_4(x_4 - 1)}x^2 + xy + \frac{x_4(1 - x_4 - 2y_4)}{2y_4(y_4 - 1)}y^2 \Bigg) \\ & \in \mathbb{K}[x,y]_{=3} \end{aligned} \tag{18}$$

has rational coefficients. If we normalize, we get Eq. (14). □

Corollary 1. *Let*

$$\mathcal{P}_1 = \{(0,0), (1,0), (0,1), R_1 \doteq (1,1)\} \in Conf(\mathbb{K}^2, 4)$$

be a four point configuration, and then $dim_{\mathbb{K}}(Proj(\mathcal{L}_3(\mathcal{P}_1))) = 1$.

We say that, $R_1 = (1,1)$ is a rhombus point; see Fig. 1.

Proof. By replacing in ϕ the points in $\mathcal{P}_1$, a direct calculation shows that the equivalent 9×8 matrix has a rank 7, where the null space of ϕ is given by the vectors $(0,0,0,-2/3,0,0,1,0,0)$ and $(-2/3,0,0,0,1,0,0,0,0)$. The linear combination of the corresponding polynomials leads to

$$f(\mathtt{a},\mathtt{d},x,y) = a\left(2x^3 - 3x^2\right) + d\left(2y^3 - 3y^2\right), \quad [a,\, d] \in \mathbb{KP}^1. \tag{19}$$

□

Remark 4. Behavior of the linear system at $\mathscr{A}$. Let $\mathcal{P} = \{(0,0), (1,0), (0,1), (x_4, y_4)\}$ be a configuration.
1. If (x_4, y_4) tends to be in a line

$$L_\alpha \subset \mathscr{A}\{\mathscr{A}\}\backslash\{R_1 = (1,1), R_2 = (-1,1), R_3 = (1,-1)\},$$

then the polynomial $f(x_4, y_4, x, y)$ in (17) has two lines of singular points in the respective pair of parallel $\mathbb{K}$-lines L_α, L_β, in the arrangement $\{\mathscr{A}(x,y) = 0\}$. Figure 4 provides a sketch up to affine transformations.
2. If (x_4, y_4) tends to be the vertex $(0,0) \in \triangle$, then the polynomial $f(x_4, y_4, x, y)$ in (16) becomes

$$f(0,0,x,y) = \frac{1}{3}(x^3 + y^3) - (x^2 y + xy^2) - \frac{1}{2}(x^2 + y^2) + xy.$$

As is expected, the curve $\{f(0,0,x,y) = 0\}$ has a cusp of multiplicity 2 at (0,0), see Fig. 4. The same is valid if (x_4, y_4) tends to be any other vertex $(1,0)$, $(0,1)$ of $\triangle$. Figure 4 shows $f(1,0,x,y)$, corresponding to $V_2 = (0,1)$ denoted as $\mathtt{V}_2$ in the figure.

Remark 5. Let $\mathcal{P}$ be any configuration of four points. Thus $\mathcal{L}_3(\mathcal{P}) \neq \emptyset$: there exists a nonconstant degree 3 polynomial with singular points at least in $\mathcal{P}$.

4.2. Affine Classification of Quadrilateral Configurations

We now study the independence of the previous results §4.1, with respect to the coordinate system.

A valuable tool in the study of polynomials of degree 3 is the action of the group of affine automorphisms of $\mathbb{K}^2$, say $\mathit{Aff}(\mathbb{K}^2)$. It is a six $\mathbb{K}$-dimensional Lie group. Let $\mathit{Aff}(\mathbb{K}^2)$ acts on the space of polynomials of degree d as

$$\mathit{Aff}(\mathbb{K}^2) \times \mathbb{K}[x,y]_{=d} \longrightarrow \mathbb{K}[x,y]_{=d}, \quad (T, f) \longmapsto f \circ T. \tag{20}$$

This action is rich enough and yet treatable. The affine group acts on configurations such as

$$\mathit{Aff}(\mathbb{K}^2) \times \mathit{Conf}(\mathbb{K}^2, n) \longrightarrow \mathit{Conf}(\mathbb{K}^2, n), \quad (T, \mathcal{P}) \longmapsto T^{-1}(\mathcal{P}). \tag{21}$$

Thus, if $f \in \mathbb{K}[x,y]_{=d}$ has n isolated singular points, say $\mathcal{P} \in \mathit{Conf}(\mathbb{K}^2, n)$, then $f \circ T$ has singular points at $T^{-1}(\mathcal{P})$. Hence, a useful associated object is the quotient space of quadrilateral configurations up to affine transformations.

Definition 5. The space of *generic quadrilateral configurations* is

$$\mathcal{Q} = \left\{ \mathcal{P}_0 = \{(x_{1\,0}, y_{1\,0}), \ldots, (x_{4\,0}, y_{4\,0})\} \;\middle|\; \begin{array}{l} \text{quadrilateral configurations} \\ \text{having no three collinear vertices} \\ \text{or determining two parallel lines} \end{array} \right\}$$
$$\subsetneq \mathit{Conf}(\mathbb{K}^2, 4). \tag{22}$$

Note that a quadrilateral configuration $\mathcal{P}_0$ does not have order. It determines several quadrilaterals, i.e., with a cyclic order in its vertices. Let

$$\triangle = \{V_1 = (0,0), V_2 = (1,0), V_3 = (0,1)\},$$
$$\Delta = \{\mathtt{V}_1 = (0,0), \mathtt{V}_2 = (1,0), \mathtt{V}_3 = (1/2, \sqrt{3}/2)\}$$

be two triangles. Consider a linear transformation $R \in GL(2,\mathbb{K})$ such that $R(\triangle) = \Delta$, $R(V_2) = \mathtt{V}_2$ and $R(V_3) = \mathtt{V}_3$, see Fig. 1. The affine symmetries of Δ,

$$Sym(3) = \{\sigma_\alpha \in \mathit{Aff}(\mathbb{K}^2) \mid \sigma_\alpha(\Delta) = \Delta,\ \alpha \in 1, \ldots, 6\}, \tag{23}$$

are isomorphic to the symmetric group of order 3: with three reflections $\sigma_2, \sigma_4, \sigma_6$ (with the axis in the lines $\mathtt{N}_1$, $\mathtt{N}_2$, $\mathtt{N}_3$) and their products $\sigma_1 = id, \sigma_3, \sigma_5$; see Fig. 1b. By abusing the notation, $Sym(3)$ also denotes the affine symmetries of $\triangle$.
Thus, we use three coordinate systems as follows. Let $\mathcal{P}_0 = \{(x_{1\,0}, y_{1\,0}), \ldots, (x_{4\,0}, y_{4\,0})\}$ as in (22). By using the affine action, we reduce $\mathcal{P}_0$ to $\{(x_4, y_4)\}$ or $\{(\mathtt{x}_4, \mathtt{y}_4)\}$. There are affine maps $T_j \in \mathit{Aff}(\mathbb{K}^2)$ as follows

$$\begin{array}{ccc} & \mathcal{P}_0 = \{(x_{1\,0}, y_{1\,0}), \ldots, (x_{4\,0}, y_{4\,0})\} & \\ \swarrow T_j & & R \circ T_j \searrow \\ \mathcal{P} = \{\underbrace{V_1, V_2, V_3}_{\triangle}, V_4 = (x_4, y_4)\} & \underset{R}{\overset{R^{-1}}{\leftrightarrows}} & \{\underbrace{\mathtt{V}_1, \mathtt{V}_2, \mathtt{V}_3}_{\Delta}, \mathtt{V}_4 = (\mathtt{x}_4, \mathtt{y}_4)\}. \end{array} \tag{24}$$

By notational simplicity, we also denote by $\mathcal{P}$ the configuration on the right side.

A key point is the number of affine maps $\{T_j\}$, depending on $\mathcal{P}_0$ to be computed in Corollary 2.

In accordance with Figs. 1 and 3, the triangles $\triangle$, Δ determine the points, line arrangements and regions below.

· Three *rhombus points* R_1, R_2, R_3 (resp. $\mathtt{R}_1$, $\mathtt{R}_2$, $\mathtt{R}_3$).
· Four *center points* C_1, C_2, C_3, C_4 (resp. $\mathtt{C}_1$, $\mathtt{C}_2$, $\mathtt{C}_3$, $\mathtt{C}_4$).
· A six line arrangement $\mathscr{A} = L_1 \cup \cdots \cup L_6$ (resp. $\mathtt{A} = \mathtt{L}_1 \cup \cdots \cup \mathtt{L}_6$) sketched as six double lines. $\mathscr{A}$ was already described in the introduction and in (13).
· A six line arrangement $\mathscr{B} = N_1 \cup \cdots \cup N_6$ (resp. $\mathtt{B} = \mathtt{N}_1 \cup \cdots \cup \mathtt{N}_6$) sketched as six blue lines, where N_1, N_2, N_3 are the axis of symmetry of $\triangle$. The lines N_1, N_2, N_3 are fixed under σ_1, σ_2, σ_3 in $\mathit{Aff}(\mathbb{R}^2)$ leaving invariant $\triangle$. The lines N_4, N_5, N_6 determine the triangle C_1, C_2, C_3.
Naturally, these points and arrangements correspond to under the map R in (24).
· In case $\mathbb{K} = \mathbb{R}$, we have two open connected regions in $\mathbb{R}^2$; convex quadrilateral configurations when $(x_4, y_4) \in Q_1$ (aquamarine) and nonconvex for Q_2 (magenta).

Analogously, we have $\mathtt{Q}_1 = R(Q_1)$ and $\mathtt{Q}_2 = R(Q_2)$. Moreover, the boundary of Q_1, Q_2 shall be described by using the isotropy of the respective configurations.

Lemma 2. *Let $\mathcal{P} \in \mathcal{Q}$ be a generic quadrilateral configuration in $\mathbb{K}^2$ as in (22). If the affine isotropy group of $\mathcal{P}$*

$$Aff(\mathbb{K}^2)_{\mathcal{P}} \doteq \{T \in Aff(\mathbb{K}^2) \mid T^{-1}(\mathcal{P}) = \mathcal{P}\}$$

is nontrivial, then it is isomorphic to one of the subgroups below.

Case 1. $Aff(\mathbb{K}^2)_{\mathcal{P}} \cong Sym(3)$ *if and only if up to affine transformation $\mathcal{P}$ has vertices in an equilateral triangle and its center.*

Case 2. $Aff(\mathbb{K}^2)_{\mathcal{P}} \cong \mathbb{Z}_2 \times \mathbb{Z}_4$ *if and only if up to affine transformation $\mathcal{P}$ is a rhombus; its vertices determine a pair of two parallel lines.*

Case 3. $Aff(\mathbb{K}^2)_{\mathcal{P}} \cong \mathbb{Z}_2$ *if and only if up to affine transformation*
(i) $\mathcal{P} = \{(0,0), (1,0), (1/2, \sqrt{3}/2), (\mathtt{x}_4, \mathtt{y}_4)\}$ *where $(\mathtt{x}_4, \mathtt{y}_4)$ is a fixed point under the reflection σ_2' with axis $\mathtt{N}_2$ in the isotropy of the triangle Δ and it is different of the center of Δ,*
(ii) Conversely, $\mathcal{P}$ is a trapezoid and its vertices determine two parallel lines, different from a rhombus. □

Corollary 2. *Let $\mathcal{P}_0$ be a generic quadrilateral configuration. The following assertions are equivalent.*

(1) *$\mathcal{P}_0$ has a trivial isotropy group $Aff(\mathbb{K}^2)_{\mathcal{P}_0} = id$.*
(2) *There are 24 affine transformations $R \circ T_j$ in (24), sending $\mathcal{P}_0$ to $\{(0,0)$, $(1,0)$, $(1/2, \sqrt{3}/2), (\mathtt{x}_4, \mathtt{y}_4)\}$.* □

Now we compute the orbit $\{R \circ T_j(\mathcal{P}_0)\}_{j=1}^{24}$ in terms of the fourth point in $\{(\mathtt{x}_4, \mathtt{y}_4)\} \in \mathbb{R}^2$. Certainly, the orbit has obvious elements given by the affine symmetries of Δ. The nonintuitive transformations between quadrilateral configurations $R \circ T_j(\mathcal{P}_0)$ are computed in the following result.

Lemma 3. *Let*

$$\{\underbrace{(0,0), (1,0), (1/2, \sqrt{3}/2),}_{\Delta} \mathtt{V}_4 = (\mathtt{x}_4, \mathtt{y}_4)\}$$

be a generic quadrilateral configuration and consider a vertex $\mathtt{V}_j \in \Delta$. There exist three $\mathbb{K}$-rational diffeomorphisms (different from the identity)

$$\mathtt{g}(\mathtt{V}_j, \quad) : \mathbb{K}^2 \backslash \mathtt{A} \longrightarrow \mathbb{K}^2 \backslash \mathtt{A}, \quad \mathtt{V}_4 \longmapsto \mathtt{g}(\mathtt{V}_j, \mathtt{V}_4), \quad j \in 1, 2, 3, \tag{25}$$

such that the quadrilateral configurations

$$\{(0,0), (1,0), (1/2, \sqrt{3}/2), \mathtt{V}_4\} \quad \textit{and} \quad \{(0,0), (1,0), (1/2, \sqrt{3}/2), \mathtt{g}(\mathtt{V}_j, \mathtt{V}_4)\}$$

are $Aff(\mathbb{K}^2)$-equivalent.

We note that $\mathtt{g}(\mathtt{V}_j, \quad)$ are nonaffine maps.

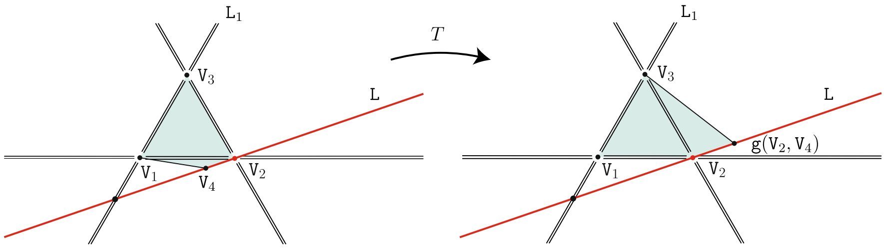

FIGURE 2. The point $\mathtt{g}(\mathtt{V}_2, \mathtt{V}_4)$ determines an affine map T between generic quadrilateral configurations

Proof. The choice of one vertex $\mathtt{V}_j \in \Delta$, determines an opposite side Δ. Without loss of generality, we consider the vertex $\mathtt{V}_2 = (1, 0) \in \Delta$ and $\mathtt{L}_1 = \{\mathtt{y} - \sqrt{3}\mathtt{x} = 0\} \subset \mathtt{A}$ is the opposite side; see Fig. 2.
For fixed $j = 2$, we consider $\mathtt{V}_4$. Let $\mathtt{L}$ be the line by $\mathtt{V}_4$ and $\mathtt{V}_2$; $\mathtt{L}$ is the red line in Fig. 2. We assume that $\mathtt{L}_1$ and $\mathtt{L}$ are nonparallel. There exists a unique $\mathbb{K}$-affine embedding

$$\mathtt{j} : \mathbb{K} \longrightarrow \mathbb{K}^2, \quad \text{with} \quad \mathtt{j}(\mathbb{K}) = \mathtt{L}, \quad \mathtt{j}(1) = \mathtt{V}_2, \quad \mathtt{j}(0) = \mathtt{L}_1 \cap \mathtt{L} \doteq \mathtt{0}.$$

The definition of the map in $\mathtt{L}$ is

$$\mathtt{g}(\mathtt{V}_2, \quad) : \mathtt{L} \backslash \mathtt{j}(0) \longrightarrow \mathtt{L} \backslash \mathtt{j}(0), \quad \mathtt{V}_4 \longmapsto \mathtt{j}\left(\frac{1}{\mathtt{j}^{-1}(\mathtt{x}_4, \mathtt{y}_4)}\right). \tag{26}$$

Secondly, we shall extend this definition for $\mathtt{V}_4 \in \mathbb{K}^2 \backslash \mathtt{L}_1$. In order to avoid cumbersome computations, the coordinates $\{(x, y)\}$ in (24) are more suitable. Assume $\mathcal{P} = \{(0, 0), (1, 0), (0, 1), (x_4, y_4)\}$, the vertex is $V_2 = (1, 0) \in \triangle$ and $L_1 = \{x_4 = 0\}$ is the opposite side. The analogous definition provides the rational map

$$\begin{aligned} g(V_2, \quad) : \mathbb{K}^2 \backslash \{x_4(x_4 - 1) = 0\} &\longrightarrow \mathbb{K}^2 \backslash \{x_4(x_4 - 1) = 0\}, \\ V_4 = (x_4, y_4) &\longmapsto \left(\frac{1}{x_4}, \frac{-y_4 + y_4 x_4}{x_4 - 1}\right). \end{aligned} \tag{27}$$

It enjoys the properties described below.

- $g(V_2, \quad)$ is a birational map of $\mathbb{K}^2$.
- $g^{-1}(V_2, \quad) = g(V_2, \quad)$, it is an involution.
- The point V_2 and the line $\{x = -1\}$ are fixed under $g(V_2, \quad)$.
- The poles of the map $g(V_2, \quad)$ are localized at $\{x = 0\}$ and $\{x - 1 = 0\} \backslash \{(0, 1)\}$. Thus, strictly speaking the map is a $\mathbb{K}$-analytic diffeomorphism on $\mathbb{K}^2 \backslash \{x(x - 1) = 0\}$. In the synthetic definition (26), $\mathtt{L}_1$ and $\mathtt{L}$ are nonparallel. This construction originates the pole of $g(V_2, \quad)$ at $\{x - 1 = 0\}$.
- A straightforward computations shows that the line arrangements $\mathscr{A}$ and $\mathscr{B}$ (double and blue lines in Fig. 3) are poles or remain invariants under $g_2(V_2, \quad)$.

In summary, we define (26) as

$$\mathtt{g}(\mathtt{V}_2, \quad) \doteq R \circ g(V_2, \quad) \circ R^{-1}.$$

Finally, given $\mathtt{V}_4$ and $\mathtt{g}(\mathtt{V}_2, \mathtt{V}_4)$, there exists a unique transformation $T \in Aff(\mathbb{K}^2)$, which leaves the line $\mathtt{L}_1$ fixed so that $T(\mathtt{V}_4) = \mathtt{g}(\mathtt{V}_2, \mathtt{V}_4)$; see Fig. 3. Under T, the quadrilateral configurations

$$\{(0,0),(1,0),(1/2,\sqrt{3}/2),\mathtt{V}_4\} \quad \text{and} \quad \{(0,0),(1,0),(1/2,\sqrt{3}/2),T(\mathtt{V}_4)\}$$

are affine equivalent.

The other vertices of the triangle Δ determine rational maps $\mathtt{g}(\mathtt{V}_1,)$, $\mathtt{g}(\mathtt{V}_3,)$, both enjoy analogous properties. □

Remark 6. Three blue lines in Fig. 3 correspond to the fixed points under the reflection symmetries $Sym(3)$ of Δ. By using (26), the complete configuration of six blue lines $N_1, \ldots, N_6$ is invariant under the three transformations $\mathtt{g}(\mathtt{V}_j, \quad)$. We leave this assertion for the reader.

Lemma 4. 1. *The quotient space of generic quadrilateral configurations up to affine transformations, given by*

$$\pi : \mathcal{Q} \longrightarrow \mathcal{Q}/Aff(\mathbb{K}^2), \quad \{(x_{1\,0}, y_{1\,0}), \ldots, (x_{4\,0}, y_{4\,0})\} \longmapsto [(\mathtt{x}_4, \mathtt{y}_4)], \tag{28}$$

is a $\mathbb{K}$-analytic surface $\mathtt{Q}$.

2. *For $\mathbb{K} = \mathbb{C}$, the quotient $\mathtt{Q}$ is a connected complex surface.*
3. *For $\mathbb{K} = \mathbb{R}$, the quotient has two connected components $\mathtt{Q} = \mathtt{Q}_1 \cup \mathtt{Q}_2$ and singular points with local models $\mathbb{K}^2/\mathbb{Z}_2$ or $\mathbb{K}^2/Sym(3)$.*

Some comments are in order. Figure 3 illustrates the fundamental domains for π over $\mathbb{K} = \mathbb{R}$. The double lines $\mathtt{A} = \mathtt{L}_1 \cup \cdots \cup \mathtt{L}_6$ in Figs. 1, 2, 3 and 4 correspond to forbidden positions for $(\mathtt{x}_4, \mathtt{y}_4)$. Moreover, $(\mathtt{x}_4, \mathtt{y}_4) \in \mathtt{Q}_1$ determines a nonconvex quadrilateral configuration; $(\mathtt{x}_4, \mathtt{y}_4) \in \mathtt{Q}_2$ determines a strictly convex quadrilateral configuration.

Proof. The set theoretical construction of the quotient is simple, and we describe its projection π in (28). Given $\mathcal{P}_0 \in \mathcal{Q}$, we apply an affine transformation $R \circ T_j$ in (24) sending it to

$$R \circ T_j(\mathcal{P}) = \{(0,0),(1,0),(1/2,\sqrt{3}/2),\mathtt{V}_4 = (\mathtt{x}_4, \mathtt{y}_4)\}.$$

Case 1. The isotropy is trivial $Aff(\mathbb{K}^2)_{\mathcal{P}} = id$. There are exactly 24 different choices for $R \circ T_j$, as in Lemma 2; we have that π has as a target $\mathbb{K}^2 = \{(\mathtt{x}_4, \mathtt{y}_4)\}$.

In order to describe its analytic properties, recall that the Klein four-group K is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. It is such that each element is self-inverse (composing it with itself produces the identity) and composing any two of the three nonidentity elements produces the third one; see [2, p. 87]. Moreover, the

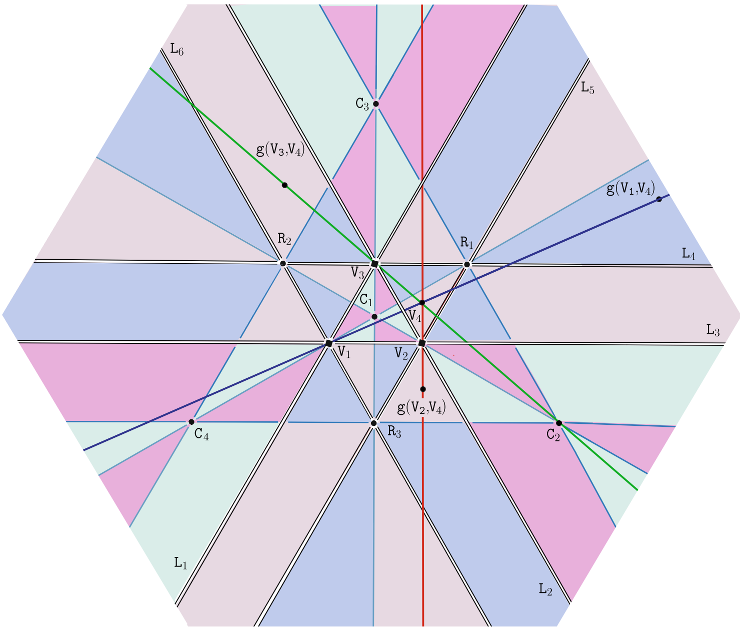

Figure 3. The plane $\mathbb{R}^2 \backslash \mathtt{A}$ with coordinates $\{\mathtt{x}_4, \mathtt{y}_4\}$ parametrizes the quadrilateral configurations $\{\mathtt{V}_2, \mathtt{V}_2, \mathtt{V}_3, \mathtt{V}_4 = (\mathtt{x}_4, \mathtt{y}_4)\}$. The pair tile $\mathtt{Q} = \mathtt{Q}_1 \cup \mathtt{Q}_2$ is a fundamental domain for the moduli space of quadrilateral configurations, up to $\mathit{Aff}(\mathbb{K}^2)$-equivalence. There are 24 copies of the fundamental region $\mathtt{Q}$. We colored $\mathtt{Q}_2$ and its copies with pink or blue (resp. $\mathtt{Q}_1$ and its copies aquamarine or magenta) tiles for strictly convex (resp. non convex) quadrilateral configurations

group $Sym(4)$ is of order 24, having a Klein four-group K as a proper normal subgroup; thus $Sym(3) = Sym(4)/K$. We recognize

$$K = \{id, \, \mathtt{g}(\mathtt{V}_j, \quad) \mid j \in 1, 2, 3\}$$

as the group in Lemma 3. Recall (23) and consider the homomorphism given by

$$\varphi : Sym(3) \longrightarrow Aut(K), \quad \sigma \longmapsto \sigma_\alpha^{-1} \circ \mathtt{g}(\mathtt{V}_j, \quad) \circ \sigma_\alpha(\mathtt{x}_4, \mathtt{y}_4).$$

The semidirect product of K and $Sym(3)$ determined by φ is $Sym(4) = K \rtimes_\varphi Sym(3)$, see [2, p. 133]. Hence, we have a representation of $Sym(4)$ in the birational transformations of $\mathbb{K}^2 \backslash \mathtt{A}$ and

$$\mathtt{Q} = \frac{\mathcal{Q}}{\mathit{Aff}(\mathbb{K}^2)} = \frac{\mathbb{K}^2 \backslash \mathtt{A}}{\mathtt{Sym(4)}} \tag{29}$$

is the quotient space. See [16] for a general theory of the quotients of complex manifolds under a discontinuous group of automorphisms. Assertion (1) is done.

For assertion (2), we assume $\mathbb{K} = \mathbb{C}$; note that $\mathbb{K}^2\backslash\mathtt{A}$ is a connected complex manifold. The local behavior of this complex quotient at the points with nontrivial isotropy $\mathbb{Z}_2$ at the lines $\mathtt{N}_1$, $\mathtt{N}_2$, $\mathtt{N}_3$ is known to be nonsingular (because of Chevalley [7], see also [12]). For $\mathtt{C}$ the isotropy is $Sym(3)$ and the same references describe the local structure of the quotient.

For assertion (3), we assume $\mathbb{K} = \mathbb{R}$, clearly the convexity or non convexity of a quadrilateral configurations are affine invariants, from where there are two connected components. At the points $\mathtt{C}, \dots, \mathtt{C}_4$ and lines $\mathtt{N}_1$, $\mathtt{N}_2$, $\mathtt{N}_3$ where the isotropy of the quadrilateral configurations is nontrivial, the quotient (29) has singularities; it is an orbifold. $\square$

As final step in the proof of Theorem 1, we consider the action on projective classes

$$\mathcal{A} : \mathit{Aff}(\mathbb{K}^2) \times \mathit{Proj}(\mathbb{K}[x, y]_{=3}) \longrightarrow \mathit{Proj}(\mathbb{K}[x, y]_{=3}), \quad (T, [f]) \longmapsto [f \circ T]. \tag{30}$$

This action provides an $\mathit{Aff}(\mathbb{K}^2)$-bundle structure on $\mathbb{K}[x, y]_{=3}$. Denote the *stabilizer* or *isotropy group* of $[f] \in \mathit{Proj}(\mathbb{K}[x, y]_{=3})$ by

$$\mathit{Aff}(\mathbb{K}^2)_{[f]} \doteq \{T \in \mathit{Aff}(\mathbb{K}^2) \mid f \circ T = \lambda f,\ \lambda \in \mathbb{K}^*\}.$$

Equations (15) and (24) provide bijective correspondence between the generic quadrilateral configuration in $(\mathtt{x}_4, \mathtt{y}_4) \in \mathbb{K}^2\backslash\mathtt{A}$ and projective classes of polynomials $[f(R^{-1}(\mathtt{x}_4, \mathtt{y}_4), x, y)]$. If $\mathcal{P} \in \mathtt{Q}$, then we verify that the isotropy of the quadrilateral configuration $\mathit{Aff}(\mathbb{K}^2)_{\mathcal{P}}$ is isomorphic to $\mathit{Aff}(\mathbb{K}^2)_{[f]}$. Thus, we have a section

$$f \circ R^{-1} : \mathbb{K}^2\backslash\{\mathtt{A}\} \longrightarrow \mathit{Proj}(\mathbb{K}[x, y]_{=3}), \quad (\mathtt{x}_4, \mathtt{y}_4) \longmapsto [f(R^{-1}(\mathtt{x}_4, \mathtt{y}_4), x, y)]$$

and a diagram

$$\begin{array}{ccc} & & \mathit{Proj}(\mathbb{K}[x, y]_{=3}) \\ & \nearrow^{[f(R^{-1}(\mathtt{x}_4, \mathtt{y}_4), x, y)]} & \downarrow \pi \\ \mathbb{K}^2\backslash\{\mathtt{A}\} & & \dfrac{\mathit{Proj}(\mathbb{K}[x, y]_{=3})}{\mathit{Aff}(\mathbb{K}^2)}, \end{array} \tag{31}$$

where π is the projection of classes from the action (30). The $\mathit{Aff}(\mathbb{K})$-orbit of a projective class $[f] \in \mathbb{K}[x, y]_{=3}$ is homeomorphic to $\mathit{Aff}(\mathbb{K}^2)/\mathit{Aff}(\mathbb{K}^2)_{[f]}$. Obviously, $\mathbb{K}[x, y]_{=3,id}$ is open and dense in $\mathbb{K}[x, y]_{=3}$.

The proof of assertion 1, Theorem 1 is done.

Remark 7. It is well known (as a seen for instance in [9, p. 53]) that if we consider

$$\mathbb{K}[x,y]_{=3,id} \doteq \{f \in \mathbb{K}[x,y]_{=3} \mid \mathit{Aff}(\mathbb{K}^2)_f = id\},$$

then the restricted action in $\mathbb{K}[x,y]_{id}$, determines a principal fiber $\mathit{Aff}(\mathbb{K}^2)$-bundle structure. In particular, the quotient $\mathbb{K}[x,y]_{=3,id}/\mathit{Aff}(\mathbb{K}^2)$ is a two dimensional $\mathbb{K}$-analytic manifold.

Remark 8. For $\mathbb{K} = \mathbb{R}$, the fundamental domain $\mathtt{Q}_1 \cup \mathtt{Q}_2$ determines the bifurcation diagram of the respective Hamiltonian vector fields, see Fig. 4. By construction, $\mathtt{Q}_1$ has two boundaries and one vertex $\mathtt{C}$ and $\mathtt{Q}_2$ has one boundary (without extreme points).

We summarize the results in Table 2.

Example 1. Relation to the classification of cubic plane curves. The Hesse pencil of cubic curves is

$$\{z^3 + x^3 + y^3 - 3\mu zxy = 0\}, \quad \text{resp. } \{x^3 + y^3 - 3\mu xy + 1 = 0\}, \quad \mu \in \mathbb{C}^*,$$

in the projective plane $\mathbb{CP}^2 = \{[z,x,y]\}$, resp. the affine plane; see [3]. The key property is that any nonsingular cubic plane is projectively equivalent to a member of the Hesse pencil. The singular points of the affine Hesse polynomial

$$f(\mu,x,y) = x^3 + y^3 - 3\mu xy + 1$$

determine a generic quadrilateral configuration

$$\big\{(0,0), (\mu,\mu), \big(-\zeta_1\mu,\, \zeta_2\mu\big), \big(\zeta_2\mu,\, -\zeta_1\mu\big)\big\} \subset \mathbb{C}^2 \backslash \mathbb{R}^2,$$

where $\{1, \zeta_2, \zeta_3\}$ are the cube roots of unity. In order to translate it to our language, up to the linear transformation $M_\mu : \mathbb{C}^2 \longrightarrow \mathbb{C}^2,\ (x,y) \longmapsto \big(\mu x - \zeta_2\mu y, \mu x + \zeta_3\mu y\big)$. The quadrilateral configuration changes to

$$\mathcal{P} = \{(0,0), (1,0), (0,1), (2\zeta_1\mu^2, \big(1+\zeta_2\big)\mu^2)\}.$$

By Theorem 1, the affine Hesse polynomial

$$f(\mu, \quad) \circ M(x,y) = \mu^3\left(2x^3 - 3x(-1+y)y - 3x^2(1+y) + y^2(-3+2y)\right) + 1$$

is essentially determined. Since these quadrilateral configurations are nonreal, they are different from those given in Fig. 4.

4.3. Nonessential Determined Polynomials of Degree 3

By completeness, we describe the polynomials arising from the configurations

$$\mathcal{P} = \{(0,0), (1,0), (0,1), (x_4,y_4)\} \in \mathit{Conf}(\mathbb{K}^2, 4),\ (x_4,y_4) \in \mathscr{A}.$$

Lemma 5. 1. *Let* $\mathcal{P} = \{(0,0), (1,0), (x_3,0), (x_4,y_4)\}$, *with* $x_3 \neq 0, 1$ *and* $y_4 \neq 0$, *then* $\dim_{\mathbb{K}}(\mathit{Proj}(\mathcal{L}_3(\mathcal{P}))) = 0$.

2. *Let* $\mathcal{P} = \{(0,0), (1,0), (x_3,0), (x_4,0)\}$ *be a configuration, then* $\dim_{\mathbb{K}}$ $(\mathit{Proj}(\mathcal{L}_3(\mathcal{P}))) = 2$.

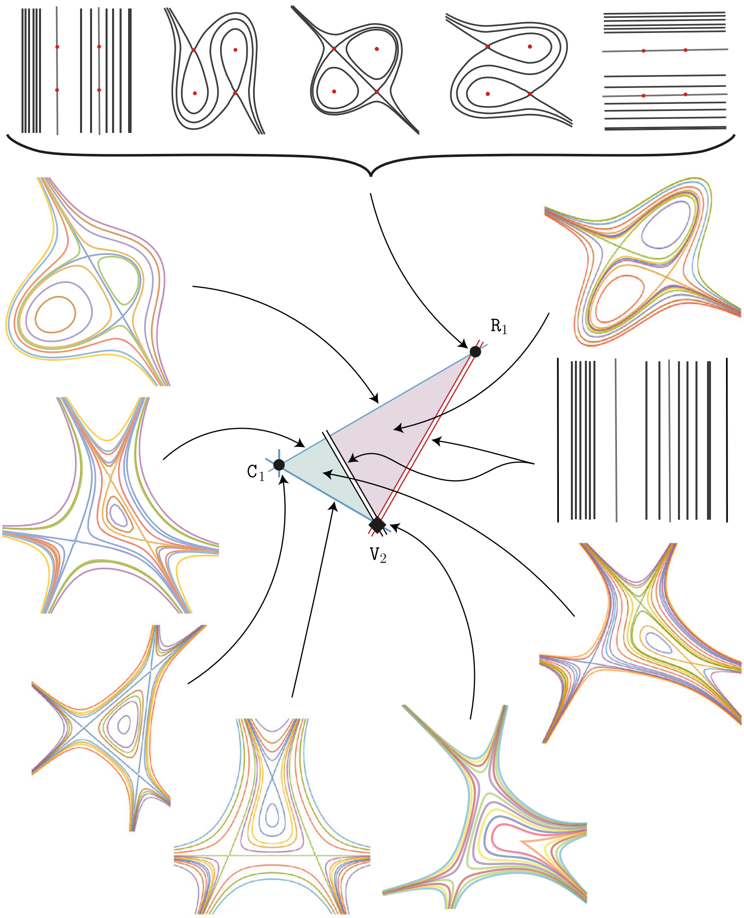

FIGURE 4. Bifurcation diagram of the real Hamiltonian vector fields $X_{f\circ R^{-1}}$ according to the position of four singular points in the fundamental region Q. At the rhombus point R_1, the configuration of four points $\mathcal{P} = \{(0,0),(1,0),(0,1),R_1 = (1,1)\} \subset \Sigma(f_\theta)$ is common; see Example 6. The upper row illustrates the topology of $\{f_\theta(x,y) \mid \theta \in [0,\pi/2]\}$. A saddle connection bifurcation occurs for $\theta = \pi/4$. See https://github.com/alexander-arredondo/Mathematica-code-for-Essentially-determined-polynomials-of-degree-3/commit/e6a08f9a20da7b23d7a72beff8290af3a23260dc for a code animation in Mathematica of this situation

TABLE 2. Dimension, generators and isotropy for $\mathcal{L}_3(\mathcal{P})$, where $\mathcal{P}$ is a configuration with 4 points (3 simple points and a double one in the last row)

Configuration $\mathcal{P}\,\{(0,0),(1,0),(0,1),(x_4,y_4)\}$	Cardinality of $\Sigma(f)$	$dim_{\mathbb{K}}(\mathcal{L}_3(\mathcal{P}))$	Generators of $\mathcal{L}_3(\mathcal{P})$	Isotropy $Aff(\mathbb{K}^2)_{\mathcal{P}}$
$(x_4,y_4)\in Q$	4	0	Equation (14)	id
$(x_4,y_4)=(1/3,1/3)=C_1$	4	0	$xy(y+x-1)$	$Sym(3)$
(x_4,x_4), $x_4\neq 0, 1$	4	0	Equation (14)	$\mathbb{Z}_2$
$(x_4,y_4)=(1,1)=R_1$	4	1	$2y^3-3y^2$, $2x^3-3x^2$ Eq. (19)	$\mathbb{Z}_2\times\mathbb{Z}_4$
$(1,y_4)\in L_5,\ y_4\neq\pm 1$	∞	0	$2x^3-3x^2$	$Aff(\mathbb{K})$
$\{(0,0),(1,0),(x_3,0),(x_4,0)\}$	∞	2	y^3, xy^2, y^2 Lemma 5.2	$\mathbb{Z}_2$
$\{(0,0),(1,0),(0,1),(0,0)\}$	3, 4 or ∞	2	$x^3-3x^2, y^3-3y^2, x^2y+xy^2-xy$	$\mathbb{Z}_2$

Proof. In assertion (1), up to an affine transformation we can assume $y_4 = 1$. The corresponding cubic polynomial takes the form $f(x, y) = a_4\left(2y^3 - 3y^2\right)$, where $a_4 \in \mathbb{K}^*$.

For assertion (2), we search for polynomials $f(x, y) \in \mathbb{K}[x, y]^0_{\leq 3}$ with at least 4 affine collinear singular points. The matrix of Eq. (17) results in the cubic polynomials

$$f(x, y) = a_3xy^2 + a_4y^3 + a_7y^2 = y^2(a_3x + a_4y + a_7), \quad [a_3, a_4, a_7] \in \mathbb{KP}^2,$$

with a line of singular points in $\{y = 0\}$. □

Example 2. The elementary methods provide an insight in the case of a double point in $\Sigma(f)$. Let $\mathcal{P}_2 = \{(0, 0), (1, 0), (0, 1), (0, 0)\}$ be such a configuration. A basis for $\mathcal{L}_3(\mathcal{P}_2)$ is

$$x^3 - 3x^2, \quad y^3 - 3y^2, \quad x^2y + xy^2 - xy.$$

The first and second polynomials have lines of singularities, while the third one has four isolated critical points. The family of polynomials is

$$f(a_1, a_2, a_4, x, y) = a_1(x^3 - 3x^2) + a_2(x^2y + xy^2 - xy) + a_4(y^3 - 3y^2),$$
$$[a_1, a_2, a_4] \in \mathbb{KP}^2.$$

As is expected, for values $\left\{(a_1, a_2, a_4 = a_2^2/9a_1\right\}$ the 2-dimensional family $f(a_1, a_2, a_4, x, y)$ determines polynomials with three isolated singular points, one of them of multiplicity 2, see Fig. 4.

5. Degree 4 Polynomials

Let

$$f(x, y) = a_1x^4 + a_2x^3y + \cdots + a_{13}x + a_{14}y \in \mathbb{K}[x, y]^0_{\leq 4} \tag{32}$$

be a polynomial as in (3). Here by notational simplicity, we have avoided the double subindex, and let $\mathcal{P} = \{(x_\iota, y_\iota) \mid \iota \in 1, \ldots, 7\}$ be a configuration of seven points. The associated linear system for Eq. (32) is

$$\begin{pmatrix} & & & & & \vdots & & & & & & & & \\ 4x_\iota^3 & 3x_\iota^2y_\iota^2 & 2x_\iota y_\iota^2 & y_\iota^3 & 0 & 3x_\iota^3 & 2x_\iota y_\iota & y_\iota^2 & 0 & 2x_\iota & y_\iota & 0 & 1 & 0 \\ 0 & x_\iota^3 & 2x_\iota^2y_\iota & 3x_\iota y_\iota^2 & 4y_\iota^3 & 0 & x_\iota^2 & 2x_\iota y_\iota & 3y_\iota^2 & 0 & x_\iota & 2y_\iota & 0 & 1 \\ & & & & & \vdots & & & & & & & & \end{pmatrix}$$
$$\begin{pmatrix} a_1 \\ \vdots \\ a_{14} \end{pmatrix} = \bar{0}, \quad \iota = 1, \ldots, 7. \tag{33}$$

The interpolation matrix ϕ, Eq. (33), is square. Hence, for an open and dense set of configurations $\{\mathcal{P}\} \subset Conf(\mathbb{K}^2, 7)$ such that $\{det(\phi) = 0\}$, the resulting space of polynomials of degree 4 with having these $\mathcal{P}$ as critical points is empty. In order to overcome this situation, we introduce the following concept.

Definition 6. Assume $\mathbb{K}[x,y]^0_{\leq d}$ with an even dimension and $\delta(d) = \frac{1}{4}\left(d^2+3d\right)$ as in (10). Given a configuration $\mathcal{P}_0 \in Conf(\mathbb{K}^2, \delta(d)-1)$, consider a point $(x,y) \in \mathbb{K}^2$ and

$$\mathcal{P}_1 = \Big\{ \underbrace{(x_1,y_1),\ldots,(x_{\delta(d)-1},y_{\delta(d)-1})}_{\mathcal{P}_0}, (x,y) \Big\} \in Conf(\mathbb{K}^2, \delta(d)).$$

The *interpolation algebraic curve of* $\mathcal{P}_0$ is

$$\mathcal{I} = \left\{ det\big(\phi(x_1,y_1,\ldots,x_{\delta(d)-1},y_{\delta(d)-1},x,y)\big) = 0 \right\} \text{ in } \mathbb{K}^2.$$

Obviously, $\mathcal{I}$ depends on $\mathcal{P}_0$, by notational simplicity we omit this dependence. Thus, we have a map

$$\mathcal{P}_0 = \{(x_1,y_1),\ldots,(x_{\delta(d)-1},y_{\delta(d)-1})\} \longmapsto \mathcal{I}.$$

Proposition 2. *Assume* $\mathbb{K}[x,y]^0_{\leq d}$ *with even dimension.*

1. *The interpolation curve* $\mathcal{I}$ *of* $\mathcal{P}_0$ *describes the position of the* $\delta(d)$*-th point such that* $dim_{\mathbb{K}}(\mathcal{L}_d(\mathcal{P}_1)) \geq 0$.
2. *There exists a Zariski open set* $\{\mathcal{P}_0\} \subset Conf(\mathbb{K}^2, \delta(d)-1)$ *such that the associated* $\{\mathcal{I}\}$ *are algebraic curves of degree* $2d-2$ *in* $\mathbb{K}^2$.

Proof. For assertion (2), we consider the degree d polynomial

$$f(x,y) = a_1 x^d + a_2 x^{d-1} y + \cdots + a_{\delta(d)-1} x + a_{\delta(d)} y.$$

After fixing the configuration $\mathcal{P}_0$, the associated linear system only has free variables x, y, and the linear system is as follows

$$\begin{pmatrix} & & & & \vdots & & & & & & & \\ (d)x^{d-1} & (d-1)x^{d-2}y & (d-2)x^{d-3}y^2 & \cdots 0 & (d-1)x^{d-2} & \cdots y^2 & 0 & 2x & y & 0 & 1 & 0 \\ 0 & x^{d-1} & 2x^{d-2}y & \cdots 4y^3 & 0 & x^2 & 2xy & 3y^2 & 0 & x & 2y & 0 \;\; 1 \\ & & & & \vdots & & & & & & & \end{pmatrix}$$

$$\begin{pmatrix} a_1 \\ \vdots \\ a_{\delta(d)} \end{pmatrix} = \bar{0}. \tag{34}$$

The determinant of this matrix has x^{2d-2} as a higher degree monomial, and we are done. □

We describe some interpolation curves $\mathcal{I}$.

Example 3. Let $f \in \mathbb{K}[x,y]^0_{\leq 4}$ be a polynomial having of degree 4 and let $\mathcal{P}_0 = \{(x_\iota, y_\iota) \mid \iota \in 1,\ldots,6\}$ be a fixed configuration of six different singular points of f.

1. If three points of $\mathcal{P}_0$ are in a line $\{x=0\}$ and two points are in $\{x=1\}$, then the interpolation curve $\mathcal{I}$, of $\mathcal{P}_0$, is given by

$$\mathcal{I}(x,y) = \left(-1152 y_4^2 y_5^2 (y_4-1)^2 x_6 (x_6-1)\right) x(x-1)(x-x_6) g(x,y). \tag{35}$$

The $\mathcal{I}$ is reducible and singular, it is the product of three parallel lines and a polynomial $g(x, y)$ that pass through the six points in $\mathcal{P}_0$.

2. Let $\mathcal{P}_0 = \{(x_\iota, y_\iota) \mid \iota \in 1, \dots, 6\}$ be any configuration of six points in the grid of nine points

$$\mathcal{G} = \{x(x-1)(x-c_1) = 0\} \cap \{y(y-1)(y-c_2) = 0\}, \quad \text{where} \quad c_1,\, c_2 \notin \{0, 1\}.$$

Therefore, the interpolation curve $\mathcal{I}$, associated with the seventh point (x_7, y_7), is the product of the six lines defining $\mathcal{G}$.

3. Let $\mathcal{P} = \{(x_\iota, y_\iota) \mid \iota \in 1, \dots, 6\}$ be a configuration of six singular points of f. If the six points are distributed in a conic Q, then the interpolation curve $\mathcal{I}$, associated to the seventh point (x_7, y_7), contains the conic, which is $\mathcal{I} = Qg$ for some $g \in \mathbb{K}[x, y]^0_{\leq 4}$.

A complete study of the interpolation curves $\mathcal{I}$ arising from configurations of six points is the goal of a future project.

6. Polynomial Vector Fields with $(d-1)^2$ Singularities

Now we will consider some special configurations of $(d-1)^2 \geq 4$ points.

Definition 7. Let $\{F(x, y) = 0\}$ and $\{G(x, y) = 0\}$ be two algebraic curves in $\mathbb{K}^2$, both of degree $d - 1$ (≥ 2). We assume that they have transversal intersections in exactly $(d-1)^2$ affine points; therefore

$$\mathcal{P}_{ci} = \{F(x, y) = 0\} \cap \{G(x, y) = 0\} \in Conf(\mathbb{K}^2, (d-1)^2) \tag{36}$$

is a *complete intersection configuration*. The associated *pencil of curves* is

$$\left\{\mu F(x, y) + \nu G(x, y) = 0 \mid [\mu, \nu] \in \mathbb{KP}^1\right\}. \tag{37}$$

$\mathcal{P}_{ci}$ is the *base locus* of the pencil of curves.

Corollary 3. *An ordered pair of polynomial functions from (37), not just curves, determines a $SL(2, \mathbb{K})$-pencil of polynomial vector fields*

$$\mathfrak{F}(\mathcal{P}_{ci}) = \left\{ X_{\mathtt{M}} = -\big(\mathtt{c}F(x, y) + \mathtt{d}G(x, y)\big)\frac{\partial}{\partial x} + \big(\mathtt{a}F(x, y) + \mathtt{b}G(x, y)\big)\frac{\partial}{\partial y} \;\Big|\; \mathtt{M} = \begin{pmatrix} -\mathtt{c} & -\mathtt{d} \\ \mathtt{a} & \mathtt{b} \end{pmatrix} \in SL(2, \mathbb{K}) \right\} \tag{38}$$

Each vector field $X_{\mathtt{M}}$ has singularities of multiplicity 1 at $\mathcal{P}_{ci}$. □

Lemma 6. *Let $\mathcal{U}_d \subseteq \mathfrak{X}(\mathbb{K}^2)_{\leq d-1}$ be the open and dense set of polynomial vector fields of degree $d-1$, with exactly $(d-1)^2$ singular points in $\mathcal{P}_{ci} \subset Conf(\mathbb{K}^2, (d-$*

$1)^2$). *Assume that* $\mathcal{P}_{ci}$ *has a trivial isotropy group in* $Aff(\mathbb{K}^2)$. *In* $\mathcal{U}_d$ *there exists an analytic* $SL(2,\mathbb{K})$*-bundle structure as follows*

$$\begin{array}{ccc} SL(2,\mathbb{K}) & \longrightarrow & \mathcal{U}_d \\ & & \Big\downarrow \pi \\ & & \dfrac{\mathcal{U}_d}{SL(2,\mathbb{K})} \subseteq Conf(\mathbb{K}^2,(d-1)^2). \end{array} \tag{39}$$

Proof. We want to show that a polynomial vector field $X \in \mathfrak{X}(\mathbb{K}^2)_{\leq d-1}$ has $(d-1)^2$ singular points exactly at $\mathcal{P}_{ci}$ as in (36) if and only if it is of the shape $X_{\mathtt{M}}$ in (38).

($\Rightarrow$) Let $X = A(x,y)\frac{\partial}{\partial x} + B(x,y)\frac{\partial}{\partial y}$ be a vector field in $\mathfrak{X}(\mathbb{K}^2)_{\leq d-1}$. The curve $\mathcal{C}_A \doteq \{A(x,y)=0\}$ has at most degree $d-1$ and would contain $\mathcal{P}_{ci}$. An open set there exists of values $\{[\mu,\nu]\} \subset \mathbb{KP}^1$ such that for each value the respective curve $\{\mu F + \nu G = 0\}$ in the pencil (37) intersects in a transversal way $\mathcal{C}_A$ at every point of $\mathcal{P}_{ci}$. By Bézout's theorem, the degree of $\mathcal{C}$ is exactly $d-1$. For any point $p \in \mathcal{C}_A \backslash \mathcal{P}_{ci} \subset \mathbb{K}^2$, there exists a value, say $[-\mathtt{c},-\mathtt{d}]$ in (37), such that its respective curve satisfies $\mathcal{C}_{-\mathtt{c}-\mathtt{d}} \cap \mathcal{C}_A \supset \widehat{\mathcal{P}} \cup \{p\}$. Hence (again by Bézout's theorem), both curves coincide as sets and $A = -\mathtt{c}F - \mathtt{d}G$ as polynomials. □

Thus, each configuration $\mathcal{P}_{ci}$ has an associated fiber $\{X_{\mathtt{M}} \mid \mathtt{M} \in SL(2,\mathbb{K})\} \subset \mathcal{U}_d$ in (39), which is a family of not necessarily Hamiltonian vector fields. A further goal is the study of the intersection

$$\{X_{\mathtt{M}} \mid \mathtt{M} \in SL(2,\mathbb{K})\} \cap Ham(\mathbb{K}^2)_{\leq d}.$$

Corollary 4. *A jump phenomena. Let* $\mathcal{P} = \{(0,0),(1,0),(1/2,\sqrt{3}/2),(x_4,y_4)\}$ *be a configuration leading to a family of vector fields* $\mathfrak{F}(\mathcal{P}) = \{X_{\mathtt{m}} \mid \mathtt{m} \in SL(2,\mathbb{K})\}$ *as in (38).*

(1) *If* $(x_4,y_4) \in \mathbb{K}^2 \backslash \mathtt{A}$, *then there exists one projective class in* $\mathfrak{F}(\mathcal{P}) \cap Ham(\mathbb{K}^2)_{\leq 2}$.
(2) *If* $(x_4,y_4) = \mathtt{R}_1$, $\mathtt{R}_2$ *or* $\mathtt{R}_3$, *then there exists a* $\mathbb{KP}^1$*-family of Hamiltonian vector fields* $\mathfrak{F}(\mathcal{P}) \cap Ham(\mathbb{K}^2)_{\leq 2}$. □

Example 4. A family $\{X_{\mathtt{M}} \mid \mathtt{M} \in SL(2,\mathbb{K})\}$ *exists in* (39) *with* $(d-1)^2 \geq 4$ *points as a base locus and such that its Hamiltonian vector fields* $Ham(\mathbb{K}^2)_{\leq d-1} = [f]$ *determine one projective class.*

Consider two algebraic curves such that

$$\mathcal{P}_{ci} = \{\underbrace{y - \mu\Pi_{\iota=1}^{d}(x - x_\iota) = 0}_{F(x,y)=0}\} \cap \{\underbrace{x - \nu\Pi_{j=1}^{d}(y - y_j) = 0}_{G(x,y)=0}\}, \quad d \geq 3$$

has exactly $(d-1)^2 \geq 4$ points.

It follows that the associated 1-form $\omega_{\mathtt{m}}$ is exact if and only if $\mathtt{m} = \begin{pmatrix} \mathtt{a} & 0 \\ 0 & \mathtt{a} \end{pmatrix}$.
In fact, suppose $f(x,y)$ such that $\omega_{\mathtt{m}} = df$, then

$$\mathtt{a}F(x,y) + \mathtt{b}G(x,y) = f_x \quad \text{and} \quad \mathtt{c}F(x,y) + \mathtt{d}G(x,y) = f_y.$$

As $f_{xy} = f_{yx}$, then $\mathtt{a} - \mathtt{b}\frac{\partial}{\partial y}\Pi_{j=1}^{d}(y - y_j) = -\mathtt{c}\frac{\partial}{\partial x}\Pi_{\iota=1}^{d}(x - x_\iota) + \mathtt{d}$, so $\mathtt{a} = \mathtt{d}$ and $\mathtt{b} = \mathtt{c} = 0$.
By assuming $\omega_{\mathtt{m}}$ is exact and defining $f_{\mathtt{m}}(x,y) = \int^{(x,y)} \omega_{\mathtt{m}}$, we conclude that

$$\mathfrak{F}(\mathcal{P}_{ci}) \cap Ham(\mathbb{K}^2)_{\leq d-1} = \mathcal{L}_d(\mathcal{P}_{ci}) = [f_{\mathtt{m}}] \quad \text{and} \quad dim_{\mathbb{K}}(\mathcal{L}_d(\mathcal{P}_{ci})) = 0. \tag{40}$$

Example 5. A fiber $\{X_{\mathtt{M}} \mid \mathtt{M} \in SL(2,\mathbb{K})\}$ *as in* (39), *with* $(d-1)^2 \geq 9$ *points as a base locus satisfying that*

$$\{X_{\mathtt{M}} \mid \mathtt{M} \in SL(2,\mathbb{K})\} \cap Ham(\mathbb{K}^2)_{=d} = \emptyset.$$

Consider two hyperelliptic curves such that

$$\widehat{\mathcal{P}} = \{F(x,y) = y^2 - \mu\Pi_{\iota=1}^{d}(x - x_\iota) = 0\} \cap \{G(x,y) = x^2 - \nu\Pi_{j=1}^{d}(y - y_j) = 0\}$$

has exactly $(d-1)^2 \geq 9$ points. It follows that $\omega_{\mathtt{m}}$ is nonexact for all $\mathtt{m} = \begin{pmatrix} \mathtt{a} & \mathtt{b} \\ \mathtt{c} & \mathtt{d} \end{pmatrix}$. We conclude that

$$\mathcal{L}_d(\widehat{\mathcal{P}}) = \emptyset \quad \text{and} \quad dim_{\mathbb{K}}(\mathcal{L}_d(\widehat{\mathcal{P}})) = -1. \tag{41}$$

In fact, if we suppose $f(x,y)$ such that $\omega_m = df$, then $2\mathtt{a}y - \mathtt{b}\frac{\partial}{\partial y}\Pi_{j=1}^{d}(y - y_j) = -\mathtt{c}\frac{\partial}{\partial x}\Pi_{\iota=1}^{d}(x - x_\iota) + 2\mathtt{d}x$, so $\mathtt{a} = \mathtt{b} = \mathtt{c} = \mathtt{d} = 0$.

Corollary 5. *There exists a fiber* $\mathfrak{F}$ *as in* (39) *having* d^2 *points as a base locus and*

$$\mathfrak{F}(\widehat{\mathcal{P}}) \cap Ham(\mathbb{K}^2)_{=d} = \mathbb{KP}^1.$$

Moreover, $\mathbb{KP}^1$ *minus a finite set determines Morse polynomials.*

The above result uses the following very particular configurations.

Definition 8. *A grid of* $(d-1)^2$ *points* $\mathcal{G}$ is determined by two sets of $d-1$ parallel lines where one set is transverse to the other: up to affine transformation

$$\mathcal{G} = \{F(x,y) = \Pi_{j=1}^{d-1}(y - y_j) = 0\} \cap \{G(x,y) = \Pi_{\iota=1}^{d-1}(x - x_\iota) = 0\}$$

with exactly $(d-1)^2 \geq 4$ points; it is a complete intersection.

Proof of the Corollary. The family $X_{\mathtt{M}}$ with a grid of $(d-1)^2$ points is Hamiltonian if and only if

$$\mathtt{M} \in \left\{ \begin{pmatrix} 0 & -\mathtt{d} \\ \mathtt{a} & 0 \end{pmatrix} \right\} \cong \mathbb{K}^2 \subset SL(2,\mathbb{K}).$$

In fact, $\omega_{\mathtt{m}} = (\mathtt{a}F(x) + \mathtt{b}G(y))dx + (\mathtt{c}F(x) + \mathtt{d}G(y))dy = 0$ is exact if and only if $\mathtt{b}G(y)_y = \mathtt{c}F(x)_x$. The equality holds only for $\mathtt{b} = \mathtt{c} = 0$.
The respective vector subspace of polynomials

$$\left\{ f(\mathtt{a},\mathtt{d},x,y) = \mathtt{a}\int^{(x,y)} \Pi_{\iota=1}^{d}(x - x_\iota)dx + \mathtt{d}\int^{(x,y)} \Pi_{j=1}^{d}(y - y_j)dy \;\Big|\; (\mathtt{a},\mathtt{d}) \in \mathbb{K}^2 \backslash \{\bar{0}\} \right\} \subset \mathbb{K}[x,y]^0_{\leq d} \tag{42}$$

shows that

$$\mathcal{L}_d(\mathcal{P}) \supset \{[f(\mathtt{a},\mathtt{d},x,y)]\} \quad \text{and} \quad dim_{\mathbb{K}}(\mathcal{L}_d(\mathcal{P})) = 1. \tag{43}$$

For $(\mathtt{a},\mathtt{d}) \neq (\mathtt{a},0), (0,\mathtt{d})$, each polynomial $f(\mathtt{a},\mathtt{d},x,y) \in \mathbb{K}[x,y]^0_{\leq d}$ in (42) has $(d-1)^2$ Morse singular points. In fact, at each point $p \in \mathcal{P}$, a very simple observation with the Taylor series shows that $f(\mathtt{a},\mathtt{d},x,y) = \widetilde{\mathtt{a}}x^2 + \widetilde{\mathtt{b}}y^2 + \mathcal{O}_3(x,y)$, where $\widetilde{\mathtt{a}}\widetilde{\mathtt{b}} \neq 0$.

On the other hand, for $(\mathtt{a},\mathtt{d}) = (\mathtt{a},0), (0,\mathtt{d})$ the polynomial $f(\mathtt{a},\mathtt{d},x,y)$ has lines of singular points in $\{P(x,y) = 0\}$ or $\{Q(x,y) = 0\}$. □

Example 6. Real rotated Hamiltonian vector fields for the grid of 4 *points.* Let $\mathcal{G} = \{(0,0),(1,0),(0,1), R = (1,1)\}$ be a grid, and its space of polynomials is

$$f(\mathtt{a},\mathtt{d},x,y) = \mathtt{a}\Big(\frac{x^3}{3} - \frac{x^2}{2}\Big) + \mathtt{d}\Big(\frac{y^3}{3} - \frac{y^2}{2}\Big).$$

In particular for $\mathbb{K} = \mathbb{R}$, we consider the family

$$R_\theta = \left\{ f_\theta(x,y) = \cos(\theta)\left(\frac{x^3}{3} - \frac{x^2}{2}\right) + \sin(\theta)\left(\frac{y^3}{3} - \frac{y^2}{2}\right) \;\Big|\; \theta \in [0, 2\pi] \right\}$$

of polynomials in (42). They originate from a family of rotated vector fields, see Fig. 4. The algebraic curve $\{f_\theta(x,y) + c = 0\}$ is reducible for $\theta = \pi/4$ and $c = 1/6$. In this case we obtain

$$\{(x + y - 1)(2y^2 - 2xy + 2x^2 - y - x - 1) = 0\}.$$

The following family of vector fields is related to the results in Ramírez [17, Sect. 5] (non-generic Hamiltonian vector fields, theorem 5); see Fig. 4, upper row.

Corollary 6. *The* 1*-dimensional holomorphic family of Hamiltonian vector fields of the polynomials*

$$\left\{ f(\mathtt{a},\mathtt{d},x,y) = \mathtt{a}\Big(\frac{x^3}{3} - \frac{x^2}{2}\Big) + \mathtt{d}\Big(\frac{y^3}{3} - \frac{y^2}{2}\Big) \;\Big|\; \mathtt{ad} = 1 \right\}$$

has singularities at $\mathcal{G} = \{(0,0),(1,0),(0,1), R = (1,1)\}$ *and spectra of eigenvalues*

$$\big[[i, -i], [1,-1], [i, -i], [1,-1]\big].$$

□

Corollary 7. *For $d \geq 3$, there exist Morse polynomials $f \in \mathbb{K}[x, y]^0_{=d}$ with $(d-1)^2$ singular points that are not essentially determined.* □

7. Closing Remarks

Let $\mathfrak{X}(\mathbb{K}^2)_{\leq d-1}$ be the space of polynomial vector fields $\{X\}$ of at most degree $d-1$ on $\mathbb{K}^2$. A general and natural question is as follows. Under what conditions is a polynomial vector field X on $\mathbb{K}^2$ *essentially determined* by its configuration of singular points, i.e., its zeros, $\mathcal{Z}(X)$ in $\mathbb{K}^2$?

In simple words, a vector field X *is essentially determined* (in $\mathfrak{X}(\mathbb{K}^2)_{\leq d-1}$) by its configuration of zeros $\mathcal{Z}(X_f)$;

$$\text{if for any } Y \in \mathfrak{X}(\mathbb{K}^2)_{\leq d-1} \text{ satisfying } \mathcal{Z}(X) \subset \mathcal{Z}(Y) \subset \mathbb{K}^2, \quad \text{then } X = \lambda Y.$$

Recalling that for affine degree d the number of isolated singularities of the associated singular holomorphic foliation $\mathcal{F}(\mathcal{X})$ on the whole $\mathbb{CP}^2$ is $(d-1)^2 + d$, the hypothesis of multiplicity 1 must be understood for all these points. Proposition 1 confirms that in the Hamiltonian case only $\delta(d) \leq (d-1)^2$ points are required.

Recall which Gómez-Mont and Kempf [13], established in the complex rational case the following deep result, that also enlightens the real case. *A meromorphic vector field $\mathcal{X}$ on $\mathbb{CP}^m$, $m \geq 2$, of degree $r \geq 2$, with singular points of multiplicity 1 is completely determined by its singular set.*

Moreover, Artes et al. [4,5] prove the following: *A polynomial vector field $\mathcal{X}$ on $\mathbb{K}^2$ of degree 2 is completely determined by the position of its 7 singular points (including the points at infinity).*

As far as we know, over $\mathbb{K} = \mathbb{C}$ the more general result is due to Campillo and Olivares [6]: *A singular holomorphic foliation $\mathcal{X}$ on $\mathbb{CP}^2$ of degree $r \geq 2$, is completely determined by its singular scheme.*

See Alcántara et al. [1] for recent developments regarding foliations with multiple points. We summarize our results as follows.

Corollary 8. *A polynomial Hamiltonian vector field X_f on $\mathbb{K}^2$ of degree 2 is completely determined (in the space of polynomial vector fields of degree 2, up to a scalar factor $\lambda \in \mathbb{K}^*$) by its zero points, when there are 4 isolated points different from $\{(0,0), (1,0), (0,1), (1,1)\}$, up to affine transformation.*

Our hope is that the explicit results in this paper can illustrate the classification of polynomials $\mathbb{K}[x, y]$ up to algebraic equivalence $Aut(\mathbb{K}^2)$; see [11,18] for this order of ideas. This potential application is the subject of a future project.

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Data availability We do not analyse or generate any datasets, because our work proceeds within a theoretical and mathematical approach. One can obtain the relevant materials from the references below.

Declarations

Conflict of interest All authors declare that they have no conflicts of interest.

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