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SU(2)-Multi-Instantons over $S^2 \times S^2$

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Abstract. Making use of the general theory of connections invariant under a symmetry group which acts transitively on fibers, explicit solutions are derived for SU(2) × SU(2)-symmetric multi-instantons over $S^2 \times S^2$, with SU(2) structure group. These multi-instantons correspond to a principal fiber bundle characterized by a second Chern number given by $2m^2$, with m an integer.

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1. Introduction

The existence of instantons over $S^2 \times S^2$ follows from the work of Soberón-Chávez [1] on the classification of stable complex bundles of rank 2 over $S^2 \times S^2$, and the correspondence between stable complex bundles and self-duality established by Donaldson [2]. However, to the extent of our knowledge, no explicit solutions for these instantons have been given in the literature.

The solutions we present here correspond to principal fiber bundles characterized by a second Chern number $C_2(P(m)) = 2m^2$, where m is an integer, and result from treating $S^2 \times S^2$ as a homogeneous base space, $S^2 \times S^2 = \mathrm{SU}(2) \times \mathrm{SU}(2)/\mathrm{U}(1) \times \mathrm{U}(1)$, of a principal fiber bundle with characteristic group SU(2). Furthermore, the connections on the bundle are required to be SU(2) × SU(2)-invariant, so our solutions are a subset of points in the total spaces of instanton solutions. (For example, for a second Chern number equal to 2, it has been shown by Donaldson that $S^2 \times S^2$-instanton solutions constitute a 10-parameter space).

Before proceeding with our construction, we shall review, for self-consistency purposes, some basic preliminary results and notation (see, e.g., [3] for a detailed discussion on the subject).

Let $\pi: P \to M$ denote a principal fiber bundle with base space $M = S^2 \times S^2$ and structure group $G = \mathrm{SU}(2)$. The symmetry group $S = \mathrm{SU}(2) \times \mathrm{SU}(2)$ acts fiber transitively on P by means of the bundle automorphism $s(pg) = (sp)g$ (where $s \in S$, $p \in P$, $g \in G$) and it induces a well-defined transformation on M given by $s\pi(p) = \pi(sp)$.

We shall say that a connection 1-form ω is locally S-invariant at x_0 if, for all $s \in S$ with $sx_0 \in N \subset M$, there exists a connected neighborhood $V_{(s)} \subset M$ of x_0, contained in $N \cap s^{-1}N$, such that $s^*\omega|_{V(s)} = \omega|_{V(s)}$.

Consider now a point $x_0 \in M$, and $p_0 \in P$ such that $\pi(p_0) = x_0$. The isotropy group of $S = \mathrm{SU}(2) \times \mathrm{SU}(2)$ at x_0 is $J_{x_0} = \mathrm{U}(1) \times \mathrm{U}(1)$. For $j \in J_{x_0}$, we have

$$\pi(jp_0) = j\pi(p_0) = jx_0 = x_0,$$

so jp_0 is on the same fiber as p_0 and we can therefore write $jp_0 = p_0\mu(j)$, where μ is a homomorphism of Lie groups, $\mu\colon J_{x_0} \to G$.

Furthermore, by virtue of Wang's theorem (see [4] and [5]), there is a bijective correspondence between S-invariant connections and linear transformations $\Lambda\colon L(S) \to L(G)$ of the associated Lie algebras given by $\Lambda(\mathbf{X}) = \omega_{p_0}(\mathbf{X})_{p_0}$, with $\hat{\mathbf{X}}_{p_0} = \mathrm{d}/\mathrm{d}t(\exp t\mathbf{X} \cdot p_0)|_{t=0}$, and such that the following conditions are satisfied:

$$\begin{aligned} &(\mathrm{A}) \quad \Lambda(\mathbf{Y}) = \mu_{p_{0*}}(\mathbf{Y}), \quad \text{for } \mathbf{Y} \text{ in } J_{x_0}, \\ &(\mathrm{B}) \quad \Lambda(a\delta_j(\mathbf{X})) = a\delta_{\mu_{p_0}(j)}(\Lambda)(\mathbf{X})), \quad \mathbf{X} \in L(S), \quad j \in J_{x_0}. \end{aligned} \tag{1.1}$$

Let now $\sigma_\alpha\colon U_\alpha \to P$ be a local section. Then, if $s \in S$, $x \in U_\alpha$, $sx \in U_\alpha$, the local action of S on P is given by

$$s\sigma_\alpha(x) = \sigma_\alpha(sx)\varphi^\alpha_x(s), \tag{1.2}$$

where $\varphi^\alpha_x(s) \in G$ is a differentiable function which describes how the action of S on U_α has been lifted to the fibers. Moreover, since $\varphi^\alpha_x(e) = e$ (the identity), the differential $(\varphi^\alpha_x)_*$ determines a linear function from $L(S)$ to $L(G)$ (which is not necessarily a morphism of Lie algebras. In fact, $\varphi^\alpha_x(ts) = \varphi^\alpha_{sx}(t)\varphi^\alpha_x(s)$. We thus have that $x \mapsto W^\alpha_x := (\varphi^\alpha_x)_*$ gives rise to a function from M to the space of linear transformations from $L(S)$ to $L(G)$.

For $\mathbf{X} \in L(S)$, denote by $\tilde{\mathbf{X}}$ the vector field on M defined by $\tilde{\mathbf{X}}_x = \mathrm{d}/\mathrm{d}t(\exp t\mathbf{X} \cdot x)|_{t=0}$. It is then easy to show, making use of (1.2), that for $x \in U_\alpha$, $sx \in U_\alpha$, the relation between $\tilde{\mathbf{X}}_x$ and $\hat{\mathbf{X}}_p$ is given by

$$\hat{\mathbf{X}}_{\sigma_\alpha(x)} = (\sigma_\alpha)_*\tilde{\mathbf{X}}_x + [W^\alpha_x(\mathbf{X})]^*_{\sigma(x)}, \tag{1.3}$$

where $[W^\alpha_x(\mathbf{X})]^*_{\sigma(x)}$ is the fundamental field associated with $W^\alpha_x(\mathbf{X}) \in L(G)$. Consequently,

$$(\sigma^*_\alpha\omega)_x(\tilde{\mathbf{X}})_x = \omega_{\sigma_\alpha(x)}(\hat{\mathbf{X}}_{\sigma_\alpha(x)}) - W^\alpha_x(\mathbf{X}). \tag{1.4}$$

Moreover, using the S-invariance of ω, we have

$$\omega_{\sigma_\alpha(sx_0)}(\hat{\mathbf{X}}_{\sigma_\alpha(sx_0)}) = a\delta_{\varphi^\alpha_{x_0}(s)}\omega_{\sigma_\alpha(x_0)}(\widehat{a\delta_{s^{-1}}\mathbf{X}})_{\sigma_\alpha(x_0)} \tag{1.5}$$

and, since S acts transitively on M, we can set $x = sx_0$, so substituting (1.5) in (1.4) yields

$$A_\alpha(\tilde{\mathbf{X}}_x) = (\sigma^*_\alpha\omega)_x(\tilde{\mathbf{X}}_x) = a\delta_{\varphi^\alpha_{x_0}(s)}\Lambda(a\delta_{s^{-1}}\mathbf{X}) - W^\alpha_x(\mathbf{X}). \tag{1.6}$$

Now, in the case where the symmetry group S acts transitively on the base manifold, it is known [3, 6] that there is a one-to-one correspondence between the equivalence classes of principal fiber bundles with gauge group G over $M = S/J_{x_0}$, admitting an S-action which projects on a given action of S on M, and the conjugate classes of homomorphisms $\mu_x: J_{x_0} \to G$. For this situation, the inequivalent liftings of the S-action are given by

$$\varphi_x^\alpha(s) = \mu_{p_0}(\tau_\alpha(sx)^{-1} s \tau_\alpha(x)), \tag{1.7}$$

where $\tau_\alpha: U_\alpha \to S$ are local sections of the bundle $S \xrightarrow{\eta} M(=S/J_{x_0})$.

Equations (1.6) and (1.7) are the basic tools that we need for our multi-instanton construction in the following section.

2. Explicit Solutions

Since for any two $\varphi_x^\alpha(s)$ mappings, given by (1.7), the resulting gauge fields will be related by a gauge transformation, all we require is to construct a specific one. This in turn implies a choice of a local section $\tau_\alpha(x)$ on the bundle $\mathrm{SU}(2) \times \mathrm{SU}(2) \xrightarrow{\eta} S^2 \times S^2$.

Furthermore, since the base manifold is a product, our construction will consist of two identical copies of sections for each sector. Thus, using the space of unit quaternions, we have

$$\mathrm{SU}(2) \approx S^3 = \{x_1 + ix_2 + jx_3 + kx_4 | x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\},$$

where i, j, k satisfy the usual rules of multiplication for quaternions, and the Lie algebra $L(\mathrm{SU}(2))$ can be identified with the quaternion vector subspace of $\mathscr{H} = \{x_1 + ix_2 + jx_3 + kx_4\}$ generated by

$$\mathbf{X}_1 = \tfrac{1}{2}i, \qquad \mathbf{X}_2 = \tfrac{1}{2}j, \qquad \mathbf{X}_3 = \tfrac{1}{2}k,$$

with $\mathbf{X}_i$ $(i = 1, 2, 3)$ satisfying the conmutation rules $[\mathbf{X}_i, \mathbf{X}_j] = \varepsilon_{ijk}\mathbf{X}_k$. In particular, we shall assume that $\mathbf{X}_1$ generates the isotropy group $J = \mathrm{U}(1)$.

A coordinate expression for the projection over S^2 can be obtained by means of $\pi: S^3 \subset \mathscr{R}^4 \to \mathscr{R}^2 = S^2 - \{\infty\}$ given by

$$(x_1, \ldots, x_4) \mapsto \left(\frac{-x_1x_3 + x_2x_4}{x_1^2 + x_2^2}, \frac{-x_1x_4 - x_2x_3}{x_1^2 + x_2^2}\right). \tag{2.1}$$

This projection has been chosen in such a way that it satisfies the requirement $\pi(pJ) = \pi(p)$.

Note that the coordinates

$$y_1 = \frac{-x_1x_3 + x_2x_4}{x_1^2 + x_2^2}, \qquad y_2 = \frac{-x_1x_4 - x_2x_3}{x_1^2 + x_2^2}$$

can be interpreted geometrically as the real and imaginary components of the slopes $(x_3 + ix_4)/(-x_1 + ix_2)$ of the complex lines in $\mathscr{C}^2$, intersecting at the origin, which we identify with S^2.

If we choose bases

$$\left\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_4}\right\}, \quad \left\{\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}\right\}$$

for TR^4 and TR^2, respectively, then it is easy to verify that the differential of the projection map (1.8) is

$$\pi_*\left(l_j \frac{\partial}{\partial x_j}\right) = \frac{\partial \pi_i}{\partial x_j} l_j \frac{\partial}{\partial y_i}$$

with

$$\frac{\partial \pi_i}{\partial x_j} = \begin{bmatrix} \dfrac{-x_3 Q - 2x_1(-x_1x_3 + x_2x_4)}{Q^2} & \dfrac{x_4 Q - 2x_2(-x_1x_3 + x_2x_4)}{Q^2} & \dfrac{-x_1}{Q} & \dfrac{x_2}{Q} \\ \dfrac{-x_4 Q + 2x_1(x_1x_4 + x_2x_3)}{Q^2} & \dfrac{-x_3 Q + 2x_2(x_1x_4 + x_2x_3)}{Q^2} & \dfrac{-x_2}{Q} & \dfrac{-x_1}{Q} \end{bmatrix}, \tag{2.2}$$

where $Q = x_1^2 + x_2^2$.

Observe also that for

$$p = x_1 + ix_2 + jx_3 + kx_4 \in S^3,$$

the kernel of (2.2) is the left invariant vector field

$$\xi_1(p) = L_{p*}\mathbf{X}_1 = \tfrac{1}{2}\left(-x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2} + x_4\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_4}\right),$$

so that $\xi_1(p)$ is vertical in TS^3 and is the fundamental vector field $[\mathbf{X}_1]_p^*$ generated by $\mathbf{X}_1$. Hence, $\mathbf{X}_1$ is indeed the generator of the isotropy subgroup $J = \mathrm{U}(1)$.

We can now choose the local section $\tau_\alpha : \mathscr{R}^2 \subset S^2 \to S^3 \subset \mathscr{R}^4$ to be given by

$$\tau_\alpha(y_1, y_2) = \frac{1}{(1 + y_1^2 + y_2^2)^{1/2}}(1 - jy_1 - ky_2). \tag{2.3}$$

Note, in fact, that τ_α satisfies the requirements $\|\tau_\alpha(y_1, y_2)\|_{\mathscr{R}^4} = 1$ and $\pi \circ \tau_\alpha = \mathrm{id}_{\mathscr{R}^2}$.

Furthermore, making use of the commutative diagram

$$\begin{array}{ccc} \tau_\alpha(y_1, y_2) & \xrightarrow{s} & s\tau_\alpha(y_1, y_2) \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ (y_1, y_2) & \xrightarrow{s} & (y_1', y_2') \end{array} \tag{2.4}$$

and setting $s = a + ib + jc + kd \in \mathscr{H}$, we find

$$\begin{aligned} y_1' &= \frac{-(c - ay_1 + by_2)(a + cy_1 + dy_2) + (d - by_1 - ay_2)(b + dy_1 - cy_2)}{(a + cy_1 + dy_2)^2 + (b + dy_1 - cy_2)^2}, \\ y_2' &= \frac{-(d - by_1 - ay_2)(a + cy_1 + dy_2) - (c - ay_1 + by_2)(b + dy_1 - cy_2)}{(a + cy_1 + dy_2)^2 + (b + dy_1 - cy_2)^2}. \end{aligned} \tag{2.5}$$

Consequently,

$$\begin{aligned}
&\tau_\alpha(y_1', y_2')^{-1} \cdot s \cdot \tau_\alpha(y_1, y_2) \\
&= \left[\frac{1}{(a + cy_1 + dy_2)^2 + (b + dy_1 - cy_2)^2}\right]^{1/2} \times \\
&\quad \times [(a + cy_1 + dy_2) + i(b + dy_1 - cy_2)],
\end{aligned} \tag{2.6}$$

which follows from substituting (2.5) in (2.3) and a somewhat lengthy but straightforward calculation with quaternions.

In order to complete the calculation of $\varphi_x(s)$ in (1.7), we choose the morphism $\mu : \mathrm{U}(1) \times \mathrm{U}(1) \to \mathrm{SU}(2)$ to be given by $\mu(j_1, j_2) = (j_1)^m (j_2)^n$, where m, n are integers and $(j_1)^m (j_2)^n$ denotes the usual product of powers in the subgroup $\mathrm{U}(1) \subset \mathrm{SU}(2)$. Note that if m and n are different from zero, then μ does not extend to a smooth morphism from $\mathrm{SU}(2) \times \mathrm{SU}(2)$ to $\mathrm{SU}(2)$ and, thus, the bundle $P_{m,n}$ associated to μ, will be nontrivial (cf. Corollary 2.3 in [3]). Furthermore, since every Abelian subgroup of SU(2) is isomorphic to either U(1) or the trivial group, every morphism from J to SU(2) is, up to conjugation, as given by the μ above.

Therefore, using two copies of the above chart for S^2 and of the section (2.3) we obtain

$$\begin{aligned}
\varphi_x^\alpha(s) &= \left[\frac{1}{(a_1 + c_1 y_1 + d_1 y_2)^2 + (b_1 + d_1 y_1 - c_1 y_2)^2}\right]^{m/2} \times \\
&\quad \times \left[\frac{1}{(a_2 + c_2 y_3 + d_2 y_4)^2 + (b_2 + d_2 y_3 - c_2 y_4)^2}\right]^{n/2} \times \\
&\quad \times [(a_1 + c_1 y_1 + d_1 y_2) + 2X_1(b_1 + d_1 y_1 - c_1 y_2)]^m \times \\
&\quad \times [(a_2 + c_2 y_3 + d_2 y_4) + 2X_1(b_2 + d_2 y_3 - c_2 y_4)],
\end{aligned} \tag{2.7}$$

where

$$s = (s_1, s_2) = (a_1 + ib_1 + jc_1 + kd_1, a_2 + ib_2 + jc_2 + kd_2) \in \mathrm{SU}(2) \times \mathrm{SU}(2),$$

and $x = (y_1, y_2, y_3, y_4) \in \mathscr{R}^2 \times \mathscr{R}^2$.

Note that for

$$s = \left(\cos\frac{t}{2} + i \sin\frac{t}{2}, 1\right) \in J$$

we get

$$\varphi_x^\alpha(s) = \left(\cos\frac{t}{2} + 2\mathbf{X}_1 \sin\frac{t}{2}\right)^m,$$

so

$$W_x^\alpha(X_1 \oplus 0) = (\varphi_x^\alpha)_*(X_1 \oplus 0) = mX_1 \tag{2.8}$$

Similarly, we find

$$W^{\alpha}_{x}(X_2 \oplus 0) = -my_2X_1, \qquad W^{\alpha}_{x}(X_3 \oplus 0) = my_1X_1,$$
$$W^{\alpha}_{x}(0 \oplus X_1) = nX_1, \qquad W^{\alpha}_{x}(0 \oplus X_2) = -ny_4X_1, \qquad W^{\alpha}_{x}(0 \oplus X_3) = ny_3X_1. \tag{2.9}$$

We now turn to the calculation of the remaining terms needed in (1.6). To this end, and with the explicit purpose of expressing the gauge potentials in terms of the local Euclidean coordinates (y_1, y_2, y_3, y_4), we first need to write the coordinate basis

$$\left\{\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial y_3}, \frac{\partial}{\partial y_4}\right\}$$

in terms of the vector fields $(\tilde{\mathbf{X}}_i)_x$ defined in Section 1. Furthermore, we already know that for the left invariant field ξ_1, we have that $\pi_*\xi_1 = 0$, so our calculation can be simplified considerably if, instead of evaluating (1.6) relative to right invariant vector fields, we calculate the gauge fields using two copies of the left-invariant vector fields

$$\begin{aligned}
\xi_1 &= \tfrac{1}{2}\left(-x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2} + x_4\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_4}\right),\\
\xi_2 &= \tfrac{1}{2}\left(-x_3\frac{\partial}{\partial x_1} - x_4\frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial x_3} + x_2\frac{\partial}{\partial x_4}\right),\\
\xi_3 &= \tfrac{1}{2}\left(-x_4\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_3} + x_1\frac{\partial}{\partial x_4}\right),
\end{aligned} \tag{2.10}$$

as a basis for $TS^3 \oplus TS^3$ and take

$$(\tilde{\mathbf{X}}_2^L)_x = (\pi_*\xi_2 \oplus 0), \qquad (\tilde{\mathbf{X}}_3^L)_x = (\pi_*\xi_3 \oplus 0),$$
$$(\tilde{\mathbf{X}}_5^L)_x = (0 \oplus \pi_*\xi_2), \qquad (\tilde{\mathbf{X}}_6^L)_x = (0 \oplus \pi_*\xi_3)$$

as the corresponding local basis in $TS^2 \oplus TS^2$.

Thus, operating with π_* on ξ_2 and ξ_3, and making use of (2.2) and of the fact that on τ_α we have $x_2 = 0$, it immediately follows that

$$\begin{aligned}
(\tilde{\mathbf{X}}_2^L)_x &= (\pi_*\xi_2 \oplus 0) = -\tfrac{1}{2}(1 + y_1^2 + y_2^2)\frac{\partial}{\partial y_1},\\
(\tilde{\mathbf{X}}_3^L)_x &= (\pi_*\xi_3 \oplus 0) = -\tfrac{1}{2}(1 + y_1^2 + y_2^2)\frac{\partial}{\partial y_2},\\
(\tilde{\mathbf{X}}_5^L)_x &= (0 \oplus \pi_*\xi_2) = -\tfrac{1}{2}(1 + y_3^2 + y_4^2)\frac{\partial}{\partial y_3},\\
(\tilde{\mathbf{X}}_6^L)_x &= (0 \oplus \pi_*\xi_3) = -\tfrac{1}{2}(1 + y_3^2 + y_4^2)\frac{\partial}{\partial y_4}.
\end{aligned} \tag{2.11}$$

To convert (1.6) to the corresponding relation for our left invariant vector fields, note that the point of isotropy on $S^2 \times S^2$ is

$$x_0 = (y_1 = y_2 = y_3 = y_4 = 0),$$

so

$$(\tau_x)_1(0,0) = 1, \qquad (\tau_x)_2(0,0) = 1.$$

It then follows from (2.3) that for $x = sx_0 = (s_1, s_2)x_0$

$$\begin{aligned}\tau_x(x) &:= ((\tau_x)_1 \times (\tau_x)_2)(y_1, y_2, y_3, y_4)\\ &= (s_1, s_2)((\tau_x)_1 \times (\tau_x)_2)(0,0,0,0) = (s_1, s_2).\end{aligned}$$

Hence

$$s \equiv (s_1, s_2) = \left(\frac{(1 - jy_1 - ky_2)}{(1 + y_1^2 + y_2^2)^{1/2}}, \frac{(1 - jy_3 - ky_4)}{(1 + y_3^2 + y_4^2)^{1/2}}\right). \tag{2.12}$$

Now, if $\hat{\mathbf{X}}$ is one of the right-invariant vector fields defined in the previous section, and ω is a locally S-invariant connection, then

$$\omega_{\tau_x(x_0)}(\hat{\mathbf{X}}_{\tau_x(x_0)}) = (s^*\omega)_{\tau_x(x_0)}(\hat{\mathbf{X}}_{\tau_x(x_0)}) = \omega_{\tau_x(x)}(s_* \hat{\mathbf{X}}_{\tau_x(x_0)}).$$

But

$$s_* \hat{\mathbf{X}}_{\tau_x(x_0)} = \frac{\mathrm{d}}{\mathrm{d}t}(s \cdot \exp t\mathbf{X} \cdot \tau_x(x_0))\big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}(s \cdot \tau_x(x_0) \exp t\mathbf{X})\big|_{t=0} = (\hat{\mathbf{X}}^L)_{\tau_x(x)}$$

Consequently,

$$\Lambda(\mathbf{X}) = \omega_{\tau_x(x_0)}(\hat{\mathbf{X}}^L_{\tau_x(x_0)}). \tag{2.13}$$

Moreover, since (1.4) also applies to left invariant fields, we get

$$A_x(\tilde{\mathbf{X}}^L_x) \equiv (\tau_x^* \omega)_x(\tilde{\mathbf{X}}^L_x) = \Lambda(\mathbf{X}) - W^x_x(\mathbf{X}). \tag{2.14}$$

Next, in order to solve the linear transformations $\Lambda: L(\mathrm{SU}(2) \times \mathrm{SU}(2)) \to L(\mathrm{SU}(2))$, we must apply Wang's theorem (cf. Eq. (1.1)), and impose on these solutions the additional constraints which result from the requirement of selfduality on the gauge fields.

Consider first the selfduality conditions which can be derived by recalling that

$$\begin{aligned}&(\tau_x^* \Omega)(\tilde{\mathbf{X}}^L, \tilde{\mathbf{Y}}^L)\\ &\quad = \tilde{\mathbf{X}}^L[A_x(\tilde{\mathbf{Y}}^L)] - \tilde{\mathbf{Y}}^L[\mathrm{A}_x(\tilde{\mathbf{X}}^L)] - A_x([\tilde{\mathbf{X}}^L, \tilde{\mathbf{Y}}^L]) + [A_x(\tilde{\mathbf{X}}^L), A_x(\tilde{\mathbf{Y}}^L)].\end{aligned} \tag{2.15}$$

Now using (2.14) and, as a basis for the gauge fields $F_x \equiv \tau_x^* \Omega$, the local basis $\{(\tilde{\mathbf{X}}^L_2)_x, (\tilde{\mathbf{X}}^L_3)_x, (\tilde{\mathbf{X}}^L_5)_x, (\tilde{\mathbf{X}}^L_6)_x\}$ found before, and also observing that

$$[\tilde{\mathbf{X}}^L_2, \tilde{\mathbf{X}}^L_3] = -y_1 \tilde{\mathbf{X}}^L_3 + y_2 \tilde{\mathbf{X}}^L_2, \qquad [\tilde{\mathbf{X}}^L_5, \tilde{\mathbf{X}}^L_6] = -y_3 \tilde{\mathbf{X}}^L_6 + y_4 \tilde{\mathbf{X}}^L_5,$$

$$[\tilde{\mathbf{X}}^L_2, \tilde{\mathbf{X}}^L_5] = [\tilde{\mathbf{X}}^L_2, \tilde{\mathbf{X}}^L_6] = [\tilde{\mathbf{X}}^L_3, \tilde{\mathbf{X}}^L_5] = [\tilde{\mathbf{X}}^L_3, \tilde{\mathbf{X}}^L_6] = 0,$$

we obtain

$$
\begin{aligned}
&F_\alpha(\tilde{\mathbf{X}}_2^L, \tilde{\mathbf{X}}_3^L)_x \\
&\quad = m\mathbf{X}_1 - y_2\Lambda(\mathbf{X}_2 \oplus 0) + y_1\Lambda(\mathbf{X}_3 \oplus 0) + [\Lambda(\mathbf{X}_2 \oplus 0), \Lambda(\mathbf{X}_3 \oplus 0)] - \\
&\qquad - [W_x^\alpha(\mathbf{X}_2 \oplus 0), \Lambda(\mathbf{X}_3 \oplus 0)] - [\Lambda(\mathbf{X}_2 \oplus 0), W_x^\alpha(\mathbf{X}_3 \oplus 0)], \\
&F_\alpha(\tilde{\mathbf{X}}_5^L, \tilde{\mathbf{X}}_6^L)_x \\
&\quad = n\mathbf{X}_1 + y_3\Lambda(0 \oplus \mathbf{X}_3) - y_4\Lambda(0 \oplus \mathbf{X}_2) + [\Lambda(0 \oplus \mathbf{X}_2), \Lambda(0 \oplus \mathbf{X}_3)] - \\
&\qquad - [\Lambda(0 \oplus \mathbf{X}_2), W_x^\alpha(0 \oplus \mathbf{X}_3)] - [W_x^\alpha(0 \oplus \mathbf{X}_2), \Lambda(0 \oplus \mathbf{X}_3)], \\
&F_\alpha(\tilde{\mathbf{X}}_2^L, \tilde{\mathbf{X}}_5^L)_x \\
&\quad = [\Lambda(\mathbf{X}_2 \oplus 0), \Lambda(0 \oplus \mathbf{X}_2)] - [\Lambda(\mathbf{X}_2 \oplus 0), W_x^\alpha(0 \oplus \mathbf{X}_2)] - \\
&\qquad - [W_x^\alpha(\mathbf{X}_2 \oplus 0), \Lambda(0 \oplus \mathbf{X}_2)], \\
&F_\alpha(\tilde{\mathbf{X}}_6^L, \tilde{\mathbf{X}}_3^L)_x \\
&\quad = [\Lambda(0 \oplus \mathbf{X}_3), \Lambda(\mathbf{X}_3 \oplus 0)] - [\Lambda(0 \oplus \mathbf{X}_3), W_x^\alpha(\mathbf{X}_3 \oplus 0)] - \\
&\qquad - [W_x^\alpha(0 \oplus \mathbf{X}_3), \Lambda(\mathbf{X}_3 \oplus 0)], \\
&F_\alpha(\tilde{\mathbf{X}}_3^L, \tilde{\mathbf{X}}_5^L)_x \\
&\quad = [\Lambda(\mathbf{X}_3 \oplus 0), \Lambda(0 \oplus \mathbf{X}_2)] - [\Lambda(\mathbf{X}_3 \oplus 0), W_x^\alpha(0 \oplus \mathbf{X}_2)] - \\
&\qquad - [W_x^\alpha(\mathbf{X}_3 \oplus 0), \Lambda(0 \oplus \mathbf{X}_2)], \\
&F_\alpha(\tilde{\mathbf{X}}_2^L, \tilde{\mathbf{X}}_6^L)_x \\
&\quad = [\Lambda(\mathbf{X}_2 \oplus 0), \Lambda(0 \oplus \mathbf{X}_3)] - [\Lambda(\mathbf{X}_2 \oplus 0), W_x^\alpha(0 \oplus \mathbf{X}_3)] - \\
&\qquad - [W_x^\alpha(\mathbf{X}_2 \oplus 0), \Lambda(0 \oplus \mathbf{X}_3)].
\end{aligned}
\tag{2.16}
$$

Notice that $\mathrm{SU}(2) \cong S^3$ has a bi-invariant Riemann metric g which can be explicitly determined by means of the orthonormal left-invariant vector fields ξ_1, ξ_2, ξ_3, given in (2.10). We thus have

$$g_{ij} = \langle \xi_i, \xi_j \rangle = \delta_{ij}, \quad i, j = 1, 2, 3.$$

In addition, since the section τ_x is an immersion of $S^2 - \{\infty\}$ in S^3, and since ξ_1 is vertical in TS^3, we can use $\{\xi_2, \xi_3\}$ as a local frame for $S^2 = \mathrm{SU}(2)/\mathrm{U}(1)$ with restricted metric

$$g|_{\mathrm{SU}(2)/\mathrm{U}(1)} = \begin{pmatrix} \langle \xi_2, \xi_2 \rangle & \langle \xi_2, \xi_3 \rangle \\ \langle \xi_3, \xi_2 \rangle & \langle \xi_3, \xi_3 \rangle \end{pmatrix} = \mathrm{diag}(1, 1). \tag{2.17}$$

This metric in turn induces a metrix $\bar{g}$ on $\mathscr{R}^2$ defined by

$$(\pi^* \bar{g})(\xi_i, \xi_j) = g(\xi_i, \xi_j), \quad i, j = 2, 3. \tag{2.18}$$

Now taking two copies of the above construction, we have

$$\bar{g}(\tilde{\mathbf{X}}_2^L, \tilde{\mathbf{X}}_2^L) = \bar{g}(\tilde{\mathbf{X}}_3^L, \tilde{\mathbf{X}}_3^L) = \bar{g}(\tilde{\mathbf{X}}_5^L, \mathbf{X}_5^L) = \bar{g}(\tilde{\mathbf{X}}_6^L, \tilde{\mathbf{X}}_6^L) = 1,$$

and (2.19)

$$\bar{g}(\tilde{\mathbf{X}}_i^L, \tilde{\mathbf{X}}_j^L) = 0, \quad i \neq j.$$

Consequently, the basis (2.11) is orthonormal with respect to $\bar{g}$, and we can use this fact together with Equation (2.16) and the notation $\Lambda(X_i \oplus 0) = \Lambda_i^j \mathbf{X}_j$, $\Lambda(0 \oplus \mathbf{X}_i) = \Lambda_{i+3}^j \mathbf{X}_j$, to arrive at the following selfduality constraints:

$$\begin{aligned}
&-y_2\Lambda_2^1 + y_1\Lambda_3^1 + (\Lambda_2^2\Lambda_3^3 - \Lambda_2^3\Lambda_3^2) + m \\
&\quad = n - y_4\Lambda_5^1 + y_3\Lambda_6^1 + (\Lambda_5^2\Lambda_6^3 - \Lambda_5^3\Lambda_6^2), \\
&-y_2\Lambda_2^2 + y_1\Lambda_3^2 + (\Lambda_2^3\Lambda_3^1 - \Lambda_2^1\Lambda_3^3) - my_2\Lambda_3^3 - my_1\Lambda_2^3 \\
&\quad = -y_4\Lambda_5^2 + y_3\Lambda_6^2 - ny_4\Lambda_6^3 - ny_3\Lambda_5^3 + (\Lambda_5^3\Lambda_6^1 - \Lambda_5^1\Lambda_6^3), \\
&-y_2\Lambda_2^3 + y_1\Lambda_3^3 + (\Lambda_2^1\Lambda_3^2 - \Lambda_2^2\Lambda_3^1) + my_2\Lambda_3^2 + my_1\Lambda_2^2 \\
&\quad = -y_4\Lambda_5^3 + y_3\Lambda_6^3 + ny_4\Lambda_6^2 + ny_3\Lambda_5^2 + (\Lambda_5^1\Lambda_6^2 - \Lambda_5^2\Lambda_6^1);
\end{aligned} \tag{2.20a}$$

$$\begin{aligned}
&\Lambda_2^2\Lambda_5^3 - \Lambda_2^3\Lambda_5^2 = \Lambda_6^2\Lambda_3^3 - \Lambda_6^3\Lambda_3^2, \\
&\Lambda_2^3\Lambda_5^1 - \Lambda_2^1\Lambda_5^3 + ny_4\Lambda_2^3 - my_2\Lambda_5^3 = \Lambda_6^3\Lambda_3^1 - \Lambda_6^1\Lambda_3^3 - my_1\Lambda_6^3 + ny_3\Lambda_3^3, \\
&\Lambda_2^1\Lambda_5^2 - \Lambda_2^2\Lambda_5^1 - ny_4\Lambda_2^2 + my_2\Lambda_5^2 = \Lambda_6^1\Lambda_3^2 - \Lambda_6^2\Lambda_3^1 + my_1\Lambda_6^2 - ny_3\Lambda_3^2;
\end{aligned} \tag{2.20b}$$

$$\begin{aligned}
&\Lambda_3^2\Lambda_5^3 - \Lambda_3^3\Lambda_5^2 = \Lambda_2^2\Lambda_6^3 - \Lambda_2^3\Lambda_6^2, \\
&\Lambda_3^3\Lambda_5^1 - \Lambda_3^1\Lambda_5^3 + ny_4\Lambda_3^3 + my_1\Lambda_5^3 = \Lambda_2^3\Lambda_6^1 - \Lambda_2^1\Lambda_6^3 - ny_3\Lambda_2^3 - my_2\Lambda_6^3, \\
&\Lambda_3^1\Lambda_5^2 - \Lambda_3^2\Lambda_5^1 - ny_4\Lambda_3^2 - my_1\Lambda_5^2 = \Lambda_2^1\Lambda_6^2 - \Lambda_2^2\Lambda_6^1 + ny_3\Lambda_2^2 + my_2\Lambda_6^2.
\end{aligned} \tag{2.20c}$$

Clearly, since Λ_i^j are constants, Equations (2.20) can only be satisfied if $\Lambda_i^j = 0$ for $i = 2, 3$ and $j = 2, 3, 5, 6$, and $m = n$.

Furthermore, since $\mu_{p_0}(j) = \varphi_{x_0}(j)$, it follows from (2.8) and (2.9) that

$$\mu_{p_0 *}(\mathbf{X}_1 \oplus 0) = m\mathbf{X}_1 \quad \text{and} \quad \mu_{p_0 *}(0 \oplus \mathbf{X}_1) = n\mathbf{X}_1,$$

so (1.1A) yields

$$\Lambda_1^1 = m, \qquad \Lambda_4^1 = n, \qquad \Lambda_1^2 = \Lambda_1^3 = \Lambda_4^2 = \Lambda_4^3 = 0.$$

Finally, note that a fairly straightforward calculation of (1.1B) for an infinitesimal $j = (1 + 2\alpha\mathbf{X}_1, 1 + 2\beta\mathbf{X}_1)$ yields

$$\begin{aligned}
&(\alpha m + \beta n)\varepsilon_{1kj}\Lambda_i^k = \alpha\varepsilon_{1ik}\Lambda_k^j, \\
&(\alpha m + \beta n)\varepsilon_{1kj}\Lambda_{i+3}^k = \beta\varepsilon_{1ik}\Lambda_{k+3}^j.
\end{aligned} \tag{2.21}$$

Therefore, $\Lambda_k^1 = \Lambda_{k+3}^1 = 0$, $k = 2, 3$.

In summary, the only S-invariant connection in $P_{m,n}$ which allows for instanton solutions is the canonical connection, defined by $\Lambda_r = 0$, where Λ_r is the restriction of Λ to r, and r is the subspace of the Lie algebra spanned by $\{\mathbf{X}_2 \oplus 0, \mathbf{X}_3 \oplus 0, 0 \oplus \mathbf{X}_2, 0 \oplus \mathbf{X}_3\}$.

In matrix notation, we have

$$\Lambda = \begin{bmatrix} m & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ m & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{2.22}$$

The gauge potentials on $S^2 \times S^2 \subset \mathscr{R}^4$ for this case can now be readily obtained from (2.9), (2.11), (2.14) and (2.22). We thus arrive at

$$\begin{aligned} A_\alpha\left(\frac{\partial}{\partial y_1}\right) &= \frac{-2my_2}{(1+y_1^2+y_2^2)}\mathbf{X}_1, & A_\alpha\left(\frac{\partial}{\partial y_2}\right) &= \frac{2my_1}{(1+y_1^2+y_2^2)}\mathbf{X}_1, \\ A_\alpha\left(\frac{\partial}{\partial y_3}\right) &= \frac{-2my_4}{(1+y_3^2+y_4^2)}\mathbf{X}_1, & A_\alpha\left(\frac{\partial}{\partial y_4}\right) &= \frac{2my_3}{(1+y_3^2+y_4^2)}\mathbf{X}_1. \end{aligned} \tag{2.23}$$

The corresponding nonvanishing local holonomic components of the gauge fields are

$$(\mathbf{F}_\alpha)_{12} = \frac{4m\mathbf{X}_1}{(1+y_1^2+y_2^2)^2}, \qquad (\mathbf{F}_\alpha)_{34} = \frac{4m\mathbf{X}_1}{(1+y_3^2+y_4^2)^2}. \tag{2.24}$$

To complete our discussion, we next calculate the second Chern class for our bundle $P(m) := P_{m,m}$. This is given by [7]

$$C_2(\mathbf{F}_\alpha) = -\frac{1}{16\pi^2}\sum_{a=1}^{3} \mathbf{F}_\alpha^a \wedge \mathbf{F}_\alpha^a. \tag{2.25}$$

Using the selfduality of $\mathbf{F}_\alpha$ and the definition of the Hodge star operator, we have

$$\mathbf{F}_\alpha^a \wedge \mathbf{F}_\alpha^a = \mathbf{F}_\alpha^a \wedge {}^*\mathbf{F}_\alpha^a = \tilde{g}(\mathbf{F}_\alpha^a, \mathbf{F}_\alpha^a)\mu, \tag{2.26}$$

where μ is the volume element of $S^2 \times S^2$ relative to the metric $\tilde{g}$ given by (2.19), and

$$\tilde{g}(\mathbf{F}_\alpha^a, \mathbf{F}_\alpha^a) = \tfrac{1}{2}\tilde{g}^{ij}\tilde{g}^{kl}(F_\alpha)_{ik}^a(F_\alpha)_{jl}^a. \tag{2.27}$$

Expressing (2.19) in terms of the holonomic basis

$$\left\{\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial y_3}, \frac{\partial}{\partial y_4}\right\}$$

by means of (2.11), we get

$$\tilde{g}_{ij} = \mathrm{diag}\left(\frac{4}{(1+y_1^2+y_2^2)^2}, \frac{4}{(1+y_1^2+y_2^2)^2}, \frac{4}{(1+y_3^2+y_4^2)^2}, \frac{4}{(1+y_3^2+y_4^2)^2}\right). \tag{2.28}$$

Hence, making use of (2.24), we obtain

$$c_2(\mathbf{F}_\gamma) = -\frac{2m^2}{16\pi^2}\mu. \tag{2.29}$$

Noting that for the canonical Riemannian product metric we have that 4-volume of $S^2 \times S^2 = [2\text{-volume of } S^2]^2 = (4\pi^2)^2$, we finally arrive at the second Chern number

$$C_2(P(m)) = -\frac{m^2}{8\pi^2}\int_{S^2 \times S^2} \mu = \pm 2m^2, \tag{2.30}$$

where the sign on the right side of the above result depends on the choice of orientation for the volume element of $S^2 \times S^2$. Moreover, since $S^2 \times S^2$ admits an orientation reversing isometry, the self-dual and the anti-self-dual connections are equivalent.

Clearly then, our solutions (2.23) correspond to multi-(anti)-instantons for our fiber bundle $P(m)$, characterized by the integer m associated to the morphism μ_{p_0}: $\mathrm{U}(1) \times \mathrm{U}(1) \to \mathrm{SU}(2)$ given by $\mu(j_1, j_2) = (j_1)^m(j_2)^m$.

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