ON THE CLASSIFICATION OF GROUP-INVARIANT CONNECTIONS

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Abstract

Connection 1–forms on principal fiber bundles with arbitrary characteristic groups are considered, and a characterization of gauge-equivalent connections in terms of their associated holonomy groups is given. We then apply these results to group-invariant connections, which gives us an algebraic procedure for obtaining solutions to the gauge field equations and for classifying them into classes of equivalence.

1. Introduction

Spacetime symmetries play an important role in the study of monopoles and instantons in non-abelian gauge theories. These symmetries, given by a group of motions of an underlying spacetime manifold M, are reflected in a gauge field A which is said to possess the group of symmetries if a group transformation has the sole effect of a gauge transformation on A (thus leaving "physical", gauge-invariant quantities the same). In the language of fiber bundles, this traduces into transformations leaving the connection 1–form ω of a principal fiber bundle P invariant.

If we let $AUT(P)$ denote the group of automorphisms of P, and $Aut(P)$ its normal subgroup which contains the identity diffeomorphism of M, then the internal symmetry group of ω is given by

$$I_\omega(P) = \{F \in Aut(P)/F^*\omega = \omega\}.$$

The study of $I_\omega(P)$ is important not only because of the dimensional reduction of Lie groups that it provides (note that if P is connected then $I_\omega(P)$ is finite-dimensional, while

$AUT_\omega(P) = \{F \in AUT(P)/F^*\omega = \omega\}$ is in general infinite-dimensional, Fischer [1987]) but, more importantly, because in Lagrangian field theories it is precisely $I_\omega(P)$ which generates global internal conservation laws.

There is little hope that the action density be physically meaningful unless it is invariant under gauge transformations (i.e. base-preserving automorphisms of P). Besides, the gauge potentials themselves are interpreted as various forms of radiation (photons, intermediate vector bosons, gluons, etc.). Therefore, physically one is interested in studying the internal symmetry group of gauge-equivalent classes of connections.

In this work, a gauge-equivalent characterization of connections, in terms of their associated holonomy groups, is presented, as well as a study of the conditions which result from imposing the additional requirement that the two gauge-related connections should both be S–invariant, for S and arbitrary group with a given action on M. Only the results, both in the local and global domains, will be presented here (due to restrictions of space), while the details will be given elsewhere.

2. Gauge-Equivalence in terms of Holonomy.

Let $P(M,G)$ denote a principal fiber bundle with structure group G and projection operator $\pi : P \to M$. Denote by $C(P,G)$ the space of all maps $\tau : P \to G$ which satisfy $\tau(pg) = g^{-1}\tau(p)g$ for all $g \in G$, $p \in P$. This space is isomorphic to the space of sections of the associated bundle $P \times_G G \to M$ with standard fiber G. A diffeomorphism $f : P \to P$ which satisfies $f(pg) = f(p)g$ for all $p \in P, g \in G$, is called a fiber bundle automorphism. Note that such an automorphism induces a diffeomorphism $\bar{f} : M \to M$ given by $\bar{f}(\pi(p)) = \pi(f(p))$. We define a gauge transformation to be an automorphism $f : P \to P$ such that $\bar{f} = 1_M$, and shall denote the group of gauge transformations on P by $GA(P) = Aut(P)$.

Now let ω be a connection 1–form on P, and $C(x,y)$ denote the collection of paths in M from x to y. Thus, $\alpha \in C(x,x)$ is a loop based at $x \in M$, i.e. $\alpha(0) = \alpha(1) = x$, and if $\hat{\alpha}(t)$ denotes the ω–horizontal lift of $\alpha(t)$ which passes through $p \in \pi^{-1}(x)$ then there exists an $h_p^\omega(\alpha) \in G$ such that $\hat{\alpha}(1) = \hat{\alpha}(0)h_p^\omega(\alpha)$. The holonomy group $Hol_p(\omega)$ of ω at p consists of all such elements for all possible loops based at $x = \pi(p)$, i.e. $Hol_p(\omega) = \{h_p^\omega(\alpha) | \alpha \in C(x,x), x = \pi(p)\}$. In the next section we shall also make use of the restricted holonomy group $Hol_p^0(\omega)$, which is the subgroup of $Hol_p(\omega)$ generated by loops at x which are homotopic to the identity.

With the notation introduced above we can then prove the following

Proposition 2.1: Let ω_1, ω_2 be two connections on a principal fiber bundle $P(M, G)$. Then a gauge transformation f with the property $f^*\omega_2 = \omega_1$ exists if and only if at some point $p \in P$ we have

$$h_p^{\omega_2} = u h_p^{\omega_1} u^{-1} \tag{2.1}$$

with $u \in C(P, G)$ such that $f(p) = pu(p)$. For a fixed p, and f such that $f^*\omega_2 = \omega_1$ and $u(p) = u$, f is unique.

This general result has as immediate corollaries two interesting results due to Fischer (1987):

Corollary 2.2: Let $p \in P$ be fixed, $f \in GA(P)$, and suppose that $f^*\omega = \omega$. There then exists $u = u(p) \in C_G(Hol_p(\omega))$ with $f(p) = pu$. Conversely, for every $u \in C_G(Hol_p(\omega))$ there exists a unique gauge transformation $f : P \to P$ such that $f^*\omega = \omega$ and $f(p) = pu$. (Here, $C_G(Hol_p(\omega))$ denotes the centralizer in G of the holonomy group of ω with reference point p.)

Corollary 2.3: $f \in GA(P)$ with associated function $\tau \in C(P, G)$. Then the following conditions are equivalent:
 i) $f^*\omega = \omega$.
 ii) τ is constant on each ω–horizontal curve in P.
 iii) τ is constant on the holonomy subbundle $P(p_0)$ of P.

3. S-invariant Connections

We shall here look at the following problem: given two connections, both required to be invariant under certain group S, what are the conditions for them to be related by a gauge transformation? The answer to this question will provide us with a means of classifying our construction of symmetric gauge fields into classes modulo gauge-equivalence. We start with two definitions:

Definition 3.1: Let $U \subset M$ be an open subset of the base manifold and ω_1, ω_2 two connection 1–forms in P. We then say that ω_2 is gauge-equivalent to ω_1 on U iff there exists a gauge transformation $f \in GA(\pi^{-1}(U))$ such that $f^*\omega_1|_{\pi^{-1}(U)}\omega_2|_{\pi^{-1}(U)}$.

Definition 3.2: Let $W \subset M$ be an open set, $x_o \in W$, and A a connection defined on $\pi^{-1}(W)$. We say that ω is locally S–invariant at x_0 iff for all $s \in S$ with $sx_0 \in W$ there exists a connected neighborhood V_s of x_0 contained in $W \cap s^{-1}W$ and such that

$$s^*\omega|_{V_s} = \omega|_{V_s}.$$

Let $\mathcal{W} = \{s \in S/sx_0 \in W\}$. Clearly, we have $\mathcal{W}x_0 = W$. Note also that given any $x \in M$, $x = \pi(p)$, there exists a neighborhood $U_0 \subset M$ of x such that $Hol_p^0(\omega) = Hol_p(\omega)(\pi^{-1}(U_0)) = Hol_p(\omega)(\pi^{-1}(V))$ for any simply connected neighborhood V of x contained in U_0. In what follows we shall take neighborhoods V of x_0 such that $Hol_{p_0}^0(\omega) = Hol_{p_o}(\omega)(\pi^{-1}(V))$, for $x_0 = \pi(p_0)$.

We may associate, to any given S–invariant connection ω, a linear transformation Λ defined as follows: if $X \in L(S)$ (the Lie algebra of S) then $\Lambda(X) = [\omega_i(\hat{X})]_{p_o}$ where $\hat{X}_p = \frac{d}{dt}(exp\ tX \cdot p)|_{t=0}$. It turns out that the answer to the question posed at the beginning is more easily dealt with in terms of these associated linear transformations. Indeed, if $J \subset S$ denotes the isotropy group which fixes x_0 (given the action of S on M), and the action of $j \in J$ on any $p \in \pi^{-1}(x_0)$ is expressed as $jp = p\mu(j)$ with $\mu(j) \in G$, then it is easy to show that $\mu(j_1, j_2) = \mu(j_1)\mu(j_2)$ (so that $\mu : J \to G$ is a morphism of groups) and one can prove the following

Proposition 3.3: Let ω_1 and ω_2 be two S-invariant connections, and let Λ_1 and Λ_2 be their associated linear transformations espectively. Then an open set $V_s \subset M$ containing x_0 exists, such that ω_1 and ω_2 are gauge-equivalent over $\pi^{-1}(V_s)$ if and only if there exists $u \in G$ with the following properties:

i) $\mu(j)^{-1}u\mu(j)u^{-1} \in C_G(Hol_{p_0}^0(\omega))$ for all $j \in J$.

ii) There exists a local section $\sigma : V_s \to Q_s(p_0) = \pi^{-1}(V_s)$.

iii) There exists a function $\nu : \mathcal{W} \to C_G(Hol_{P_0}^0(\omega))$ satisfying the following conditions: Given $x \in V_s$ and $s \in S$, with $sx \in V_s$, and writing $s\sigma(x) = \sigma(sx)\varphi_x(s)$ for some $\varphi_x(s) \in G$, then

$$\nu(rt) = \varphi_{x_0}(t)^{-1}\nu(r)\varphi_{x_0}(t)\nu(t), \text{ for } r \in \mathcal{W}, t \in V_s \tag{3.1}$$

$$\nu(j) = \mu(j)^{-1}u\mu(j)u^{-1}, \text{ for } j \in J \tag{3.2}$$

iv) $\Lambda_2 = u^{-1}(\Lambda_1 + \nu_*|_e)u$ (3.3)

The local result obtained above may be extended to all of M. Indeed the global version of Proposition 3.3 is

Proposition 3.4: Let ω_1 and ω_2 be two S–invariant connection 1–forms and Λ_1, Λ_2 their respective associated linear transformations. Then ω_1 and ω_2 are gauge-equivalent iff there exists $u \in G$ such that

i) $\mu(j)^{-1}u\mu(j)u^{-1} \in C_G(Hol_{p_0}(\omega))$ for all $j \in J$.

ii) there exists a map $\nu : S \to C_G(Hol_{p_0}(\omega))$ such that $\nu(st) = \tilde{\varphi}(t)^{-1}\nu(s)\tilde{\varphi}(t)\nu(t)$, where $\tilde{\varphi}_x(S) : M \times S \to N_G(Hol_{p_0}(\omega))/Hol_{p_0}(\omega)$ satisfies $\tilde{\varphi}_x(ts) = \tilde{\varphi}_{sx}(t)\tilde{\varphi}_x(s)$, and one can show that, for $s \in S$ and $x, y \in M$, $\tilde{\varphi}_x(s) = \tilde{\varphi}_y(S) \equiv \tilde{\varphi}(s)$.

iii) $\nu(j) = \mu(j)^{-1}u\mu(j)u^{-1}$ for all $j \in J$.

iv) $\Lambda_2 = u^{-1}(\Lambda_1 + \nu_*|_e)u$.

When the connection 1–forms are generic, ν_* is a Lie algebra homomorphism from $L(S)$ onto an abelian subgroup of $L(G)$. If S or G are simple, then in this case $\nu = e$ and the necessary and sufficient conditions for gauge-equivalence of ω_1 and ω_2 reduce to $\Lambda_2 = u^{-1}\Lambda_1 u$, with $u \in C_G(\mu(J))$.

4. An Integro-differential version.

The finding of $u \in G$ and $\nu : \mathcal{W} \to C_G(Hol^0_{p_0}(\omega))$ with the properties required in Proposition 3.3 may prove untractable in certain circunstances. Nevertheless, and alternative approach in terms of integro–differential conditions, which may prove more amenable to actual calculations in such cases, can be obtained in the local domain.

Let $U \subset M$ be an open neighborhood of x_o, and let X_i, X_j be any two elements of a coordinate basis for $\Xi(U)$, the space of vector fields on U. (If U is a coordinate neighborhood, with coordinates $(x^1, \ldots, x^n)$, then we may take $X_i = \partial/\partial x^i$, etc.) Starting at a point $y_1 \in U$, move a distance $\varepsilon > 0$ along the integral curve of X_i passing through y_1, reaching a point y_2. From there move a distance ε along the integral curve of X_j passing through y_2; and then back along X_i and X_j to form a "rectangle" which we call $\gamma : [0,1] \to U$. Then, making use of Ado's and Frobenius' theorems we obtain, after a very lengthy proof, the desired result:

Proposition 4.1: Let ω_1 and ω_2 be two S-invariant connection 1-forms in $\pi^{-1}(U)$. Then ω_1 and ω_2 are locally gauge-equivalent iff there exists $u \in G$ such that, for $\tau(\sigma(x)g) = g^{-1}\varphi_{x_0}(s)u\varphi_{x_0}(s)^{-1}g$, with $\sigma(x)$ a local section in the ω_1-holonomy subbundle and $\sigma(x)g \in \pi^{-1}(U)$, one has

i) $$-\lim_{\varepsilon\to 0}\left(\frac{1}{\varepsilon^2}\oint_\gamma(\sigma^*\omega_2)(\gamma'(t))dt\right)+[(\sigma^*\omega_2)X_j,(\sigma^*\omega_2)X_i]=$$
$$=\tau(\sigma(x))^{-1}\{\lim_{\varepsilon\to 0}\left(\frac{1}{\varepsilon^2}\oint_\gamma(\sigma^*\omega_1)(\gamma'(t))dt\right)+[(\sigma^*\omega_1)X_j,(\sigma^*\omega_1)X_i]\}\tau(\sigma(x)),$$

ii) $\tau(\sigma(x))^{-1}\tau(\sigma(x))_*\sigma_*X_k+a\delta_{\tau(\sigma(x))^{-1}}\omega_1(\sigma_*X_k)-\omega_2(\sigma_*X_k)\in C_G(Hol_{\sigma(x)}(\omega_2))$, for $k=i,j$.

References

Fischer, A. E. (1987) : Comm Math. Phys. 113, 231.