ABELIAN INTEGRALS FOR POLYNOMIALS WITH TRIVIAL GLOBAL MONODROMY ON $\mathbb{C}^2$

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Abstract. We consider infinitesimal perturbations of Hamiltonian differential equations $dH + \varepsilon\omega = 0$ on the complex plane $\mathbb{C}^2$, where H is a polynomial of degree $m+1$ and ω is a non-exact polynomial 1-form of degree n. In order to study these perturbed differential equations, the associated Abelian integrals $I(c) = \int_{\gamma(c)} \omega$ are valuable tools. We assume that the polynomials H are primitive with trivial global monodromy. W. D. Neumann and P. Norbury provided a classification up to algebraic equivalence of these polynomials. The knowledge of the families of these polynomials allows us to prove that the respective Abelian integrals $I(c)$ are polynomial functions of the variable c, and to find sharp upper bounds for the number of their zeros that depend on m, n and the topology of the generic fibers of H. These upper bounds are explicit for several new families of infinitesimal perturbations of Hamiltonian differential equations.

Contents

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1. Introduction

A complex polynomial function H on $\mathbb{C}^2$ determines a Hamiltonian differential equation $dH = 0$. Given a complex polynomial 1-form ω and $\varepsilon \in (\mathbb{C}, 0)$, we consider the infinitesimal perturbation of the Hamiltonian differential equation

$$dH + \varepsilon\omega = 0 \quad \text{on} \;\; \mathbb{C}^2. \tag{1}$$

As usual for the study of Abelian integrals, let c_0 be a generic value of H and let $\gamma_{\mathtt{i}}(c_0)$ be a *cycle of* H in a basis of the homology $H_1(L_{c_0}, \mathbb{Z})$ of the generic fiber $L_{c_0} \doteq \{H(u,v) = c_0\}$. We denote the bifurcation set of H as $\mathfrak{B}(H) \subset \mathbb{C}$ and throughout our work a generic value of H means $c_0 \in \mathbb{C}\backslash\mathfrak{B}(H)$. The *Abelian integral* defined by H, c_0, $\gamma_{\mathtt{i}}(c_0)$ and ω is the holomorphic function germ

$$I_{\mathtt{i}}(c) = \int_{\gamma_{\mathtt{i}}(c)} \omega \colon (\mathbb{C}, c_0) \longrightarrow \mathbb{C}, \tag{2}$$

where $\gamma_{\mathtt{i}}(c)$ is obtained from $\gamma_{\mathtt{i}}(c_0)$ by local monodromy, that is, continuous transport of the cycle in the fibers of H. Abusing the notation, we only write explicitly the subindex $1 \le \mathtt{i} \le \mathfrak{r} = \dim H_1(L_c, \mathbb{Z})$ in the germ (2).

From a dynamical point of view, the study of the zeros of the integrals (2) is related to the bifurcation of limit cycles in equation (1). More precisely, recall that (1) is a non-conservative perturbation of $dH = 0$, when $I_{\mathtt{i}}(c) \not\equiv 0$. In this case, the well known Poincaré–Pontryagin criterion implies that the number of limit cycles of (1) generated from the cycles $\gamma_{\mathtt{i}}(c)$ is bounded by the number of isolated zeros, counting multiplicities, of the Abelian integral $I_{\mathtt{i}}(c)$, see [17, §2, §7, §8]. Certainly, there are examples of differential equations (1) with limit cycles generated from cycles of H when the Abelian integrals $I_{\mathtt{i}}(c)$ vanish identically or with limit cycles generated from singular fibers of H; see for instance [19, 27, 28].

One of the most general results regarding the maximal number of isolated zeros of Abelian integrals has been achieved by G. Binyamini *et al.* [5], who provided as an explicit upper bound a double exponential in the maximum degree of H and ω, such upper bound is far from being optimal. This result is actually the most general solution to the *weak infinitesimal Hilbert's 16th problem.* See [1, 9, 14, 17, 18, 20, 21, 25, 30] and references therein for several aspects of this subject. Naturally, the richness of the weak infinitesimal Hilbert's 16th problem suggests looking at particular families of polynomials H, which could be accessible intermediate steps towards accurate upper bounds.

In this work, we consider the family of polynomials H with trivial global monodromy, see for instance E. Artal-Bartolo *et al.* [3], A. Dimca [10], W. D. Neumann and P.

Norbury [23], as well as references therein. If H has trivial global monodromy and $\dim H_1(L_c, \mathbb{Z}) = \mathfrak{r} \geq 1$, then each $I_{\mathtt{i}}(c)$ is a univalued complex analytic function on $\mathbb{C} \backslash \mathfrak{B}(H)$. Recall that H is *primitive* when its generic fibers L_c are connected. We prove the following result.

Theorem 1. *Let H be a primitive polynomial on $\mathbb{C}^2$ with trivial global monodromy of degree at most $m+1$, suppose $\dim H_1(L_c, \mathbb{Z}) = \mathfrak{r} \geq 1$, and let ω be a polynomial 1-form of degree at most n.*

1) For each cycle $\gamma_{\mathtt{i}}(c_0)$ of H in a generic fiber L_{c_0}, the Abelian integral (2) extends to the complex line

$$I_{\mathtt{i}}(c) = \int_{\gamma_{\mathtt{i}}(c)} \omega \colon \mathbb{C} \longrightarrow \mathbb{C}, \tag{3}$$

as a polynomial function.

2) The degree of $I_{\mathtt{i}}(c)$ is as follows:

$$\deg(I_{\mathtt{i}}(c)) \leq \begin{cases} \left[\dfrac{n+1}{2}\right] & \text{if } m = 1, \\ (n+1)(m-1) - 1 & \text{if } 2 \leq m \leq 8, \\ \left((n+1)\left[\dfrac{m-\mathfrak{r}}{\mathfrak{r}}\right] - 1\right)(m - \mathfrak{r} - 2) - \mathfrak{r} + 1 & \text{if } m \geq 9. \end{cases} \tag{4}$$

The significance of this theorem lies in the polynomial nature of the extension of the Abelian integrals (2), which may not be immediately apparent under the given hypothesis of the theorem. Additionally, the theorem provides explicit upper bounds for the degree of the polynomial Abelian integrals, emphasizing the emergence of the homology dimension of the generic fibers of the polynomial H at these bounds.

Concerning the weak infinitesimal Hilbert's 16th problem, we will search for the following two bounds:

- $Z(I_{\mathtt{i}}(c))$ *the maximal number of zeros of* $I_{\mathtt{i}}(c)$ *in* $\mathbb{C} \backslash \mathfrak{B}(H)$ counted with multiplicities, and
- $\mathscr{N}(H, \omega)$ *the total number of limit cycles of a non-conservative perturbation* (1), *generated from the cycles in the generic fibers of* H.

Our result is as follows.

Theorem 2. *Let H be a primitive polynomial on $\mathbb{C}^2$ with trivial global monodromy of degree at most $m+1$, suppose $\dim H_1(L_c, \mathbb{Z}) = \mathfrak{r} \geq 1$, and let ω be a polynomial 1-form of degree at most n such that (1) is a non-conservative perturbation.*

1) Then

$$Z(I_{\mathtt{i}}(c)) \leq \deg(I_{\mathtt{i}}(c)). \tag{5}$$

2) Moreover,

$$\mathscr{N}(H, \omega) \leq Z(I_1(c)) + \cdots + Z(I_{\mathfrak{r}}(c)) \leq \mathfrak{r} \max_{1 \leq \mathtt{i} \leq \mathfrak{r}} Z(I_{\mathtt{i}}(c)). \tag{6}$$

3) If μ is the number of vanishing cycles of H, then

$$\mathscr{N}(H,\omega) \leq \mathfrak{r} \max_{1\leq \mathtt{i} \leq \mathfrak{r}} Z(I_{\mathtt{i}}(c)) - \mu. \tag{7}$$

Note that our bounds in (5), (6), (7) consider a basis of cycles of $H_1(L_c,\mathbb{Z})$ and the total sum over $c \in \mathbb{C}\backslash\mathfrak{B}(H)$.

Recall that for H with non-trivial global monodromy the Abelian integral $I_{\mathtt{i}}(c)$ is usually a multivalued function on $\mathbb{C}\backslash\mathfrak{B}(H)$, hence, $Z(I_{\mathtt{i}}(c))$ only makes sense for $I_{\mathtt{i}}(c)$ as the germ in (2).

The work is organized as follows. In Section 2, we recall notations and definitions concerning our problem. As a novel aspect, we introduce the group of algebraic automorphisms $\text{Aut}(\mathbb{C}^2) \times \text{Aut}(\mathbb{C})$ as a valuable tool for the study of Abelian integrals. As a contribution, the infinitesimal Hilbert's 16th problem is invariant under algebraic automorphisms, see Corollary 3. This suggests we should study certain families of normal forms $\mathcal{H}$ for polynomials H under $\text{Aut}(\mathbb{C}^2) \times \text{Aut}(\mathbb{C})$, see §2.2. Thus, we change the study of the infinitesimal perturbation $dH + \varepsilon\omega = 0$ into the study of $d\mathcal{H} + \varepsilon\vartheta = 0$, where ϑ is the push-forward of ω under the algebraic automorphism that transforms H into $\mathcal{H}$. In order to prove our main result, in Section 3, we propose a *Program*, which consists of four steps. The first concerns the algebraic classification; to find the normal form $\mathcal{H}$ of an original primitive polynomial with trivial global monodromy H. In the second step, we regard a birational map $\mathcal{R}$ which globally rectifies the Hamiltonian differential equation $d\mathcal{H} = 0$, that allows us to perform a great simplification in our study. The third step recognizes the Abelian integrals for the infinitesimal perturbation of the rectified differential equation. We present an invariance of Abelian integrals under the birational maps $\mathcal{R}$, see Corollary 13 for complete details. Finally, as a fourth step we compute the Abelian integrals (2) by their rectified simplifications; that is, on the rectified foliation and by using the residue theorem at the punctures, that arise from the corresponding basis of homology classes. In Section 4, we recall the Neumann–Norbury classification, which provides three families of normal forms $\mathcal{H}$ of primitive polynomials with trivial global monodromy, as well as a result that controls the degree of the transformed objects $\mathcal{H}$ and ϑ under algebraic equivalence. We give an accurate description of the rectifying maps for the normal form polynomials, which allows us to prove the rational invariance of the infinitesimal Hilbert's 16th problem. In Section 5, we introduce some characteristic examples of the application of our Program. In Section 6, we study the properties of the Abelian integrals defined by the normal form polynomials with trivial global monodromy. The proofs of Theorems 1 and 2 are given in Section 7.

2. Generalities, algebraic equivalence and normal forms

2.1. **Notations and definitions.** As usual, $\mathbb{C}[u,v]$ denotes the vector space of complex polynomials and $\Omega^1(\mathbb{C}^2)$ is the vector space of polynomial 1-forms on $\mathbb{C}^2$.

Because of a classical result of R. Thom [29], given a polynomial $H(u,v)$ its *bifurcation set* (critical value set) is

$$\mathfrak{B}(H) = \mathfrak{B}_{fin}(H) \cup \mathfrak{B}_{inf}(H) \subset \mathbb{C}. \tag{8}$$

This includes the subset of finite critical values $\mathfrak{B}_{fin}(H)$ from critical points in $\mathbb{C}^2$, as well as the critical values at infinity $\mathfrak{B}_{inf}(H)$ corresponding to the critical points in the line at infinity. This second subset arises from the extension of H as a rational function on $\mathbb{C}^2 \cup \mathbb{CP}^1_\infty$, see [11]. The map

$$H\colon \mathbb{C}^2 \backslash H^{-1}(\mathfrak{B}(H)) \longrightarrow \mathbb{C}\backslash\mathfrak{B}(H) \tag{9}$$

is a locally trivial smooth fibration, see [6, 15]. By definition, $c \in \mathbb{C}\backslash\mathfrak{B}(H)$ is a *generic value* of H and the associated affine non-singular algebraic curve

$$L_c \doteq \{H(u,v) = c\} \subset \mathbb{C}^2$$

is a *generic fiber* of H. Moreover, a polynomial H is *primitive of type* (g,κ) when its generic fibers $\{L_c\}$ are irreducible and homeomorphic to a compact Riemann surface of genus $g \geq 0$ which is punctured at $\kappa \geq 1$ points. In that case, the first homology group $H_1(L_c,\mathbb{Z})$ of any generic fiber L_c of H is a free Abelian group of dimension $\mathfrak{r} = 2g+\kappa-1$.

Let H be a primitive polynomial of type (g,κ) and let $\omega \in \Omega^1(\mathbb{C}^2)$ be a complex polynomial 1-form. We consider $c_0 \in \mathbb{C}\backslash\mathfrak{B}(H)$ to be a generic value of H and a non-trivial cycle $\gamma_{\mathtt{i}}(c_0)$ of H, this is $[\gamma_{\mathtt{i}}(c_0)]$ belongs to a basis

$$\mathcal{B}(c_0) \doteq \big\{[\gamma_{\mathtt{i}}(c_0)] \,|\, \mathtt{i} = 1,2,\ldots,\mathfrak{r}\big\}$$

of $H_1(L_{c_0},\mathbb{Z})$. On the one hand, if $\mathbb{D}(c_0,\rho) \subset \mathbb{C}\backslash\mathfrak{B}(H)$ is an open disk of generic values of H, with center c_0 and radius ρ, then $\gamma_{\mathtt{i}}(c_0)$ can be continuously transported, by using the fibration (9) into a unique cycle $\gamma_{\mathtt{i}}(c)$ in L_c for each $c \in \mathbb{D}(c_0,\rho)$. Thus, the *Abelian integral defined by H, c_0, $\gamma_{\mathtt{i}}(c_0)$ and* ω is a holomorphic function germ

$$I_{\mathtt{i}}(c) = \int_{\gamma_{\mathtt{i}}(c)} \omega\,,$$

as in (2). For sake of simplicity, we will refer to $I_{\mathtt{i}}(c)$ only as an Abelian integral defined by H and ω. On the other hand, there is a homomorphism $\mathcal{M}$ from the fundamental group $\pi_1(\mathbb{C}\backslash\mathfrak{B}_H, c_0)$ in the topological automorphism group $\mathrm{Aut}(H_1(L_{c_0},\mathbb{Z}))$ of $H_1(L_{c_0},\mathbb{Z})$. The *monodromy group of H* is the image $\mathscr{M}_H$ of $\mathcal{M}$, see [2, Chapter 2], [10]. We say that H has *trivial global monodromy* when $\mathscr{M}_H$ is the identity map, see [3, 23].

Remark 1. *In general, by using the monodromy group $\mathscr{M}_H$ the Abelian integral $I_{\mathtt{i}}(c)$ extends to $\mathbb{C}\backslash\mathfrak{B}_H$ as a multivalued holomorphic function.*

We recall that two complex polynomials H and $\mathcal{H} \in \mathbb{C}[u,v]$ are *algebraically equivalent* or *$(\psi,\sigma)_*$-equivalent* if there are polynomial automorphisms $\psi \in \mathrm{Aut}(\mathbb{C}^2)$ and $\sigma \in \mathrm{Aut}(\mathbb{C})$ such that

$$\mathcal{H} = \sigma \circ H \circ \psi^{-1}. \tag{10}$$

Moreover, we convene the notation

$$\sigma : \mathbb{C} \longrightarrow \mathbb{C}, \quad c \longmapsto \mathfrak{c}\,. \tag{11}$$

Additionally, we say that two complex polynomial 1-forms $\omega, \vartheta \in \Omega^1(\mathbb{C}^2)$ are *algebraically equivalent* when

$$\vartheta = \sigma' \psi_*(\omega), \tag{12}$$

where $\sigma' \in \mathbb{C}^*$ is the derivative of the affine map σ.

Remark 2. *1. Since we must look at regular and critical values of H or $\mathcal{H}$, the convention* (11) *will be useful.*
2. Equation (10) *says that the group* $\mathrm{Aut}(\mathbb{C}^2) \times \mathrm{Aut}(\mathbb{C}) = \{(\psi, \sigma)\}$ *of polynomial automorphisms acts on the space of polynomials as*

$$\begin{aligned} \mathbb{C}[u,v] \times \mathrm{Aut}(\mathbb{C}^2) \times \mathrm{Aut}(\mathbb{C}) &\longrightarrow \mathbb{C}[u,v] \\ \big(H, (\psi, \sigma)\big) &\longmapsto \mathcal{H} = \sigma \circ H \circ \psi^{-1}. \end{aligned}$$

Thus, each orbit of this action is a family of algebraically equivalent polynomials.
3. Trivial global monodromy is an algebraic invariant property of polynomials in $\mathbb{C}[u,v]$.
4. If H and $\mathcal{H}$ are algebraically equivalent, then $\dim H_1(L_c, \mathbb{Z}) = \dim H_1(\mathcal{L}_{\mathfrak{c}}, \mathbb{Z})$.

2.2. **Algebraic invariance of the infinitesimal Hilbert's 16th problem.** Let H and $\mathcal{H}$ be two algebraically equivalent polynomials as in equation (10). Consider a complex polynomial 1-form ω and its algebraically equivalent polynomial 1-form $\vartheta = \sigma' \psi_*(\omega)$, as in (12). Therefore, equations (10)–(12) allow us to transform the holomorphic germ of the Abelian integral (2) into the holomorphic germ

$$\mathcal{I}_{\mathtt{i}}(\mathfrak{c}) = \int_{\delta_{\mathtt{i}}(\mathfrak{c})} \vartheta : (\mathbb{C}, \mathfrak{c}_0) \longrightarrow \mathbb{C}, \quad \mathfrak{c}_0 = \sigma(c_0),\ \delta_{\mathtt{i}}(\mathfrak{c}) = \psi\big(\gamma_{\mathtt{i}}(\sigma^{-1}(\mathfrak{c}))\big). \tag{13}$$

These algebraic equivalences imply the following result.

Corollary 3 (Algebraic invariance of the infinitesimal Hilbert's 16th problem)**.** *Let* $H, \mathcal{H} \in \mathbb{C}[u,v]$ *be polynomials as in equation* (10)*, and let* $\omega, \vartheta \in \Omega^1(\mathbb{C}^2)$ *be polynomial 1-forms as in equation* (12)*: algebraic equivalent objects in both cases.*

1) The corresponding infinitesimal perturbed Hamiltonian differential equations are algebraically equivalent[1]*, that is,*

$$\sigma' \psi_*(dH + \varepsilon\omega) = d\mathcal{H} + \varepsilon\vartheta.$$

2) The Abelian integrals $I_{\mathtt{i}}(c)$ *and* $\mathcal{I}_{\mathtt{i}}(\mathfrak{c})$ *are algebraically equivalent, that is,*

$$I_{\mathtt{i}}(c) = \frac{1}{\sigma'} \mathcal{I}_{\mathtt{i}}(\sigma(c)), \qquad \textit{denoted as } (\psi, \sigma)_* I_{\mathtt{i}} = \mathcal{I}_{\mathtt{i}},$$

even if they are multivalued functions.
3) The cardinality of the zeros (counted with multiplicities) of $I_{\mathtt{i}}(c)$ *in* $\mathbb{C}\backslash\mathfrak{B}(H)$ *and of* $\mathcal{I}_{\mathtt{i}}(\mathfrak{c})$ *in* $\mathbb{C}\backslash\mathfrak{B}(\mathcal{H})$ *coincide. In particular, if H has trivial global monodromy, then*

$$Z(I_{\mathtt{i}}(c)) = Z(\mathcal{I}_{\mathtt{i}}(\mathfrak{c})) \qquad \textit{and} \quad \mathscr{N}(H, \omega) = \mathscr{N}(\mathcal{H}, \vartheta).$$

Proof. Assertion 1) requires the accurate factor σ' in equation (12). Assertions 2) and 3) are straightforward. □

2.3. **Normal forms of Hamiltonians with respect to the degree.** In order to simplify the study of the infinitesimal Hilbert's 16th problem, clearly, Corollary 3 suggests searching for a normal form for $H(u,v)$ up to algebraic equivalence and with the property of minimal degree.

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[1] Our equivalence is of 1-forms, which obviously implies the equivalence of the associated differential equations and their foliations.

Lemma 4. *Let $H(u,v)$ be a polynomial and let*
$$\left\{\sigma \circ H \circ \psi^{-1} \mid (\psi,\sigma) \in \mathrm{Aut}(\mathbb{C}^2) \times \mathrm{Aut}(\mathbb{C})\right\} \subset \mathbb{C}[u,v]$$
be its $\mathrm{Aut}(\mathbb{C}^2) \times \mathrm{Aut}(\mathbb{C})$*-orbit. A minimum degree is reached in the orbit, that is, there exists a non unique polynomial* $\mathcal{H}$ *in the orbit such that*
$$\deg(\mathcal{H}) \leq \deg\left(\sigma \circ H \circ \psi^{-1}\right).$$

Proof. In [31, pp. 357-358], P. G. Wightwick studied the behavior of the degree of polynomials $H \in \mathbb{C}[u,v]$ under $\mathrm{Aut}(\mathbb{C}^2)$. For each H, Wightwick constructs an algorithm that depends on a finite numbers of choices that reduce the degree. Necessarily, the suitable choices produce the required $\mathcal{H}$. The polynomial $\mathcal{H}$ is not unique, since under the actions of the affine group $\mathrm{Aff}(\mathbb{C}^2)$ and $\mathrm{Aut}(\mathbb{C})$ the degree of $\mathcal{H}$ remains constant. □

By abusing the language, we convene the next concept.

Definition 1. Let $H(u,v)$ be a polynomial and its $\mathrm{Aut}(\mathbb{C}^2) \times \mathrm{Aut}(\mathbb{C})$-orbit. A *normal form* of $H(u,v)$ is a minimal degree polynomial, denoted by $\mathcal{H}(x,y)$, in this orbit.

Remark 3. *In all that follows, we reserve variables (x,y) for a normal form $\mathcal{H}(x,y)$ of $H(u,v)$ and use subscripts for ψ in equation* (10)*; that is,*
$$\psi : \mathbb{C}^2_{u\,v} \longrightarrow \mathbb{C}^2_{x\,y}.$$
It will be appropriate for our study of Abelian integrals.

If for a family of polynomials, say $\{H\} \subset \mathbb{C}[u,v]$, their normal forms can be found, then the properties and bounds of the number of zeros of the Abelian integrals of the family could be probably stated in a simpler way. In this scenario, however, two main difficulties appear:

- D.1 The algebraic classification of polynomials $H \in \mathbb{C}[u,v]$ is a difficult and challenging open problem, for example J. Fernández de Bobadilla [12].
- D.2 The degrees of H and ω are not invariant under polynomial automorphisms.

In order to analyze difficulty D.2, we will denote by $\mathbb{C}[u,v]_{\leq m}$ the vector spaces of complex polynomials of degree at most m and the vector space of polynomial 1-forms on $\mathbb{C}^2_{u\,v}$ of degree at most n by
$$\Omega^1(\mathbb{C}^2_{u\,v})_{\leq n} \doteq \{\omega = Adu + Bdv \mid A, B \in \mathbb{C}[u,v]_{\leq n}\}.$$
In accordance with the Introduction §1, we consider
$$H \in \mathbb{C}[u,v]_{\leq m+1} \quad \text{and} \quad \omega \in \Omega^1(\mathbb{C}^2_{u\,v})_{\leq n}. \tag{14}$$
Let $\mathcal{H}$ be a normal form of H through $(\psi,\sigma) \in \mathrm{Aut}(\mathbb{C}^2) \times \mathrm{Aut}(\mathbb{C})$ and let $\vartheta = \sigma' \psi_*(\omega)$ be the associated 1-form. From Definition 1, we have
$$\mathfrak{m} + 1 \doteq \deg(\mathcal{H}) \leq \deg(H) \leq m + 1. \tag{15}$$
By definition of ϑ and the fact $\deg(\psi^{-1}) \leq \deg(\psi)$, as seen in [7], we obtain
$$\mathfrak{n} \doteq \deg(\vartheta) \leq (n+1)\deg(\psi^{-1}) - 1 \leq (n+1)\deg(\psi) - 1.$$
The degree of ϑ could be greater than the degree of ω, since it depends on the degree of the polynomial automorphism ψ. As an advantage, such a degree could be bound.

Indeed, from [12, Proposition 4.17] we know that

$$\deg(\psi) \leq \frac{(m+1)!}{(\mathfrak{m})!}.$$

Thus,

$$\mathfrak{n} = \deg(\vartheta) \leq (n+1)\deg(\psi) - 1 \leq (n+1)\frac{(m+1)!}{(\mathfrak{m})!} - 1. \tag{16}$$

Hence, equations (15) and (16) imply that the set of all ϑ coming from $\Omega^1(\mathbb{C}^2_{u\,v})_{\leq n}$ through the pairs (ψ, σ) is a subset of $\Omega^1(\mathbb{C}^2_{x\,y})_{\leq (n+1)(m+1)!-1}$. Thus, although the degrees of the original objects H and ω are not invariant, the degrees of the transformed objects $\mathcal{H}$ and ϑ are well understood.

Lemma 5. *Consider $H \in \mathbb{C}[u, v]_{\leq m+1}$ and $\omega \in \Omega^1(\mathbb{C}^2_{u\,v})_{\leq n}$. If $\mathcal{H}$ is a normal form of H under $(\psi, \sigma) \in \mathrm{Aut}(\mathbb{C}^2) \times \mathrm{Aut}(\mathbb{C})$ and $\vartheta = \sigma'\psi_*(\omega)$, then*

$$\deg(\mathcal{H}) \leq m+1 \quad \textit{and} \quad \deg(\vartheta) \leq (n+1)(m+1)! - 1.$$

□

By considering the aforementioned, we have the following result.

Proposition 6. *Consider a primitive polynomial $H \in \mathbb{C}[u, v]_{\leq m+1}$ and a polynomial 1-form $\omega \in \Omega^1(\mathbb{C}^2_{u\,v})_{\leq n}$. Then*

$$\left\{ \begin{array}{c} \textit{maximal number of} \\ \textit{zeros of } I_{\mathtt{i}}(c) = \int_{\gamma_{\mathtt{i}}(c)} \omega \end{array} \right\} \leq \left\{ \begin{array}{c} \textit{maximal number of} \\ \textit{zeros of } \mathcal{I}_{\mathtt{i}}(\mathfrak{c}) = \int_{\delta_{\mathtt{i}}(c)} \vartheta \end{array} \right\},$$

where the right-hand side considers a normal form $\mathcal{H} \in \mathbb{C}[x, y]_{\leq \mathfrak{m}+1}$ of H and all polynomial 1-forms $\vartheta \in \Omega^1(\mathbb{C}^2_{x\,y})_{\leq \mathfrak{n}}$, with $\mathfrak{m} = m$ and $\mathfrak{n} = (n+1)(m+1)! - 1$. □

In simple words, for each pair $(m, n) \in \mathbb{N} \times \mathbb{N}$ and by considering only Abelian integrals in (2), defined by primitive polynomials in normal form $\mathcal{H}$ of degree at most $m+1$ and polynomial 1-forms of degree at most $(n+1)(m+1)! - 1$; we can always obtain an estimation from above for the maximal number of zeros of the Abelian integrals defined by primitive polynomials $H \in \mathbb{C}[u, v]_{\leq m+1}$ and polynomial 1-forms $\omega \in \Omega^1(\mathbb{C}^2_{u\,v})_{\leq n}$.

Moreover, looking at the family of primitive polynomials with trivial global monodromy, we will have explicit normal forms; see the next two sections.

3. The *Program*

In this work, we restrict ourselves to primitive polynomials $H(u, v)$ with trivial global monodromy. A cornerstone result behind our assertions is due to Neumann and Norbury [24], see Theorem 8 in Section 4. Very roughly speaking, these authors provide us with the following two key facts.

i) Each primitive polynomial with trivial global monodromy $H(u, v)$ on $\mathbb{C}^2_{u\,v}$ has an explicit normal form polynomial $\mathcal{H}(x, y)$ on $\mathbb{C}^2_{x\,y}$ according to Definition 1.

ii) Moreover, each normal form polynomial $\mathcal{H}(x, y)$ admits a birational map

$$\mathcal{R} : \mathbb{C}^2_{x\,y} \longrightarrow \mathbb{C}^2_{t\,\mathfrak{c}} \tag{17}$$

that sends the generic fibers of $\mathcal{H}$ into punctured horizontal lines in $\mathbb{C}^2_{t\,\mathfrak{c}}$.

Thus, $\mathcal{R}$ is a *rectifying map for* $\mathcal{H}$ and $\mathfrak{c}$ coincides with equation (11) according to $\mathcal{H} \circ \mathcal{R}^{-1}(t, \mathfrak{c}) = \mathfrak{c}$. Each map $\mathcal{R}$ transforms polynomials, 1-forms and differential equations under push-forward denoted as $\mathcal{R}_*$. We say that it induces an $\mathcal{R}$–*equivalence*.

As an advantage for study the Abelian integrals (1) and for proving Theorem 1, we propose the following *Program*. Let H be a primitive polynomial on $\mathbb{C}^2$ with trivial global monodromy of degree at most $m+1$ and let ω be a polynomial 1-form of degree at most n.

Step 1. According to Corollary 3, a suitable pair $(\psi, \sigma) \in \mathrm{Aut}(\mathbb{C}^2) \times \mathrm{Aut}(\mathbb{C})$ allows us to transform the original polynomial differential equation (1) into a differential equation

$$d\mathcal{H} + \varepsilon\vartheta = 0 \quad \text{on} \quad \mathbb{C}^2_{x\,y}, \quad \vartheta \doteq \sigma' \psi_*(\omega). \tag{18}$$

The pair (ψ, σ) is not explicit in general; however, by Proposition 6 we will obtain explicitly tighter upper bounds for the degrees of ψ, $\mathcal{H}$ and ϑ.

Step 2. The corresponding rectifying map $\mathcal{R}$ in (17) transforms this last equation into a *rational* differential equation

$$d\mathfrak{c} + \varepsilon\eta = 0 \quad \text{on} \quad \mathbb{C}^2_{t\,\mathfrak{c}}, \quad \eta \doteq \mathcal{R}_*(\vartheta), \tag{19}$$

with the advantage that the foliation of $d\mathfrak{c} = 0$ on $\mathbb{C}^2_{t\,\mathfrak{c}}$ is topologically trivial.

Step 3. We consider the infinitesimal perturbed Hamiltonian differential equations (1), (18) and (19). The corresponding three Abelian integrals are well defined in the corresponding generic values sets in $\mathbb{C}$ and satisfy

$$\begin{gathered} I_{\mathtt{i}}(c) = \int_{\gamma_{\mathtt{i}}(c)} \omega \, = \frac{1}{\sigma'} \mathcal{I}_{\mathtt{i}}(\mathfrak{c}) = \frac{1}{\sigma'} \int_{\delta_{\mathtt{i}}(\mathfrak{c})} \vartheta \, = \frac{1}{\sigma'} J_{\mathtt{i}}(\mathfrak{c}) = \frac{1}{\sigma'} \int_{\alpha_{\mathtt{i}}(\mathfrak{c})} \eta \,, \\ \mathfrak{c} = \sigma(c),\ \delta_{\mathtt{i}}(\mathfrak{c}) \doteq \psi\big(\gamma_{\mathtt{i}}(c)\big), \quad \text{and} \quad \alpha_{\mathtt{i}}(\mathfrak{c}) \doteq \mathcal{R}(\delta_{\mathtt{i}}(\mathfrak{c})). \end{gathered} \tag{20}$$

The left equality of the integrals follows from Corollary 3. The right equality of the integrals will be given in Corollary 13.

Step 4. The maximal number of isolated zeros, counted with multiplicities, for each integral in (20) is well defined and

$$Z(I_{\mathtt{i}}(c)) = Z(\mathcal{I}_{\mathtt{i}}(\mathfrak{c})) = Z(J_{\mathtt{i}}(\mathfrak{c})). \tag{21}$$

Moreover, according to (6), we archive the equalities

$$\mathcal{N}(H, \omega) = \mathcal{N}(\mathcal{H}, \vartheta) = \mathcal{N}(\mathfrak{c}, \eta). \tag{22}$$

In fact, the rectifying map $\mathcal{R}$ and the residue theorem for η, allow us to compute the upper bound given in (4).

The following diagram illustrates the Program, to be descriptive, the vertical arrows must be understood as implications:

$$
\begin{array}{ccccc}
dH + \varepsilon\omega = 0 & \xrightarrow{(\psi,\sigma)_*} & d\mathcal{H} + \varepsilon\vartheta = 0 & \overset{\mathcal{R}_*}{\dashrightarrow} & d\mathfrak{c} + \varepsilon\eta = 0 \\
\downarrow & & \downarrow & & \downarrow \\
I_{\mathtt{i}}(c) & \xrightarrow{(\psi,\sigma)_*} & \mathcal{I}_{\mathtt{i}}(\mathfrak{c}) & \overset{\mathcal{R}_*}{\dashrightarrow} & J_{\mathtt{i}}(\mathfrak{c}) \\
\downarrow & & \downarrow & & \downarrow \\
Z(I_{\mathtt{i}}(c)) & \xrightarrow{=} & Z(\mathcal{I}_{\mathtt{i}}(\mathfrak{c})) & \xrightarrow{=} & Z(J_{\mathtt{i}}(\mathfrak{c})) \\
\downarrow & & \downarrow & & \downarrow \\
\mathscr{N}(H,\omega) & \xrightarrow{=} & \mathscr{N}(\mathcal{H},\vartheta) & \xrightarrow{=} & \mathscr{N}(\mathfrak{c},\eta)\,.
\end{array}
\tag{23}
$$

4. Normal forms of polynomials with trivial global monodromy

Concerning the difficulty D.1 in §2.3, we recall the explicit normal forms of polynomials with trivial global monodromy and their associated birational rectifying maps. These properties allow us to perform our diagram (23) of the Program.

4.1. **Algebraic classification.** The simplest case for the weak infinitesimal Hilbert's 16th problem concerns the algebraic classification of primitive polynomials $H(u,v)$ of type $(0,2)$, that have generic fiber L_c bihomolorphic to $\mathbb{C}^*$. This classification was archived by M. Miyanishi and T. Sugie [22] and can be stated as follows.

Theorem 7 ([22])**.** *A primitive polynomial of type* $(0,2)$ *is algebraically equivalent to a polynomial that belongs to the family*

$$
\left\{ \mathcal{H}(x,y) = x^k \Big(x^l y + P(x) \Big)^r \;\middle|\; \begin{array}{l} k, r \in \mathbb{N},\ (k,r) = 1,\ l \in \mathbb{N} \cup \{0\}, \\ \deg(P) \leq l-1,\ P(0) \neq 0 \ \textit{if}\ l > 0, \\ \textit{and}\ P(x) \equiv 0 \ \textit{if}\ l = 0 \end{array} \right\}.
$$

Each polynomial in this family has trivial global monodromy. In general, E. Artal-Bartolo *et al.* [3] proved in [3, Corollary 2] that a primitive polynomial function $H \in \mathbb{C}[u,v]$ has trivial global monodromy if and only if it is "rational of simple type" in the terminology of Miyanishi and Sugie [22]. This result was refined by Neumann and Norbury in [23], where they pointed out a gap in the Miyanishi–Sugie classification of such polynomials, since [22, p. 346, lines 10-11] implicitly assumes trivial *geometric* monodromy. Trivial geometric monodromy implies isotriviality; the generic fibers are pairwise isomorphic as punctured compact Riemann surfaces. In [23] Neumann and Norbury provided non-isotrivial examples. Finally, Neumann and Norbury in [24] gave the algebraic classification of rational polynomials of simple type. A gap in the proof of the main result of [24] was indicated in [8], where it was filled up. It did not modify the algebraic classification. Therefore, the normal forms of primitive polynomials with trivial global monodromy can be expressed as follows.

Theorem 8 (Neumann–Norbury algebraic classification [23, 24])**.** *Each primitive polynomial H with trivial global monodromy is algebraically equivalent to a polynomial* $\mathcal{H}_\iota$*,* *for $\iota = 1, 2,$ or 3, which belongs to one of the following three families:*

$$\mathfrak{F}_1 = \left\{ \mathcal{H}_1(x,y) = x^{q_1}\mathcal{S}(x,y)^q + x^{p_1}\mathcal{S}(x,y)^p \prod_{\mathtt{i}=1}^{r-1} \left(\beta_{\mathtt{i}} - x^{q_1}\mathcal{S}(x,y)^q\right)^{a_{\mathtt{i}}} \,\Big|\, r \geq 2 \right\},$$

$$\mathfrak{F}_2 = \left\{ \mathcal{H}_2(x,y) = x^{p_1}\mathcal{S}(x,y)^p \prod_{\mathtt{i}=1}^{r-1} \left(\beta_{\mathtt{i}} - x^{q_1}\mathcal{S}(x,y)^q\right)^{a_{\mathtt{i}}} \,\Big|\, r \geq 1 \right\},$$

$$\mathfrak{F}_3 = \left\{ \mathcal{H}_3(x,y) = y \prod_{\mathtt{i}=1}^{r-1} (\beta_{\mathtt{i}} - x)^{a_{\mathtt{i}}} + h(x) \,\Big|\, r \geq 1 \right\},$$

where

- $a_1, \dots, a_{r-1}$ *are positive integers,*
- $\beta_1, \dots, \beta_{r-1}$ *are distinct points of* $\mathbb{C}^*$,
- $h(x)$ *is a polynomial of degree less than* $\sum_{\mathtt{i}=1}^{r-1} a_{\mathtt{i}}$,
- $0 \leq p_1 < p$, $0 \leq q_1 < q$, *and* $(pq_1 - qp_1) = \pm 1$,
- $\mathcal{S}(x,y) = x^k y + P(x)$, *with* $k \geq 1$ *and* $P(x) \in \mathbb{C}[x]_{\leq k-1}$.

Moreover, if $\mathcal{G}_1(x,y) = \mathcal{G}_2(x,y) = x^{q_1}\mathcal{S}(x,y)^q$ *and* $\mathcal{G}_3(x,y) = x$, *then*

$$\mathcal{R}_\iota = (\mathcal{G}_\iota, \mathcal{H}_\iota) : \mathbb{C}^2_{x\,y} \longrightarrow \mathbb{C}^2_{t\,\mathfrak{c}} \tag{24}$$

is a birational map for $\iota = 1, 2, 3$. *In fact,* $\mathcal{G}_\iota(x,y)$ *maps a generic fiber* $\mathcal{H}_\iota^{-1}(\mathfrak{c})$ *biholomorphically to*

$$\mathbb{C}\backslash\{0, \beta_1, \dots, \beta_{r-1}, \mathfrak{c}\}, \quad \mathbb{C}\backslash\{0, \beta_1, \dots, \beta_{r-1}\} \quad \text{or} \quad \mathbb{C}\backslash\{\beta_1, \dots, \beta_{r-1}\}, \tag{25}$$

according as $\iota = 1, 2, 3$. *Thus,* $\mathcal{H}_1 \in \mathfrak{F}_1$ *is not isotrivial, but* $\mathcal{H}_2 \in \mathfrak{F}_2$, $\mathcal{H}_3 \in \mathfrak{F}_3$ *are isotrivial.*

Remark 4. *If we consider* $\mathcal{H}_2 \in \mathfrak{F}_2$ *and* $r = 1$, *then* $\mathcal{H}_2(x,y) = x^{p_1}\left(x^k y + P(x)\right)^p$. *Though the parameters* q_1 *and* q *do not appear explicitly in this case, the birational map in* (24) *exists, where* q_1 *and* q *are suitable positive integers with* $q_1 \leq q$ *such that* $pq_1 - qp_1 = 1$, *that is, the conditions* $0 \leq p_1 < p$ *and* $(p_1, p) = 1$ *must be satisfied.*

Concerning the concept of normal form given in Definition 1, we have the following result.

Lemma 9. *Each polynomial* $\mathcal{H}_\iota \in \mathfrak{F}_1 \cup \mathfrak{F}_2 \cup \mathfrak{F}_3$ *is a normal form, that is,* $\mathcal{H}_\iota$ *attains the minimum degree in its* $\mathrm{Aut}(\mathbb{C}^2) \times \mathrm{Aut}(\mathbb{C})$*–orbit.*

Proof. Since the degree of a polynomial remains invariant under $\mathrm{Aut}(\mathbb{C})$, it is sufficient to prove that for each $\psi = (\psi_1, \psi_2) \in \mathrm{Aut}(\mathbb{C}^2)$ we have the inequality

$$\deg(\mathcal{H}_\iota \circ \psi) \geq \deg(\mathcal{H}_\iota). \tag{26}$$

Let $n_i = \deg(\psi_i) \geq 1$ for $i = 1, 2$. If $\mathcal{H}_3(x,y) = y \prod_{\mathtt{i}=1}^{r-1} (\beta_{\mathtt{i}} - x)^{a_{\mathtt{i}}} + h(x) \in \mathfrak{F}_3$, then

$$\mathcal{H}_3 \circ \psi = \psi_2 \prod_{\mathtt{i}=1}^{r-1} (\beta_{\mathtt{i}} - \psi_1)^{a_{\mathtt{i}}} + h(\psi_1).$$

Since $\deg(h(x)) < \sum_{\mathtt{i}=1}^{r-1} a_{\mathtt{i}}$ and $n_1, n_2 \geq 1$,

$$\deg(\mathcal{H}_3 \circ \psi) = n_2 + n_1 \sum_{\mathtt{i}=1}^{r-1} a_{\mathtt{i}} \geq 1 + \sum_{\mathtt{i}=1}^{r-1} a_{\mathtt{i}} = \deg(\mathcal{H}_3).$$

Now let $\mathcal{H}_2(x, y) = x^{p_1}(x^k y + P(x))^p \prod_{\mathtt{i}=1}^{r-1}(\beta_{\mathtt{i}} - x^{q_1}(x^k y + P(x))^q)^{a_{\mathtt{i}}} \in \mathfrak{F}_2$. Therefore,

$$\mathcal{H}_2 \circ \psi = \psi_1^{p_1}\left(\psi_1^k \psi_2 + P(\psi_1)\right)^p \prod_{\mathtt{i}=1}^{r-1} \left(\beta_{\mathtt{i}} - \psi_1^{q_1}\left(\psi_1^k \psi_2 + P(\psi_1)\right)^q\right)^{a_{\mathtt{i}}}.$$

Hence, $\deg(\mathcal{H}_2 \circ \psi) = p_1 n_1 + p(kn_1 + n_2) + (q_1 n_1 + q(kn_1 + n_2)) \sum_{\mathtt{i}=1}^{r-1} a_{\mathtt{i}}$. As $n_1, n_2 \geq 1$, we obtain

$$\deg(\mathcal{H}_2 \circ \psi) \geq p_1 + p(k+1) + (q_1 + q(k+1)) \sum_{\mathtt{i}=1}^{r-1} a_{\mathtt{i}} = \deg(\mathcal{H}_2).$$

Finally, for $\mathcal{H}_1(x, y) \in \mathfrak{F}_1$ we have an analogous computation as in the previous case, which we leave to the reader. □

Remark 5. *If we consider $\mathcal{H}_3 \in \mathfrak{F}_3$ with $r = 1$, then we have $\mathcal{H}_3(x, y) = y$. This polynomial is of type $(0, 1)$, that is, $\dim H_1(\mathcal{L}_{\mathfrak{c}}, \mathbb{Z}) = 0$. Hence, from now on we will consider only polynomials in $\mathfrak{F}_3$ with $r \geq 2$.*

The Miyanishi–Sugie and Neumann–Norbury classifications of primitive polynomials of type $(0, 2)$ are equivalent, our accurate assertion is as follows.

Lemma 10. *The normal forms of primitive polynomials of type $(0, 2)$ provided by Theorem 7 and Theorem 8 are algebraically equivalent.*

Proof. In Theorem 8, the normal forms of polynomials of type $(0, 2)$ are given by $\mathcal{H}_2 \in \mathfrak{F}_2$ with $r = 1$ and $\mathcal{H}_3 \in \mathfrak{F}_3$ with $r = 2$.

Consider $\mathcal{H}_2$ with $r = 1$, thus

$$\mathcal{H}_2(x, y) = x^{p_1}\Big(x^k y + P(x)\Big)^p. \tag{27}$$

We have the following four cases below.

Case 1. Assume that $0 < p_1 < p$ and $P(x) \equiv 0$. Then

$$\mathcal{H}_2(x, y) = x^{p_1 + pk} y^p. \tag{28}$$

If we rename the parameters $p_1 + pk$ and p by k and r, respectively, we then obtain polynomials in Theorem 7 with $1 < r < k$ and $l = 0$. Moreover, each polynomial in Theorem 7 with $1 < r < k$ and $l = 0$ can be obtained from (28). Indeed, we have $k = \mu r + \nu > r > 1$, with $\mu \geq 1$ and $0 < \nu < r$, then by using $p_1 = \nu$, $p = r$ and $k = \mu$ in (28), we get the desired polynomial. We note that polynomials $x^k y^r$, with $k < r$, in Theorem 7 are algebraically equivalent to $x^k y^r$, with $r < k$, by interchanging the variables.

Case 2. Assume that $0 < p_1 < p$, $P(x) \not\equiv 0$ and $P(0) = 0$. Thus, $P(x) = x^s \widetilde{P}(x)$, with $1 \leq s \leq k - 1$, $\widetilde{P}(x) \in \mathbb{C}[x]_{\leq k-s-1}$ and $\widetilde{P}(0) \neq 0$, then

$$\mathcal{H}_2(x, y) = x^{p_1 + ps}\Big(x^{k-s} y + \widetilde{P}(x)\Big)^p. \tag{29}$$

If we rename the parameters p_1+ps, p, $k-s$ and $\widetilde{P}(x)$ as k, r, l and $P(x)$, respectively, we then obtain polynomials in Theorem 7 with $1<r<k$ and $l\neq 0$. Moreover, each polynomial in Theorem 7 with $1<r<k$ and $l\neq 0$ can be obtained from (29). Indeed, we have $k=\mu r+\nu>r>1$, with $\mu\geq 1$ and $0<\nu<r$, and then by using $p_1=\nu$, $p=r$, $k=l+\mu$, $s=\mu$ and $\widetilde{P}(x)=P(x)$ in (29), we get the desired polynomial.

Case 3. Assume that $0<p_1<p$, $P(x)\not\equiv 0$ and $P(0)\neq 0$. Then the resulting polynomials in (27) are in correspondence with the polynomials in Theorem 7, satisfying $1\leq k<r$ and $l\neq 0$, by renaming the parameters p_1, p and k as k, r and l, respectively.

Case 4. Assume that $0=p_1$, so $p=1$. If $P(x)\equiv 0$, then $\mathcal{H}_2(x,y)=x^k y$. Thus, we obtain all polynomials in Theorem 7 with $1=r\leq k$ and $l=0$. If $P(x)\not\equiv 0$ and $P(0)=0$, then $P(x)=x^s\widetilde{P}(x)$, with $1\leq s\leq k-1$, $\widetilde{P}(x)\in\mathbb{C}[x]_{\leq k-s-1}$ and $\widetilde{P}(0)\neq 0$. Thus, $\mathcal{H}_2(x,y)=x^s\left(x^{k-s}y+\widetilde{P}(x)\right)$. Hence, if we rename s, $k-s$ and $\widetilde{P}(x)$ as k, l and $P(x)$, respectively, then we obtain the polynomials in Theorem 7 with $1=r\leq k$ and $l\neq 0$. Finally, if $P(x)\not\equiv 0$ and $P(0)\neq 0$, then by using the automorphism $\sigma(c)=c-P(0)$, the normal form $\mathcal{H}_2(x,y)$ reduces to one of the two previous situations considered in this case.

We now consider $\mathcal{H}_3\in\mathfrak{F}_3$ with $r=2$, thus

$$\mathcal{H}_3(x,y)=y(\beta_1-x)^{a_1}+h(x).$$

If we take $\mathcal{H}_2\in\mathfrak{F}_2$ with $r=1$, $p_1=0$ and $p=1$, then $\mathcal{H}_2(x,y)=x^k y+P(x)$. Hence, by taking $a_1=k$ and a translation in the x-axis, these two polynomials $\mathcal{H}_2(x,y)$ and $\mathcal{H}_3(x,y)$ are algebraically equivalent. This completes the proof. □

4.2. **Degree of the transformed polynomials and 1-forms.** In order to get an upper bound for $\mathscr{N}(H,\omega)$ through $\mathscr{N}(\mathcal{H},\vartheta)$, we must control the degrees of the transformed $\mathcal{H}$ and ϑ. For primitive polynomials with trivial global monodromy on $\mathbb{C}^2$, we can control the degree of the transformed objects explicitly. More precisely, we have the next result, which represents an improvement of Lemma 5.

Proposition 11. *Let $H(u,v)$ be a primitive polynomial with trivial global monodromy of degree $m+1$, with $\mathfrak{r}=\dim H_1(L_c,\mathbb{Z})\geq 1$, and let $\omega\in\Omega^1(\mathbb{C}^2_{u\,v})_{\leq n}$. Consider the normal form $\mathcal{H}(x,y)\in\mathfrak{F}_1\cup\mathfrak{F}_2\cup\mathfrak{F}_3$ of $H(u,v)$ through the automorphisms (ψ,σ), and the 1-form $\vartheta=\sigma'\psi_*(\omega)$ as in (12).*

1) If $\mathcal{H}\in\mathfrak{F}_1$, then $r+1=\dim H_1(\mathcal{L}_c,\mathbb{Z})=\mathfrak{r}\geq 3$ and

$$7\leq\deg(\mathcal{H})\leq m+1,\qquad \deg(\vartheta)\leq(n+1)\left[\frac{m-\mathfrak{r}}{\mathfrak{r}}\right]-1.$$

2) If $\mathcal{H}\in\mathfrak{F}_2$, then $r=\dim H_1(\mathcal{L}_c,\mathbb{Z})=\mathfrak{r}\geq 1$ and

$$\begin{cases} 2\leq\deg(\mathcal{H})\leq m+1, & \deg(\vartheta)\leq(n+1)(m)-1, \qquad \text{if } \mathfrak{r}=1;\\ 7\leq\deg(\mathcal{H})\leq m+1, & \deg(\vartheta)\leq(n+1)\left[\dfrac{m-\mathfrak{r}-1}{\mathfrak{r}+1}\right]-1, \quad \text{if } \mathfrak{r}\geq 2.\end{cases}$$

3) If $\mathcal{H}\in\mathfrak{F}_3$, then $r-1=\dim H_1(\mathcal{L}_c,\mathbb{Z})=\mathfrak{r}\geq 1$ and

$$\mathfrak{r}+1 \leq \deg(\mathcal{H}) \leq m+1, \qquad \deg(\vartheta) \leq (n+1)(m+1-\mathfrak{r})-1.$$

Proof. Let $(\psi,\sigma) \in \mathrm{Aut}(\mathbb{C}^2) \times \mathrm{Aut}(\mathbb{C})$ be a pair of polynomial automorphisms such that

$$H = \sigma^{-1} \circ \mathcal{H} \circ \psi,$$

as in equation (10). Since σ^{-1} is an affine automorphism,

$$m+1 = \deg(H) = \deg(\mathcal{H} \circ \psi) \tag{30}$$

and

$$\deg(\vartheta) = \deg(\psi_*(\omega)) = \deg(\omega)\deg(\psi^{-1}) + \deg(\psi^{-1}) - 1. \tag{31}$$

Now, let ψ_1 and ψ_2 be the two polynomial components of ψ with degrees $n_1 \geq 1$ and $n_2 \geq 1$, respectively.

For simplicity, we begin proving statement 3). Assume that $\mathcal{H}(x,y) \in \mathfrak{F}_3$, then

$$\mathcal{H} \circ \psi = \psi_2 \prod_{\mathtt{i}=1}^{r-1} (\beta_{\mathtt{i}} - \psi_1)^{a_{\mathtt{i}}} + h(\psi_1). \tag{32}$$

Thus, from equations (30) and (32) we get

$$m+1 = n_2 + n_1 \sum_{\mathtt{i}=1}^{r-1} a_{\mathtt{i}}. \tag{33}$$

Since $r-1 = \dim H_1(\mathcal{L}_c,\mathbb{Z}) = \dim H_1(L_c,\mathbb{Z}) = \mathfrak{r} \geq 1$, $r \geq 2$. Thus, $\sum_{\mathtt{i}=1}^{r-1} a_{\mathtt{i}} \geq \mathfrak{r} \geq 1$. Moreover, as $n_1, n_2 \geq 1$, from equation (33) it follows that

$$m+1 \geq 1 + \sum_{\mathtt{i}=1}^{r-1} a_{\mathtt{i}} = \deg(\mathcal{H}) \geq 1+\mathfrak{r} \qquad \text{and} \qquad m+1 \geq n_2 + n_1\mathfrak{r}.$$

Thus, $n_1 \leq m/\mathfrak{r}$ and $n_2 \leq m+1-\mathfrak{r}$. Hence, the degree of ψ is at most $m+1-\mathfrak{r}$. This implies that $\deg(\psi^{-1}) \leq m+1-\mathfrak{r}$; see [4, 7]. Therefore, from (31) we have

$$\deg(\vartheta) \leq (n+1)(m+1-\mathfrak{r})-1.$$

Now we will prove statement 2). Assume that $\mathcal{H}(x,y) \in \mathfrak{F}_2$, and then

$$\mathcal{H} \circ \psi = \psi_1^{p_1}\big(\psi_1^k\psi_2 + P(\psi_1)\big)^p \prod_{\mathtt{i}=1}^{r-1} \Big(\beta_{\mathtt{i}} - \psi_1^{q_1}\big(\psi_1^k\psi_2 + P(\psi_1)\big)^q\Big)^{a_{\mathtt{i}}}.$$

Hence, from equations (30) and (32)

$$m+1 = p_1n_1 + p(kn_1+n_2) + \big(q_1n_1 + q(kn_1+n_2)\big)\sum_{\mathtt{i}=1}^{r-1} a_{\mathtt{i}}. \tag{34}$$

In this case, $r = \dim H_1(\mathcal{L}_c,\mathbb{Z}) = \dim H_1(L_c,\mathbb{Z}) = \mathfrak{r} \geq 1$ implies that $r \geq 1$. We will consider two possibilities: $r = \mathfrak{r} = 1$ and $r = \mathfrak{r} \geq 2$.

In the former, as n_1 and n_2 are positive integers, $k \geq 1$ and $0 \leq p_1 < p$, $0 \leq q_1 < q$ with $pq_1 - qp_1 = \pm 1$, so equation (34) then yields

$$m+1 \geq p_1 + p(k+1) = \deg(\mathcal{H}) \geq 2 \quad \text{and} \quad m+1 \geq n_1+n_2.$$

Thus, $n_1 \leq m$ and $n_2 \leq m$. Hence, $\deg(\psi) \leq m$. This implies that $\deg(\psi^{-1}) \leq m$. Therefore, from equation (31) we have

$$\deg(\vartheta) \leq (n+1)(m)-1.$$

In the latter, $\sum_{\mathtt{i}=1}^{r-1} a_{\mathtt{i}} \geq r-1=\mathfrak{r}-1 \geq 1$. As n_1 and n_2 are positive integers, $k \geq 1$ and $0 \leq p_1 < p$, $0 \leq q_1 < q$ with $pq_1 - qp_1 = \pm 1$, then equation (34) gives

$$m+1 \geq p_1 + p(k+1) + (q_1 + q(k+1))(r-1) = \deg(\mathcal{H}) \geq 7. \tag{35}$$

Moreover, if $p_1 = 0$ or $q_1 = 0$, then from equation (34), together with the conditions $pq_1 - qp_1 = \pm 1$ and $r \geq 2$, we obtain

$$m+1 \geq (r+2)n_1 + (r+1)n_2.$$

Thus, since $n_1, n_2 \geq 1$,

$$n_1 \leq \left[\frac{m-r}{r+2}\right] \qquad \text{and} \qquad n_2 \leq \left[\frac{m-r-1}{r+1}\right].$$

In addition, by supposing $n_1 = n_2 = 1$, we obtain $m \geq 2r+2$, which implies

$$\left[\frac{m-r}{r+2}\right] \leq \left[\frac{m-r-1}{r+1}\right].$$

Now, if $p_1 \geq 1$ and $q_1 \geq 1$, then $p \geq 2$ and $q \geq 2$. Thus, from equation (34) we get

$$m+1 \geq n_1 + 2(n_1+n_2) + (n_1 + 2(n_1+n_2))(r-1) = 3n_1 r + 2n_2 r.$$

Again, since $n_1, n_2 \geq 1$

$$n_1 \leq \left[\frac{m+1-2r}{3r}\right] \qquad \text{and} \qquad n_2 \leq \left[\frac{m+1-3r}{2r}\right].$$

Moreover, by supposing $n_1 = n_2 = 1$, we get $m \geq 5r \geq 2r+2$. Hence, we have

$$\left[\frac{m+1-2r}{3r}\right] \leq \left[\frac{m-r-1}{r+1}\right] \qquad \text{and} \qquad \left[\frac{m+1-3r}{2r}\right] \leq \left[\frac{m-r-1}{r+1}\right].$$

Therefore, in any case $\deg(\psi) \leq [(m-r-1)/(r+1)]$. This implies that $\deg(\psi^{-1}) \leq [(m-r-1)/(r+1)]$; see [4, 7]. From (31) we obtain

$$\deg(\vartheta) \leq (n+1)\left[\frac{m-r-1}{r+1}\right] - 1 = (n+1)\left[\frac{m-\mathfrak{r}-1}{\mathfrak{r}+1}\right] - 1.$$

Finally, if $H \in \mathfrak{F}_1$, we then have the same situation as in the second part of the previous case, with $r+1=\mathfrak{r}$. Thus, the degree of ψ^{-1} is at most $[(m-\mathfrak{r})/\mathfrak{r}]$. Therefore,

$$\deg(\vartheta) \leq (n+1)\left[\frac{m-\mathfrak{r}}{\mathfrak{r}}\right] - 1.$$

This completes the proof. □

4.3. **Rectifying birational maps.** The birational map $\mathcal{R}_\iota$, in (24), satisfies the commutative diagram

$$\begin{array}{ccc} \mathbb{C}^2_{x\,y} & \overset{\mathcal{R}_\iota}{\dashrightarrow} & \mathbb{C}^2_{t\,\mathfrak{c}} \\ {\scriptstyle \mathcal{H}}\downarrow & & \downarrow{\scriptstyle \mathfrak{c}} \\ \mathbb{C}_{\mathfrak{c}} & \underset{id}{\longrightarrow} & \mathbb{C}_{\mathfrak{c}}, \end{array}$$

where $\mathfrak{c}$ is the projection in the second component. Thus, $\mathcal{R}_\iota$ rectifies the generic fibers of $\mathcal{H}_\iota$ into punctured horizontal lines in $\mathbb{C}^2_{t\,\mathfrak{c}}$. The accurate study of this property is the next step towards the proof of Theorem 1. Furthermore, $\mathcal{R}_\iota$ will allow us to establish the equivalence between the Abelian integral $\mathcal{I}_{\mathtt{i}}(\mathfrak{c})$ defined by the pair $(\mathcal{H}_\iota, \vartheta)$ and the

Abelian integral $J_{\mathtt{i}}(\mathfrak{c})$ defined by the pair $(\mathfrak{c}, \eta)$, where η is the corresponding rational 1-form. Owing to the relevance of these consequences in the proof of our main result, we will state some properties of $\mathcal{R}_\iota = (\mathcal{G}_\iota, \mathcal{H}_\iota)$ as follows.
Notation. For the sake of simplicity, we omit the subscript ι of $\mathcal{G}_\iota$, $\mathcal{H}_\iota$ and $\mathcal{R}_\iota$ when appropriate.

Lemma 12. *Let $\mathcal{R} = (\mathcal{G}, \mathcal{H})$ be a birational map, as in* (24).
1) There is a suitable algebraic subset $\mathfrak{D}$ of $\mathbb{C}^2_{t\,\mathfrak{c}}$ such that $\mathcal{R}$ is a biholomorphic map as follows

$$\begin{array}{ccccc} \mathbb{C}^2_{x\,y}\backslash\Sigma(\mathcal{R}) & \xrightarrow{\mathcal{R}} & \mathbb{C}^2_{t\,\mathfrak{c}}\backslash\mathfrak{D} & \xrightarrow{\mathcal{R}^{-1}} & \mathbb{C}^2_{x\,y}\backslash\Sigma(\mathcal{R}) \\ (x,y) & \longmapsto & (\mathcal{G}(x,y), \mathcal{H}(x,y)) & \longmapsto & \left(\frac{M(t,\mathfrak{c})}{N(t,\mathfrak{c})}, \frac{S(t,\mathfrak{c})}{T(t,\mathfrak{c})}\right), \end{array} \tag{36}$$

where $\Sigma(\mathcal{R}) \doteq \{\mathcal{G}_x\mathcal{H}_y - \mathcal{G}_y\mathcal{H}_x = 0\} \subset \mathbb{C}^2_{x\,y}$ is the ramification locus of $\mathcal{R}$, and M, N, S, T are suitable polynomials. Moreover,

$$\mathcal{H}(\Sigma(\mathcal{R})) \subset \mathfrak{B}(\mathcal{H}). \tag{37}$$

2) On $\mathbb{C}^2_{x\,y}\backslash\Sigma(\mathcal{R})$, the map $\mathcal{R}$ rectifies the foliations $d\mathcal{H} = 0$ and $d\mathcal{G} = 0$, as follows

$$\mathcal{R}_*(d\mathcal{H}(x,y)) = d\mathfrak{c} \quad \text{and} \quad \mathcal{R}_*(d\mathcal{G}(x,y)) = dt. \tag{38}$$

3) For each generic value $\mathfrak{c} \in \mathbb{C}\backslash\mathfrak{B}(\mathcal{H})$ of $\mathcal{H}$, the map $\mathcal{R}$ rectifies the corresponding fiber $\mathcal{L}_\mathfrak{c}$ biholomorphically into the punctured horizontal line

$$\begin{gathered} \mathbb{C}\backslash\{0, \beta_1, \ldots, \beta_{r-1}, \mathfrak{c}\} \times \{\mathfrak{c}\}, \qquad \mathbb{C}\backslash\{0, \beta_1, \ldots, \beta_{r-1}\} \times \{\mathfrak{c}\}, \\ \text{or} \;\; \mathbb{C}\backslash\{\beta_1, \ldots, \beta_{r-1}\} \times \{\mathfrak{c}\} \end{gathered} \tag{39}$$

in $\mathbb{C}^2_{t\,\mathfrak{c}}$, associated to the families $\mathfrak{F}_1$, $\mathfrak{F}_2$, $\mathfrak{F}_3$, respectively.

Figure 1 illustrates the rectified foliations on $\mathbb{C}^2_{t\,\mathfrak{c}}$.

Proof. We begin by recalling in Table 1 the expressions $(\mathcal{G}, \mathcal{H})$ that define the maps $\mathcal{R}$.

	$\mathcal{G}$	$\mathcal{H}$
$\mathcal{H} \in \mathfrak{F}_1,\ r \geq 1$	$x^{q_1}\mathcal{S}(x,y)^q$	$\mathcal{G}(x,y) + x^{p_1}\mathcal{S}(x,y)^p \prod_{\mathtt{i}=1}^{r-1} \left(\beta_{\mathtt{i}} - \mathcal{G}(x,y)\right)^{a_{\mathtt{i}}}$
$\mathcal{H} \in \mathfrak{F}_2,\ r \geq 1$	$x^{q_1}\mathcal{S}(x,y)^q$	$x^{p_1}\mathcal{S}(x,y)^p \prod_{\mathtt{i}=1}^{r-1} \left(\beta_{\mathtt{i}} - \mathcal{G}(x,y)\right)^{a_{\mathtt{i}}}$
$\mathcal{H} \in \mathfrak{F}_3,\ r \geq 2$	x	$y \prod_{\mathtt{i}=1}^{r-1} (\beta_{\mathtt{i}} - \mathcal{G}(x,y))^{a_{\mathtt{i}}} - h(x)$

Table 1. Components of the rectifying map $\mathcal{R} = (\mathcal{G}, \mathcal{H})$.

Since $\mathcal{R}$ is a rational map, some fibers of $\mathcal{H}$ a priori can be contracted, our interest is in the behavior under $\mathcal{R}$ of the generic fibers. If we prove equation (37), then $\mathcal{R}$ will map each generic fiber $\mathcal{L}_\mathfrak{c}$ of $\mathcal{H}$ into a horizontal line in $\mathbb{C}^2_{t\,\mathfrak{c}}$, which has punctures because $\mathcal{H}$ is of type $(0, \kappa)$ with $\kappa \geq 2$. The proof of equation (37) is as follows. In

Table 2, we provided the expression of the critical set $\Sigma(\mathcal{R})$. By using the last column of Tables 1 and 2, we then obtained $\mathcal{H}(\Sigma(\mathcal{R}))$, which appears in Table 3.

	$\Sigma(\mathcal{R})$
$\mathcal{H} \in \mathfrak{F}_1 \cup \mathfrak{F}_2$	$\left\{x^{k+p_1+q_1-1}\mathcal{S}(x,y)^{p+q-1}\prod_{\mathtt{i}=1}^{r-1}(\beta_{\mathtt{i}}-\mathcal{G}(x,y))^{a_{\mathtt{i}}}=0\right\}$
$\mathcal{H} \in \mathfrak{F}_3,\ r \geq 2$	$\left\{\prod_{\mathtt{i}=1}^{r-1}(\beta_{\mathtt{i}}-x)^{a_{\mathtt{i}}}=0\right\}$

TABLE 2. Computation of the critical set $\Sigma(\mathcal{R})$

		$\mathcal{H}(\Sigma(\mathcal{R}))$
$\mathcal{H} \in \mathfrak{F}_1$	$p_1, q_1 \geq 1$	$\{0, \beta_1, \dots, \beta_{r-1}\}$
	$q_1 = 0$	$\{P(0), 0, \beta_1, \dots, \beta_{r-1}\}$
	$p_1 = 0$	$\left\{P(0)\prod_{\mathtt{i}=1}^{r-1}\beta_{\mathtt{i}}^{a_{\mathtt{i}}}, 0, \beta_1, \dots, \beta_{r-1}\right\}$
$\mathcal{H} \in \mathfrak{F}_2$	$p_1 \geq 1$	$\{0\}$
	$p_1 = 0$	$\left\{P(0)\prod_{\mathtt{i}=1}^{r-1}\beta_{\mathtt{i}}^{a_{\mathtt{i}}}, 0\right\}$
$\mathcal{H} \in \mathfrak{F}_3,\ r \geq 2$		$\{h(\beta_1), \dots, h(\beta_{r-1})\}$

TABLE 3. Image of $\Sigma(\mathcal{R})$ under $\mathcal{H}$

Assume that $\mathcal{H}(x,y) \in \mathfrak{F}_1$. Since $p, q \geq 1$, the polynomial $\mathcal{S}(x,y)$ divides $\mathcal{H}(x,y)$. Moreover, for each $\mathtt{i} = 1, \dots, r-1$, we have that $\beta_{\mathtt{i}} - \mathcal{G}(x,y)$ divides $\mathcal{H}(x,y) - \beta_{\mathtt{i}}$. That implies that $\{0, \beta_1, \dots, \beta_{r-1}\} \subset \mathfrak{B}(\mathcal{H})$. In addition, if $q_1 = 0$, then $p_1 = 1$. Furthermore, x divides $\mathcal{G}(x,y) - P(0)$, which implies that x divides $\mathcal{H}(x,y) - P(0)$, whence $P(0) \in \mathfrak{B}(\mathcal{H})$. If $p_1 = 0$, then $p = q_1 = 1$ and x divides $\mathcal{S}(x,y)\prod_{\mathtt{i}=1}^{r-1}(\beta_{\mathtt{i}} - \mathcal{G}(x,y))^{a_{\mathtt{i}}} - P(0)\prod_{\mathtt{i}=1}^{r-1}\beta_{\mathtt{i}}^{a_{\mathtt{i}}}$, which implies that x divides $\mathcal{H}(x,y) - P(0)\prod_{\mathtt{i}=1}^{r-1}\beta_{\mathtt{i}}^{a_{\mathtt{i}}}$, from which $P(0)\prod_{\mathtt{i}=1}^{r-1}\beta_{\mathtt{i}}^{a_{\mathtt{i}}} \in \mathfrak{B}(\mathcal{H})$. This proves that if $\mathcal{H}(x,y) \in \mathfrak{F}_1$, then $\mathcal{H}(\Sigma(\mathcal{R})) \subset \mathfrak{B}(\mathcal{H})$.

Analogously, we can prove this last property for $\mathcal{H}(x,y) \in \mathfrak{F}_2$ and $\mathcal{H}(x,y) \in \mathfrak{F}_3$.

In order to compute the explicit inverse map $\mathcal{R}^{-1}$, we use $t = \mathcal{G}(x,y)$ and $\mathfrak{c} = \mathcal{H}(x,y)$. We start with the simplest case in which $\mathcal{H} \in \mathfrak{F}_3$ and $\mathcal{G}(x,y) = x$. We then have $t = x$ and $\mathfrak{c} = y\prod_{\mathtt{i}=1}^{r-1}(\beta_{\mathtt{i}} - x)^{a_{\mathtt{i}}} + h(x)$, from which we get

$$x = t \quad \text{and} \quad y = \frac{\mathfrak{c} - h(t)}{\prod_{\mathtt{i}=1}^{r-1}(\beta_{\mathtt{i}} - t)^{a_{\mathtt{i}}}}.$$

In this case, $\Sigma(\mathcal{R}) = \{\prod_{\mathtt{i}=1}^{r-1}(\beta_{\mathtt{i}} - x) = 0\}$, $\mathfrak{D} = \{\prod_{\mathtt{i}=1}^{r-1}(\beta_{\mathtt{i}} - t) = 0\}$ and $\mathcal{R}^{-1}$ is well defined in $\mathbb{C}^2_{t\,\mathfrak{c}}\backslash\mathfrak{D}$. In addition, the diagram (36) takes the explicit form

$$\begin{array}{ccccc} \mathbb{C}^2_{x\,y}\backslash\Sigma(\mathcal{R}) & \xrightarrow{\mathcal{R}} & \mathbb{C}^2_{t\,\mathfrak{c}}\backslash\mathfrak{D} & \xrightarrow{\mathcal{R}^{-1}} & \mathbb{C}^2_{x\,y}\backslash\Sigma(\mathcal{R}) \\ (x,y) & \longmapsto & (\mathcal{G}(x,y),\mathcal{H}(x,y)) & \longmapsto & \left(t, \dfrac{\mathfrak{c}+h(t)}{\prod_{\mathtt{i}=1}^{r-1}(\beta_{\mathtt{i}}-t)^{a_{\mathtt{i}}}}\right). \end{array} \tag{40}$$

Thus, each generic fiber $\mathcal{L}_{\mathfrak{c}}$ of $\mathcal{H}$ is biholomorphically mapped into the punctured horizontal line

$$\big(\mathbb{C}\backslash\{\beta_1,\ldots,\beta_{r-1}\}\big)\times\{\mathfrak{c}\}.$$

Analogously, straightforward computations show that for $\mathcal{H}\in\mathfrak{F}_2$ and $pq_1 - qp_1 = 1$, we obtain

$$\begin{array}{ccccc} \mathbb{C}^2_{x\,y}\backslash\Sigma(\mathcal{R}) & \xrightarrow{\mathcal{R}} & \mathbb{C}^2_{t\,\mathfrak{c}}\backslash\mathfrak{D} & \xrightarrow{\mathcal{R}^{-1}} & \mathbb{C}^2_{x\,y}\backslash\Sigma(\mathcal{R}) \\ (x,y) & \longmapsto & \big(\mathcal{G}(x,y),\mathcal{H}(x,y)\big) & \longmapsto & \left(\dfrac{t^p\Pi(t)^q}{\mathfrak{c}^q}, \dfrac{\mathfrak{c}^q S_1(t,\mathfrak{c})}{t^{pk+p_1}\Pi(t)^{qk+q_1}}\right), \end{array} \tag{41}$$

where $\mathcal{G}(x,y)$ is in Table 1, $\Sigma(\mathcal{R})$ is in Table 2, whence $\mathfrak{D} = \{\mathfrak{c}\,t\,\Pi(t) = 0\}$,

$$\Pi(t) \doteq \prod_{\mathtt{i}=1}^{r-1}(\beta_{\mathtt{i}} - t)^{a_{\mathtt{i}}} \tag{42}$$

and

$$S_1(t,\mathfrak{c}) \doteq \mathfrak{c}^{q(k-1)}\Big(\mathfrak{c}^{q_1} + t^{p_1}\Pi(t)^{q_1} P\left(t^p\Pi(t)^q\mathfrak{c}^{-q}\right)\Big), \tag{43}$$

which is polynomial because P has degree at most $k-1$. Thus, each generic fiber $\mathcal{L}_{\mathfrak{c}}$ of $\mathcal{H}\in\mathfrak{F}_2$ is biholomorphically mapped into the punctured horizontal line

$$\big(\mathbb{C}\backslash\{0,\beta_1,\ldots,\beta_{r-1}\}\big)\times\{\mathfrak{c}\}.$$

For $\mathcal{H}\in\mathfrak{F}_1$ and $pq_1 - qp_1 = 1$, we obtain

$$\begin{array}{ccccc} \mathbb{C}^2_{x\,y}\backslash\Sigma(\mathcal{R}) & \xrightarrow{\mathcal{R}} & \mathbb{C}^2_{t\,\mathfrak{c}}\backslash\mathfrak{D} & \xrightarrow{\mathcal{R}^{-1}} & \mathbb{C}^2_{x\,y}\backslash\Sigma(\mathcal{R}) \\ (x,y) & \longmapsto & \big(\mathcal{G}(x,y),\mathcal{H}(x,y)\big) & \longmapsto & \left(\dfrac{t^p\Pi(t)^q}{(\mathfrak{c}-t)^q}, \dfrac{(\mathfrak{c}-t)^q S_2(t,\mathfrak{c})}{t^{pk+p_1}\Pi(t)^{qk+q_1}}\right). \end{array} \tag{44}$$

The elements are $\mathcal{G}(x,y)$ in Table 1, $\Sigma(\mathcal{R})$ in Table 2, $\mathfrak{D} = \{(\mathfrak{c}-t)\,t\,\Pi(t) = 0\}$ and

$$S_2(t,\mathfrak{c}) \doteq (\mathfrak{c}-t)^{q(k-1)}\Big((\mathfrak{c}-t)^{q_1} + t^{p_1}\Pi(t)^{q_1} P\left(t^p\Pi(t)^q(\mathfrak{c}-t)^{-q}\right)\Big), \tag{45}$$

which is polynomial because P has degree at most $k-1$. Thus, each generic fiber $\mathcal{L}_{\mathfrak{c}}$ of $\mathcal{H}\in\mathfrak{F}_1$ is biholomorphically mapped into the punctured horizontal line

$$\big(\mathbb{C}\backslash\{0,\beta_1,\ldots,\beta_{r-1},\mathfrak{c}\}\big)\times\{\mathfrak{c}\}.$$

For $\mathcal{H}\in\mathfrak{F}_1\cup\mathfrak{F}_2$ and $pq_1 - qp_1 = -1$, we obtain expressions similar to (41) and (44). See (90) and (93) for details. This completes the proof of statement 1).

Let $\mathcal{X}_{\mathcal{G}}$ and $\mathcal{X}_{\mathcal{H}}$ be the Hamiltonian vector fields associated with $\mathcal{G}$ and $\mathcal{H}$, respectively. These vector fields satisfy the following identities:

$$\mathcal{R}_*\left(\frac{\mathcal{X}_{\mathcal{G}}}{\Sigma(\mathcal{R})}\right) = \frac{1}{\Sigma(\mathcal{R})}\begin{pmatrix}\mathcal{G}_x & \mathcal{G}_y\\ \mathcal{H}_x & \mathcal{H}_y\end{pmatrix}\begin{pmatrix}-\mathcal{G}_y\\ \mathcal{G}_x\end{pmatrix} = \frac{\partial}{\partial\mathfrak{c}} \tag{46}$$

and

$$\mathcal{R}_*\left(\frac{\mathcal{X}_{\mathcal{H}}}{\Sigma(\mathcal{R})}\right) = \frac{1}{\Sigma(\mathcal{R})}\begin{pmatrix}\mathcal{G}_x & \mathcal{G}_y\\ \mathcal{H}_x & \mathcal{H}_y\end{pmatrix}\begin{pmatrix}-\mathcal{H}_y\\ \mathcal{H}_x\end{pmatrix} = -\frac{\partial}{\partial t}. \tag{47}$$

Clearly, the above equations (46) and (47) prove assertion 2).

Finally, statement 3) also follows from identities (46) and (47), together with the proof of statement 1). □

Remark 6. *As a fortunate geometric situation, the punctures in the horizontal lines* $\mathcal{R}(\mathcal{L}_{\mathfrak{c}})$ *determine an arrangement of lines* $\mathcal{A}_\iota$ *for each family* $\mathfrak{F}_\iota$*, as follows:*

$$\mathcal{A}_1 \doteq \cup_{\mathtt{i}=0}^{r}\{t=\beta_{\mathtt{i}}\} \subset \mathbb{C}^2_{t\,\mathfrak{c}}, \quad \text{with } \beta_0 = 0,\ \beta_r = \mathfrak{c},$$

$$\mathcal{A}_2 \doteq \cup_{\mathtt{i}=0}^{r-1}\{t=\beta_{\mathtt{i}}\} \subset \mathbb{C}^2_{t\,\mathfrak{c}} \quad \text{and} \quad \mathcal{A}_3 \doteq \cup_{\mathtt{i}=1}^{r-1}\{t=\beta_{\mathtt{i}}\} \subset \mathbb{C}^2_{t\,\mathfrak{c}},$$

see Figure 1. For notational simplicity, we will omit the subscript $\iota = 1, 2, 3$ *in* $\mathcal{A}_\iota$*.*

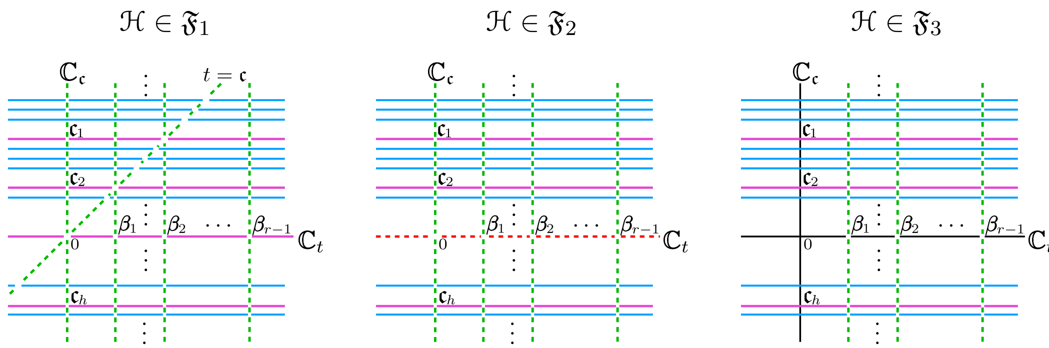

FIGURE 1. A sketch of the global rectification of the foliation $d\mathcal{H} = 0$ under $\mathcal{R}\colon \mathbb{C}^2_{x\,y} \dashrightarrow \mathbb{C}^2_{t\,\mathfrak{c}}$, for the Neumann–Norbury families $\mathfrak{F}_1$, $\mathfrak{F}_2$ and $\mathfrak{F}_3$. The blue horizontal lines are the image under $\mathcal{R}$ of the generic fibers of $\mathcal{H}$. The dashed (red and green) lines have been removed from $\mathbb{C}^2_{t\,\mathfrak{c}}$ so that $\mathcal{R}^{-1}$ is well defined. The green lines correspond to the arrangement $\mathcal{A}$ and determine punctures in the blue horizontal lines. The magenta horizontal lines are the image under $\mathcal{R}$ of some connected components of singular fibers coming from values in the bifurcation set $\mathfrak{B}(\mathcal{H}) = \{\mathfrak{c}_1, \mathfrak{c}_2, \ldots, \mathfrak{c}_h\}$.

4.4. **Rational invariance of the weak infinitesimal Hilbert's 16th problem.** According to our Program, below we provide the accurate statement about the $\mathcal{R}$-equivalence between the differential equations given in (18) and (19), as well as between the corresponding Abelian integrals (20) and their number of zeros (21). In fact, the following result remains true without the hypothesis of trivial global monodromy.

Corollary 13 (Rational invariance of the weak infinitesimal Hilbert's 16th problem)**.** *Consider* $\mathcal{H}(x, y) \in \mathfrak{F}_1 \cup \mathfrak{F}_2 \cup \mathfrak{F}_3$ *a primitive polynomial with trivial global monodromy in normal form and a polynomial 1-form* $\vartheta \in \Omega^1(\mathbb{C}^2_{x\,y})$*. Let* $\mathcal{R}$ *be the rational rectifying map for* $\mathcal{H}(x, y)$ *and let* $\eta = \mathcal{R}_*(\vartheta)$*.*

1) The corresponding infinitesimal perturbed Hamiltonian differential equations are rationally equivalent, that is,

$$\mathcal{R}_*(d\mathcal{H} + \varepsilon\vartheta) = d\mathfrak{c} + \varepsilon\eta.$$

2) The Abelian integrals

$$\mathcal{I}_{\mathtt{i}}(\mathfrak{c}) = \int_{\delta_{\mathtt{i}}(\mathfrak{c})} \vartheta : \mathbb{C}\backslash\mathfrak{B}(\mathcal{H}) \longrightarrow \mathbb{C}$$

and

$$J_{\mathtt{i}}(\mathfrak{c}) = \int_{\alpha_{\mathtt{i}}(\mathfrak{c})} \eta : \mathbb{C}\backslash\mathfrak{B}(\mathcal{H}) \longrightarrow \mathbb{C}, \quad \eta = \mathcal{R}_*(\vartheta), \quad \alpha_{\mathtt{i}}(\mathfrak{c}) = \mathcal{R}\big(\delta_{\mathtt{i}}(\mathfrak{c})\big) \tag{48}$$

are rationally equivalent; moreover,

$$\mathcal{I}_{\mathtt{i}}(\mathfrak{c}) = J_{\mathtt{i}}(\mathfrak{c}) \quad \textit{for all } \mathfrak{c} \in \mathbb{C}_{\mathfrak{c}}\backslash\mathfrak{B}(\mathcal{H}).$$

3) The number of isolated zeros, counted with multiplicities, of $\mathcal{I}_{\mathtt{i}}(\mathfrak{c})$ and $J_{\mathtt{i}}(\mathfrak{c})$ in $\mathbb{C}_{\mathfrak{c}}\backslash\mathfrak{B}(\mathcal{H})$ coincides

$$Z(\mathcal{I}_{\mathtt{i}}(\mathfrak{c})) = Z(J_{\mathtt{i}}(\mathfrak{c})).$$

Proof. The first assertion follows immediately from the definition of η and equation (38). The second and third assertions follows from the construction of $J_{\mathtt{i}}(\mathfrak{c})$ and the diagram

$$\begin{array}{ccc} \mathbb{C}_{\mathfrak{c}} & \hookrightarrow & \mathbb{C}_{\mathfrak{c}} \subset \mathbb{C}^2_{t\,\mathfrak{c}} \\ \downarrow \mathcal{I}_{\mathtt{i}} & \swarrow J_{\mathtt{i}} & \\ \mathbb{C}. & & \end{array} \tag{49}$$

□

Therefore, Corollaries 3 and 13 yield our diagram (23).

The third Abelian integral $J_{\mathtt{i}}(\mathfrak{c})$, should be understood as a family of integrals of rational 1-forms on the complex lines of the horizontal foliation $d\mathfrak{c} = 0$. Thus, we will apply the residue theorem to compute $J_{\mathtt{i}}(\mathfrak{c})$ explicitly.

Given $\mathcal{H}(x, y) \in \mathfrak{F}_1 \cup \mathfrak{F}_2 \cup \mathfrak{F}_3$, we will construct a basis of cycles $\{\delta_{\mathtt{i}}(\mathfrak{c})\}$ for $H_1(\mathcal{L}_{\mathfrak{c}}, \mathbb{Z})$ on each generic fiber $\mathcal{L}_{\mathfrak{c}}$ of $\mathcal{H}(x, y)$. The choice of this basis is simple when we apply the rectifying map $\mathcal{R}$.

According to Lemma 12, for each generic value $\mathfrak{c} \in \mathbb{C}\backslash\mathfrak{B}(\mathcal{H})$, we define a basis of $H_1(\mathcal{R}(\mathcal{L}_{\mathfrak{c}}), \mathbb{Z})$ by loops

$$\{\alpha_{\mathtt{i}}(\mathfrak{c})\}$$

that enclose only one puncture $(\beta_{\mathtt{i}}, \mathfrak{c}) \in \mathbb{C}^2_{t\,\mathfrak{c}}$ in the corresponding punctured line $\mathcal{R}(\mathcal{L}_{\mathfrak{c}}) \subset \mathbb{C} \times \{\mathfrak{c}\}$, see Figure 1 again, which assumes one of the types

$$\big(\mathbb{C}\backslash\{0, \beta_1, \ldots, \beta_{r-1}, \mathfrak{c}\}\big) \times \{\mathfrak{c}\}, \quad \big(\mathbb{C}\backslash\{0, \beta_1, \ldots, \beta_{r-1}\}\big) \times \{\mathfrak{c}\},$$
$$\text{or} \quad \big(\mathbb{C}\backslash\{\beta_1, \ldots, \beta_{r-1}\}\big) \times \{\mathfrak{c}\}.$$

Secondly, according to (20), the above basis is canonically related to a basis $\{\delta_{\mathtt{i}}(\mathfrak{c})\}$ for $H_1(\mathcal{L}_{\mathfrak{c}}, \mathbb{Z})$ under the rectifying map:

$$\alpha_{\mathtt{i}}(\mathfrak{c}) = \mathcal{R}(\delta_{\mathtt{i}}(\mathfrak{c})).$$

Remark 7. *The homology basis for the fibers of* $\mathcal{H}$ *is*

$$\mathcal{B}(\mathfrak{c}) = \begin{cases} \left\{[\delta_{\mathtt{i}}(\mathfrak{c})] \mid \mathtt{i} = 0, \dots, r\right\} & \text{if } \mathcal{H} \in \mathfrak{F}_1, \\ \left\{[\delta_{\mathtt{i}}(\mathfrak{c})] \mid \mathtt{i} = 0, \dots, r-1\right\} & \text{if } \mathcal{H} \in \mathfrak{F}_2, \\ \left\{[\delta_{\mathtt{i}}(\mathfrak{c})] \mid \mathtt{i} = 1, \dots, r-1\right\} & \text{if } \mathcal{H} \in \mathfrak{F}_3. \end{cases} \tag{50}$$

Thus, there is a bijection between the Abelian integrals $\mathcal{I}_{\mathtt{i}}(\mathfrak{c})$ *and the Abelian integrals* $J_{\mathtt{i}}(\mathfrak{c})$.

4.5. **Non-exact 1-forms and the study of $\mathcal{I}_{\mathtt{i}}(\mathfrak{c})$.** As usual, $d\,\mathbb{C}[x,y]_{\leq \mathfrak{n}+1}$ denotes the *vector space of exact polynomial 1-forms with degree at most* $\mathfrak{n}$, they provide the exact perturbations for the Hamiltonian equations. Hence, it is sufficient to consider Abelian integrals for polynomial 1-forms in the *vector space of non-exact polynomial 1-forms on* $\mathbb{C}^2_{x,y}$ *of degree at most* $\mathfrak{n}$, say

$$\Omega^1_{ne}(\mathbb{C}^2_{x\,y})_{\mathfrak{n}} \doteq \frac{\Omega^1(\mathbb{C}^2_{x\,y})_{\leq \mathfrak{n}}}{d\,\mathbb{C}[x,y]_{\leq \mathfrak{n}+1}}.$$

Lemma 14. *The set*

$$B^1(\mathbb{C}^2_{x\,y}, \mathfrak{n}) \doteq \left\{\vartheta_{ij} = x^i y^j\, dx \mid j \in 1, 2, \dots, \mathfrak{n};\ i \in 0, 1, \dots, \mathfrak{n} - j\right\} \tag{51}$$

is a basis for the vector space of non-exact polynomial 1-forms $\Omega^1_{ne}(\mathbb{C}^2_{x\,y})_{\mathfrak{n}}$.

Proof. Let

$$\vartheta = \sum_{i+j=0}^{\mathfrak{n}} a_{ij} x^i y^j\, dx + \sum_{i+j=0}^{\mathfrak{n}} b_{ij} x^i y^j\, dy \in \Omega^1(\mathbb{C}^2_{x\,y})_{\leq \mathfrak{n}}$$

be a non–exact 1-form. We have that

$$b_{ij} x^i y^j\, dy = d\left(\frac{b_{ij} x^i y^{j+1}}{j+1}\right) - \frac{i b_{ij}}{j+1} x^{i-1} y^{j+1}\, dx.$$

Hence, ϑ can be written as

$$\vartheta = \sum_{i+j=0}^{\mathfrak{n}} a_{ij} x^i y^j\, dx + d\left(\sum_{i+j=0}^{\mathfrak{n}} \frac{b_{ij} x^i y^{j+1}}{j+1}\right) - \sum_{i+j=0}^{\mathfrak{n}} \frac{i b_{ij}}{j+1} x^{i-1} y^{j+1}\, dx.$$

By reordering terms, we get

$$\vartheta = d\underbrace{\left(\sum_{i=0}^{\mathfrak{n}} \frac{a_{i0} x^{i+1}}{i+1} + \sum_{i+j=0}^{\mathfrak{n}} \frac{b_{ij} x^i y^{j+1}}{j+1}\right)}_{Q(x,y)} + \sum_{j=1}^{\mathfrak{n}} \sum_{i=0}^{\mathfrak{n}-j} \tilde{a}_{ij} \underbrace{\left(x^i y^j\, dx\right)}_{\vartheta_{ij}}, \tag{52}$$

where $Q(x,y) \in \mathbb{C}[x,y]_{\leq \mathfrak{n}+1}$. The classes in the quotient arising from the 1-forms ϑ_{ij} provide the required basis. □

Remark 8. *Clearly, the basis in* (51) *is "symmetric" with respect to the choice of the variable* x *or* y. *Thus, we would also consider the basis*

$$\left\{x^i y^j\, dy \mid i \in 1, 2, \dots, \mathfrak{n};\ j \in 0, 1, \dots, \mathfrak{n} - i\right\}.$$

Let $\mathcal{H}(x,y)$ be a primitive polynomial with trivial global monodromy, it is a normal form which is furnished with the basis $\mathcal{B}(\mathfrak{c})=\{\delta_{\mathtt{i}}(\mathfrak{c})\}$ of $H_1(\mathcal{L}_c,\mathbb{Z})$, according to (50). Let $\vartheta\in\Omega^1_{ne}(\mathbb{C}^2_{x\,y})_{\mathfrak{n}}$ be an non-exact polynomial 1-form of degree $\mathfrak{n}$. Lemma 14 implies that

$$\mathcal{I}_{\mathtt{i}}(\mathfrak{c})=\int_{\delta_{\mathtt{i}}(\mathfrak{c})}\vartheta=\sum_{j=1}^{\mathfrak{n}}\sum_{i=0}^{\mathfrak{n}-j}\int_{\delta_{\mathtt{i}}(\mathfrak{c})}\vartheta_{ij}. \tag{53}$$

The rectifying map $\mathcal{R}$ for $\mathcal{H}$ gives the rational 1-form $\eta_{ij}\doteq\mathcal{R}_*(\vartheta_{ij})$, and by using the explicit expression given in (36), we have

$$\eta_{ij}=\left(\frac{M}{N}\right)^i\left(\frac{S}{T}\right)^j\left(\frac{N\frac{\partial M}{\partial t}-M\frac{\partial N}{\partial t}}{N^2}\,dt+\frac{N\frac{\partial M}{\partial\mathfrak{c}}-M\frac{\partial N}{\partial\mathfrak{c}}}{N^2}\,d\mathfrak{c}\right),$$

which can be written in the form

$$\eta_{ij}=\eta^t_{ij}+\eta^{\mathfrak{c}}_{ij}\doteq\frac{P(t,\mathfrak{c})}{N(t,\mathfrak{c})^{2+i}T(t,\mathfrak{c})^j}\,dt+\frac{Q(t,\mathfrak{c})}{N(t,\mathfrak{c})^{2+i}T(t,\mathfrak{c})^j}\,d\mathfrak{c}, \tag{54}$$

where $P(t,\mathfrak{c})$ and $Q(t,\mathfrak{c})$ are polynomials. Recalling the basis for homology (50), in $\mathbb{C}^2_{t\,\mathfrak{c}}$ the integration of η_{ij} is considered along the cycles $\{\alpha_{\mathtt{i}}(\mathfrak{c})\}$. In addition, the integral of the second part on the right-hand side of equation (54) vanishes identically. Thus,

$$\mathcal{I}_{\mathtt{i}}(\mathfrak{c})=\int_{\delta_{\mathtt{i}}(\mathfrak{c})}\vartheta=J_{\mathtt{i}}(\mathfrak{c})=\int_{\alpha_{\mathtt{i}}(\mathfrak{c})}\eta=\sum_{j=1}^{\mathfrak{n}}\sum_{i=0}^{\mathfrak{n}-j}\int_{\alpha_{\mathtt{i}}(\mathfrak{c})}\eta^t_{ij}. \tag{55}$$

The *divisor of poles of the 1-form* η^t_{ij} is

$$\left\{N(t,\mathfrak{c})^{2+i}T(t,\mathfrak{c})^j=\prod_{\mathtt{i}=0}^{r}(t-\beta_{\mathtt{i}})^{\nu(\mathtt{i})}=0\right\}\subset\mathbb{C}^2_{t\,\mathfrak{c}}, \tag{56}$$

where the appearance of the factors $(t-\beta_{\mathtt{i}})^{\nu(\mathtt{i})}$ depends on the Neumann–Norbury family $\mathfrak{F}_\iota$.

Remark 9. *The divisor of poles of η^t_{ij} is contained in the arrangement of lines $\mathcal{A}$; Figure 1 illustrates this.*

The exponent $\nu(\mathtt{i})$, where $\mathtt{i}$ enumerates the homology classes, is the maximum positive integer value such that $(t-\beta_{\mathtt{i}})^{\nu(\mathtt{i})}$ divides $N(t,\mathfrak{c})^{2+i}T(t,\mathfrak{c})^j$; that is, $\nu(\mathtt{i})$ is the multiplicity of the pole at $\{t-\beta_{\mathtt{i}}=0\}$.

In order to compute the integral of η^t_{ij} along $\alpha_{\mathtt{i}}(\mathfrak{c})$, we simplify our notation by using $\beta=\beta_{\mathtt{i}}$, $\nu=\nu(\mathtt{i})$ and $\alpha(\mathfrak{c})=\alpha_{\mathtt{i}}(\mathfrak{c})$. Thus, at $\{t-\beta=0\}$ the term η^t_{ij} has the following representation:

$$\eta^t_{ij}=\frac{R(t,\mathfrak{c})}{(t-\beta)^\nu}dt, \tag{57}$$

where $R(t,\mathfrak{c})$ is holomorphic in a small enough two-dimensional polydisc $\Delta((\beta,\mathfrak{c}_0),(\rho_1,\rho_2))$ centered at $(\beta,\mathfrak{c}_0)$, where $\mathfrak{c}_0$ is a generic value of $\mathcal{H}$.

Proposition 15. *The Abelian integral of η^t_{ij} is*

$$\int_{\alpha(\mathfrak{c})}\eta^t_{ij}=\frac{2\pi\sqrt{-1}}{(\nu-1)!}\cdot\frac{\partial^{\nu-1}R(t,\mathfrak{c})}{\partial t^{\nu-1}}\Big|_{t=\beta}. \tag{58}$$

Proof. The function $R(t,\mathfrak{c})$ in equation (57) is written as

$$R(t,\mathfrak{c}) = R_0(\mathfrak{c}) + R_1(\mathfrak{c})(t-\beta) + \cdots + R_{\nu-1}(\mathfrak{c})(t-\beta)^{\nu-1} + \widehat{R}(t,\mathfrak{c})(t-\beta)^{\nu},$$

where $\widehat{R}(t,\mathfrak{c})$ is a holomorphic function. Thus,

$$R_{\nu-1}(\mathfrak{c}) = \frac{1}{(\nu-1)!}\frac{\partial^{\nu-1}R(t,\mathfrak{c})}{\partial t^{\nu-1}}\Big|_{t=\beta}$$

and

$$\eta^t_{ij} = \left(\frac{R_0(\mathfrak{c})}{(t-\beta)^{\nu}} + \frac{R_1(\mathfrak{c})}{(t-\beta)^{\nu-1}} + \cdots + \frac{R_{\nu-1}(\mathfrak{c})}{(t-\beta)}\right) dt + \widehat{R}(t,\mathfrak{c})dt.$$

Hence, by the residue theorem, we obtain equation (58). □

5. A list of significant examples

5.1. **The harmonic oscillator.** The Hamiltonian differential equation determined by the polynomial

$$H(u,v) = (u^2+v^2)/2$$

is the *harmonic oscillator*. We will apply the four steps of our Program §3 to study the infinitesimal perturbed Hamiltonian differential equation $dH + \varepsilon\omega = 0$, where $\omega \in \Omega^1_{ne}(\mathbb{C}^2_{u\,v})_{\leq n}$.

Step 1. According to our Program, by using the linear automorphisms

$$\psi(u,v) = \Big(\tfrac{1}{\sqrt{2}}(\sqrt{2} - u - \sqrt{-1}v), \tfrac{1}{\sqrt{2}}(u - \sqrt{-1}v)\Big) \quad \text{and} \quad \sigma(c) = \mathfrak{c},$$

we get

$$\mathcal{H}(x,y) \doteq \big(\sigma \circ H \circ \psi^{-1}\big)(x,y) = y(1-x).$$

This shows that $H(u,v)$ is algebraically equivalent to $\mathcal{H}(x,y)$, which belongs to the Neumann–Norbury family $\mathfrak{F}_3$, with $r=2$, $a_1 = 1$, $\beta_1 = 1$ and $h(x) \equiv 0$. According to Corollary 3, the corresponding differential equations $dH + \varepsilon\omega = 0$ and $d\mathcal{H} + \varepsilon\vartheta = 0$ are algebraically equivalent:

$$\sigma'\psi_*(dH + \varepsilon\omega) = d\mathcal{H} + \varepsilon\vartheta, \quad \vartheta = \sigma'\psi_*(\omega). \tag{59}$$

Since ψ is linear, the degree of ϑ coincides with the degree of ω. Thus

$$(\psi,\sigma)_*\Big(\Omega^1_{ne}(\mathbb{C}^2_{u\,v})_{\leq n}\Big) = \Omega^1_{ne}(\mathbb{C}^2_{x\,y})_{\leq n} \quad \text{and} \quad \mathscr{N}(H,\omega) = \mathscr{N}(\mathcal{H},\vartheta).$$

Step 2. We now use the rectification technique. From Lemma 12, if $\mathcal{G}(x,y) = x$, then $\mathcal{R} = (\mathcal{G}(x,y), \mathcal{H}(x,y))$ is a rectifying map for $\mathcal{H}$. In this case, $\Sigma(\mathcal{R}) = \{1-x=0\}$ and equation (36) becomes

$$\begin{array}{ccccc}
\mathbb{C}^2_{x\,y}\backslash\{1-x=0\} & \xrightarrow{\mathcal{R}} & \mathbb{C}^2_{t\,\mathfrak{c}}\backslash\{1-t=0\} & \xrightarrow{\mathcal{R}^{-1}} & \mathbb{C}^2_{x\,y}\backslash\{1-x=0\} \\
(x,y) & \longmapsto & (x, y(1-x)) & \longmapsto & \big(t, \frac{\mathfrak{c}}{1-t}\big).
\end{array}$$

Thus, equation (59) is transformed into

$$\mathcal{R}_*\big(\sigma'\psi_*(dH + \varepsilon\omega)\big) = \mathcal{R}_*\big(d\mathcal{H} + \varepsilon\vartheta\big) = d\mathfrak{c} + \varepsilon\eta, \quad \eta = \mathcal{R}_*(\vartheta). \tag{60}$$

In this way, each fiber L_c of H, with $c \neq 0$, is biholomorphically mapped by (ψ,σ) into the fiber $\mathcal{L}_c$ of $\mathcal{H}$; which through $\mathcal{R}$ is biholomorphically mapped into the horizontal line $(\mathbb{C}\backslash\{1\}) \times c)$ in $\mathbb{C}^2_{t\,\mathfrak{c}}$ with one puncture. See Figure 2. Hence, H is of type $(0, 2)$. This implies that for $c \neq 0$,

$$\dim H_1(L_c, \mathbb{Z}) = \dim H_1(\mathcal{L}_c, \mathbb{Z}) = 1. \tag{61}$$

Here we are using that $\mathfrak{c} = c$ because σ is the identity. Therefore, there exists only one Abelian integral $I_1(c)$ defined by the pair (H, ω), which is algebraically equivalent to the unique Abelian integral $\mathcal{I}_1(\mathfrak{c})$ defined by the pair $(\mathcal{H}, \vartheta)$. Thus,

$$\mathscr{N}(H, \omega) = Z(I_1(c)) = \mathscr{N}(\mathcal{H}, \vartheta) = Z(\mathcal{I}_1(\mathfrak{c})).$$

Another consequence of this second step is that $\mathfrak{B}_{inf}(H) = \mathfrak{B}_{inf}(\mathcal{H}) = \emptyset$. On the other hand, $H(u, v)$ has a unique critical point at $(0, 0)$ with $H(0, 0) = 0$. Thus, $\mathfrak{B}_{fin}(H) = \mathfrak{B}_{fin}(\mathcal{H}) = \{0\}$, since $\sigma(c) = c$. Hence, $\mathfrak{B}(H) = \mathfrak{B}(\mathcal{H}) = \{0\}$.

Step 3. Let $\alpha_1(\mathfrak{c})$ be a small cycle around the puncture $(1, \mathfrak{c})$ in the line $\big(\mathbb{C}\backslash\{1\}\big) \times \{\mathfrak{c}\}$, we have the Abelian integral

$$J_1(\mathfrak{c}) = \int_{\alpha_1(\mathfrak{c})} \eta.$$

Therefore, we have obtained that

$$I_{\mathtt{i}}(c) = \mathcal{I}_{\mathtt{i}}(\mathfrak{c}) = J_{\mathtt{i}}(\mathfrak{c}), \quad \mathfrak{c} = \sigma(c) = c.$$

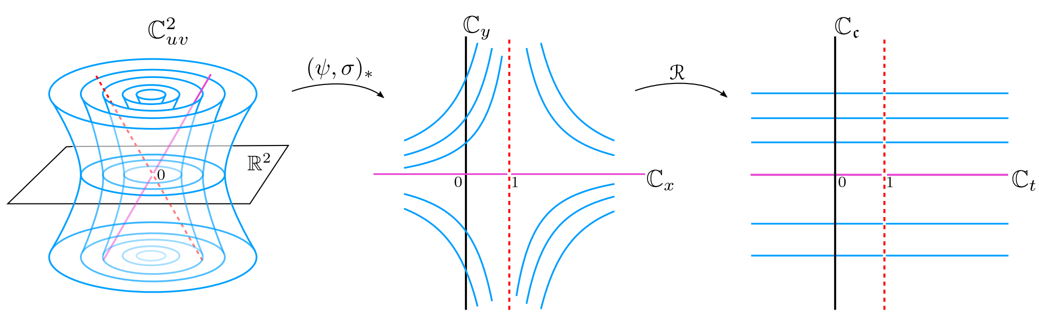

FIGURE 2. Let $H(u, v) = (u^2 + v^2)/2$, we sketch of the leaves of the foliations $dH = 0$, $d\mathcal{H} = 0$ and $d\mathfrak{c} = 0$. In a) the blue curves correspond to the generic fibers of H and the magenta curve is the connected component $\{v - \sqrt{-1}u = 0\}$ of the singular fiber L_0 of H. In b) and c) the blue and magenta curves are the image under (ψ, σ) and $\mathcal{R}$, respectively, of the blue and magenta curves in a). In a), b and c), the dashed red curves mean that they have been removed from the respective planes.

Step 4. From Lemma 14, for computing $\mathcal{I}_1(\mathfrak{c})$ it is sufficient to consider the basis $B^1(\mathbb{C}^2_{x\,y}, n) = \{\vartheta_{ij} = x^i y^j dx\}$ of non-exact 1-forms of degree n. Then

$$\eta_{ij} = \mathcal{R}_*(\vartheta_{ij}) = t^i \left(\frac{\mathfrak{c}}{1 - t}\right)^j [1\, dt + 0\, d\mathfrak{c}] = \frac{(-1)^j t^i \mathfrak{c}^j}{(t - 1)^j} dt.$$

By using the criterion given in equation (58), we obtain

$$\int_{\alpha_1(\mathfrak{c})} \eta_{ij} = \left(\frac{\left(2\pi\sqrt{-1}\right)}{(j-1)!} \frac{\partial^{j-1}(-1)^j t^i}{\partial t^{j-1}}\Big|_{t=1} \right) \mathfrak{c}^j .$$

Thus, the last expression is a polynomial function in $\mathfrak{c}$, of degree j if and only if $i \geq j-1$. This condition and $i+j \leq n$ imply that $2j-1 \leq n$, or equivalently

$$j \leq \left[\frac{n+1}{2}\right].$$

Hence, $J_1(\mathfrak{c})$ is a polynomial of degree at most $\left[\frac{n+1}{2}\right]$. Moreover, $\mathfrak{c} = 0 \in \mathfrak{B}(\mathcal{H})$ always is a zero of $J_1(\mathfrak{c})$. Therefore, according to equation (55) and diagram (23), we have

$$Z(I_1(c)) = Z(\mathfrak{I}_1(\mathfrak{c})) = Z(J_1(\mathfrak{c})) \leq \left[\frac{n-1}{2}\right],$$

from which

$$\mathscr{N}(H,\omega) = \mathscr{N}(\mathcal{H},\vartheta) = \mathscr{N}(\mathfrak{c},\eta) \leq \left[\frac{n-1}{2}\right].$$

In order to find the optimal upper bound for $Z(I_1(c))$, let $c_1,\ldots,c_s \in \mathbb{C}^*$ be generic values of H. We define $A(c) = \mu(c-c_1)\cdots(c-c_s)$, $\mu \in \mathbb{C}^*$, and the polynomial 1-form $\omega_s \doteq \psi^*\left(A\big(y(1-x)\big)\, y\, dx\right)$ of degree $n = 2s+1$. The corresponding integral

$$\int_{\gamma_1(c)} \omega_s = -\int_{\alpha_1(c)} \frac{A(c)}{t-1}\, dt = -\left(2\pi\sqrt{-1}\right) c(c-c_1)\cdots(c-c_s)$$

has $s = [(n-1)/2]$ zeros in $\mathbb{C}\backslash\mathfrak{B}(H)$.

In conclusion, if $H(u,v) = (u^2+v^2)/2$ and ω is a polynomial 1-form of degree n, then the maximal number of zeros of the Abelian integral

$$I_1(c) = \int_{\gamma_1(c)} \omega : \mathbb{C}\backslash\mathfrak{B}_H \longrightarrow \mathbb{C}$$

is

$$Z(I_1(c)) = \left[\frac{\deg(\omega)-1}{\deg(H)}\right] = \left[\frac{n-1}{2}\right]. \tag{62}$$

Recall that, $\mathscr{N}(H,\omega) = Z(I_1(c))$ is the total number of limit cycles, generated from the cycles in the generic fibers of H, of a non-conservative infinitesimal perturbation $dH + \varepsilon\omega = 0$ of the harmonic oscillator. This coincides with previous results in [13, 16].

5.2. **Broughton's polynomial.** Let

$$H(u,v) = u(uv-1)$$

be the polynomial given by S. A. Broughton in [6], and let $\omega \in \Omega^1_{ne}(\mathbb{C}^2_{u\,v})_{\leq n}$. We apply the four steps of our Program §3 to study the infinitesimal perturbed Hamiltonian differential equation $dH + \varepsilon\omega = 0$.

Step 1. By using the linear automorphisms $\psi(u,v) = (1-u,v)$ and $\sigma(c) = \mathfrak{c}$, whose inverses are where $\psi^{-1}(x,y) = (1-x,y)$, and $\sigma^{-1}(\mathfrak{c})$ the identity, we get

$$\mathcal{H}(x,y) \doteq \left(\sigma \circ H \circ \psi^{-1}\right)(x,y) = y(1-x)^2 + (x-1).$$

In accordance with equation (12),

$$\sigma'\psi_*(dH + \varepsilon\omega) = d\mathcal{H} + \varepsilon\vartheta, \quad \vartheta = \sigma'\psi_*(\omega). \tag{63}$$

This proves that $H(u,v)$ is algebraically equivalent to $\mathcal{H}(x,y)$ belonging to the family $\mathfrak{F}_3$, with $r=2$, $a_1=2$, $\beta_1=1$ and $h(x)=x-1$. Moreover, since ψ is linear, the degrees of $\mathcal{H}$ and ϑ are the same as the degrees of H and ω, thus

$$(\psi,\sigma)_*\,\Omega^1_{ne}(\mathbb{C}^2_{u\,v})_{\leq n}=\Omega^1_{ne}(\mathbb{C}^2_{x\,y})_{\leq n}\quad\text{and}\quad \mathscr{N}(H,\omega)=\mathscr{N}(\mathcal{H},\vartheta).$$

Step 2. We now use the rectification technique. From Lemma 12 and according to the Tables 1 and 2, the rectifying map for $\mathcal{H}$ is

$$\begin{array}{ccccc}\mathbb{C}^2_{x\,y}\backslash\{1-x=0\} & \xrightarrow{\mathcal{R}} & \mathbb{C}^2_{t\,\mathfrak{c}}\backslash\{1-t=0\} & \xrightarrow{\mathcal{R}^{-1}} & \mathbb{C}^2_{x\,y}\backslash\{1-x=0\}\\ (x,y) & \longmapsto & (x,y(1-x)^2+(x-1)) & \longmapsto & \big(t,\frac{\mathfrak{c}+1-t}{(1-t)^2}\big).\end{array}$$

In addition, equation (63) transforms to

$$\mathcal{R}_*\big(\sigma'\psi_*(dH+\varepsilon\omega)\big)=\mathcal{R}_*\big(d\mathcal{H}+\varepsilon\vartheta\big)=d\mathfrak{c}+\varepsilon\eta=0,\quad \eta=\mathcal{R}_*(\vartheta). \tag{64}$$

Moreover, each original fiber L_c of H, with $c\neq 0$, is mapped through (ψ,σ) into the fiber $\mathcal{L}_\mathfrak{c}$ of $\mathcal{H}$, with $\mathfrak{c}=\sigma(c)=c$, which is biholomorphically mapped under $\mathcal{R}$ into the puntured horizontal line $\big(\mathbb{C}\backslash\{1\}\big)\times\{\mathfrak{c}\}\subset\mathbb{C}^2_{t\,\mathfrak{c}}$. See Figure 3. Hence, H is of type $(0,2)$. Thus, for $c=\mathfrak{c}\neq 0$,

$$\dim H_1(L_c,\mathbb{Z})=\dim H_1(\mathcal{L}_\mathfrak{c},\mathbb{Z})=\dim H_1\left(\big(\mathbb{C}\backslash\{1\}\big)\times\{\mathfrak{c}\},\mathbb{Z}\right)=1. \tag{65}$$

Therefore, there exists one Abelian integral $I_1(c)$ defined by the pair (H,ω), which is algebraically equivalent to the unique Abelian integral $\mathcal{I}_1(\mathfrak{c})$ defined by the pair $(\mathcal{H},\vartheta)$. Thus,

$$\mathscr{N}(H,\omega)=Z(I_1(c))=\mathscr{N}(\mathcal{H},\vartheta)=Z(\mathcal{I}_1(\mathfrak{c})).$$

The polynomial H does not have finite critical points, in symbols $\mathfrak{B}_{fin}(H)=\emptyset$. The fiber $L_0=\{u(uv-1)=0\}$ has a critical point at infinity, thus $\mathfrak{B}_{inf}(H)=\{0\}$. We have $\mathfrak{B}_{fin}(H)=\mathfrak{B}_{fin}(\mathcal{H})=\emptyset$. Hence, $\mathfrak{B}(H)=\mathfrak{B}(\mathcal{H})=\{0\}$. See Figure 3.

Step 3. Let $\alpha_1(\mathfrak{c})$ be the cycle around the puncture $(1,\mathfrak{c})$ of the line $\big(\mathbb{C}\backslash\{1\}\big)\times\{\mathfrak{c}\}$. We have the Abelian integral

$$J_1(\mathfrak{c})=\int_{\alpha_1(\mathfrak{c})}\eta.$$

Clearly,

$$I_1(c)=\mathcal{I}_1(\mathfrak{c})=J_1(\mathfrak{c}),\quad \mathfrak{c}=c.$$

Step 4. From Lemma 14, for computing $\mathcal{I}_1(\mathfrak{c})$ it is sufficient to consider the basis $B^1(\mathbb{C}^2_{x\,y},n)=\{\vartheta_{ij}=x^iy^j dx\}$ of non-exact 1-forms of degree n. Then

$$\eta_{ij}=\mathcal{R}_*(\vartheta_{ij})=t^i\left(\frac{\mathfrak{c}+1-t}{(1-t)^2}\right)^j dt=\sum_{\mu=0}^{j}\binom{j}{\mu}\frac{(-1)^\mu t^i\mathfrak{c}^{j-\mu}}{(t-1)^{2j+\mu}}\,dt\,. \tag{66}$$

Since $2j+\mu\geq 1$, $t_1=1$ is a pole of η_{ij}, then the evaluation is

$$\int_{\alpha_1(\mathfrak{c})}\frac{(-1)^\mu t^i\mathfrak{c}^{j-\mu}}{(1-t)^{2j-\mu}}\,dt=\frac{(-1)^\mu(2\pi\sqrt{-1})}{(2j-\mu-1)!}\,\frac{\partial^{2j-\mu-1}\left(t^i\mathfrak{c}^{j-\mu}\right)}{\partial t^{2j-\mu-1}}\Big|_{t=1},$$

by applying the criterion of equation (58). This integral is a polynomial function in $\mathfrak{c}$ of degree $j-\mu$ if and only if $i\geq 2j-\mu-1$.

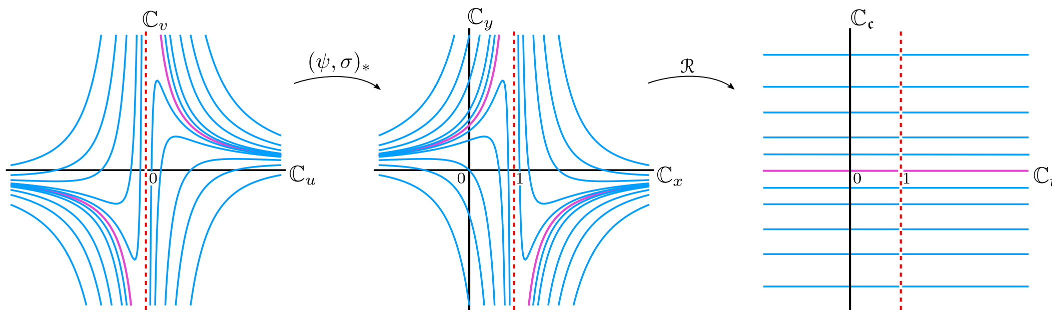

FIGURE 3. Let $H(u,v) = u(uv-1)$, we sketch the leaves of the foliations $dH = 0$, $d\mathcal{H} = 0$ and $d\mathfrak{c} = 0$. The singular fiber is $L_0 = \{u(uv-1) = 0\}$. We follow the conventions given in Figure 2. The magenta curves arise from the irreducible component $\{uv-1=0\}$. The dashed red curve $\{u=0\}$ and its images have been removed from the respective planes.

The maximum of $j-\mu$ is reached if and only if $\mu = 0$ and $i \geq 2j-1$. This last condition and $i+j \leq n$ imply $3j-1 \leq n$, or equivalently

$$j \leq \left[\frac{n+1}{3}\right].$$

The polynomial $J_1(\mathfrak{c})$ is of degree at most $\left[\frac{n+1}{3}\right]$. In conclusion, according to (55) and diagram (23), we have

$$Z(I_1(c)) = Z(\mathcal{I}_1(\mathfrak{c})) = Z(J_1(\mathfrak{c})) \leq \left[\frac{n+1}{3}\right],$$

from which

$$\mathscr{N}(H,\omega) = \mathscr{N}(\mathcal{H},\vartheta) = \mathscr{N}(\mathfrak{c},\eta) \leq \left[\frac{n+1}{3}\right].$$

We show that this bound is optimal. Let $s \doteq [(n+1)/3]$. If we consider the polynomial 1-form

$$\omega_n = -\Big(v^n - s\left(u^{2s-1}v^s - v\right)\Big)\,du$$

of degree n, then

$$\vartheta_n = \sigma'\psi_*(\omega) = \Big(y^n - s\left((1-x)^{2s-1}y^s - y\right)\Big)\,dx$$

and

$$\eta_n = \mathcal{R}_*(\vartheta_n) = \left(\frac{(\mathfrak{c}+1-t)^n}{(1-t)^{2n}} - s\left(\frac{(\mathfrak{c}+1-t)^s}{1-t} - \frac{(\mathfrak{c}+1-t)}{(1-t)^2}\right)\right)\,dt.$$

By using the criterion given in equation (58), we obtain

$$\int_{\alpha_1(\mathfrak{c})} \frac{(\mathfrak{c}+1-t)^n}{(1-t)^{2n}} = \frac{2\pi\sqrt{-1}}{(2n-1)!}\,\frac{\partial^{2n-1}\,(\mathfrak{c}+1-t)^n}{\partial t^{2n-1}}\Big|_{t=1} = 0,$$

$$\int_{\alpha_1(\mathfrak{c})} \frac{(\mathfrak{c}+1-t)^s}{1-t} = -\left(2\pi\sqrt{-1}\right)\,(\mathfrak{c}+1-t)^s\,\Big|_{t=1} = -2\pi\sqrt{-1}\,\mathfrak{c}^s$$

and

$$\int_{\alpha_1(\mathfrak{c})} \frac{(\mathfrak{c}+1-t)}{(1-t)^2} = \left(2\pi\sqrt{-1}\right) \left.\frac{\partial\ (\mathfrak{c}+1-t)}{\partial t}\right|_{t=1} = -2\pi\sqrt{-1}.$$

Therefore,

$$I_1(c) = \int_{\gamma_1(c)} \omega_n = \left(2\pi\sqrt{-1}\right)\ s\,(c^s - 1)$$

has $s = [(n+1)/3]$ zeros in $\mathbb{C}\backslash\mathfrak{B}(H)$. We have obtained the following result.

Lemma 16. *Let $H(u,v) = u(uv-1)$ and let ω be a polynomial 1-form of degree n. The maximal number of zeros of the polynomial Abelian integral*

$$I_1(c) = \int_{\gamma_1(c)} \omega : \mathbb{C} \longrightarrow \mathbb{C} \quad in \quad \mathbb{C}\backslash\mathfrak{B}_H$$

is

$$Z(I_1(c)) = \left[\frac{\deg(\omega)+1}{\deg(H)}\right] = \left[\frac{n+1}{3}\right]. \tag{67}$$

□

5.3. **A polynomial of type** $(0,3)$ **in family** $\mathfrak{F}_2$**.** We consider the polynomial

$$\mathcal{H}(x,y) = (xy-1)(1-x(xy-1)^2),$$

which belongs to the Neumann–Norbury family $\mathfrak{F}_2$, with $r=2$, $a_1=1$, $\beta_1=1$, $p_1=0$, $p=1$, $q_1=1$, $q=2$, and $\mathcal{S}(x,y) = xy-1$. We will apply steps 2-4 of the Program §3 for the study of the infinitesimal perturbed Hamiltonian differential equation $d\mathcal{H}+\varepsilon\vartheta = 0$, where $\vartheta \in \Omega^1_{ne}(\mathbb{C}^2_{x\,y})_{\leq\mathfrak{n}}$.

From Lemma 12 and according to Tables 1, 2, the rectifying map for $\mathcal{H}$ is

$$\begin{array}{ccccc} \mathbb{C}^2_{x\,y}\backslash\Sigma(\mathcal{R}) & \overset{\mathcal{R}}{\longrightarrow} & \mathbb{C}^2_{t\,\mathfrak{c}}\backslash\{\mathfrak{c}\,t(1-t)=0\} & \overset{\mathcal{R}^{-1}}{\longrightarrow} & \mathbb{C}^2_{x\,y}\backslash\Sigma(\mathcal{R}) \\ (x,y) & \longmapsto & (\mathcal{G}(x,y),\mathcal{H}(x,y)) & \longmapsto & \left(\frac{t(1-t)^2}{\mathfrak{c}^2}, \frac{\mathfrak{c}^2(\mathfrak{c}+1-t)}{t(1-t)^3}\right). \end{array}$$

Hence,

$$\mathcal{R}_*\big(d\mathcal{H}+\varepsilon\vartheta\big) = d\mathfrak{c}+\varepsilon\eta = 0, \quad \eta = \mathcal{R}_*(\vartheta). \tag{68}$$

The polynomial $\mathcal{H}$ has a unique finite critical point at $(0,-1)$, with critical value $\mathcal{H}(0,-1) = -1$, and its fiber $\mathcal{L}_0$ is the disjoint union of the algebraic curves $\{xy-1=0\}$ and $\{1-x(xy-1)^2=0\}$, thus $\mathfrak{B}_{fin}(\mathcal{H}) = \{-1\}$ and $0 \in \mathfrak{B}_{inf}(\mathcal{H})$.

In addition, from the rectification step, each fiber $\mathcal{L}_\mathfrak{c}$, with $\mathfrak{c} \notin \{0,-1\}$, of $\mathcal{H}$ is biholomorphically mapped, through $\mathcal{R}$, into the horizontal line $\big(\mathbb{C}\setminus\{0,1\}\big)\times\{\mathfrak{c}\} \subset \mathbb{C}^2_{t\,\mathfrak{c}}$, with punctures $(0,\mathfrak{c})$ and $(1,\mathfrak{c})$. Figure 4 illustrates this. Hence, $\mathcal{H}$ is of type $(0,3)$ and for $\mathfrak{c} \neq \{0,-1\}$,

$$\dim H_1(\mathcal{L}_\mathfrak{c},\mathbb{Z}) = \dim H_1\left(\big(\mathbb{C}\setminus\{0,1\}\big)\times\{\mathfrak{c}\},\mathbb{Z}\right) = 2. \tag{69}$$

In addition, $\mathfrak{B}_{inf}(\mathcal{H}) = \{0\}$.

Therefore, there are two Abelian integrals $\mathcal{I}_1(c)$ and $\mathcal{I}_2(c)$ defined by the pair $(\mathcal{H},\vartheta)$. Thus,

$$\mathscr{N}(\mathcal{H},\vartheta) = Z(\mathcal{I}_1(\mathfrak{c})) + Z(\mathcal{I}_2(\mathfrak{c})).$$

For each $\mathtt{i} \in \{1,2\}$, we take a small cycle $\alpha_{\mathtt{i}}(\mathfrak{c})$ around the puncture $(t_{\mathtt{i}}, \mathfrak{c})$ in the line $\big(\mathbb{C} \setminus \{0,1\}\big) \times \{\mathfrak{c}\} \subset \mathbb{C}^2_{t\,\mathfrak{c}}$ and we have the Abelian integrals

$$\mathcal{J}_{\mathtt{i}}(\mathfrak{c}) = J_{\mathtt{i}}(\mathfrak{c}) = \int_{\alpha_{\mathtt{i}}(\mathfrak{c})} \eta, \quad \mathtt{i} \in \{1,2\}.$$

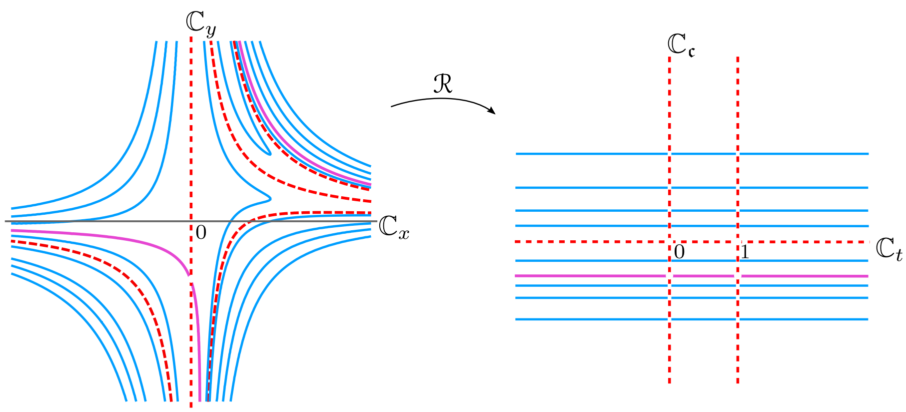

FIGURE 4. Let $\mathcal{H}(x,y) = (xy-1)(1-x(xy-1)^2)$, we sketch the leaves of the foliations $d\mathcal{H} = 0$ and $d\mathfrak{c} = 0$. On the left, the blue curves correspond to the generic fibers of $\mathcal{H}$ and the magenta curve is the connected component $\{y + (1-xy)^3 = 0\}$ of the singular fiber $\mathcal{L}_{-1}$ of $\mathcal{H}$. On the right, the blue and magenta straight lines are the image under $\mathcal{R}$ of the blue and the magenta curves in the left, respectively. The dashed red curves mean that they have been removed from the respective planes.

From Lemma 14 we know that for computing $\mathcal{J}_1(\mathfrak{c})$ it is sufficient to consider the basis $B^1(\mathbb{C}^2_{x\,y}, n) = \{\vartheta_{ij} = x^i y^j dx\}$ of non-exact 1-forms of degree n. Then

$$\eta_{ij} = \mathcal{R}_*(\vartheta_{ij}) = \left(\frac{t(1-t)^2}{\mathfrak{c}^2}\right)^i \left(\frac{\mathfrak{c}^2(\mathfrak{c}+1-t)}{t(1-t)^3}\right)^j \left[\frac{(1-t)(1-3t)}{\mathfrak{c}^2}\,dt - \frac{t(1-t)^2}{\mathfrak{c}^4}\,d\mathfrak{c}\right].$$

Thus, we get

$$\int_{\alpha_{\mathtt{i}}(\mathfrak{c})} \eta_{ij} = \int_{\alpha_{\mathtt{i}}(\mathfrak{c})} \eta^t_{ij} = \int_{\alpha_{\mathtt{i}}(\mathfrak{c})} \frac{(-1)^{3j}\mathfrak{c}^{2(j-i-1)}(\mathfrak{c}+1-t)^j(3t-1)}{t^{j-i}(t-1)^{3j-2i-1}}\,dt. \tag{70}$$

Case $\mathtt{i} = 1$. If $j - i \leq 0$, then $t_1 = 0$ is not a pole of η^t_{ij}. Hence, $\int_{\alpha_1(\mathfrak{c})} \eta_{ij} \equiv 0$. If $j - i \geq 1$, then $t_1 = 0$ is a pole of η^t_{ij} and it is clear that $\int_{\alpha_1(\mathfrak{c})} \eta_{ij}$ a polynomial function in $\mathfrak{c}$ of degree at most $3j - 2i - 2 \leq 3j - 2 \leq 3\mathfrak{n} - 2$. Hence, $J_1(\mathfrak{c})$ is a polynomial of degree at most $3\mathfrak{n} - 2$.

Case $\mathtt{i} = 2$. If $3j - 2i - 1 \leq 0$, then $t_2 = 1$ is not a pole of η^t_{ij}. Hence, $\int_{\alpha_2(\mathfrak{c})} \eta_{ij} \equiv 0$. If $3j - 2i - 1 > 0$, then $t_2 = 1$ is a pole of η^t_{ij}. Hence, from (70) and the criterion given in (58), we have

$$\int_{\alpha_2(\mathfrak{c})} \eta_{ij} = \frac{(-1)^{3j}\left(2\pi\sqrt{-1}\right)\, \mathfrak{c}^{2(j-i-1)}}{(3j-2i-2)!} \frac{\partial^{3j-2i-2}}{\partial t^{3j-2i-2}} \frac{(\mathfrak{c}+1-t)^j(3t-1)}{t^{j-i}}\Big|_{t=1}.$$

It is clear that that

$$\frac{\partial^{3j-2i-2}}{\partial t^{3j-2i-2}} \frac{(\mathfrak{c}+1-t)^j(3t-1)}{t^{j-i}}\Big|_{t=1} = \sum_{\mu=0}^{j} \binom{j}{\mu} \mathfrak{c}^{\mu} \frac{\partial^{3j-2i-2}}{\partial t^{3j-2i-2}} \frac{(3t-1)(1-t)^{j-\mu}}{t^{j-i}}\Big|_{t=1}.$$

From the general Leibniz rule, we know that the derivative

$$\frac{\partial^{3j-2i-2}}{\partial t^{3j-2i-2}} \frac{(3t-1)(1-t)^{j-\mu}}{t^{j-i}}\Big|_{t=1} \tag{71}$$

can be written as

$$\sum_{\nu=0}^{3j-2i-2} \binom{3j-2i-2}{\nu} \left((1-t)^{j-\mu}\right)^{(\nu)} \left((3t-1)(t^{i-j})\right)^{(3j-2i-2-\nu)}\Big|_{t=1},$$

which is different from zero only if

$$\nu = j-\mu \quad \text{and} \quad 3j-2i-3-\nu \le i-j$$

or equivalently

$$-2(j-i-1) \le \mu \le 3(i-j+1).$$

Thus, we have two cases: $j-i-1 \ge 0$ and $i-j+1 > 0$. In the first one, analogously to the previous paragraph, $\int_{\alpha_2(\mathfrak{c})} \eta_{ij}$ is a polynomial of degree at most $3\mathfrak{n}-2$. In the second one, we obtain that $\mathfrak{c}$ in $\int_{\alpha_2(\mathfrak{c})} \eta_{ij}$ has degree $2(j-i-1)+\mu$ and according to previous equation

$$0 \le 2(j-i-1)+\mu \le i-j+1.$$

Hence, also in this case $\int_{\alpha_2(\mathfrak{c})} \eta_{ij}$ is a polynomial function of maximum degree $[(\mathfrak{n}+2)/6]$, which can be deduced from the conditions $3j-2i-1>0$, $i-j+1>0$ and $i+j \le \mathfrak{n}$. Therefore, $J_2(\mathfrak{c})$ is a polynomial of degree at most $3\mathfrak{n}-2$.

In conclusion,

$$Z(\mathcal{I}_{\mathtt{i}}(\mathfrak{c})) = Z(J_{\mathtt{i}}(\mathfrak{c})) \le 3\mathfrak{n}-2, \quad \mathtt{i} \in \{1,2\},$$

from which

$$\mathscr{N}(\mathcal{H},\vartheta) = \mathscr{N}(\mathfrak{c},\eta) =\le 2(3\mathfrak{n}-2).$$

As an explicit example we take the polynomial 1-form

$$\vartheta_0 = y(y^2-108xy-66)\,dx$$

of degree $\mathfrak{n}=3$, then

$$\mathcal{I}_1(\mathfrak{c}) = J_1(\mathfrak{c}) = 3\left(2\pi\sqrt{-1}\right)(c+1)(4c^6+3c^5-36c-58),$$

$$\mathcal{I}_2(\mathfrak{c}) = J_2(\mathfrak{c}) = -3\left(2\pi\sqrt{-1}\right)(c-1)(c+2)(4c^5+3c^4+8c^3-2c^2+18c-58).$$

Hence, $\mathcal{I}_1(\mathfrak{c})$ has $6 = 3\mathfrak{n}-3$ zeros in $\mathbb{C}\backslash\mathfrak{B}(\mathcal{H})$ and $\mathcal{I}_2(\mathfrak{c})$ has $7 = 3\mathfrak{n}-2$ zeros in $\mathbb{C}\backslash\mathfrak{B}(\mathcal{H})$. The zeros of $\mathcal{I}_1(\mathfrak{c})$ are different from the zeros of $\mathcal{I}_2(\mathfrak{c})$. Therefore, we have

$$\mathscr{N}(\mathcal{H},\vartheta_0) = 6+7 = 13.$$

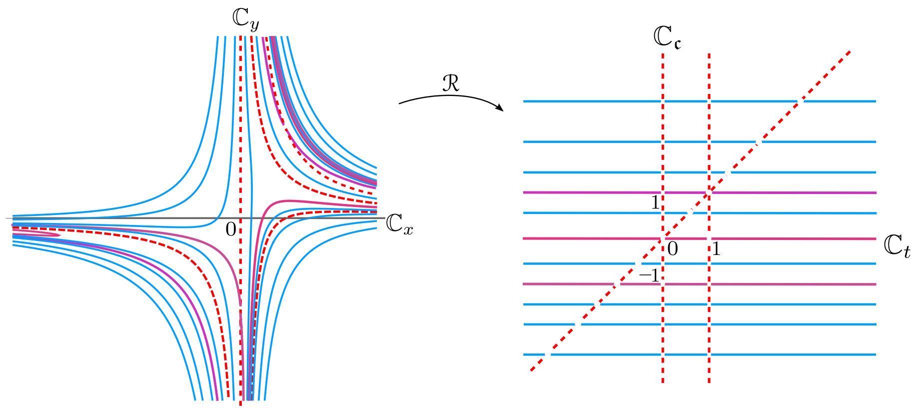

FIGURE 5. Let $\mathcal{H}(x,y) = x(xy-1)^2 + (xy-1)(1-x(xy-1)^2)$, we sketch of the leaves of the foliations $d\mathcal{H} = 0$ and $d\mathfrak{c} = 0$. The magenta curves correspond to (complex) connected components of the singular fibers of $\mathcal{H}$. The dashed red curves mean that they have been removed from the respective planes.

5.4. **A polynomial of type** $(0,4)$ **in family** $\mathfrak{F}_1$**.** We consider the polynomial

$$\mathcal{H}(x,y) = x(xy-1)^2 + (xy-1)(1-x(xy-1)^2),$$

which belongs to the Neumann–Norbury family $\mathfrak{F}_1$, with $r=2$, $a_1=1$, $\beta_1=1$, $p_1=0$, $p=1$, $q_1=1$, $q=2$ and $\mathcal{S}(x,y) = xy-1$. Again, we will apply steps 2–4 of the Program for the study of the infinitesimal perturbed Hamiltonian differential equation $d\mathcal{H} + \varepsilon\vartheta = 0$, where $\vartheta \in \Omega^1_{ne}(\mathbb{C}^2_{x\,y})_{\leq \mathfrak{n}}$.

From Lemma 12 and (24), let $\mathcal{G}(x,y) = x(xy-1)^2$, then $\mathcal{R} = (\mathcal{G}(x,y), \mathcal{H}(x,y))$ is a rectifying map for $\mathcal{H}$. In this case, $\Sigma(\mathcal{R}) = \{x(xy-1)^2\left(1-x(xy-1)^2\right) = 0\}$ and (36) becomes

$$\begin{array}{ccccc} \mathbb{C}^2_{x\,y}\backslash\Sigma(\mathcal{R}) & \xrightarrow{\mathcal{R}} & \mathbb{C}^2_{t\,\mathfrak{c}}\backslash\{t(1-t)(\mathfrak{c}-t)=0\} & \xrightarrow{\mathcal{R}^{-1}} & \mathbb{C}^2_{x\,y}\backslash\Sigma(\mathcal{R}) \\ (x,y) & \longmapsto & (\mathcal{G}(x,y),\mathcal{H}(x,y)) & \longmapsto & \left(\frac{t(1-t)^2}{(c-t)^2}, \frac{(c+1-2t)(c-t)^2}{t(1-t)^3}\right). \end{array}$$

Thus,

$$\mathcal{R}_*\big(d\mathcal{H} + \varepsilon\vartheta\big) = d\mathfrak{c} + \varepsilon\eta = 0, \quad \eta = \mathcal{R}_*(\vartheta). \tag{72}$$

The polynomial $\mathcal{H}$ has only two finite critical points at $(0,-1)$ and $(1,2)$, with critical values $\mathcal{H}(0,-1) = -1$ and $\mathcal{H}(1,2) = 1$, and its fiber $\mathcal{L}_0$ is the disjoint union of two algebraic curves , thus $\mathfrak{B}_{fin}(\mathcal{H}) = \{-1,1\}$ and $0 \in \mathfrak{B}_{inf}(\mathcal{H})$.

In addition, each fiber $\mathcal{L}_{\mathfrak{c}}$, with $\mathfrak{c} \notin \{0,-1,1\}$, of $\mathcal{H}$ is biholomorphically mapped, through $\mathcal{R}$, into the horizontal line $\big(\mathbb{C}\setminus\{0,1,\mathfrak{c}\}\big)\times\{\mathfrak{c}\} \subset \mathbb{C}^2_{t,\mathfrak{c}}$, with the points $(0,\mathfrak{c})$, $(1,\mathfrak{c})$ and $(\mathfrak{c},\mathfrak{c})$ removed. See Figure 5. Hence, H is of type $(0,4)$ and for $\mathfrak{c} \notin \{0,-1,1\}$, we have

$$\dim H_1(\mathcal{L}_{\mathfrak{c}},\mathbb{Z}) = \dim H_1\left(\big(\mathbb{C}\setminus\{0,1,\mathfrak{c}\}\big)\times\{\mathfrak{c}\},\mathbb{Z}\right) = 3. \tag{73}$$

Therefore, there are three Abelian integrals defined by the pair $(\mathcal{H},\vartheta)$. Thus,

$$\mathscr{N}(\mathcal{H},\vartheta)=Z(\mathcal{I}_1(\mathfrak{c}))+Z(\mathcal{I}_2(\mathfrak{c}))+Z(\mathcal{I}_3(\mathfrak{c})).$$

For each $\mathtt{i}\in\{1,2,3\}$, we consider a small cycle $\alpha_{\mathtt{i}}(\mathfrak{c})$ around the puncture $(t_{\mathtt{i}},\mathfrak{c})$ in the line $\big(\mathbb{C}\setminus\{0,1,\mathfrak{c}\}\big)\times\{\mathfrak{c}\}$, with $t_1=0, t_2=1$ and $t_3=\mathfrak{c}$. Thus, we have the Abelian integrals

$$\mathcal{I}_{\mathtt{i}}(\mathfrak{c})=J_{\mathtt{i}}(\mathfrak{c})=\int_{\alpha_{\mathtt{i}}(\mathfrak{c})}\eta,\quad \mathtt{i}\in\{1,2,3\}.$$

For computing $\mathcal{I}_1(\mathfrak{c})$ it is sufficient to consider the basis $B^1(\mathbb{C}^2_{x\,y},n)=\{\vartheta_{ij}=x^iy^jdx\}$ of non-exact 1-forms of degree n, by Lemma 14. The push-forward $\mathcal{R}_*(\vartheta_{ij})$ is then

$$\eta_{ij}=\frac{t^i(1-t)^{2i}}{(c-t)^{2i}}\,\frac{(c+1-2t)^j(c-t)^{2j}}{t^j(1-t)^{3j}}\left[\frac{(1-t)(t^2+t+\mathfrak{c}-3t\mathfrak{c})}{(\mathfrak{c}-t)^3}\,dt-\frac{2t(1-t)^2}{(c-t)^3}\,d\mathfrak{c}\right].$$

Thus,

$$\eta^t_{ij}=\frac{(\mathfrak{c}+1-2t)^j(t^2+t+\mathfrak{c}-3t\mathfrak{c})}{t^{j-i}(1-t)^{3j-2i-1}(\mathfrak{c}-t)^{2i-2j+3}}\,dt.$$

By applying Newton's binomial theorem, we have

$$\eta^t_{ij}=\sum_{\mu=0}^{j}\binom{j}{\mu}\frac{(t^2+t+\mathfrak{c}-3t\mathfrak{c})}{t^{j-i}(t-1)^{2j-2i-1+\mu}(t-\mathfrak{c})^{2i-2j+3-\mu}}\,dt. \tag{74}$$

Case $\mathtt{i}=1$. If $j-i\le 0$, then $t_1=0$ is not a pole of η^1_{ij}. Thus, $\int_{\alpha_1(\mathfrak{c})}\eta_{ij}\equiv 0$. If $j-i=1$, then $t_1=0$ is a pole of order one of η^1_{ij} and form (74), we get

$$\int_{\alpha_1(\mathfrak{c})}\eta_{i,i+1}=\sum_{\mu=0}^{i+1}\binom{i+1}{\mu}\int_{\alpha_1(\mathfrak{c})}\frac{(t-\mathfrak{c})^{\mu-1}(t^2+t+\mathfrak{c}-3t\mathfrak{c})}{t(t-1)^{1+\mu}}\,dt.$$

It is clear that for $\mu\ge 1$, every integral on the right-hand side is a polynomial function in $\mathfrak{c}$. Moreover, for $\mu=0$ and according to the criterion given in (58), we have

$$\int_{\alpha_1(\mathfrak{c})}\frac{(t-\mathfrak{c})^{-1}(t^2+t+\mathfrak{c}-3t\mathfrak{c})}{t(t-1)}\,dt=\big(2\pi\sqrt{-1}\big)\left.\frac{t^2+t+\mathfrak{c}-3t\mathfrak{c}}{(t-1)(t-\mathfrak{c})}\right|_{t=0}=2\pi\sqrt{-1}.$$

Hence, $\int_{\alpha_1(\mathfrak{c})}\eta_{i,i+1}$ is a polynomial function of degree at most $i+1=[(\mathfrak{n}+1)/2]$. If $j-i>1$, then $2i-2j+3-\mu\le 0$. Thus, from (74) is clear that in this case $\int_{\alpha_1(\mathfrak{c})}\eta_{ij}$ is a polynomial, of degree at most $3j-2i-2\le 3j-2\le 3\mathfrak{n}-2$. Hence, we have that $J_1(\mathfrak{c})$ is a polynomial function of degree at most $3\mathfrak{n}-2$.

Case $\mathtt{i}=2$. If $2j-2i-1+\mu\le 0$, then $t_2=1$ is not a pole of the corresponding term in (74). Thus, $\int_{\alpha_2(\mathfrak{c})}\eta_{ij}\equiv 0$. If $2j-2i-1+\mu=1$, then $t_2=1$ is a pole of order one of the corresponding term in (74), whence

$$\int_{\alpha_2(\mathfrak{c})}\frac{(t^2+t+\mathfrak{c}-3t\mathfrak{c})}{t^{1-\mu/2}(t-1)(t-\mathfrak{c})}\,dt=\big(2\pi\sqrt{-1}\big)\left.\frac{(t^2+t+\mathfrak{c}-3t\mathfrak{c})}{t^{1-\mu/2}(t-\mathfrak{c})}\right|_{t=1}=4\pi\sqrt{-1},$$

by applying the criterion given in (58). If $2j-2i-1+\mu>1$, then $2i-2j+3-\mu<1$. Thus, the integral of the corresponding term in (74) is a polynomial function of degree at most $2j-2i-2+\mu\le 3j-2\le 3\mathfrak{n}-2$. Hence, $J_2(\mathfrak{c})$ is a polynomial function of degree at most $3\mathfrak{n}-2$.

Case $\mathtt{i}=3$. If $2i-2j+3-\mu\leq 0$, then $t_3=\mathfrak{c}$ is not a pole of the corresponding term in (74). Thus, $\int_{\alpha_3(\mathfrak{c})}\eta_{ij}\equiv 0$. If $2i-2j+3-\mu=1$, then μ is even and $t_3=\mathfrak{c}$ is a pole of order one of the corresponding term in (74), whose integral is

$$\int_{\alpha_3(\mathfrak{c})}\frac{(t^2+t+\mathfrak{c}-3t\mathfrak{c})}{t^{1-\mu/2}(t-1)(t-\mathfrak{c})}\,dt=\left(2\pi\sqrt{-1}\right)\left.\frac{(t^2+t+\mathfrak{c}-3t\mathfrak{c})}{t^{1-\mu/2}(t-1)}\right|_{t=\mathfrak{c}}=4\pi\sqrt{-1}c^{\frac{\mu}{2}},$$

applying the criterion given in (58) and it is a polynomial of degree at most $[\mathfrak{n}/2]$. If $2i-2j+3-\mu>1$, then $2j-2i-1+\mu<1$ and $2(i-j)>\mu-2$. Thus $2j-2i-1+\mu\leq 0$ and $i-j\geq 0$. Moreover, $t_3=\mathfrak{c}$ is a pole of order greater than one of the corresponding term in (74), whose integral is

$$\int_{\alpha_3(\mathfrak{c})}\frac{(t^2+t+\mathfrak{c}-3t\mathfrak{c})t^{i-j}(t-1)^{2i-2j+1-\mu}}{(t-\mathfrak{c})^{2i-2j+3-\mu}}\,dt.$$

By the criterion given in (58), the previous integral is equal to

$$\frac{2\pi\sqrt{-1}}{(2i-2j+2-\mu)!}\left.\frac{\partial^{2i-2j+2-\mu}}{\partial t^{2i-2j+2-\mu}}(t^2+t+\mathfrak{c}-3t\mathfrak{c})t^{i-j}(t-1)^{2i-2j+1-\mu}\right|_{t=\mathfrak{c}},$$

which is a polynomial function of degree at most $i-j+1\leq i\leq\mathfrak{n}-1$. Therefore, $J_3(\mathfrak{c})$ is a polynomial function of degree at most $\mathfrak{n}-1$.

In conclusion,

$$Z(\mathcal{I}_{\mathtt{i}}(\mathfrak{c}))=Z(J_{\mathtt{i}}(\mathfrak{c}))\leq 3\mathfrak{n}-2,\quad \mathtt{i}\in\{1,2\}$$

and

$$Z(\mathcal{I}_3(\mathfrak{c}))=Z(J_3(\mathfrak{c}))\leq\mathfrak{n}-1,$$

whence

$$\mathscr{N}(\mathcal{H},\vartheta)=\mathscr{N}(\mathfrak{c},\eta)=\leq 2(3\mathfrak{n}-2)+\mathfrak{n}-1.$$

As an explicit example, we take the polynomial 1-form

$$\vartheta_0=y(y^2-96x^2+1008)\,dx$$

of degree $\mathfrak{n}=3$, then

$$\begin{aligned}
\mathcal{I}_1(\mathfrak{c})&=J_1(\mathfrak{c})=6\left(2\pi\sqrt{-1}\right)(c+1)(2c^6-2c^5-c^4+c^3+168),\\
\mathcal{I}_2(\mathfrak{c})&=J_2(\mathfrak{c})=-6\left(2\pi\sqrt{-1}\right)(c-2)(2c^6+4c^5+5c^4+10c^3+21c^2+42c+252),\\
\mathcal{I}_3(\mathfrak{c})&=J_3(\mathfrak{c})=96\left(2\pi\sqrt{-1}\right)(2c+5)(c-4).
\end{aligned}$$

Hence, $\mathcal{I}_1(\mathfrak{c})$ has $6=3\mathfrak{n}-3$ zeros in $\mathbb{C}\backslash\mathfrak{B}(\mathcal{H})$, $\mathcal{I}_2(\mathfrak{c})$ has $7=3\mathfrak{n}-2$ zeros in $\mathbb{C}\backslash\mathfrak{B}(\mathcal{H})$ and $\mathcal{I}_3(\mathfrak{c})$ has $2=\mathfrak{n}-1$ zeros in $\mathbb{C}\backslash\mathfrak{B}(\mathcal{H})$. The zeros of the three Abelian integrals are different. Therefore, we have

$$\mathscr{N}(\mathcal{H},\vartheta_0)=6+7+2=15.$$

5.5. **Primitive polynomials of type** $(0,2)$**.** Recall that the simplest non-trivial case for the study of Abelian integrals is when the polynomial H is primitive of type $(0,2)$. The Abelian integrals for the family of these polynomial were studied in [26]. In order to show the advantage of the Program §3, we will finish this section by studying this family of polynomials.

Let $H(u,v)$ be a primitive polynomial of type $(0,2)$ of degree $m+1$. Consider a polynomial 1-form $\omega\in\Omega^1_{ne}(\mathbb{C}^2_{u\,v})_{\leq n}$ and the infinitesimal perturbed Hamiltonian differential equation $dH+\varepsilon\omega=0$.

By using the notation of the Neumann–Norbury families, the Miyanishi–Sugie classification given in Theorem 7 can be expressed as

$$\left\{ \mathcal{H}(x,y) = x^{p_1}\Big(x^k y + P(x)\Big)^p \;\Big|\; \begin{array}{l} p_1, p \in \mathbb{N},\ (p_1,p)=1,\ k\in\mathbb{N}_0,\ P(x)\in\mathbb{C}[x]_{\le k-1}, \\ P(0)\neq 0 \text{ if } k>0, \text{ and } P(x)\equiv 0 \text{ if } k=0 \end{array} \right\}.$$

We know from Theorem 7 and Lemma 10 that the polynomials of the previous family are normal forms of the primitive polynomials of type $(0,2)$.

Step 1. There exists a pair $(\psi,\sigma)\in \mathrm{Aut}(\mathbb{C}^2)\times\mathrm{Aut}(\mathbb{C})$ such that $H(u,v)$ and

$$\mathcal{H}(x,y) = x^{p_1}\big(x^k y + P(x)\big)^p$$

are algebraically equivalent. Since H and $\mathcal{H}$ are of type $(0,2)$, there exists a unique Abelian integral $I_1(c)$ defined by H and ω, which is algebraically equivalent to the unique Abelian integral $\mathcal{I}_1(\mathfrak{c})$ defined by $\mathcal{H}$ and ϑ. Thus, the two first columns of diagram (23) hold. In particular, according to Corollary 3 and (12),

$$\sigma'\psi_*(dH+\varepsilon\omega) = d\mathcal{H}+\varepsilon\vartheta, \qquad \vartheta = \sigma'\psi_*(\omega).$$

and

$$\mathscr{N}(H,\omega) = Z(I_1(c)) = \mathscr{N}(\mathcal{H},\vartheta) = Z(\mathcal{I}_1(\mathfrak{c})).$$

From Lemma 10 and Proposition 11, we get that

$$\deg(\vartheta)\le (n+1)(m)-1.$$

Thus,

$$(\psi,\sigma)_*\Big(\Omega^1_{ne}(\mathbb{C}^2_{u\,v})_{\le n}\Big) \subset \Omega^1_{ne}(\mathbb{C}^2_{x\,y})_{\le (n+1)(m)-1}.$$

Step 2. We now suppose that $\mathcal{H}$ is of degree $\mathfrak{m}+1$ and that the polynomial 1-form $\vartheta\in\Omega^1_{ne}(\mathbb{C}^2_{x\,y})_{\le\mathfrak{n}}$. Of course, we have $2\le\mathfrak{m}+1\le m+1$ and $\mathfrak{n}\le(n+1)(m)-1$.

From Lemmas 10 and 12, there exists a rectifying map for $\mathcal{H}$. Since $(p_1,p)=1$, there are positive integers q_1 and q such that $pq_1-qp_1=1$. Moreover, by Remark 4, if we take $\mathcal{G}(x,y)=x^{q_1}\left(x^l y+P(x)\right)^q$, then (36) becomes

$$\begin{array}{ccccc} \mathbb{C}^2_{x\,y}\backslash\{\mathcal{H}=0\} & \xrightarrow{\mathcal{R}} & \mathbb{C}^2_{t\,\mathfrak{c}}\backslash\{t\mathfrak{c}=0\} & \xrightarrow{\mathcal{R}^{-1}} & \mathbb{C}^2_{x\,y}\backslash\{\mathcal{H}=0\} \\ (x,y) & \longmapsto & (\mathcal{G}(x,y),\mathcal{H}(x,y)) & \longmapsto & \left(\dfrac{t^p}{\mathfrak{c}^q}, \dfrac{\mathfrak{c}^{qk}\left(\mathfrak{c}^{q_1}-t^{p_1}P(t^p\mathfrak{c}^{-q})\right)}{t^{p_1+pk}}\right). \end{array}$$

Since $P(x)\in\mathbb{C}[x]_{\le k-1}$, $\mathfrak{c}^{qk}\left(\mathfrak{c}^{q_1}-t^{p_1}P(t^p\mathfrak{c}^{-q})\right)$ is a polynomial. Moreover, we have

$$\mathcal{R}_*\big(\sigma'\psi_*(dH+\varepsilon\omega)\big) = \mathcal{R}_*\big(d\mathcal{H}+\varepsilon\vartheta\big) = d\mathfrak{c}+\varepsilon\eta = 0, \quad \eta=\mathcal{R}_*(\vartheta).$$

For $k>0$, the fiber $\mathcal{L}_0$ of $\mathcal{H}$ is the disjoint union of the algebraic curves $\{x=0\}$ and $\{x^k y+P(x)=0\}$, thus 0 is a critical value. If $k=0$, then $(0,0)$ is a critical point of $\mathcal{H}$, with critical value $\mathcal{H}(0,0)=0$. Hence, in any case, $0\in\mathfrak{B}(\mathcal{H})$. Moreover, $\mathcal{R}$ maps biholomorphically each fiber $\mathcal{L}_\mathfrak{c}$ of $\mathcal{H}$, with $\mathfrak{c}\neq 0$, into the horizontal line $\mathbb{C}^*\times\{\mathfrak{c}\}$ in $\mathbb{C}^2_{t\,\mathfrak{c}}$. Thus $\mathfrak{B}(\mathcal{H})=\{0\}$ and

$$\dim H_1(L_c,\mathbb{Z}) = \dim H_1(\mathcal{L}_\mathfrak{c},\mathbb{Z}) = \dim H_1(\mathbb{C}^*\times\{\mathfrak{c}\},\mathbb{Z}) = 1.$$

Step 3. Let $\alpha_1(\mathfrak{c})$ be a small cycle around the puncture $(0,\mathfrak{c})$ in the line $\mathbb{C}^*\times\{\mathfrak{c}\}$. Thus, we obtain

$$\mathcal{I}_1(\mathfrak{c}) = J_1(\mathfrak{c}) = \int_{\alpha_1(\mathfrak{c})}\eta.$$

According to Corollary 3,

$$I_1(c) = \frac{1}{\sigma'}\mathcal{I}_1(\mathfrak{c}) = \frac{1}{\sigma'}J_1(\mathfrak{c}), \quad \mathfrak{c} = \sigma(c),$$

as in diagram (23).

Step 4. From Lemma 14, we know that for computing $\mathcal{I}_1(\mathfrak{c})$ it is sufficient to consider the basis $B^1(\mathbb{C}^2_{x\,y}, n) = \{\vartheta_{ij} = x^i y^j dx\}$ of non-exact 1-forms of degree n. Then

$$\eta_{ij} = \mathcal{R}_*(\vartheta_{ij}) = \left(\frac{t^p}{\mathfrak{c}^q}\right)^i \left(\frac{\mathfrak{c}^{qk}\left(\mathfrak{c}^{q_1} - t^{p_1}P(t^p\mathfrak{c}^{-q})\right)}{t^{p_1+pk}}\right)^j \left[\frac{p\,t^{p-1}}{\mathfrak{c}^q}\,dt - \frac{q\,t^p}{\mathfrak{c}^{q+1}}\,d\mathfrak{c}\right].$$

Thus,

$$\eta^t_{ij} = \frac{p\,\mathfrak{c}^{q(jk-i-1)}\left(\mathfrak{c}^{q_1} - t^{p_1}P(t^p\mathfrak{c}^{-q})\right)^j}{t^{p(jk-i-1)+p_1 j+1}}\,dt.$$

The binomial theorem implies

$$\left(\mathfrak{c}^{q_1} - t^{p_1}P(t^p\mathfrak{c}^{-q})\right)^j = \sum_{\mu=0}^{j}\binom{j}{\mu}\mathfrak{c}^{q_1(j-\mu)}t^{p_1\mu}P\left(t^p\mathfrak{c}^{-q}\right)^\mu.$$

We can assume that $P(x) = \lambda_0 + \cdots + \lambda_s x^s$, with $s \le k-1$ and $\lambda_s \ne 0$, then the multinomial theorem gives

$$P\left(t^p\mathfrak{c}^{-q}\right)^\mu = \sum \frac{\mu!}{n_0!\cdots n_s!}\lambda_0^{n_0}\cdots\lambda_s^{n_s}t^{pN_s}\mathfrak{c}^{-qN_s},$$

where the sum is over all lists of $s+1$ non-negative integers $(n_0, \ldots, n_s)$ such that

$$n_0 + \cdots + n_s = \mu \quad \text{and} \quad N_s = n_1 + \cdots + sn_s. \tag{75}$$

Therefore, by simplifying we obtain

$$\eta^t_{ij} = \sum_{\mu=0}^{j}\sum A^\mu_{n_0\cdots n_s}\frac{p\,\mathfrak{c}^{q_1(j-\mu)-q\widetilde{N}_s}}{t^{p_1(j-\mu)-p\widetilde{N}_s+1}}\,dt, \tag{76}$$

where

$$A^\mu_{n_0\ldots n_s} = \frac{j!}{n_0!\cdots n_s!(j-\mu)!}\lambda_0^{n_0}\cdots\lambda_s^{n_s} \quad \text{and} \quad \widetilde{N}_s = N_s - jk + i + 1. \tag{77}$$

The integral along $\alpha_1(\mathfrak{c})$ of each term in the sum on the right-hand side of (76) is different from zero if and only if $p_1(j-\mu) = p\widetilde{N}_s$. Since $(p_1, p) = 1$, there exists a positive integer $q_{s\mu}$ such that $\widetilde{N}_s = p_1 q_{s\mu}$ and $j - \mu = pq_{s\mu}$. We have,

$$q_{s\mu}(p_1 + p(k+1)) = \widetilde{N}_s + (k+1)(j-\mu).$$

Moreover, $\widetilde{N}_s = N_s - jk + i + 1 \le (k-1)\mu - jk + i + 1$, it follows that

$$\widetilde{N}_s + (k+1)(j-\mu) \le i + j + 1 - 2\mu.$$

Thus $q_{s\mu}(p_1 + p(k+1)) \le i + j + 1 \le \mathfrak{n} + 1$, whence

$$q_{s\mu} \le \left[\frac{\mathfrak{n}+1}{p_1 + p(k+1)}\right] = \left[\frac{\mathfrak{n}+1}{\mathfrak{m}+1}\right]. \tag{78}$$

In addition, the degree of $\mathfrak{c}$ is

$$q_1(j-\mu) - q\widetilde{N}_s = q_1(pq_{s\mu}) - q(p_1 q_{s\mu}) = q_{s\mu}.$$

Hence, according to (55) and (78), the Abelian integral $\mathcal{I}_1(\mathfrak{c})$ is a polynomial of degree at most $[(\mathfrak{n}+1)/(\mathfrak{m}+1)]$, because of (78). Therefore

$$Z(\mathcal{I}_1(\mathfrak{c})) = Z(J_1(\mathfrak{c})) \leq \left[\frac{\mathfrak{n}+1}{\mathfrak{m}+1}\right]. \tag{79}$$

Since $2 \leq \mathfrak{m}+1 \leq m+1$ and $\mathfrak{n} \leq (n+1)(m)-1$,

$$Z(I_1(c)) = \mathscr{N}(H,\omega) \leq \left[\frac{(n+1)(m)}{2}\right].$$

In conclusion, we have proven the following result.

Theorem 17 ([26])**.** *Let $H(u,v)$ be a primitive polynomial of type $(0,2)$ of degree $m+1$ and let ω be a polynomial 1-form of degree n on $\mathbb{C}^2$.*

1) The Abelian integral $I_1(c) = \int_{\gamma_1(c)} \omega \colon \mathbb{C} \longrightarrow \mathbb{C}$ is a polynomial.

2) Morever, $I_1(c)$ has at most $\left[\frac{(n+1)\,m}{2}\right]$ isolated zeros in $\mathbb{C}\backslash\mathfrak{B}(H)$. □

6. Abelian integrals for the Neumann–Norbury classification

In this section, we provide general properties of Abelian integrals for the Neumann–Norbury algebraic classification, that is, for the normal forms of primitive polynomials with trivial global monodromy given in the families $\mathfrak{F}_1$, $\mathfrak{F}_2$ and $\mathfrak{F}_3$ of Theorem 8. In order to establish the result on this issue, we split the Neumann–Norbury families $\mathfrak{F}_1$ and $\mathfrak{F}_2$ into disjoint families:

$$\mathfrak{F}_\iota^+ \doteq \{\mathcal{H}(x,y) \in \mathfrak{F}_\iota \mid pq_1 - qp_1 = 1\} \quad \textit{and} \quad \mathfrak{F}_\iota^- \doteq \{\mathcal{H}(x,y) \in \mathfrak{F}_\iota \mid pq_1 - qp_1 = -1\},$$

where $\iota = 1, 2$. For the sake of brevity, in all that follows

- $\delta_{\mathtt{i}}(\mathfrak{c})$ denotes a cycle in the basis $\mathcal{B}(\mathfrak{c})$ of $H_1(\mathcal{L}_\mathfrak{c}, \mathbb{Z})$, as in equation (50), and
- ϑ is a non exact polynomial 1-form in $\Omega^1_{ne}(\mathbb{C}^2_{x\,y})_{\leq \mathfrak{n}}$, recall equation (51).

Our main result concerning the Abelian integrals for normal forms of polynomials with trivial global monodromy is the following.

Theorem 18. *Let $\mathcal{H}(x,y)$ be a polynomial of degree $\mathfrak{m}+1$ in the Neumann–Norbury families $\mathfrak{F}_1, \mathfrak{F}_2$ or $\mathfrak{F}_3$. If $\vartheta \in \Omega^1_{ne}(\mathbb{C}^2_{x\,y})_{\leq \mathfrak{n}}$, then for each cycle $\delta_{\mathtt{i}}(\mathfrak{c})$ the Abelian integral*

$$\mathcal{I}_{\mathtt{i}}(\mathfrak{c}) = \int_{\delta_{\mathtt{i}}(\mathfrak{c})} \vartheta \colon \mathbb{C} \longrightarrow \mathbb{C}$$

is a polynomial, in addition:

1) If $\mathcal{H} \in \mathfrak{F}_1$, then $r+1 = \dim H_1(\mathcal{L}_\mathfrak{c}, \mathbb{Z}) \geq 3$ and

$$\deg(\mathcal{I}_{\mathtt{i}}(\mathfrak{c})) \leq \begin{cases} \mathfrak{n}\left(\left[\dfrac{\mathfrak{m}-1}{r-1}\right]-2\right)-2 & \textit{for } \mathcal{H} \in \mathfrak{F}_1^+ \textit{ and } 0 \leq \mathtt{i} \leq r-1, \\[2ex] (\mathfrak{n}-1)\left[\dfrac{\mathfrak{m}-r-2}{2}\right] & \textit{for } \mathcal{H} \in \mathfrak{F}_1^+ \textit{ and } \mathtt{i} = r, \\[2ex] (\mathfrak{n}-1)\left[\dfrac{\mathfrak{m}-4}{2(r-1)}\right] & \textit{for } \mathcal{H} \in \mathfrak{F}_1^- \textit{ and } 0 \leq \mathtt{i} \leq r-1, \\[2ex] \mathfrak{n}\,(\mathfrak{m}-1-r)-r & \textit{for } \mathcal{H} \in \mathfrak{F}_1^- \textit{ and } \mathtt{i} = r. \end{cases} \tag{80}$$

2) If $\mathcal{H} \in \mathfrak{F}_2$, then $r = \dim H_1(\mathcal{L}_{\mathfrak{c}}, \mathbb{Z}) \geq 1$ and

$$\deg(\mathcal{I}_{\mathtt{i}}(\mathfrak{c})) \leq \begin{cases} \left[\dfrac{\mathfrak{n}+1}{\mathfrak{m}+1}\right] & \text{for } r=1, \\ \mathfrak{n}\left(\left[\dfrac{\mathfrak{m}-1}{r-1}\right]-2\right)-2 & \text{for } \mathcal{H} \in \mathfrak{F}_2^+ \text{ and } r>1, \\ (\mathfrak{n}-1)\left[\dfrac{\mathfrak{m}-4}{2(r-1)}\right] & \text{for } \mathcal{H} \in \mathfrak{F}_2^- \text{ and } r>1. \end{cases} \tag{81}$$

3) If $\mathcal{H} \in \mathfrak{F}_3$, then $r-1 = \dim H_1(\mathcal{L}_{\mathfrak{c}}, \mathbb{Z}) \geq 0$ and

$$\deg(\mathcal{I}_{\mathtt{i}}(\mathfrak{c})) \leq \begin{cases} 0 & \text{for } r=1, \\ \left[\dfrac{\mathfrak{n}+1}{\mathfrak{m}+1}\right] & \text{for } r=2, \\ \mathfrak{n} & \text{for } r>2. \end{cases} \tag{82}$$

Recalling the equation (23), $\mathscr{N}(\mathcal{H}, \vartheta)$ denotes the total number of limit cycles of the non-conservative perturbation $d\mathcal{H} + \varepsilon\vartheta = 0$.

Corollary 19. *Let $\mathcal{H}(x,y)$ be a polynomial of degree $\mathfrak{m}+1$ in the Neumann–Norbury families $\mathfrak{F}_1, \mathfrak{F}_2$ or $\mathfrak{F}_3$ and let $\vartheta \in \Omega^1_{ne}(\mathbb{C}^2_{x\,y})_{\leq \mathfrak{n}}$, with $\mathcal{I}_{\mathtt{i}}(\mathfrak{c}) \not\equiv 0$. The following assertions hold:*

1) If $\mathcal{H} \in \mathfrak{F}_1$, then $r+1 = \dim H_1(\mathcal{L}_{\mathfrak{c}}, \mathbb{Z}) \geq 3$ and

$$\mathscr{N}(\mathcal{H}, \vartheta) \leq \begin{cases} r\left(\mathfrak{n}\left[\dfrac{\mathfrak{m}+1-2r}{r-1}\right]-2\right)+(\mathfrak{n}-1)\left[\dfrac{\mathfrak{m}-r-2}{2}\right] & \text{for } \mathcal{H} \in \mathfrak{F}_1^+, \\ \mathfrak{n}\,(\mathfrak{m}-r-1)+r\left((\mathfrak{n}-1)\left[\dfrac{\mathfrak{m}-4}{2(r-1)}\right]-1\right) & \text{for } \mathcal{H} \in \mathfrak{F}_1^-. \end{cases} \tag{83}$$

2) If $\mathcal{H} \in \mathfrak{F}_2$, then $r = \dim H_1(\mathcal{L}_{\mathfrak{c}}, \mathbb{Z}) \geq 1$ and

$$\mathscr{N}(\mathcal{H}, \vartheta) \leq \begin{cases} \left[\dfrac{\mathfrak{n}+1}{\mathfrak{m}+1}\right] & \text{for } r=1, \\ r\left(\mathfrak{n}\left[\dfrac{\mathfrak{m}+1-2r}{r-1}\right]-2\right) & \text{for } \mathcal{H} \in \mathfrak{F}_1^+ \text{ and } r>1, \\ r(\mathfrak{n}-1)\left[\dfrac{\mathfrak{m}-4}{2(r-1)}\right] & \text{for } \mathcal{H} \in \mathfrak{F}_1^- \text{ and } r>1. \end{cases} \tag{84}$$

3) If $\mathcal{H} \in \mathfrak{F}_3$, then $r-1 = \dim H_1(\mathcal{L}_{\mathfrak{c}}, \mathbb{Z}) \geq 0$ and

$$\mathscr{N}(\mathcal{H}, \vartheta) \leq \begin{cases} 0 & \text{for } r=1, \\ \left[\dfrac{\mathfrak{n}+1}{\mathfrak{m}+1}\right] & \text{for } r=2, \\ (r-1)\mathfrak{n}, & \text{for } r>2. \end{cases} \tag{85}$$

Proof. The computation of the total number $\mathscr{N}(\mathcal{H},\vartheta)$ requires the addition over the number of cycles in the basis. Thus, equations (83), (84) and (85) follow from equations (80), (81) and (82), respectively. $\square$

Scheme for the proof of Theorem 18: The upper bound in (82) for case $r=1$, that is, $\dim H_1(\mathcal{L}_{\mathfrak{c}},\mathbb{Z})=0$, follows from Remark 5. The upper bounds in (82) for $r=2$ and in (81) for $r=1$, that is, $\dim H_1(\mathcal{L}_{\mathfrak{c}},\mathbb{Z})=1$, follow from (79). Therefore, to complete the proof, it is sufficient to consider $\mathcal{H}\in\mathfrak{F}_1\cup\mathfrak{F}_2\cup\mathfrak{F}_3$ with $\dim H_1(\mathcal{L}_{\mathfrak{c}},\mathbb{Z})\geq 2$. The rest of the proof follows from the next three propositions. More precisely, Proposition 20 will give the upper bound in (82) for $r>2$. Proposition 21 will provide the upper bounds in (81) for $r>1$, and finally Proposition 22 will give the the upper bounds in (80).

Proposition 20. *Let $\mathcal{H}(x,y)$ be a polynomial of degree $\mathfrak{m}+1$ in Neumann–Norbury family $\mathfrak{F}_3$, with $r-1=\dim H_1(\mathcal{L}_{\mathfrak{c}},\mathbb{Z})\geq 2$. If $\vartheta\in\Omega^1_{ne}(\mathbb{C}^2_{x\,y})_{\leq\mathfrak{n}}$, then for each cycle $\delta_{\mathtt{i}}(\mathfrak{c})$ the Abelian integral*

$$\mathcal{I}_{\mathtt{i}}(\mathfrak{c})=\int_{\delta_{\mathtt{i}}(\mathfrak{c})}\vartheta\colon\mathbb{C}\longrightarrow\mathbb{C}$$

is a polynomial of degree at most $\mathfrak{n}$. Moreover, this upper bound is reached.

Proof. Consider $\mathcal{H}\in\mathfrak{F}_3$ of degree $\mathfrak{m}+1$ and $r-1=\dim H_1(\mathcal{L}_{\mathfrak{c}},\mathbb{Z})\geq 2$, that is,

$$\mathcal{H}(x,y)=y\prod_{\mathtt{i}=1}^{r-1}(\beta_{\mathtt{i}}-x)^{a_{\mathtt{i}}}-h(x)\,,$$

where $r\geq 3$, $a_1,\ldots,a_{r-1}$ are positive integers, $\beta_1,\ldots,\beta_{r-1}$ are distinct points in $\mathbb{C}^*$, and $h(x)$ is a polynomial of degree at most $\mathfrak{m}=a_1+\cdots+a_{r-1}$.

Consider the rectifying map $\mathcal{R}$ for $\mathcal{H}$ given in (40) and the basis $B^1(\mathbb{C}^2_{x\,y},\mathfrak{n})=\{\vartheta_{ij}=x^iy^j dx\}$ of non-exact 1-forms of degree $\mathfrak{n}$. Then,

$$\eta_{ij}=\mathcal{R}_*(\vartheta_{ij})=\eta^t_{ij}=\frac{t^i\,(\mathfrak{c}+h(t))^j}{\prod_{\mathtt{i}=1}^{r-1}(\beta_{\mathtt{i}}-t)^{ja_{\mathtt{i}}}}\,dt.$$

Thus, η^t_{ij} admits a representation at $t=\beta_{\mathtt{i}}$ of the form (57) as follows

$$\eta^t_{ij}=\frac{R_1(t,\mathfrak{c})}{(t-\beta_{\mathtt{i}})^{ja_{\mathtt{i}}}}\,dt,\qquad\text{where}\quad R_1(t,\mathfrak{c})=\frac{(-1)^{ja_{\mathtt{i}}}t^i\,(\mathfrak{c}+h(t))^j}{\prod_{s=1,s\neq\mathtt{i}}^{r-1}(\beta_s-t)^{ja_s}}.$$

We use the criterion given in (58) to obtain

$$\int_{\alpha_{\mathtt{i}}(\mathfrak{c})}\eta^t_{ij}=\frac{2\pi\sqrt{-1}}{(ja_{\mathtt{i}}-1)!}\cdot\frac{\partial^{ja_{\mathtt{i}}-1}}{\partial t^{ja_{\mathtt{i}}-1}}\left(\frac{(-1)^{ja_{\mathtt{i}}}t^i\,(\mathfrak{c}+h(t))^j}{\prod_{\nu=1,\nu\neq\mathtt{i}}^{r-1}(\beta_\nu-t)^{ja_\nu}}\right)\Big|_{t=\beta_{\mathtt{i}}},$$

which is a polynomial in $\mathfrak{c}$ of degree at most j.

Hence, according to (55), the Abelian integral $\mathcal{I}_{\mathtt{i}}(\mathfrak{c})$ is a polynomial of degree at most $\mathfrak{n}$. This completes the proof of the first part of the proposition.

We now show that the upper bound is reached. For each integer $\mathfrak{m}\geq 2$, the polynomial $\mathcal{H}(x,y)=y(-1-x)(1-x)^{\mathfrak{m}-1}$ of degree $\mathfrak{m}+1$ belongs to family $\mathfrak{F}_3$ and equation

(40) becomes

$$\begin{array}{ccccc} \mathbb{C}^2_{x\,y}\backslash\{1-x^2=0\} & \xrightarrow{\mathcal{R}} & \mathbb{C}^2_{t\,\mathfrak{c}}\backslash\{1-t^2=0\} & \xrightarrow{\mathcal{R}^{-1}} & \mathbb{C}^2_{x\,y}\backslash\{1-x^2=0\} \\ (x,y) & \longmapsto & (x,\mathcal{H}(x,y)) & \longmapsto & \left(t, \dfrac{(-1)^{\mathfrak{m}}\mathfrak{c}}{(t+1)(t-1)^{\mathfrak{m}-1}}\right). \end{array} \tag{86}$$

We consider the polynomial 1-form $\vartheta = (\mathfrak{a}_1 y + \mathfrak{a}_2 y^2 + \cdots + \mathfrak{a}_{\mathfrak{n}} y^{\mathfrak{n}})\, dx$ of degree $\mathfrak{n}$. Thus,

$$\eta = \mathcal{R}_*(\vartheta) = \sum_{\nu=1}^{\mathfrak{n}} \mathfrak{a}_\nu \left(\frac{(-1)^{\nu\mathfrak{m}}\,\mathfrak{c}^\nu}{(t+1)^\nu (t-1)^{\nu(\mathfrak{m}-1)}}\right) dt$$

and

$$\int_{\delta_{\mathtt{i}}(\mathfrak{c})} \vartheta = \int_{\alpha_{\mathtt{i}}(\mathfrak{c})} \eta = \sum_{\nu=1}^{\mathfrak{n}} \mathfrak{a}_\nu \left(\int_{\alpha_{\mathtt{i}}(\mathfrak{c})} \frac{(-1)^{\nu\mathfrak{m}}\, dt}{(t+1)^\nu (t-1)^{\nu(\mathfrak{m}-1)}}\right) \mathfrak{c}^\nu.$$

By criterion (58), we have

$$\xi_{1,\nu} \doteq \int_{\alpha_1(\mathfrak{c})} \frac{(-1)^{\nu\mathfrak{m}}\, dt}{(t+1)^\nu (t-1)^{\nu(\mathfrak{m}-1)}} = \frac{2\pi\sqrt{-1}}{(\nu-1)!} \frac{\partial^{\nu-1}}{\partial t^{\nu-1}} \left(\frac{(-1)^{\nu\mathfrak{m}}}{(t-1)^{\nu(\mathfrak{m}-1)}}\right)\Big|_{t=-1}$$

and

$$\xi_{2,\nu} \doteq \int_{\alpha_2(\mathfrak{c})} \frac{(-1)^{\nu\mathfrak{m}}\, dt}{(t+1)^\nu (t-1)^{\nu(\mathfrak{m}-1)}} = \frac{2\pi\sqrt{-1}}{(\nu(\mathfrak{m}-1)-1)!} \frac{\partial^{\nu(\mathfrak{m}-1)-1}}{\partial t^{\nu(\mathfrak{m}-1)-1}} \left(\frac{(-1)^{\nu\mathfrak{m}}}{(t+1)^{\nu}}\right)\Big|_{t=1}.$$

A straightforward computation gives

$$\xi_{\mathtt{i},\nu} = \frac{\left(2\pi\sqrt{-1}\right)(-1)^{\nu+1-\mathtt{i}}}{2^{\nu\mathfrak{m}-1}} \binom{\nu\mathfrak{m}-2}{\nu(\mathfrak{m}-1)-1} \neq 0.$$

Therefore, for $\mathtt{i}=1,2$, the Abelian integral $\mathcal{I}_{\mathtt{i}}(\mathfrak{c})$ is the polynomial of degree $\mathfrak{n}$

$$\mathfrak{c}\left(\mathfrak{a}_1\xi_{\mathtt{i},1} + \mathfrak{a}_2\xi_{\mathtt{i},2}\mathfrak{c} + \cdots + \mathfrak{a}_{\mathfrak{n}}\xi_{\mathtt{i},\mathfrak{n}}\mathfrak{c}^{\mathfrak{n}-1}\right).$$

Furthermore, we can find suitable values of $\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_{\mathfrak{n}}$ such that the respective integral $\mathcal{I}_{\mathtt{i}}(\mathfrak{c})$ has zeros at $0, \mathfrak{c}_1, \mathfrak{c}_2, \ldots, \mathfrak{c}_{\mathfrak{n}-1} \in \mathbb{C}$. □

Remark 10. *There are polynomials in $\mathfrak{F}_3$ of degree $\mathfrak{m}+1 \geq 3$, which do not reach the upper bound of the previous result. For instance, the polynomial $\mathcal{H}(x,y) = y(1-x)^{\mathfrak{m}}$, with $\mathfrak{m} \geq 2$, belongs to the family $\mathfrak{F}_3$ and is clearly algebraically equivalent to $yx^{\mathfrak{m}}$. Thus, from* [26, Theorem 2]*, we know that the Abelian integral defined by $\mathcal{H}$ and a polynomial 1-form of degree $\mathfrak{n}$ has at most $[(\mathfrak{n}+1)/(\mathfrak{m}+1)]$ isolated zeros in $\mathbb{C}\backslash\mathfrak{B}(\mathcal{H})$.*

Proposition 21. *Let $\mathcal{H}(x,y)$ be a polynomial of degree $\mathfrak{m}+1$ in Neumann–Norbury family $\mathfrak{F}_2$, with $r = \dim H_1(\mathcal{L}_{\mathfrak{c}}, \mathbb{Z}) \geq 2$. If $\vartheta \in \Omega^1_{ne}(\mathbb{C}^2_{x\,y})_{\leq\mathfrak{n}}$, then for each cycle $\delta_{\mathtt{i}}(\mathfrak{c})$ the Abelian integral*

$$\mathcal{I}_{\mathtt{i}}(\mathfrak{c}) = \int_{\delta_{\mathtt{i}}(\mathfrak{c})} \vartheta \colon \mathbb{C} \longrightarrow \mathbb{C}$$

is a polynomial of degree at most

$$\begin{cases} \mathfrak{n}\left(\left[\dfrac{\mathfrak{m}-1}{r-1}\right] - 2\right) - 2 & \textit{for } \mathcal{H} \in \mathfrak{F}_2^+, \\ (\mathfrak{n}-1)\left[\dfrac{\mathfrak{m}-4}{2(r-1)}\right] & \textit{for } \mathcal{H} \in \mathfrak{F}_2^-. \end{cases}$$

Proof. Consider $\mathcal{H} \in \mathfrak{F}_2$ of degree $\mathfrak{m}+1$, with $r = \dim H_1(\mathcal{L}_\mathfrak{c}, \mathbb{Z}) \geq 2$, that is,

$$\mathcal{H}(x,y) = x^{p_1}\mathcal{S}(x,y)^p \prod_{\mathtt{i}=1}^{r-1} \left(\beta_\mathtt{i} - x^{q_1}\mathcal{S}(x,y)^q\right)^{a_\mathtt{i}},$$

where $0 \leq p_1 < p$, $0 \leq q_1 < q$ and $pq_1 - qp_1 = \pm 1$, $r \geq 2$, $a_1, \ldots, a_{r-1}$ are positive integers, $\beta_1, \ldots, \beta_{r-1}$ are distinct points of $\mathbb{C}^*$, $\mathcal{S}(x,y) = x^k y - P(x)$, with k a positive integer and $P(x) \in \mathbb{C}[x]_{\leq k-1}$, and

$$\mathfrak{m}+1 = p_1 + p(k+1) + (q_1 + q(k+1))(a_1 + \cdots + a_{r-1}). \tag{87}$$

Case $pq_1 - qp_1 = 1$. We use the rectifying map $\mathcal{R}$ for $\mathcal{H}$ given in (41) and the basis $B^1(\mathbb{C}^2_{x\,y}, \mathfrak{n}) = \{\vartheta_{ij} = x^i y^j dx\}$ of non-exact 1-forms of degree $\mathfrak{n}$. Then,

$$\eta_{ij} = \mathcal{R}_*(\vartheta_{ij}) = \frac{t^{pi}\Pi(t)^{qi}}{\mathfrak{c}^{qi}} \, \frac{\mathfrak{c}^{jq} S_1(t,\mathfrak{c})^j}{t^{j(pk+p_1)}\Pi(t)^{j(qk+q_1)}} \, d\left(\frac{t^p\Pi(t)^q}{\mathfrak{c}^q}\right).$$

Thus,

$$\eta_{ij}^t = \frac{\mathfrak{c}^{q(j-i-1)} S_1(t,\mathfrak{c})^j \left(qt\Pi'(t) + p\Pi(t)\right)}{t^{j(pk+p_1)-p(i+1)+1} \prod_{\mathtt{i}=1}^{r-1} (\beta_\mathtt{i} - t)^{(j(qk+q_1)-q(i+1)+1)a_\mathtt{i}}} \, dt.$$

Recall that $\Pi(t)$ and $S_1(t,\mathfrak{c})$ are given in (42) and (43). The binomial theorem then yields

$$S_1(t,\mathfrak{c})^j = \mathfrak{c}^{jq(k-1)} \sum_{\mu=0}^{j} \binom{j}{\mu} \mathfrak{c}^{q_1(j-\mu)} t^{p_1\mu} \Pi(t)^{q_1\mu} P\left(t^p\Pi(t)^q\mathfrak{c}^{-q}\right)^\mu.$$

Following the same idea as in subsection 5.5, we get

$$P\left(t^p\Pi(t)^q\mathfrak{c}^{-q}\right)^\mu = \sum \frac{\mu!}{n_0!\cdots n_s!} \lambda_0^{n_0}\cdots\lambda_s^{n_s} t^{pN_s}\Pi(t)^{qN_s}\mathfrak{c}^{-qN_s},$$

where the sum is over all lists of $s+1$ non-negative integers $(n_0, \ldots, n_s)$ that satisfy (75). Therefore, by using the two previous equalities and simplifying, we obtain

$$\eta_{ij}^t = \sum_{\mu=0}^{j} \sum A^\mu_{n_0\cdots n_s} \frac{\mathfrak{c}^{q_1(j-\mu)-q\widetilde{N}_s}\left(qt\Pi'(t) + p\Pi(t)\right)}{t^{p_1(j-\mu)-p\widetilde{N}_s+1}\Pi(t)^{q_1(j-\mu)-q\widetilde{N}_s+1}} \, dt, \tag{88}$$

where $A^\mu_{n_0\ldots n_s}$ and $\widetilde{N}_s$ are the same as in (77).

We will now prove that if $q_1(j-\mu) - q\widetilde{N}_s < 0$, then the integral along $\alpha_\mathtt{i}(\mathfrak{c})$ of the corresponding term in the sum on the right-hand side of (88) vanishes identically, which implies, according to (55) and the criterion given in (58), that the Abelian integral $\mathcal{I}_\mathtt{i}(\mathfrak{c})$ is a polynomial of degree at most

$$q_1(j-\mu) - q\widetilde{N}_s = j(qk+q_1) - q_1\mu - q(N_s+i+1) \leq j(qk+q_1) - q \leq \mathfrak{n}(qk+q_1) - q. \tag{89}$$

Indeed, if $q_1(j-\mu) - q\widetilde{N}_s < 0$ and $p\widetilde{N}_s - p_1(j-\mu) \leq 0$, then by using that $p > 0$ and $q > 0$, we get

$$pq_1(j-\mu) - pq\widetilde{N}_s < 0 \quad \text{and} \quad pq\widetilde{N}_s - qp_1(j-\mu) \leq 0,$$

whence $(pq_1 - qp_1)(j-\mu) < 0$, which is a contradiction because $pq_1 - qp_1 = 1$ and $j - \mu \geq 0$. Hence, if $q_1(j-\mu) - q\widetilde{N}_s < 0$, then $p\widetilde{N}_s - p_1(j-\mu) > 0$. This implies that

$$p_1(j-\mu) - p\widetilde{N}_s + 1 \leq 0 \quad \text{and} \quad q_1(j-\mu) - q\widetilde{N}_s + 1 \leq 0.$$

Hence, the 1-form in the sum on the right-hand side of (88) does not have any pole. This proves our assertion.

From (87) it follows that $qk + q_1 \le [(\mathfrak{m}-1)/(r-1)] - q$. Thus, according to (89), the degree of $\mathcal{I}_{\mathtt{i}}(\mathfrak{c})$ is at most

$$\mathfrak{n}\left(\left[\frac{\mathfrak{m}-1}{r-1}\right]-q\right)-q \le \mathfrak{n}\left(\left[\frac{\mathfrak{m}-1}{r-1}\right]-2\right)-2.$$

Case $pq_1 - qp_1 = -1$. The rectifying map $\mathcal{R}$ for $\mathcal{H}$ and its inverse are

$$\begin{array}{ccccc} \mathbb{C}^2_{x\,y}\backslash\Sigma(\mathcal{H}) & \xrightarrow{\mathcal{R}} & \mathbb{C}^2_{t\,\mathfrak{c}}\backslash\mathfrak{D}_2 & \xrightarrow{\mathcal{R}^{-1}} & \mathbb{C}^2_{x\,y}\backslash\Sigma(\mathcal{H}) \\ (x,y) & \longmapsto & \big(G(x,y),\mathcal{H}(x,y)\big) & \longmapsto & \left(\dfrac{\mathfrak{c}^q}{t^p\Pi(t)^q},\dfrac{t^p\Pi(t)^q S_1^-(t,\mathfrak{c})}{\mathfrak{c}^{qk+q_1}}\right). \end{array} \tag{90}$$

where $\mathcal{G}(x,y)$, $\mathcal{H}(x,y)$ and $\Sigma(\mathcal{R})$ are according to Tables 1 and 2, $\mathfrak{D}_2 = \{\mathfrak{c}\,t\,\Pi(t)=0\}$, $\Pi(t)$ as in (42) and

$$S_1^-(t,\mathfrak{c}) = t^{p(k-1)}\Pi(t)^{q(k-1)}\Big(t^{p_1}\Pi(t)^{q_1} + \mathfrak{c}^{q_1}P\left(\mathfrak{c}^q t^{-p}\Pi(t)^{-q}\right)\Big),$$

which is polynomial because P has degree at most $k-1$.

Consider the basis $B^1(\mathbb{C}^2_{x\,y},\mathfrak{n}) = \{\vartheta_{ij} = x^i y^j\,dx\}$ of non-exact 1-forms of degree $\mathfrak{n}$. As a result,

$$\eta_{ij} = \mathcal{R}_*(\vartheta_{ij}) = \frac{\mathfrak{c}^{qi}}{t^{pi}\Pi(t)^{qi}}\,\frac{t^{jp}\Pi(t)^{jq}S_1^-(t,\mathfrak{c})^j}{\mathfrak{c}^{j(qk+q_1)}}\,d\left(\frac{\mathfrak{c}^q}{t^p\Pi(t)^q}\right).$$

Thus,

$$\eta^t_{ij} = -\frac{\mathfrak{c}^{q(i-jk+1)-jq_1}S_1^-(t,\mathfrak{c})^j\big(qt\Pi'(t)+p\Pi(t)\big)}{t^{p(i-j+1)+1}\Pi(t)^{q(i-j+1)+1}}\,dt.$$

Analogously to the previous case, after applying the binomial theorem to $S_1^-(t,\mathfrak{c})^j$ and the multinomial theorem to $P\left(\mathfrak{c}^q t^{-p}\Pi(t)^{-q}\right)^\mu$, we obtain

$$\eta^t_{ij} = -\sum_{\mu=0}^{j}\sum A^{\mu}_{n_0\cdots n_s}\frac{\mathfrak{c}^{q\widetilde{N}_s-q_1(j-\mu)}\big(qt\Pi'(t)+p\Pi(t)\big)}{t^{p\widetilde{N}_s+1-p_1(j-\mu)}\Pi(t)^{q\widetilde{N}_s+1-q_1(j-\mu)}}\,dt,$$

where $A^{\mu}_{n_0\ldots n_s}$ and $\widetilde{N}_s$ are the same as in (77). We can prove that if $q\widetilde{N}_s - q_1(j-\mu) < 0$, then the integral along $\alpha_{\mathtt{i}}(\mathfrak{c})$ of the corresponding term in the sum on the right-hand side of previous equation vanishes identically, which implies, according to (55) and (58), that the Abelian integral $\mathcal{I}_{\mathtt{i}}(\mathfrak{c})$ is a polynomial of degree at most

$$q\widetilde{N}_s - q_1(j-\mu) \le q(N_s - jk + i + 1) \le q((k-1)j - jk + i + 1) \le q(\mathfrak{n}-1). \tag{91}$$

Since from (87) it follows that $q \le [(\mathfrak{m}-4)/(2(r-1))]$, we obtain that the degree of $\mathcal{I}_{\mathtt{i}}(\mathfrak{c})$ is at most

$$(\mathfrak{n}-1)\left[\frac{\mathfrak{m}-4}{2(r-1)}\right].$$

We are done. □

Proposition 22. *Let $\mathcal{H}(x,y)$ be a polynomial of degree $\mathfrak{m}+1$ in Neumann–Norbury family $\mathfrak{F}_1$, with $r+1 = \dim H_1(\mathcal{L}_{\mathfrak{c}},\mathbb{Z}) \ge 3$. If $\vartheta \in \Omega^1(\mathbb{C}^2_{x\,y})_{\le\mathfrak{n}}$, then for each cycle $\delta_{\mathtt{i}}(\mathfrak{c})$*

the Abelian integral

$$\mathcal{I}_{\mathtt{i}}(\mathfrak{c})=\int_{\delta_{\mathtt{i}}(\mathfrak{c})}\vartheta\colon\mathbb{C}\longrightarrow\mathbb{C}$$

is a polynomial of degree at most

$$\begin{cases}\mathfrak{n}\left(\left[\dfrac{\mathfrak{m}-1}{r-1}\right]-2\right)-2 & \textit{for } \mathcal{H}\in\mathfrak{F}_1^{+} \textit{ and } 0\le \mathtt{i}\le r-1,\\ (\mathfrak{n}-1)\left[\dfrac{\mathfrak{m}-r-2}{2}\right] & \textit{for } \mathcal{H}\in\mathfrak{F}_1^{+} \textit{ and } \mathtt{i}=r,\\ (\mathfrak{n}-1)\left[\dfrac{\mathfrak{m}-4}{2(r-1)}\right] & \textit{for } \mathcal{H}\in\mathfrak{F}_1^{-} \textit{ and } 0\le \mathtt{i}\le r-1,\\ \mathfrak{n}\,(\mathfrak{m}-1-r)-r & \textit{for } \mathcal{H}\in\mathfrak{F}_1^{-} \textit{ and } \mathtt{i}=r.\end{cases}$$

Proof. Consider $\mathcal{H}\in\mathfrak{F}_1$ of degree $\mathfrak{m}+1$, with $r+1=\dim H_1(\mathcal{L}_{\mathfrak{c}},\mathbb{Z})\ge 3$, that is,

$$H(x,y)=x^{q_1}\mathcal{S}(x,y)^q+x^{p_1}\mathcal{S}(x,y)^p\prod_{\mathtt{i}=1}^{r-1}(\beta_{\mathtt{i}}-x^{q_1}\mathcal{S}(x,y)^q)^{a_{\mathtt{i}}}$$

where; $0\le p_1<p$, $0\le q_1<q$ and $pq_1-qp_1=\pm1$; $r\ge 2$, $a_1,\dots,a_{r-1}$ are positive integers; $\beta_1,\dots,\beta_{r-1}$ are distinct points of $\mathbb{C}^*$; $\mathcal{S}(x,y)=x^ky-P(x)$, with k a positive integer, $P(x)\in\mathbb{C}[x]_{\le k-1}$; and $\mathfrak{m}+1=p_1+p(k+1)+(q_1+q(k+1))(a_1+\cdots+a_{r-1})$.

Case $pq_1-qp_1=1$. We can use the rectifying map $\mathcal{R}$ for $\mathcal{H}$ given in (44) and the basis $B^1(\mathbb{C}^2_{x\,y},\mathfrak{n})=\{\vartheta_{ij}=x^iy^jdx\}$ of non-exact 1-forms of degree $\mathfrak{n}$. Then,

$$\eta_{ij}=\mathcal{R}_*(\vartheta_{ij})=\frac{t^{pi}\Pi(t)^{qi}}{(c-t)^{qi}}\,\frac{(c-t)^{jq}S_2(t,c)^j}{t^{j(pk+p_1)}\Pi(t)^{j(qk+q_1)}}\,d\left(\frac{t^p\Pi(t)^q}{(c-t)^q}\right).$$

Thus,

$$\eta^t_{ij}=\frac{S_2(t,c)^j\big(qt\Pi(t)+(c-t)(qt\Pi'(t)+p\Pi(t))\big)}{t^{j(pk+p_1)-p(i+1)+1}\Pi(t)^{j(qk+q_1)-q(i+1)+1}(c-t)^{q(i-j+1)+1}}\,dt.$$

Recall that $S_2(t,\mathfrak{c})$ is given in (45). Therefore, by following the same idea as in subsection 5.5 and the same steps as in the proof of Proposition 21, we obtain

$$\eta^t_{ij}=\sum_{\mu=0}^{j}\sum A^{\mu}_{n_0\cdots n_s}\frac{(\mathfrak{c}-t)^{q_1(j-\mu)-q\widetilde{N}_s-1}\big(qt\Pi(t)+(c-t)(qt\Pi'(t)+p\Pi(t))\big)}{t^{p_1(j-\mu)-p\widetilde{N}_s+1}\Pi(t)^{q_1(j-\mu)-q\widetilde{N}_s+1}}\,dt,$$

where the second sum is over all lists of $s+1$ non-negative integers $(n_0,\dots,n_s)$ that satisfy (75), and $A^{\mu}_{n_0\dots n_s}$, $\widetilde{N}_s$ are the same as in (77). Hence, η^t_{ij} could have poles at $t=\beta_{\mathtt{i}}$, with $\mathtt{i}=0,1,\dots,r$.

As in the proof of Proposition 21, if $q_1(j-\mu)-q\widetilde{N}_s<0$, then η^t_{ij} does not have poles at $t=\beta_{\mathtt{i}}$, for $\mathtt{i}=0,1,\dots,r-1$ because

$$p_1(j-\mu)-p\widetilde{N}_s+1\le 0\quad\text{and}\quad q_1(j-\mu)-q\widetilde{N}_s+1\le 0.$$

Hence, according to (55) and the criterion given in (58), that for $\mathtt{i}=0,1,\dots,r-1$ each Abelian integral $\mathcal{I}_{\mathtt{i}}(\mathfrak{c})$ is a polynomial of degree at most

$$q_1(j-\mu)-q(N_s-jk+i+1)\le j(qk+q_1)-q\le\mathfrak{n}(qk+q_1)-q. \tag{92}$$

In the remaining case $\mathtt{i} = r$, each term in the sum on the right-hand side of the previous equation for η_{ij}^t admits the following representation at $t = \mathfrak{c}$,

$$\eta_{ij}^t(\mu, n_0, \ldots, n_s) \doteq \frac{(-1)^{q\widetilde{N}_s - q_1(j-\mu)}\, R(t,\mathfrak{c})}{(t-\mathfrak{c})^{q\widetilde{N}_s - q_1(j-\mu)+1}}\, dt,$$

where $R(t,\mathfrak{c}) = t^{p\widetilde{N}_s - p_1(j-\mu)-1}\Pi(t)^{q\widetilde{N}_s - q_1(j-\mu)-1}\big(qt\Pi(t) + (\mathfrak{c}-t)(qt\Pi'(t) + p\Pi(t))\big)$.

Hence, if $q\widetilde{N}_s - q_1(j-\mu)+1 \le 0$, then $t = \mathfrak{c}$ is not a pole of $\eta_{ij}^t(\mu, n_0, \ldots, n_s)$, whence the integral $\int_{\alpha_r(\mathfrak{c})} \eta_{ij}^t(\mu, n_0, \ldots, n_s)$ vanishes identically. If $q\widetilde{N}_s - q_1(j-\mu)+1 \ge 1$, then $R(t,\mathfrak{c})$ is a polynomial. Thus, according to the criterion given in (58), we have

$$\int_{\alpha_r(\mathfrak{c})} \eta_{ij}^t(\mu, n_0, \ldots, n_s) = \frac{\big(2\pi\sqrt{-1}\big)}{(q\widetilde{N}_s - q_1(j-\mu))!} \cdot \frac{\partial^{q\widetilde{N}_s - q_1(j-\mu)}}{\partial t^{q\widetilde{N}_s - q_1(j-\mu)}} \left(R(t,\mathfrak{c})\right)\Big|_{t=\mathfrak{c}},$$

which is a polynomial function in $\mathfrak{c}$. Moreover, by recalling from equation (42) that $\Pi(t) = \prod_{\mathtt{i}=1}^{r-1}(\beta_{\mathtt{i}} - t)^{a_{\mathtt{i}}}$, then the degree of t in $R(t,\mathfrak{c})$ is at most

$$p\widetilde{N}_s - p_1(j-\mu) + \Big(q\widetilde{N}_s - q_1(j-\mu)\Big)\Big(\sum_{\mathtt{i}=1}^{r-1} a_{\mathtt{i}}\Big).$$

Thus, the maximum degree of t in the previous derivative of $R(t,\mathfrak{c})$ is

$$p\widetilde{N}_s - p_1(j-\mu) + \Big(q\widetilde{N}_s - q_1(j-\mu)\Big)\Big(\sum_{\mathtt{i}=1}^{r-1} a_{\mathtt{i}} - 1\Big),$$

which is then the maximum degree of $\mathfrak{c}$ in $\int_{\alpha_r(\mathfrak{c})} \eta_{ij}^t(\mu, n_0, \ldots, n_s)$.

The previous expression can be written as

$$\Big(p + q\Big(\sum_{\mathtt{i}=1}^{r-1} a_{\mathtt{i}} - 1\Big)\Big)\widetilde{N}_s - (j-\mu)\Big(p_1 + q_1\Big(\sum_{\mathtt{i}=1}^{r-1} a_{\mathtt{i}} - 1\Big)\Big).$$

Thus, it is bounded from above by

$$\Big(p + q\Big(\sum_{\mathtt{i}=1}^{r-1} a_{\mathtt{i}} - 1\Big)\Big)\widetilde{N}_s,$$

which, following (91), is bounded by

$$\Big(p + q\Big(\sum_{\mathtt{i}=1}^{r-1} a_{\mathtt{i}} - 1\Big)\Big)(i-j+1) + 1 \le (\mathfrak{n}-1)\Big(p + q\Big(\sum_{\mathtt{i}=1}^{r-1} a_{\mathtt{i}} - 1\Big)\Big).$$

The degree of $\mathcal{H}$ is the same as in (87), from which we obtain

$$qk + q_1 \le \left[\frac{\mathfrak{m}-1}{r-1}\right] - q \quad \text{and} \quad p + q\Big(\sum_{\mathtt{i}=1}^{r-1} a_{\mathtt{i}} - 1\Big) \le \left[\frac{\mathfrak{m}-r-2}{2}\right].$$

Therefore, according to (92), for each $\mathtt{i} = 0, \ldots, r-1$ the Abelian integral $\mathcal{I}_{\mathtt{i}}(\mathfrak{c})$ is a polynomial of degree at most

$$\mathfrak{n}(qk + q_1) - q \le \mathfrak{n}\left(\left[\frac{\mathfrak{m}-1}{r-1}\right] - 2\right) - 2$$

and the Abelian integral $\mathcal{I}_r(\mathfrak{c})$ is a polynomial of degree at most

$$(\mathfrak{n}-1)\left[\frac{\mathfrak{m}-r-2}{2}\right].$$

Case $pq_1 - qp_1 = -1$. The rectifying map $\mathcal{R}$ for $\mathcal{H}$ and its inverse are

$$\begin{array}{ccccc} \mathbb{C}^2_{x\,y}\backslash\Sigma(\mathcal{H}) & \xrightarrow{\mathcal{R}} & \mathbb{C}^2_{t\,\mathfrak{c}}\backslash\mathfrak{D}_2 & \xrightarrow{\mathcal{R}^{-1}} & \mathbb{C}^2_{x\,y}\backslash\Sigma(\mathcal{H}) \\ (x,y) & \longmapsto & \big(\mathcal{G}(x,y),\mathcal{H}(x,y)\big) & \longmapsto & \left(\dfrac{(\mathfrak{c}-t)^q}{t^p\Pi(t)^q}, \dfrac{t^p\Pi(t)^q S_2^-(t,\mathfrak{c})}{(\mathfrak{c}-t)^{kq+q_1}}\right), \end{array} \tag{93}$$

where $\mathcal{G}(x,y)$, $\mathcal{H}(x,y)$ and $\Sigma(\mathcal{R})$ agree with Tables 1 and 2,

$$\mathfrak{D}_2 = \{(\mathfrak{c}-t)\,t\,\Pi(t)=0\},$$

and

$$S_2^-(t,\mathfrak{c}) = (t^p\Pi(t)^q)^{k-1}\left(t^{p_1}\Pi(t)^{q_1} + (\mathfrak{c}-t)^{q_1} P\left(\frac{(\mathfrak{c}-t)^q}{t^p\Pi(t)^q}\right)\right)$$

which is polynomial because P has degree at most $k-1$. The rest of the proof is analogous to the previous case. □

7. Proof of Theorems 1 and 2

Proof of Theorem 1. The result follows in a rather straightforward way from Proposition 11, which deals with the relationship between the degrees of the original pair (H,ω) and the degree of the transformed $(\mathcal{H},\vartheta)$, and Theorem 18, concerning the maximal number of zeros of Abelian integrals defined by $(\mathcal{H},\vartheta)$, with $\mathcal{H}$ in the Neumann–Norbury families.

We consider $H \in \mathbb{C}[u,v]_{\leq m+1}$ and $\omega \in \Omega^1(\mathbb{C}^2_{u\,v})_{\leq n}$, where H is a primitive polynomial with trivial global monodromy and $\mathfrak{r} = \dim H_1(L_c,\mathbb{Z}) \geq 1$.

The first two steps of our Program are guaranteed by Theorem 8, Proposition 11 and Lemma 12. Hence, let $\mathcal{H}(x,y)$ be a normal form of $H(u,v)$, through the pair $(\psi,\sigma) \in \mathrm{Aut}(\mathbb{C}^2)\times\mathrm{Aut}(\mathbb{C})$, and let ϑ be the 1-form as in (12), that is, $\vartheta = \sigma'\psi_*(\omega)$. Since $\mathcal{H}$ belongs to one of the Neumann–Norbury families $\mathfrak{F}_1$, $\mathfrak{F}_2$ or $\mathfrak{F}_3$, the assertion 1) of Theorem 1 follows directly from Corollary 3 and first part of Theorem 18.

Regarding the assertion 2) of Theorem 1, we will consider the following cases.

Case 1. If $\mathcal{H}(x,y) \in \mathfrak{F}_3$ with $r=2$ or $\mathcal{H}(x,y)\in\mathfrak{F}_2$ with $r=1$, then $\mathcal{H}$ and H are of type $(0,2)$. Hence, from Theorem 17, we obtain the upper bound given in assertion 2) for $m=1$.

Case 2. If $\mathcal{H}(x,y)\in\mathfrak{F}_3$ with $r>2$, then $r-1=\mathfrak{r}=\dim H_1(\mathcal{L}_c,\mathbb{Z}) = \dim H_1(L_c,\mathbb{Z}) \geq 2$ and $\mathfrak{m}+1 = \deg(\mathcal{H}) \geq 3$. Hence, from Corollary 3, Theorem 18.3) and Proposition 11.3), we obtain that each Abelian integral $I_{\mathtt{i}}(c)$, $1\leq \mathtt{i} \leq \dim H_1(L_c,\mathbb{Z})$, satisfies

$$\deg(I_{\mathtt{i}}(c)) = \deg(\mathcal{I}_{\mathtt{i}}(\sigma(c))) = \deg(\mathcal{I}_{\mathtt{i}}(\mathfrak{c})) \leq (n+1)(m+1-\mathfrak{r})-1.$$

Case 3. If $\mathcal{H}(x,y)\in\mathfrak{F}_2$ with $r\geq 2$, then $r=\mathfrak{r}=\dim H_1(\mathcal{L}_c,\mathbb{Z}) = \dim H_1(L_c,\mathbb{Z})\geq 2$ and $\mathfrak{m}+1=\deg(\mathcal{H})\geq 7$. In addition, we have that $r\leq [\mathfrak{m}/2]-1$, which follows from equation (87). A simple but cumbersome computation shows that for $\mathfrak{m} \geq 6$, and $\mathfrak{n} \geq 1$, we have

$$\mathfrak{n}\left(\left[\frac{\mathfrak{m}-1}{r-1}\right]-2\right)-2 \geq (\mathfrak{n}-1)\left[\frac{\mathfrak{m}-4}{2(r-1)}\right].$$

Thus, if $\vartheta \in \Omega^1(\mathbb{C}^2_{x\,y})_{\leq \mathfrak{n}}$, then from Theorem 18.2) we conclude that

$$\deg(\mathcal{I}_{\mathtt{i}}(\mathfrak{c})) \leq \mathfrak{n}\left(\left[\frac{\mathfrak{m}-1}{r-1}\right]-2\right)-2.$$

Therefore, this inequality, Corollary 3 and Proposition 11.2) imply

$$\deg(I_{\mathtt{i}}(c)) \leq \left((n+1)\left[\frac{m-\mathfrak{r}-1}{\mathfrak{r}+1}\right]-1\right)\left(\left[\frac{m-1}{\mathfrak{r}-1}\right]-2\right)-2.$$

Case 4. If $\mathcal{H}(x,y) \in \mathfrak{F}_1$, then $r+1 = \mathfrak{r} = \dim H_1(\mathcal{L}_c,\mathbb{Z}) = \dim H_1(L_c,\mathbb{Z}) \geq 3$, $\mathfrak{m}+1 = \deg(\mathcal{H}) \geq 7$, and as in the previous case $r \leq [\mathfrak{m}/2]-1$. Again, simple but cumbersome computations show that in this case and for $\mathfrak{n} \geq 1$, the number $\mathfrak{n}\,(\mathfrak{m}-1-r)-r$ is the biggest of the four upper bounds given in Theorem 18.1). Thus, if $\vartheta \in \Omega^1(\mathbb{C}^2_{x\,y})_{\leq \mathfrak{n}}$, then from Theorem 18.1) we conclude that

$$\deg(\mathcal{I}_{\mathtt{i}}(\mathfrak{c})) \leq \mathfrak{n}\,(\mathfrak{m}-1-r)-r = \mathfrak{n}\,(\mathfrak{m}-\mathfrak{r}-2)-\mathfrak{r}+1.$$

Therefore, this inequality, Corollary 3 and Proposition 11.1) imply

$$\deg(I_{\mathtt{i}}(c)) \leq \left((n+1)\left[\frac{m-\mathfrak{r}}{\mathfrak{r}}\right]-1\right)(m-\mathfrak{r}-2)-\mathfrak{r}+1.$$

Simple computations show that for $m = 6,7,8$ the upper bound given in Case 2 is the biggest one of the last three cases, which yields the upper bound given in assertion 2) for $2 \leq m \leq 8$. Finally, by comparing the upper bounds given in Cases 2, 3 and 4, it is clear that for $m \geq 9$, the biggest bound is the provided in the last case, which gives the remainder upper bound of assertion 2). □

Proof of Theorem 2. Statement 1) follows from second part of Theorem 1. The Poincaré–Pontryagin criterion implies statements 2) and 3). □

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