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# Spontaneous compactification and coupling constants in a geometric model for SU(2)×U(1) with gravity

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A fiber-bundle treatment for Kaluza-Klein-type geometric unification of gravitation with the bosonic sector of the standard electroweak theory was presented by Rosenbaum *et al.* Here we show that it admits spontaneously compactified solutions where the dimensions of the internal space are of the order of the Planck length. Furthermore, the model is able to predict a numerical value for the ratio of the SU(2) and U(1) coupling constants at the energy where both compactification and the unification of gravitational with electroweak interactions would occur, and this value is in agreement with that obtained from applying the renormalization group to the standard model.

## I. INTRODUCTION

In a previous paper[1] (hereafter referred to as paper I) we developed a fiber-bundle treatment for a Kaluza-Klein-type geometric unification of gravitation and the bosonic sector of the standard electroweak theory. By allowing $G$-invariant quadratic Lagrangians and a non-Levi-Civita connection on the bundle of frames, we showed that the torsion on the frame acquires dynamics and acts a source for the scalar-field Lagrangian. It also generates the symmetry-breaking potential.

The most general $G$-invariant action resulting from our theory is [cf. Eq. (3.41) in I]

$$I=\frac{1}{V_I}\int\sqrt{|\mathcal{g}|}\left[-\kappa\underline{R}+\alpha_1\underline{R}^2+\alpha_2\underline{R}_{ijkm}\underline{R}^{ijkm}+\alpha_6\underline{R}_{ij}R^{ij}\right.$$
$$-\tfrac{1}{4}F^{\alpha}{}_{ij}F_{\alpha}{}^{ij}-\frac{1}{4}F_{ij}F^{ij}+\tfrac{1}{2}(D_i\Phi_A)(D^i\Phi^A)$$
$$+\tfrac{1}{2}m^2\Phi_A\Phi^A-\tfrac{1}{4}\lambda(\Phi_A\Phi^A)^2$$
$$\left.+\frac{n}{2(n-1)}\lambda\underline{R}\Phi_A\Phi^A-\kappa\Lambda\right]d^nx\ ,\quad(1.1)$$

where everything has already been pulled down to the $n$-dimensional base manifold $M$, $V_I$ is the volume of the extra $n-4$ dimensions, and, except for the nonminimal coupling term proportional to $\underline{R}\Phi_A\Phi^A$ (which has been used on some inflationary models to induce cosmological-constant damping), the remainder in (1.1) has the usual interpretation. In particular, the Riemann tensor $\underline{R}_{ijkm}$ is derived from a Levi-Civita connection, so there are no torsion terms in the base manifold.

There has been some work reported in the literature[2,3] on unified approaches to the Weinberg-Salam model based on pure Yang-Mills theories in six dimensions, where the components of the gauge fields in the extra dimensions play the role of the Higgs fields. By embedding SU(2)×U(1) in a larger gauge group (appropriately selected), both Fairlie[2] and Manton[3] have been able to make predictions on some of the parameters of the electroweak model, including the Weinberg angle. None of these approaches, however, include gravitation, and in fact the radius of the compactified two-sphere in Manton's model turns out to be of the order of $10^{-16}$ cm, which is far too large.

Also, although a fairly general existence theorem for compactification of solutions to Einstein-Yang-Mills equations was developed by Luciani[4] working with linear Lagrangians, and even if one could extend his proof to theories with nonlinear Lagrangians, these solutions are contingent on some constraints on the gauge and symmetry groups which are not satisfied for SU(2)×U(1). Thus, it is not at all obvious from that work that compact solutions exist for the case under consideration.

Here we show that the Lagrangian (1.1) indeed admits spontaneously compactified solutions where the dimensions of the internal space are of the order of the Planck length. Furthermore, our model is able to predict a numerical value for the ratio of the SU(2) and U(1) coupling constants at the energy where both compactification and unification of gravity with the electroweak interactions would occur (our model does not as yet include strong interactions), and this value is in agreement with that predicted by the standard model via the renormalization group.

Our procedure is substantially different from the one followed in the papers mentioned above. First, the Higgs scalars in our formalism stem from the torsion on the fibers of the bundle (so they also have a geometric origin) and not from the connections. In addition, the symmetry group $S$ of our homogeneous internal space is the same as the gauge group $G$ of the theory: $S=G=\mathrm{SU}(2)\times\mathrm{U}(1)$. Following an approach based on the work by Wang[5] and Kobayashi and Nomizu[6] we find a family of $S$-invariant connections and $S$-symmetric Higgs fields, which, when substituted into the field equations resulting from (1.1), lead [as in the case of the original paper of Cremmer and Scherk[7] with gauge group SO(3)] to a system of algebraic relations for the parameters of the metric of the internal space, those of the gauge and Higgs fields, and the remaining parameters in (1.1). It turns out that these equations allow nontrivial solutions only if general quadratic terms in the curvature are admitted in the Lagrangian and, in this case, the range of permissible values for the ratio of coupling constants is quite restricted, so that the theory *is* predictive. Furthermore, as mentioned earlier, the characteristic length parameters of the metric must have a magnitude of the order of the Planck length.

Since terms quadratic in the curvature appear naturally in the low-energy limit of superstring theories, the investigation of spontaneous compactification for such Lagrangians seems worthwhile. We need to stress, however, that our analysis is at the classical level and should therefore be seen as intended to contribute to the semiquantitative understanding of the spontaneous compactification phenomenon, in the hope that such a mechanism will still occur in the domain of whichever the correct finite theory for quantum gravity will be.

Our formalism obtains all the gauge fields and necessary Higgs bosons, which ultimately trigger the $SU(2)\times U(1)$ breaking, as part of the metric tensor. Even though for the purposes of spontaneous compactification, i.e., the process of transforming the base manifold into the form $M^4\times B$ (with $B$ compact) induced by the structure of the vacuum or ground state, fermion fields do not contribute to the ground-state solution at the classical level (due to Lorentz invariance), and it is thus sufficient to consider only the bosonic sector of the theory,[8,9] a serious candidate model for describing nature must eventually also include fermionic matter. This would consist of essentially zero-mass leptons (extremely light compared to the energy scale of gravitation), and quarks if SU(3) were considered.

We should perhaps stress here that in ordinary Kaluza-Klein theories one considers a principal fiber bundle where the base manifold is taken to be the four-dimensional space-time, each fiber is group isomorphic to the gauge group, and the fibers themselves (as manifolds) are compactified into what is called the internal space. The Kaluza-Klein point of view, in its purest form, is to attribute all interactions other than gravity, as well as the spectrum of elementary particles, to the structure of this internal manifold. In this framework, there are severe obstructions to the incorporation of chiral *metric* fermions. The problem stems from the fact that left-handed fermions transform differently than right-handed ones. In other words, fermions of given helicity form a complex representation of the gauge group. One may write Dirac's equation for a massless particle in $n$ dimensions as

$$0=\not{D}\Psi=\not{D}^{(4)}\Psi+\not{D}^{(n-4)}\Psi\ , \tag{1.2}$$

so that the eigenvalues of $\not{D}^{(n-4)}$ are observed in four dimensions as the particle's mass. One needs, then, to look for zero modes of $\not{D}^{(n-4)}$, and these will acquire the small mass we see through the Higgs mechanism of $SU(2)\times U(1)$-symmetry breaking. However, the Atiyah-Hirzebruch[10] theorem states that for any continuous symmetry group the Dirac zero modes form a *real* representation. In multidimensional simple ($N=1$) supergravity or in superstring theories there are no fundamental spin-$\frac{1}{2}$ fields, but only the Rarita-Schwinger spin-$\frac{3}{2}$ field. But Witten[11] has extended the result of Atiyah and Hirzebruch to show that, on homogeneous spaces, the Rarita-Schwinger zero modes always lead to a real representation. What happens for nonhomogeneous spaces we do not know, but almost all of the work on Kaluza-Klein theories has been done using homogeneous spaces, presumably in response to a requirement of *minimality,* and because they admit a structure of a real analytic manifold. One could, of course, try to incorporate chiral fermions using as internal manifold a nonhomogeneous space, but from the work of Alvarez-Gaumé and Witten[12] in 11 dimensions one expects one-loop anomalies that spoil general covariance and cannot be canceled.

The usual way out is to consider *elementary* (i.e., not arising from components of the metric in $n$ dimensions) fermion fields. This may be done by the use of spin structures and the introduction of fermions as particle fields which are naturally isomorphic to the space of sections of an associated bundle. Such a procedure, however, not only goes against the philosophy of Kaluza-Klein theories, but is also a much less ambitious program: one may no longer hope to *unify* (but only amalgamate) all interactions in nature, nor to predict the observed values of the coupling constants.

Another alternative for introducing massless fermions in a theory is based on a modification of the spin connection to accommodate torsion in the internal manifold, as was done in interesting works by Wu and Zee,[13] and Orzalesi and co-workers.[14] Unfortunately, for the group manifolds they use, they obtain right- and left-handed fermions in equal numbers. However, there are still some possibilities of generalizing this approach, both by analyzing appropriate quotient spaces and by using more general forms of torsion (nonparallelizable).

The formalism on which this paper is based is different from the "fiber-bundle—over spacetime" just described. We build a fiber-bundle formalism, with fiber $G=SU(2)\times U(1)$, over a base manifold which itself is of the form $M^4\times B$, with $B$ a homogeneous compact space of the form $SU(2)\times U(1)/I$ (here $I$ is the isotropy group for a certain action of $G$ on $B$). In other words, the *internal manifold* is in some way another copy of the manifold determined by the fibers. The gauge fields arise from a connection in the bundle, but once pulled down to the base manifold one ends up with fields defined over the entire $M^4\times B$, which may then be reduced to an effective four-dimensional theory. In this way our fields *do* arise from the geometry, but from the point of view of the four-dimensional spacetime they are seen as "given", i.e., as elementary fields put in by hand in the higher-dimensional $M^4\times B$. It is possible that by such a procedure, either within the topology of the internal manifold investigated in this paper or the alternative ones suggested, the small oscillations of the fermion fields resulting from their interaction with the compactifying gauge fields may not only allow zero modes to exist for the internal Dirac operator, but may also hopefully avoid the *no-go* theorems of Atiyah-Hirzebruch and Witten described above, by allowing the introduction of chiral fermions in the model.

The paper is organized as follows: Section II deals with the possible topologies for the base manifold $M$, and contains an outline of a general procedure for obtaining $S$-invariant connections. We then specialize the formalism to the specific case $S=\mathrm{SU}(2)\times\mathrm{U}(1)$ in order to arrive at a family of possible solutions for the gauge fields. Making use of work already done in paper I, we also obtain the SU(2)×U(1)-symmetric solutions for the Higgs fields. Section III gives a presentation of the Einstein-Yang-Mills-Higgs field equations in a coordinate-free form. (This is most appropriate for the nonholonomic basis of right-invariant vector fields that we use for our calculations, as it leads to considerable simplifications.) We next use the results from Sec. II to generate solutions to the field equations which lead, in turn, to a system of nonlinear coupled algebraic equations. This is solved in Sec. IV, resulting in predictions for the numerical values of the Yang-Mills coupling constants and the orders of magnitude of the characteristic parameters of the compactified internal space. Section V concludes with some general remarks about fine-tuning, and observations regarding some terms in our Lagrangian which also appear in other authors' works[15–17] as possible means to approach the cosmological-constant problem. We also give some additional remarks on how the inclusion of fermions may be dealt with in future extensions of the present work.

## II. BASE-SPACE TOPOLOGY. $S$-INVARIANT CONNECTIONS AND HIGGS FIELDS

As pointed out in the Introduction, the base space of the fiber bundle [onto which the Lagrangian (1.1) has been pulled down] is of topology $M=\mathcal{M}^4\times S_I$, where $S_I$ is a compact manifold. One of the purposes of this section is to construct a connection (gauge fields) on $M$. To this end, consider the trivial principal fiber bundle $P=\mathrm{SU}(2)\times\mathrm{U}(1)\times S_I$ on $m_0\times S_I$, with $m_0\in\mathcal{M}^4$ and gauge group $G=\mathrm{SU}(2)\times\mathrm{U}(1)$. We will generate an $S$ symmetry (or $S$ action) on $P$, with $S=\mathrm{SU}(2)\times\mathrm{U}(1)$, with the property that our connection on $P$ should be invariant under this $S$ symmetry. (Note that we are dealing here with a situation where the symmetry group of the compact base space is the same as the gauge group of the bundle.) Once the connection on $m_0\times S_I$ has been constructed, it is a trivial matter to extend it to the whole of $M$.

Since the simplest of all possible actions are the transitive ones, we choose the action of SU(2)×U(1) on $S_I$ to be of this type. In this case $S_I$ turns out to be a homogeneous space of the form $S/I(x_0)$, where $I(x_0)$ is the isotropy subgroup of $S$ which fixes the point $x_0$ of $S_I$. In particular, since all continuous Lie subgroups $H$ of SU(2) are isomorphic to U(1), we have the following possibilities for $S/I(x_0)$:

$$
\begin{aligned}
&(1)\ \mathrm{SU}(2)\times\mathrm{U}(1)/\mathrm{SU}(2)=S^1\ ,\\
&(2)\ \mathrm{SU}(2)\times\mathrm{U}(1)/H\times\mathrm{U}(1)=S^2\ ,\\
&(3)\ \mathrm{SU}(2)\times\mathrm{U}(1)/H=S^2\times S^1\ ,\\
&(4)\ \mathrm{SU}(2)\times\mathrm{U}(1)/\mathrm{U}(1)=S^3\ ,\\
&(5)\ \mathrm{SU}(2)\times\mathrm{U}(1)/\tilde{H}=S^3\ ,\\
&(6)\ \mathrm{SU}(2)\times\mathrm{U}(1)/\{e\}=S^3\times S^1\ ,
\end{aligned}
\tag{2.1}
$$

where $\tilde{H}=\{(h,h^{-1})\in\mathrm{SU}(2)\times\mathrm{U}(1)|h\in H,h^{-1}\in U(1)\simeq H\}$, and $e\in\mathrm{U}(1)$ is the identity.

Cases (1), (2), and (4) may be immediately discarded since the quotients in these imply a trivial action of one of the factors SU(2) or U(1). By further requiring that $S_I$ be of minimal dimension and of simplest homology, we are left with case (5). In what follows we restrict ourselves to this case, i.e., $S_I$ will have the topology of the three-sphere.

Next we identify $S^3$ with the space of unitary quaternions in $\mathcal{R}^4$:

$$
\begin{aligned}
S^3&=\{x_1+ix_2+jx_3+kx_4|x_1^2+x_2^2+x_3^2+x_4^2=1\}\\
&=\mathrm{SU}(2)\ ,
\end{aligned}
$$

where $i,j,k$ satisfy the usual multiplicative rules of quaternions. Consequently, the Lie algebra su(2) of SU(2) can be identified with the quaternion vector subspace of $\mathcal{H}=\{x_1+ix_2+jx_3+kx_4\}$ generated by $X_1=\frac{1}{2}k,X_2=-\frac{1}{2}j,X_3=\frac{1}{2}i$, with $X_i$ $(i=1,2,3)$ satisfying the commutation rules

$$[X_i,X_j]=\varepsilon_{ijk}X_k\ . \tag{2.2}$$

If we now define the right action on quaternions $R_{\exp(tX_i)}$: $\mathcal{H}\to\mathcal{H}$ by $R_{\exp(tX_i)}s=s\cdot e^{tX_i}$, where the center "dot" operation denotes multiplication of quaternions, we have that, for $s=x_1+ix_2+jx_3+kx_4$, the left-invariant vector fields $\xi_i(s)=(d/dt)(R_{\exp(tX_i)}s)|_{t=0}=L_{s*}X_i$ on $S^3$, are given by

$$\xi_1=\tfrac{1}{2}(-x_4\partial_1+x_3\partial_2-x_2\partial_3+x_i\partial_4)\ , \tag{2.3}$$

$$\xi_2=\tfrac{1}{2}(x_3\partial_1+x_4\partial_2-x_1\partial_3-x_2\partial_4)\ , \tag{2.4}$$

$$\xi_3=\tfrac{1}{2}(-x_2\partial_1+x_1\partial_2+x_4\partial_3-x_3\partial_4)\ . \tag{2.5}$$

Similarly, for the right-invariant vector fields $\tilde{\xi}_i(s)=(d/dt)(L_{\exp(tX_i)}s)|_{t=0}=R_{s*}X_i$ on $S^3$ we get

$$\tilde{\xi}_1=\tfrac{1}{2}(-x_4\partial_1-x_3\partial_2+x_2\partial_3+x_1\partial_4)\ , \tag{2.6}$$

$$\tilde{\xi}_2=\tfrac{1}{2}(x_3\partial_1-x_4\partial_2-x_1\partial_3+x_2\partial_4)\ , \tag{2.7}$$

$$\tilde{\xi}_3=\tfrac{1}{2}(-x_2\partial_1+x_1\partial_2-x_4\partial_3+x_3\partial_4)\ . \tag{2.8}$$

It follows from the general theory, or by direct calculation making use of (2.3)–(2.8), that

$$[\xi_i,\xi_j]=\epsilon_{ijk}\xi_k,\quad i,j,k\in\{1,2,3\}\ , \tag{2.9}$$

$$[\tilde{\xi}_i,\tilde{\xi}_j]=-\epsilon_{ijk}\tilde{\xi}_k\ , \tag{2.10}$$

$$[\xi_i,\tilde{\xi}_j]=0\ . \tag{2.11}$$

We now associate the one-parameter subgroup $H$ of SU(2) with the left translations which generate the integral curve of $\tilde{\xi}_3$ which passes through $e$. That is,

$$H=\{\exp[t(\tfrac{1}{2}i)]\}=\{(\cos\tfrac{1}{2}t,\ \sin\tfrac{1}{2}t,\ 0,\ 0)\in S^3|t\in\mathcal{R}\}\ .$$

Clearly, $H$ is isomorphic to U(1).

For the construction of the $S$ action on $S^3$, we make the identification

$$S=\mathrm{SU}(2)\times\mathrm{U}(1)=\{(g,h)|g\in S^3,h\in H\}\ . \tag{2.12}$$

Thus the transitive right action of $S$ on $S^3$ is given by

$$x(g,h)=h\cdot x\cdot g \quad \text{for } x\in S^3\ . \tag{2.13}$$

It is obvious that the isotropy group of $x_0=e=(1,0,0,0)$ is $I(e)=\{(h^{-1},h)\in\mathrm{SU}(2)\times\mathrm{U}(1)|h\in H\}$. Consequently, by case (5) of Eq. (2.1),

$$S_I=\mathrm{SU}(2)\times\mathrm{U}(1)/I(e)\simeq S^3\ . \tag{2.14}$$

Up to this point we have determined our compact manifold $S_I$ as the homogeneous space $\mathrm{SU}(2)\times\mathrm{U}(1)/I(e)$ with the topology of a three-sphere, but without a given shape. In order to give a metric $\mathbf{g}$ to $S_I$, we note that we want it to be SU(2)×U(1) invariant, i.e., Iso(**g**) =SU(2)×U(1), so that we can take $\xi_1,\xi_2,\xi_3,\tilde{\xi}_3$, as Killing vector fields for the SU(2)×U(1) action on $S_I$ and obtain

$$\mathbf{g}=-\frac{\rho_1}{4}(\tilde{\sigma}^1\otimes\tilde{\sigma}^1)-\frac{\rho_1}{4}(\tilde{\sigma}^2\otimes\tilde{\sigma}^2)-\frac{\rho_2}{4}(\tilde{\sigma}^3\otimes\tilde{\sigma}^3)\ , \tag{2.15}$$

where $\tilde{\sigma}^i$ $(i=1,2,3)$ are the one-forms dual to $\tilde{\xi}_i$, and $\rho_1,\rho_2$ are parameters with units of (length)$^2$. Indeed, it can be readily verified that $\mathcal{L}_{\xi_i}\mathbf{g}=0$ for $i=1,2,3$ and $\mathcal{L}_{\tilde{\xi}_3}\mathbf{g}=0$.

*S-invariant connections.* In order to proceed with the construction of the SU(2)×U(1)-invariant connections on $P$, we shall first recall some results of a general character.

Let $\pi:P\to M$ denote a principal fiber bundle with structure group $G$. Let $S$ be a Lie group and $\mathcal{S}$ its corresponding Lie algebra. We say that $S$ acts from the right on $P$ if $S$ acts differentiably from the right on $P$ and

$$(pg)s=(ps)g \quad \text{for } s\in S,\ \ g\in G,\ \ p\in P\ . \tag{2.16}$$

The action of $S$ on $P$ induces an action of $S$ on the base space $M$ as follows. For $x\in M$ choose $p\in P$ such that $\pi(p)=x$, and write $xs=\pi(ps)$. Note that (2.16) guarantees that $xs$ does not depend on the location of $p$ on the fiber. We shall also say that $S$ acts *orbit transitively* on $P$ if the induced action on the base space $M$ is transitive.

Consider now a point $x_0\in M$, and a $p_0\in P$ such that $\pi(p_0)=x_0$. Let $I(x_0)$ be the isotropy group of $x_0$ relative to the action $S$. Then, for $j\in I(x_0)$ we have $\pi(p_0j^{-1})=\pi(p_0)j^{-1}=x_0j^{-1}=x_0$. Consequently, $p_0j^{-1}$ is on the same fiber as $p_0$, so that $p_0j^{-1}=p_0\lambda(j)$ with $\lambda(j)\in G$. Observe that $p_0(j_2j_1)^{-1}=p_0j_1^{-1}j_2^{-1}$ $=p_0\lambda(j_1)j_2^{-1}=p_0j_2^{-1}\lambda(j_1)=p_0\lambda(j_2)\lambda(j_1)$; hence, $\lambda(j_2j_1)=\lambda(j_2)\lambda(j_1)$ so that $\lambda$ is a homomorphism of Lie groups, $\lambda:I(x_0)\to G$.

Now, according to Wang[5] (see also Ref. 6) there is a bijective correspondence between $S$-invariant connections and linear transformations $\Lambda:\mathcal{S}\to\mathcal{G}$ of Lie algebras which satisfy the following conditions:

$$\begin{aligned}&(\mathrm{A})\quad \Lambda(Y)=-\lambda_*(Y) \quad \text{for } Y\in\mathcal{I}(x_0)\\ &\qquad\qquad [\text{the Lie algebra of } I(x_0)]\ ,\\ &(\mathrm{B})\quad \Lambda[a\delta_j(X)]=a\delta_{\lambda(j)}(\Lambda(X))\\ &\qquad\qquad \text{for } X\in\mathcal{S},\ \ j\in I(x_0)\ .\end{aligned} \tag{2.17}$$

If $w$ is the $S$-invariant connection corresponding to $\Lambda$, we have

$$\Lambda(X)=\omega_{p_0}(\hat{X}_{p_0}) \quad \text{with } \hat{X}_{p_0}=\frac{d}{dt}(p_0e^{tX})|_{t=0},\quad X\in\mathcal{S}\ . \tag{2.18}$$

We choose now a global section $\sigma:M\to P$ (i.e., we suppose that $P$ is a trivial bundle) for the bundle $\pi:P\to M$, that is, a morphism $\sigma$ of manifolds with $\pi\sigma=id_M$. Associated to the action of $S$ on $P$ and the section $\sigma$, we have a differentiable function $\phi:M\times S\to G$ such that if we let $\phi(x,s)=\phi_x(s)$, then $\sigma(x)s=\sigma(xs)\phi_x(s)$. For a fixed $x$, $\phi_x$ is a function from $S$ to $G$. In addition, $\phi_x(e)=e$, so its differential $\phi_{x*}$ determines a linear function from $\mathcal{S}=T_e(S)$ to $T_e(G)=\mathcal{G}$, though this function is not necessarily a morphism of Lie algebras, because $\phi_x$ is not necessarily a morphism of Lie groups. We thus have that $W=\phi_*:M\to\mathcal{L}(\mathcal{S},\mathcal{G})=$ *linear transformations from* $\mathcal{S}$ *to* $\mathcal{G}$. (Note that our $W_x$ is minus the $W_x$ obtained by Forgács and Manton[18] in a somewhat different approach to the problem.)

For $X\in\mathcal{S}$ denote by $\bar{X}$ the vector field on $M$ defined by $\bar{X}_x=(d/dt)[x\cdot\exp(tX)]|_{t=0}$. The relation between $\bar{X}$ and $\hat{X}$, the corresponding field defined on $P$, is

$$\begin{aligned}\hat{X}_{\sigma(x)}=\frac{d}{dt}[\sigma(x)\cdot\exp(tX)]|_{t=0}&=\frac{d}{dt}[\sigma(x\cdot\exp(tX))\phi_x(\exp(tX))]|_{t=0}\\ &=\frac{d}{dt}[\sigma(x\cdot\exp(tX))]|_{t=0}+\frac{d}{dt}\{\sigma(x)\phi[\exp(tX)]\}|_{t=0}\\ &=\sigma_*(\bar{X}_x)+[W_x(X)]^*_{\sigma(x)}\ ,\end{aligned} \tag{2.19}$$

where $[W_x(X)]^*$ is the fundamental field associated with $W_x(X)\in\mathcal{G}$. Consequently,

$$\omega[\sigma_*(\bar{X}_x)]=(\sigma^*\omega)_x(\bar{X}_x)=\omega_{\sigma(x)}(\hat{X}_{\sigma(x)})-W_x(X)\ . \tag{2.20}$$

Making now use of the $S$ invariance of $\omega$, we have

$$\omega_{\sigma(x_0)}(\hat{X})_{\sigma(x_0)}=(s^*\omega)_{\sigma(x_0)}(\hat{X})_{\sigma(x_0)}=\omega_{\sigma(x_0)s}(s_*\hat{X})_{\sigma(x_0)s}=\omega_{\sigma(x_0)s}(\widehat{s^{-1}Xs})_{\sigma(x_0)s}\ . \tag{2.21}$$

Moreover, from the property of connections under right translations, we also have [using (2.16)]

$$\begin{aligned}\omega_{[\sigma(x_0)s]a}(\widehat{s^{-1}Xs})_{[\sigma(x_0)s]a}&=\omega_{[\sigma(x_0)s]a}[R_{a*}(\widehat{s^{-1}Xs})_{\sigma(x_0)s}]\\&=a\delta_{a^{-1}}\omega_{\sigma(x_0)s}(\widehat{s^{-1}Xs})_{\sigma(x_0)s}\quad\text{for } a\in G\ .\end{aligned} \tag{2.22}$$

On the other hand, since $S$ acts transitively on $M$, we have $x=x_0s$ for some $s\in S$, and $\sigma(x)=\sigma(x_0s)=[\sigma(x_0)s]\phi_{x_0}(s)^{-1}$. Thus (2.21) and (2.22) imply

$$\begin{aligned}\omega_{\sigma(x_0)}(\hat{X})_{\sigma(x_0)}&=a\delta_a\omega_{[\sigma(x_0)s]a}(\widehat{s^{-1}Xs})_{[\sigma(x_0)s]a}\\&=a\delta_{\phi_{x_0}(s)^{-1}}\omega_{\sigma(x)}(\widehat{s^{-1}Xs})_{\sigma(x)}\ ,\end{aligned} \tag{2.23}$$

after the identification $a\equiv\phi_{x_0}(s)^{-1}$. It then follows from (2.23) and (2.18) that

$$\begin{aligned}\omega_{\sigma(x)}(\hat{X})_{\sigma(x)}&=a\delta_{\phi_{x_0}}(s)\omega_{\sigma(x_0)}(\widehat{s^{-1}Xs})_{\sigma(x_0)}\\&=a\delta_{\phi_{x_0}(s)}\Lambda(a\delta_sX)\ .\end{aligned} \tag{2.24}$$

If we now write $A=(\sigma^*\omega)_x$, and substitute (2.4) in (2.20), we finally get for our $S$-invariant gauge fields the general expression

$$A(\bar{X})_x=a\delta_{\phi_{x_0}(s)}\Lambda(a\delta_sX)-W_x(X)\ . \tag{2.25}$$

*Explicit solutions.* Our main objective in this section has been to establish a topology for our base space and to find $S$-invariant connections for $S=\mathrm{SU}(2)\times\mathrm{U}(1)$. However, in identifying $S^3$ with the unitary quaternions we have been using the Cartesian coordinates $x_1,\ldots,x_4$, which are not the natural coordinates on the three-dimensional sphere. Nonetheless, the use of these coordinates facilitates the obtainment of solutions to (2.25) as well as many of the calculations in the following sections. Instead of $S^3$ it therefore proves more convenient to resort to the space of non-null quaternions $\mathcal{H}^\times=\mathcal{R}^4\setminus\{0\}$, where $x_1,\ldots,x_4$ are the natural coordinates.

If we further let $S'=\mathrm{SU}(2)\times\mathrm{U}(1)\times\mathcal{R}^+$, where $\mathcal{R}^+$ is the set of positive real numbers, to be the group that operates from the right on $\mathcal{H}^\times$, and we introduce in addition the transformtion $\psi:\mathcal{H}^\times\to S^3$ such that $\psi(x)=x/\|x\|$, then we can identify the $S$-invariant connections $\omega$ on $S^3$ with the $S'$-invariant connections on $\mathcal{H}^\times$ which are of the type $\psi^*\omega$.

In order to use the general theory described above, it is desirable to treat $\mathcal{H}^\times$ as a homogeneous space. This may be accomplished by letting $S'$ act transitively on $\mathcal{H}^\times$ according to the rule $q(s,h,r)=(h\cdot q\cdot s)r$, where, as before the "centerdot" operation denotes multiplication of quaternions. Taking $q=e$, the isotropy group of $e$ turns out to be $I(e)=\{(h^{-1},h,1)|h\in H\}$. (Throughout we will use the symbols $e$ or 1 indistinctly to denote the group identity.) Consequently, $\mathcal{H}^\times$ is the homogeneous space $\mathcal{H}^\times=S'/I(e)$.

Now consider the trivial principal fiber bundle $\mathcal{H}^\times\times\mathrm{SU}(2)\times\mathrm{U}(1)$ with base space $\mathcal{H}^\times$ and global section $\sigma(x)=(x,(1,1))$. As we have seen in our general considerations above, the action of $S'$ on $\mathcal{H}^\times\times\mathrm{SU}(2)\times\mathrm{U}(1)$ which induces the given action of $S'$ on $\mathcal{H}^\times$ is determined by the corresponding transformations $\phi_q$. The analysis of possible choices for these transformations and their implications in terms of the $S$-invariant connections is beyond the scope of the present paper and is the subject of work by the authors, on the general theory, which will be published separately. Here we shall make the following natural choice for $\phi_q:S'\to G=\mathrm{SU}(2)\times\mathrm{U}(1)$:

$$\phi_q(s,h,r)=(s^{-1},h^{-1})\ . \tag{2.26}$$

Since

$$\begin{aligned}\sigma(q)(s,h,r)&=\sigma(q(s,h,r))\phi_q(s,h,r)\\&=\sigma(h\cdot q\cdot sr)\phi_q(s,h,r)\end{aligned} \tag{2.27}$$

we have

$$(q,(1,1))(s,h,r)=(h\cdot q\cdot rs,(s^{-1},h^{-1}))\ . \tag{2.28}$$

On the trivial principal bundle $S^3\times\mathrm{SU}(2)\times\mathrm{U}(1)$ we also choose the section $\sigma$ given by $\sigma(x)=(x(1,1))$ and the $S$ action given by the transformations $\phi_x:G\to G$, $\phi_x(s,h)=(s^{-1},h^{-1})$.

Thus, for our choice of action of $S'$ on $\mathcal{H}^\times\times S'$, the resulting transformation $\phi_q$ is an antihomomorphism which maps each element of $\mathrm{SU}(2)\times\mathrm{U}(1)$ onto its inverse and acts trivially on $\mathcal{R}^+$ by sending all elements to the identity. It is important to remark here that the form of the action (2.26), with the inverses of $s$ and $h$ occurring on the right-hand side, is made necessary by the fact that $\phi_{q_0}|_{I(e)}=\lambda$ has to be an homomorphism of Lie groups. Indeed, note that if we let $p_0=\sigma(q_0)$ and recall that $p_0j^{-1}=p_0\lambda(j)$ for $j\in I(e)$, then $p_0j^{-1}=\sigma(q_0)j^{-1}=\sigma(q_0j^{-1})\phi_q(j^{-1})=p_0\phi_{q_0}(j^{-1})$. Consequently, $\lambda(j)=\phi_{q_0}(j^{-1})$. But $j=(h^{-1},h,1)$ so $\lambda(h^{-1},h,1)=\phi_{q_0}(h,h^{-1},1)=(h^{-1},h)$ [by (2.26)]. Hence, $\lambda|_{I(e)}$ is the identity and it clearly follows that $\lambda(j_1)\lambda(j_2)=\lambda(j_1j_2)$.

In order to describe $W:\mathcal{H}^{\times}\to\mathcal{L}(\mathcal{S}',\mathcal{G})$ we note that $\mathcal{S}'=\mathrm{su}(2)\oplus i\mathcal{R}\oplus X_5\mathcal{R}$, where su(2) is the Lie algebra of SU(2), $i$ is the basis of the Lie algebra of U(1), and $X_5$ is the generator of the Lie algebra of $\mathcal{R}^+$. Since $W_q=\phi_{q*}$ it follows from (2.26) that

$$W_q(X)=-X, \quad \forall X\in \mathrm{su}(2)\oplus i\mathcal{R} \ , \tag{2.29a}$$

$$W_q(X_5)=0 \ . \tag{2.29b}$$

Note also that $(\bar{X}_5)_q=(d/dt)[q\exp(tX_5)]|_{t=0}=x_1\partial_1+x_2\partial_2+x_3\partial_3+x_4\partial_4$, with $q=x_1+ix_2+jx_3+kx_4\in\mathcal{H}^{\times}$. So $(\bar{X}_5)_q$ is a radial vector.

We are now ready to solve (2.25) for the particular case under consideration. Choosing $\underline{s}=(s,e,e)\in \mathrm{SU}(2)\times\mathrm{U}(1)\times\mathcal{R}^+$ and making use of (2.26) we have $\phi_{x_0}(\underline{s})=(s^{-1},e)$. Hence

$$a\delta_{\phi_{x_0}(\underline{s})}=\begin{bmatrix} a\delta_{s^{-1}} & 0\\ 0 & I_{i\mathcal{R}}\end{bmatrix},\quad a\delta_s=\begin{bmatrix} a\delta_s & 0\\ 0 & I\end{bmatrix},$$

where $I$ is the identity in $i\mathcal{R}+X_5\mathcal{R}$.

Furthermore, given $X\in \mathrm{su}(2)\oplus i\mathcal{R}\oplus X_5\mathcal{R}$ we can write $X=Y+V+Z$, $Y\in\mathrm{su}(2)$, $V\in i\mathcal{R}$, $Z\in X_5\mathcal{R}$, so that $a\delta_{\underline{s}}X=s\cdot Y\cdot s^{-1}+V+Z$. Similarly, for $X'=Y+V$, $Y\in\mathrm{su}(2)$, $V\in i\mathcal{R}$, we have $a\delta_{\phi_{x_0}(\underline{s})}X'=a\delta_{s^{-1}}(Y)+V=s^{-1}\cdot Y\cdot s+V$. A simple calculation with quaternions, taking $s=x_1+ix_2+jx_3+kx_4\in\mathrm{SU}(2)$ and the basis $\{X_i\}$ for su(2) defined at the beginning of this section, yields the following expression for the matrix corresponding to $a\delta_s$ relative to this basis:

$$a\delta_s=\begin{bmatrix} x_1^2-x_2^2-x_3^2+x_4^2 & -2(x_3x_4+x_1x_2) & 2(x_2x_4-x_1x_3)\\ -2(x_3x_4-x_1x_2) & x_1^2-x_2^2+x_3^2-x_4^2 & -2(x_2x_3+x_1x_4)\\ 2(x_2x_4+x_1x_3) & -2(x_2x_3-x_1x_4) & x_1^2+x_2^2-x_3^2-x_4^2\end{bmatrix}. \tag{2.30}$$

Before proceeding with the calculation of the matrix $\Lambda$ in (2.25) recall that we are actually interested in the $S$-invariant connections of the trivial bundle $S^3\times\mathrm{SU}(2)\times\mathrm{U}(1)$ over $S^3$, where the action on $S^3\times\mathrm{SU}(2)\times\mathrm{U}(1)$ is given (as explained before) by the functions $\phi_x$. If we now define $\bar{\psi}:\mathcal{H}^{\times}\times\mathrm{SU}(2)\times\mathrm{U}(1)\to S^3\times\mathrm{SU}(2)\times\mathrm{U}(1)$ by $\bar{\psi}(q,(h,s))=(\psi(q),(h,s))$, with $\psi$ being the projection operator introduced above, then the correspondence $\omega\to\bar{\psi}^*\omega$ gives a one-to-one relation between the $S$-invariant connections on $S^3\times\mathrm{SU}(2)\times\mathrm{U}(1)$ and the $S'$-invariant connections $\omega'$ on $\mathcal{H}^{\times}\times\mathrm{SU}(2)\times\mathrm{U}(1)$ with $\omega'(\hat{X}_5)=0$. Then, from (2.25) it follows that relative to the basis $X_1,X_2,X_3,X_4=i,X_5$ the matrix representation of the linear transformation $\Lambda$: $\mathcal{S}'\to\mathcal{S}$ has to be of the form

$$\Lambda=(\Lambda_0,0)\quad\text{where } \Lambda_0:\mathcal{S}\to\mathcal{S} \ . \tag{2.31}$$

According to Wang's theorem $\Lambda_0$ must satisfy the conditions (A) and (B) in (2.17). Moreover, in our case $\lambda=id$, so by (B) $\Lambda_0$ must commute with the matrix

$$a\delta_{(\psi(\underline{h})^{-1},\psi(\underline{h}))}=\begin{bmatrix} a\delta_{h^{-1}} & 0\\ 0 & 1\end{bmatrix},$$

with $h^{-1}=\cos t-i\sin t$ for some $t\in\mathcal{R}$. Setting $x_1=\cos t$, $x_2=-\sin t$, it readily follows from (2.30) that

$$a\delta_{(\psi(\underline{h})^{-1},\psi(\underline{h}))}=\begin{bmatrix} x_1^2-x_2^2 & -2x_1x_2 & 0 & 0\\ 2x_1x_2 & x_1^2-x_2^2 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} =\begin{bmatrix} A & 0\\ 0 & 1\end{bmatrix}, \tag{2.32}$$

with

$$A=\begin{bmatrix}\alpha & \beta\\ -\beta & \alpha\end{bmatrix}.$$

From the equality

$$\Lambda_0\begin{bmatrix} A & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix} A & 0\\ 0 & 1\end{bmatrix}\Lambda_0 ,$$

we find

$$\Lambda_0=\begin{bmatrix}\omega & -z & 0 & 0\\ z & \omega & 0 & 0\\ 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{L}}\\ 0 & 0 & & \end{bmatrix}. \tag{2.33}$$

On the other hand, the generator of the Lie algebra of the isotropy group $I(e)$ is $-X_3+X_4$, so condition (A) of (2.17) implies

$$L\begin{bmatrix}-1\\ 1\end{bmatrix}=-\begin{bmatrix}-1\\ 1\end{bmatrix}. \tag{2.34}$$

Consequently,

$$L=\begin{bmatrix} m & 1+m\\ n & n-1\end{bmatrix}. \tag{2.35}$$

We thus have all the ingredients needed to evaluate (2.25), which by virtue of (2.28) and (2.29) reduces to

$$A(\bar{X}_a)=a\delta_{\psi(\underline{s})^{-1}}\Lambda_0(a\delta_{\psi(\underline{s})}X_a)+X_a\quad (a=1,2,3,4)\ , \tag{2.36}$$

where, for $a=i=1,2,3$, $\bar{X}_i=\xi_i$ as given in (2.3)–(2.5), and $\bar{X}_4=\tilde{\xi}_3$ [cf. Eq. (2.8)].

The calculation is simplified considerably if instead of evaluating relative to the left invariant vector fields we calculate the gauge fields using the right invariant fields $(\tilde{X}_i)_x=(d/dt)(L_{\exp(tX_i)}x)|_{t=0}$ as basis. In fact, taking

$x = x_0\psi(\underline{s})$ with $x_0 = e \in S^3$, one has

$$(\tilde{X}_i)_x = \frac{d}{dt}[e^{tX_i}\cdot\psi(\underline{s})]|_{t=0} = \frac{d}{dt}[\psi(\underline{s})\cdot\psi(\underline{s})^{-1}\cdot e^{tX_i}\cdot\psi(\underline{s})]|_{t=0} = \overline{(a\delta_{\psi(\underline{s})^{-1}}X_i)} \ . \tag{2.37}$$

Substituting now (2.37) into (2.36) we get

$$\begin{aligned} A(\tilde{X}_i)_x &= A\overline{(a\delta_{\psi(\underline{s})^{-1}}X_i)}_x \\ &= a\delta_{\psi(\underline{s})^{-1}}\Lambda_0(a\delta_{\psi(\underline{s})}\cdot a\delta_{\psi(\underline{s})^{-1}}X_i) \\ &\quad + a\delta_{\psi(\underline{s})^{-1}}X_i \\ &= a\delta_{\psi(\underline{s})^{-1}}(\Lambda_0+I_0)X_i \ . \end{aligned} \tag{2.38}$$

From (2.33) and (2.35) it is evident that the matrix $\Lambda_0+I_0$ is of the form

$$\Lambda_0+I_0 = \begin{pmatrix} \frac{a}{2} & -\frac{b}{2} & 0 & 0 \\ \frac{b}{2} & \frac{a}{2} & 0 & 0 \\ 0 & 0 & \frac{e}{2} & \frac{e}{2} \\ 0 & 0 & \frac{-c}{2} & \frac{-c}{2} \end{pmatrix} \ . \tag{2.39}$$

Therefore, the components of $A(\tilde{X}_i)_x$ relative to the basis $X_1, X_2, X_3, X_4$, are given by the $i$th column of the matrix $a\delta_{s^{-1}}(\Lambda_0+I_0)$. A straightforward calculation using (2.30) and (2.39), and denoting the $a$th coordinate of $A(\tilde{X}_j)$ by $\tilde{A}\,^a_j$ $(a=1,2,3,4)$, leads to

$$\begin{aligned} &\tilde{A}\,^1_1 = \tfrac{1}{2}a(x_1^2-x_2^2-x_3^2+x_4^2) - b(x_4x_3-x_1x_2) \ , \\ &\tilde{A}\,^2_1 = -a(x_3x_4+x_1x_2) + \tfrac{1}{2}b(x_1^2-x_2^2+x_3^2-x_4^2) \ , \\ &\tilde{A}\,^3_1 = a(x_2x_4-x_1x_3) - b(x_2x_3+x_1x_4) \ , \\ &\tilde{A}\,^4_1 \equiv \tilde{B}_1 = 0 \ , \\ &\tilde{A}\,^1_2 = -\tfrac{1}{2}b(x_1^2-x_2^2-x_3^2+x_4^2) - a(x_3x_4-x_1x_2) \ , \\ &\tilde{A}\,^2_2 = b(x_3x_4+x_1x_2) + \tfrac{1}{2}a(x_1^2-x_2^2+x_3^2-x_4^2) \ , \\ &\tilde{A}\,^4_2 = -b(x_2x_4-x_1x_3) - a(x_2x_3+x_1x_4) \ , \\ &\tilde{A}\,^4_2 \equiv \tilde{B}_2 = 0 \ , \quad \tilde{A}\,^1_3 = e(x_2x_4+x_1x_3) \ , \\ &\tilde{A}\,^2_3 = -e(x_2x_3-x_1x_4) \ , \\ &\tilde{A}\,^3_3 = \tfrac{1}{2}e(x_1^2+x_2^2-x_3^2-x_4^2) \ , \quad \tilde{A}\,^4_3 \equiv \tilde{B}_3 = -\tfrac{1}{2}c \ . \end{aligned} \tag{2.40}$$

To complete our discussion on $S$-invariant connections for $S = \mathrm{SU}(2)\times\mathrm{U}(1)$, we evaluate the components of the gauge fields in the Euclidean coordinate basis $x_1, x_2, x_3, x_4$. To this end recall that $\tilde{X}_5 = x_1\partial_1 + x_2\partial_2 + x_3\partial_3 + x_4\partial_4$. Inverting this equation simultaneously with (2.6)–(2.8) one gets

$$\begin{aligned} \partial_1 &= 2(-x_4\tilde{X}_1 + x_3\tilde{X}_2 - x_2\tilde{X}_3 + x_1\tilde{X}_5) \ , \\ \partial_2 &= 2(-x_3\tilde{X}_1 - x_4\tilde{X}_2 + x_1\tilde{X}_3 + x_2\tilde{X}_5) \ , \\ \partial_3 &= 2(x_2\tilde{X}_1 - x_1\tilde{X}_2 - x_4\tilde{X}_3 + x_3\tilde{X}_5) \ , \\ \partial_4 &= 2(x_1\tilde{X}_1 + x_2\tilde{X}_2 + x_3\tilde{X}_3 + x_4\tilde{X}_5) \ . \end{aligned} \tag{2.41}$$

Note that

$$\tilde{A}_5 = A(\tilde{X}_5) = a\delta_{\underline{s}^{-1}}\Lambda(a\delta_{\underline{s}}X_5) - W(X_5) \ , \tag{2.42}$$

which, by (2.29) and (2.31) vanishes as required in our construction. Hence, letting $A(\partial_a) \equiv A_a^b X_b$ $(a,b = 1,2,3,4)$, we obtain

$$\begin{aligned} A_1 &= 2(-x_4\tilde{A}_1 + x_3\tilde{A}_2 - x_2\tilde{A}_3) \ , \\ A_2 &= 2(-x_3\tilde{A}_1 - x_4\tilde{A}_2 + x_1\tilde{A}_3) \ , \\ A_3 &= 2(x_2\tilde{A}_1 - x_1\tilde{A}_2 - x_4\tilde{A}_3) \ , \\ A_4 &= 2(x_1\tilde{A}_1 + x_2\tilde{A}_2 + x_3\tilde{A}_3) \ . \end{aligned} \tag{2.43}$$

Substituting the expressions given in (2.40) into (2.43) gives the components $A_a^b$ of the gauge fields in the Euclidean coordinate basis. Here we only remark the fact that these calculations readily show that linear expressions for $A_a^b$ in terms of the Euclidean coordinates $\{x_a\}$ are obtained if and only if $b=0$ and $a=e$. For simplicity we shall adopt this ansatz $(b=0, a=e)$ in the following sections, leaving extensions to the more general case to future work.

*S-symmetric Higgs fields.* We conclude this section with the corresponding analysis of the possible solutions for the Higgs fields which display $\mathrm{SU}(2)\times\mathrm{U}(1)$ symmetry. In this case the discussion will be facilitated considerably by the fact that a large portion of the work has already been done in paper I.

Indeed, we showed in I [cf. Eqs. (4.11) and (2.49)] that the Higgs fields $\Phi_A$ must satisfy the differential equation

$$\mathcal{L}_{\xi_m}\Phi_A = -\rho(X_m)_A{}^C\Phi_C, \quad m = 1,2,3,4 \ , \tag{2.44}$$

where the $\rho(X_m)$ are matrix representations of the generators of $\mathrm{SU}(2)\times\mathrm{U}(1)$, and $\xi_4 = \tilde{\xi}_3$. If we now note that (2.3)–(2.5) and (2.8) can be written in terms of the $\rho(X_m)$ matrices as

$$\begin{aligned} &\xi_1 = -x_A\rho(X_1)_A{}^B\partial_B, \quad \xi_2 = x_A\rho(X_2)_A{}^B\partial_B \ , \\ &\xi_3 = -x_A\rho(X_3)_A{}^B\partial_B, \quad \xi_4 = -x_A\rho(X_4)_A{}^B\partial_B \ , \end{aligned} \tag{2.45}$$

then (2.44) results in the set

$$\begin{aligned} x_A\rho(X_1)_A{}^B\partial_B\Phi_C &= \rho(X_1)_C{}^D\Phi_D \ , \\ x_A\rho(X_2)_A{}^B\partial_B\Phi_C &= -\rho(X_2)_C{}^D\Phi_D \ , \\ x_A\rho(X_3)_A{}^B\partial_B\Phi_C &= \rho(X_3)_C{}^D\Phi_D \ , \\ x_A\rho(X_4)_A{}^B\partial_B\Phi_C &= \rho(X_4)_C{}^D\Phi_D \ . \end{aligned} \tag{2.46}$$

If we require that $\Phi_A$ be nonsingular for all points in $S^3$, and therefore write $\Phi_A$ as a series in positive powers of $x_A$, then (2.46) implies that they have to be linear in the Euclidean coordinates $x_A$, and that they are of the form

$$\Phi_1 = x_1 d + x_2 f, \quad \Phi_2 = x_1 f - x_2 d \ ,$$
$$\Phi_3 = x_3 d + x_4 f, \quad \Phi_4 = x_3 f - x_4 d \ . \tag{2.47}$$

## III. SOLUTIONS TO THE EINSTEIN-YANG-MILLS-HIGGS FIELD EQUATIONS

In this section we shall derive the Einstein-Yang-Mills-Higgs field equations for our Lagrangian (1.1), and seek for solutions to this system in terms of the parameters of the metric (2.15) and those of the connections and the Higgs fields, as given by Eqs. (2.40) and (2.47), respectively. Since the calculations are simplified considerably when one uses the nonholonomic basis of right-invariant vector fields (2.6)–(2.8), we begin by obtaining the field equations in the language of forms (which is most suitable for that purpose).

Following the notation introduced in I, and resorting to the results of Bleecker[19] (cf. chapters, 5, 7, and 9), the Higgs and Yang-Mills field equations are given by

$$\delta^{\mathring{\omega}_1+\mathring{\omega}_2}\frac{\partial\mathcal{L}}{\partial(D^{\mathring{\omega}_1+\mathring{\omega}_2}\Phi^A)}+\frac{\partial\mathcal{L}}{\partial\Phi^A}=0 \ , \tag{3.1a}$$

$$(\mathscr{g}\,\mathrm{Tr})(\delta^{\omega_1}(\mathbf{F}_{(1)})-J^{\omega_1}(\Phi^A),\tau_1)=0 \ , \tag{3.1b}$$

$$(\mathscr{g}\,\mathrm{Tr})(\delta^{\omega_2}(\mathbf{F}_{(2)})-J^{\omega_2}(\Phi^A),\tau_2)=0 \ , \tag{3.1c}$$

where $\delta^{\omega}:\bar{\Lambda}^k(P_1\circ P_2, V)\to\bar{\Lambda}^{k-1}(P_1\circ P_2,V)$ is the covariant codifferential relative to the connection $\omega$, $J^{\omega_i}(\Phi^A)\in\bar{\Lambda}^1(P_i,\mathcal{G}_i)$ is the current defined by

$$(\mathscr{g}\,\mathrm{Tr})_{(p_i)}\left[\frac{\delta\mathcal{L}}{\delta D\Phi},g_{(i)}\tau_i\cdot\Phi\right]=(\mathscr{g}k)_{(p_i)}(J_{(p_i)}{}^{\omega_i}\cdot\tau_i) \tag{3.2}$$

for all $\tau_i\in\bar{\Lambda}^1(P_i,\mathcal{G}_i)_{(p_i)}$, and $\mathbf{F}_{(i)}$ are the field strengths, associated with $\omega_i$, i.e., $\mathbf{F}_{(1)}=(1/g')d(\sigma_2^*\omega_2)=d\mathbf{B}$,

$$\mathbf{F}_{(2)}=(1/g)(d(\sigma_1^*\omega_1)+\tfrac{1}{2}[\sigma_1^*\omega_1,\sigma_1^*\omega_1])$$
$$=d\mathbf{W}+\tfrac{1}{2}g[\mathbf{W},\mathbf{W}] \ .$$

Substitute the Lagrangian density from (1.1) into (3.1a) to get

$$\delta^{\mathring{\omega}_1+\mathring{\omega}_2}D^{\mathring{\omega}_1+\mathring{\omega}_2}\Phi^A+m^2\Phi_A-\lambda(\Phi_B\Phi^B)\Phi_A+\tfrac{7}{6}\lambda\underline{R}\Phi_A$$
$$=0 \ , \tag{3.3}$$

and recall that for any $\phi\in\bar{\Lambda}^k(P,V)$, the covariant codifferential is defined by

$$\delta^{\mathring{\omega}_1+\mathring{\omega}_2}(\phi)=-(-1)^{\mathscr{g}}(-1)^{n(k+1)}\bar{*}D^{\mathring{\omega}_1+\mathring{\omega}_2}(\bar{*}\phi) \ , \tag{3.4}$$

where $(-1)^{\mathscr{g}}$=sign of det $\mathscr{g}$,$n=\dim M$, and the star operator "$\bar{*}$" which maps $V$-valued $k$-forms onto $V$-valued $(n-k)$-forms vanishing on vertical vectors in the tangent space $T_{(p_1,p_2)}P_1\circ P_2$, is a unique extension of the usual Hodge star operator acting on $\mathcal{R}$-valued forms. In our case $(-1)^{\mathscr{g}}=1$ [signature of base-space metric is $(+,-,\ldots,-)$] and $n=\dim M=7$, so that $\delta^{\mathring{\omega}_1+\mathring{\omega}_2}(D^{\mathring{\omega}_1+\mathring{\omega}_2}\Phi_A)=-\bar{*}D^{\mathring{\omega}_1+\mathring{\omega}_2}(\bar{*}D^{\mathring{\omega}_1+\mathring{\omega}_2}\Phi_A)$. Using this in Eq. (3.3) and pulling down to $M$ yields, after some manipulations,

$$\delta\, d\Phi_A+\delta(g\mathbf{W}^\alpha\rho(X_\alpha)_A{}^C\Phi_C+g'\mathbf{B}\rho(X_4)_A{}^C\Phi_C)-{*}\{[g\mathbf{W}^\alpha\rho(X_\alpha)_A{}^C+g'\mathbf{B}\rho(\mathbf{X}_4)_A{}^C]\wedge{*}d\Phi^C\}$$
$$-{*}\{[g\mathbf{W}^\alpha\rho(X_\alpha)_A{}^C+g'\mathbf{B}\rho(X_4)_A{}^C]\wedge{*}[gW^\beta\rho(X_\beta)_C{}^D\Phi_D+g'\mathbf{B}\rho(X_4)_C{}^D\Phi_D]\}$$
$$+m^2\Phi_A-\lambda(\Phi_B\Phi^B)\Phi_A+\tfrac{7}{6}\lambda\underline{R}\Phi_A=0 \ , \tag{3.5}$$

where the matrices $\rho(X_\alpha)$ and $\rho(X_4)$ are the same as those defined in Eqs. (2.49) of paper I.

To calculate the currents in (3.1b) and (3.1c) we make use of (3.2). We have

$$(\mathscr{g}k)_{p_1}(J^{\omega_1}_{(p_1)}(\Phi),\tau_1)=(\mathscr{g}\,\mathrm{Tr})_{p_1}(\mathbf{D}\Phi,g\tau_1\cdot\Phi) \ , \tag{3.6}$$

and writing

$$\tau_1=(\tau_1)^\gamma\rho(X_\gamma), \quad (\tau_1)^\gamma=\mathrm{Tr}[\tau_1\rho(X^\gamma)] \ , \tag{3.7}$$

where $\rho(X^\gamma)\equiv\rho(X_\gamma)^{-1}=-\rho(X_\gamma)$, we get $\tau_1\cdot\Phi=(\tau_1)^\gamma\rho(X_\gamma)_A{}^C\Phi_C$. Consequently, (3.6) may be reexpressed as

$$(\mathscr{g}k)_{p_1}(J^{\omega_1}_{(p_1)}(\Phi),\tau_1)$$
$$=g\mathscr{g}^{ij}[D_i\Phi^A\rho(X_\alpha)_A{}^C\Phi_C(\tau_1)^\alpha_j]$$
$$=g\ \mathrm{Tr}[\mathscr{g}(D\Phi^A\rho(X^\alpha)_A{}^C\Phi_C\rho(X_\alpha),\tau_1)] \ . \tag{3.8}$$

Similarly, for $J^{\omega_2}$ we get

$$(\mathscr{g}k)_{p_2}(J^{\omega_2}_{(p_2)}(\Phi),\tau_2)$$
$$=g'\mathrm{Tr}[\mathscr{g}(D\Phi^A\rho(X^4)_A{}^C\Phi_C\rho(X_4),\tau_2)] \ . \tag{3.9}$$

Substitution of (3.8) and (3.9) into (3.1b) and (3.1c), respectively, and noting that $\tau_1$ and $\tau_2$ are arbitrary, yields

$$\delta^{\omega_1}(\mathbf{F}_{(1)})^\alpha=\delta\mathbf{F}^\alpha_{(1)}+g\epsilon_{\alpha\beta\gamma}{*}(\mathbf{W}^\beta\wedge{*}\mathbf{F}^\gamma_{(1)})$$
$$=-g(D\Phi^A)\rho(X_\alpha)_A{}^C\Phi_C \ , \tag{3.10}$$

and

$$\delta^{\omega_2}\mathbf{F}_{(2)}=\delta\, d\mathbf{B}=-g'(D\Phi^A)\rho(X_4)_A{}^C\Phi_C \ . \tag{3.11}$$

Equations (3.5), (3.10), and (3.11) are the coordinate-free expressions of the field equations for the Higgs and Yang-Mills fields. They can be evaluated in terms of the nonholonomic basis of right invariant $\tilde{\xi}_i$, the calculation being fairly straightforward. Equation (3.5) then becomes

$$-\mathscr{g}^{ij}\tilde{\xi}_i[\tilde{D}_j\Phi_A]-\mathscr{g}^{ij}(g\tilde{W}_i^\alpha\rho(X_\alpha)_A{}^C\tilde{\xi}_j[\Phi_C]+g'\tilde{B}_i\rho(X_4)_A{}^C\tilde{\xi}_j(\Phi_C))$$
$$-\mathscr{g}^{ij}\{[g\tilde{W}_i^\alpha\rho(X_\alpha)_A{}^C+g'\tilde{B}_i\rho(X_4)_A{}^C][g\tilde{W}_j^\beta\rho(X_\beta)_C{}^D\Phi_D+g'\tilde{B}_j\rho(X_4)_C{}^D\Phi_D]\}$$
$$+m^2\Phi_A-\lambda(\Phi_B\Phi^B)\Phi_A+\tfrac{7}{6}\lambda\underline{R}\Phi_A=0\ , \qquad (3.12)$$

where $\tilde{W}_i^\alpha\equiv W^\alpha(\tilde{\xi}_i)$, and $\tilde{B}_i=\mathbf{B}(\tilde{\xi}_i)$. On the other hand, after pulling down to $M$, Eqs. (3.10) and (3.11) result in

$$\frac{|\mathscr{g}|}{2}\Sigma(\tilde{\xi}_l[\tilde{F}^{\alpha ij}]\epsilon_{ijk}\mathscr{g}^{lr}\mathscr{g}^{ks}+g\epsilon_{\alpha\beta\gamma}\tilde{F}^{\gamma ij}\tilde{W}_l^\beta\epsilon_{ijk}\mathscr{g}^{lr}\mathscr{g}^{ks}+\tilde{F}^{\alpha ij}\mathscr{g}^{ir}\mathscr{g}^{js})\epsilon_{rst}=-g(\tilde{D}_t\Phi^A)\rho(X_\alpha)_A{}^C\Phi_c\ , \qquad (3.13)$$

and

$$\frac{|\mathscr{g}|}{2}\Sigma(\tilde{\xi}_l[\tilde{F}^{ij}]\epsilon_{ijk}\mathscr{g}^{lr}\mathscr{g}^{ks}+\tilde{F}^{ij}\mathscr{g}^{ir}\mathscr{g}^{js})\epsilon_{rst}$$
$$=-g'(\tilde{D}_l\Phi^A)\rho(X_4)_A{}^C\Phi_C\ . \qquad (3.14)$$

In order to derive the Einstein field equations for the Lagrangian (1.1) we note that the topology of the base space is $\mathcal{M}^4\times S^3$ with metric given by

$$ds^2=\eta_{\mu\nu}(x)dx^\mu dx^\nu-(\rho_1/4)(\tilde{\sigma}^1\otimes\tilde{\sigma}^1)$$
$$-(\rho_1/4)(\tilde{\sigma}^2\otimes\tilde{\sigma}^2)-(\rho_2/4)(\tilde{\sigma}^3\otimes\tilde{\sigma}^3)\ .$$

The space-time components of the Riemann tensor therefore vanish, as well as those with mixed (space-time and internal) indices. The calculation of the components relative to $S^3$ is greatly simplified if we recall that, in a three-dimensional space,

$$\underline{R}_{ijkl}=\underline{R}_{ik}\mathscr{g}_{jl}-\underline{R}_{il}\mathscr{g}_{jk}+\underline{R}_{jl}\mathscr{g}_{ik}-\underline{R}_{jk}\mathscr{g}_{il}$$
$$+\tfrac{1}{2}\underline{R}(\mathscr{g}_{il}\mathscr{g}_{jk}-\mathscr{g}_{ik}\mathscr{g}_{jl})\ . \qquad (3.15)$$

Consequently, $\underline{R}_{ijkl}\underline{R}^{ijkl}=4\underline{R}_{kl}\underline{R}^{kl}-\underline{R}^2$, and (1.1) reduces to

$$I=\frac{1}{V_I}\int\left[-\kappa\underline{R}+\alpha_8\underline{R}^2+\alpha_9\underline{R}^{ij}\underline{R}_{ij}-\tfrac{1}{4}\tilde{F}^{\alpha ij}\tilde{F}_\alpha{}^{ij}+\tfrac{1}{2}(\tilde{D}_i\Phi_A)(\tilde{D}^i\Phi^A)+\frac{m^2}{2}\Phi_A\Phi^A-\frac{\lambda}{4}(\Phi_A\Phi^A)^2+\tfrac{7}{12}\lambda\underline{R}(\Phi_A\Phi^A)-\kappa\Lambda\right]\mu_{\mathscr{g}}\ , \qquad (3.16)$$

where we have made use of the relation $\alpha_6=-\frac{343}{144}\lambda-7\alpha_1-\frac{1}{3}\alpha_2$ [cf. Eq. (3.44) in paper I], and defined the new parameters $\alpha_8$ and $\alpha_9$ by

$$\alpha_8\overset{\text{def}}{=}\alpha_1-\alpha_2,\quad \alpha_9\overset{\text{def}}{=}4\alpha_2-\tfrac{343}{144}\lambda-7\alpha_1-\tfrac{1}{3}\alpha_2\ . \qquad (3.17)$$

Varying (3.16) with respect to $\mathscr{g}_{ik}$ one gets

$$-\kappa(\tfrac{1}{2}\underline{R}\mathscr{g}^{ms}-\underline{R}^{ms})+\alpha_8(\tfrac{1}{2}\underline{R}^2\mathscr{g}^{ms}-2\underline{R}\underline{R}^{ms}+2\underline{R}^{|m|s}-2\mathscr{g}^{ms}\underline{R}_{|h}{}^{|h})$$
$$+\alpha_9(-\tfrac{1}{2}\underline{R}_{ik}\underline{R}^{ik}\mathscr{g}^{ms}+\underline{R}^s{}_k\underline{R}^{km}-\tfrac{3}{2}\underline{R}\underline{R}^{sm}+\tfrac{1}{2}\underline{R}^2\mathscr{g}^{sm}-\underline{R}^{ms}{}_{|i}{}^{|i}+\underline{R}^s{}_k{}^{|m|k}-\mathscr{g}^{ms}\underline{R}^k{}_{h|k}{}^{|h}+\underline{R}^{km}{}_{|k}{}^{|s})$$
$$+\tfrac{7}{12}\lambda(\Phi_a\Phi^A)(-\underline{R}^{ms}+\tfrac{1}{2}\underline{R}\mathscr{g}^{ms})-\tfrac{1}{2}\kappa\Lambda\mathscr{g}^{ms}$$
$$+\left[-\tfrac{1}{8}\tilde{F}^\alpha{}_{ij}\tilde{F}_\alpha{}^{ij}-\tfrac{1}{8}\tilde{F}_{ij}\tilde{F}^{ij}+\tfrac{1}{4}(\tilde{D}_i\Phi_A)(\tilde{D}^i\Phi^A)+\frac{m^2}{4}\Phi_A\Phi^A-\frac{\lambda}{8}(\Phi_A\Phi^A)^2\right]\mathscr{g}^{ms}$$
$$+\tfrac{1}{2}\tilde{F}^{am}{}_j\tilde{F}_\alpha{}^{sj}+\tfrac{1}{2}\tilde{F}^m{}_j\tilde{F}^{sj}-\tfrac{1}{2}(\tilde{D}^m\Phi_A)(\hat{D}^s\Phi^A)=0\ . \qquad (3.18)$$

Relative to the basis $\tilde{\xi}_1,\tilde{\xi}_2,\tilde{\xi}_3$, the internal space metric is simply

$$\mathscr{g}_{ij}\equiv\mathscr{g}(\tilde{\xi}_i,\tilde{\xi}_j)=-\frac{1}{4}\begin{bmatrix}\rho_1&0&0\\0&\rho_1&0\\0&0&\rho_2\end{bmatrix}\ , \qquad (3.19)$$

and the only nonvanishing components of the Riemann tensor are

$$\underline{R}^2{}_{112}=-\left[1-\frac{3}{4}\frac{\rho_2}{\rho_1}\right],\quad \underline{R}^1{}_{212}=\left[1-\frac{3}{4}\frac{\rho_2}{\rho_1}\right],$$
$$\underline{R}^3{}_{113}=-\frac{1}{4}\frac{\rho_2}{\rho_1},\quad \underline{R}^1{}_{313}=\frac{1}{4}\left[\frac{\rho_2}{\rho_1}\right]^2, \qquad (3.20)$$
$$\underline{R}^3{}_{223}=-\frac{1}{4}\frac{\rho_2}{\rho_1},\quad \underline{R}^2{}_{323}=\frac{1}{4}\left[\frac{\rho_2}{\rho_1}\right]^2.$$

It follows, in turn, that the Ricci tensor components and the curvature scalar are given by

$$\underline{R}_{11}=\underline{R}_{22}=1-\frac{1}{2}\frac{\rho_2}{\rho_1},\quad \underline{R}_{33}=\frac{1}{2}\left[\frac{\rho_2}{\rho_1}\right]^2,$$
$$\underline{R}_{12}=\underline{R}_{13}=\underline{R}_{23}=0\ , \qquad (3.21)$$

and

$$\underline{R}=-\frac{2}{\rho_1}\left[4-\frac{\rho_2}{\rho_1}\right]\ . \qquad (3.22)$$

Furthermore, in this basis

$$\tilde{F}_{ij}=\tilde{\xi}_i[\tilde{B}_j]-\tilde{\xi}_j[\tilde{B}_i]-B([\tilde{\xi}_i,\tilde{\xi}_j]) , \tag{3.23}$$

$$\tilde{F}^{\alpha}{}_{ij}=\tilde{\xi}_i[\tilde{W}^{\alpha}_j]-\tilde{\xi}_j[\tilde{W}^{\alpha}_i]+\epsilon_{ijk}\tilde{W}^{\alpha}_k+g\epsilon_{\alpha\beta\gamma}\tilde{W}^{\beta}_i\tilde{W}^{\gamma}_j , \tag{3.24}$$

and

$$\tilde{D}_i\Phi_A=\tilde{\xi}_i[\Phi_A]+g\tilde{W}^{\alpha}_i\rho(X_\alpha)_A{}^C\Phi_C+g'\tilde{B}_i\rho(X_4)_A{}^C\Phi_C . \tag{3.25}$$

We can now substitute, in the expressions above, the $S$-invariant solutions for the connection and Higgs fields as given in Eqs. (2.40) (identifying $\tilde{A}^{\alpha}_i$ with $\tilde{W}^{\alpha}_i$) and (2.47). We get

$$\tilde{F}_{12}=-\frac{c}{2},\quad \tilde{F}_{13}=\tilde{F}_{23}=0 ; \tag{3.26}$$

$$\begin{aligned}
&\tilde{F}^{1}{}_{12}=-a(1-\tfrac{1}{2}ag)(x_1x_3+x_2x_4),\\
&\tilde{F}^{1}{}_{13}=a(1-\tfrac{1}{2}ag)(x_1x_2-x_3x_4) ,\\
&\tilde{F}^{1}{}_{23}=\frac{a}{2}(1-\tfrac{1}{2}ag)(x_2^2+x_3^2-x_1^2-x_4^2) ,\\
&\tilde{F}^{2}{}_{12}=-a(1-\tfrac{1}{2}ag)(x_1x_4-x_2x_3) ,\\
&\tilde{F}^{2}{}_{13}-\frac{a}{2}(1-\tfrac{1}{2}ag)(x_2^2+x_4^2-x_1^2-x_3^2) ,\\
&\tilde{F}^{2}{}_{23}=a(1-\tfrac{1}{2}ag)(x_1x_2+x_3x_4) ,\\
&\tilde{F}^{3}{}_{12}=-\frac{a}{2}(1-\tfrac{1}{2}ag)(x_1^2+x_2^2-x_3^2-x_4^2) ,\\
&\tilde{F}^{3}{}_{13}=-a(1-\tfrac{1}{2}ag)(x_1x_4+x_2x_3) ,\\
&\tilde{F}^{3}{}_{23}=-a(1-\tfrac{1}{2}ag)(x_2x_4-x_1x_3) ;
\end{aligned} \tag{3.27}$$

$$\begin{aligned}
&\tilde{D}_1\Phi_1=-\tilde{D}_2\Phi_2=-\tfrac{1}{2}(1-\tfrac{1}{2}ag)(x_4d+x_3f) ,\\
&\tilde{D}_2\Phi_1=\tilde{D}_1\Phi_2=\tfrac{1}{2}(1-\tfrac{1}{2}ag)(x_3d-x_4f) ,\\
&\tilde{D}_3\Phi_1=\tfrac{1}{2}(1-\tfrac{1}{2}ag+\tfrac{1}{2}cg')(x_1f-x_2d) ,\\
&\tilde{D}_3\Phi_2=-\tfrac{1}{2}(1-\tfrac{1}{2}ag+\tfrac{1}{2}cg')(x_2f+x_1d) ,\\
&\tilde{D}_1\Phi_3=\tfrac{1}{2}(1-\tfrac{1}{2}ag)(x_2d+x_1f) ,\\
&\tilde{D}_2\Phi_3=\tilde{D}_1\Phi_4=-\tfrac{1}{2}(1-\tfrac{1}{2}ag)(x_1d-x_2f) ,\\
&\tilde{D}_3\Phi_3=\tfrac{1}{2}(1-\tfrac{1}{2}ag+\tfrac{1}{2}cg')(x_3f-x_4d) ,\\
&\tilde{D}_2\Phi_4=-\tfrac{1}{2}(1-\tfrac{1}{2}ag)(x_1f+x_2d) ,\\
&\tilde{D}_3\Phi_4=-\tfrac{1}{2}(1-\tfrac{1}{2}ag+\tfrac{1}{2}cg')(x_4f+x_3d) .
\end{aligned} \tag{3.28}$$

Equations (3.19), (3.26), (3.27) and (3.28) lead in turn to

$$\tilde{F}_{ij}\tilde{F}^{ij}=\frac{8c^2}{\rho_1^2} , \tag{3.29}$$

$$\tilde{F}^{\alpha}{}_{ij}\tilde{F}_{\alpha}{}^{ij}=\frac{8a^2}{\rho_1^2}\left[1+2\frac{\rho_1}{\rho_2}\right](1-\tfrac{1}{2}ag)^2 , \tag{3.30}$$

$$(\tilde{D}_i\Phi_A)(\tilde{D}^i\Phi^A)=-\frac{1}{\rho_1}(d^2+f^2)\times\left[2(1-\tfrac{1}{2}ag)^2+\frac{\rho_1}{\rho_2}(1-\tfrac{1}{2}ag+\tfrac{1}{2}cg')^2\right] . \tag{3.31}$$

The field equations (3.12)–(3.14) and (3.18) may now be solved using the results above and the expressions for $\tilde{W}^{\alpha}_i$, $\tilde{B}_i$, and $\Phi_A$. The four partial differential equations (3.12) yield the unique algebraic condition

$$m^2-\lambda(d^2+f^2)=\frac{1}{\rho_1}\left[\epsilon(1-\tfrac{1}{2}ag+\tfrac{1}{2}cg')^2+2(1-\tfrac{1}{2}ag)^2+\tfrac{7}{3}\lambda\left[4-\frac{1}{\epsilon}\right]\right] , \tag{3.32}$$

where

$$\epsilon=\frac{\rho_1}{\rho_2} . \tag{3.33}$$

The three differential equations (3.13) give rise to two algebraic conditions relating the parameters of the gauge and Higgs fields as follows

$$(f^2+d^2)=\frac{8}{\rho_1}\frac{a}{g}[1-\tfrac{1}{2}ag(1+\epsilon)] , \tag{3.34}$$

$$\frac{8}{\rho_1}a(1-\tfrac{1}{2}ag)(1-\tfrac{1}{2}ag-\frac{1}{2\epsilon})=\tfrac{1}{2}g(1-\tfrac{1}{2}ag+\tfrac{1}{2}cg')(f^2+d^2) . \tag{3.35}$$

Finally, Eq. (3.14) leads to the unique condition

$$\frac{c}{\rho_1}=-\tfrac{1}{8}g'\epsilon(1-\tfrac{1}{2}ag+\tfrac{1}{2}cg')(d^2+f^2) . \tag{3.36}$$

To obtain solutions to the Einstein equations we first note that, due to the Cartesian product nature of our base space $M$, Eq. (3.18) breaks up naturally into two sets. One of these comes from taking the free indices as *space-time indices,* and results in the condition

$$\frac{\kappa}{\rho_1}\left[4-\frac{1}{\epsilon}\right]-\frac{7}{12}\frac{\lambda}{\rho_1}\left[4-\frac{1}{\epsilon}\right](d^2+f^2)-\tfrac{1}{2}\kappa\Lambda-\frac{a^2}{\rho_1^2}(1-\tfrac{1}{2}ag)^2(1+2\epsilon)-\frac{c^2}{\rho_1^2}$$
$$-\frac{1}{4\rho_1}(d^2+f^2)[2(1-\tfrac{1}{2}ag)^2+\epsilon(1-\tfrac{1}{2}ag+\tfrac{1}{2}cg')^2]+\tfrac{1}{4}m^2(d^2+f^2)-\tfrac{1}{8}\lambda(d^2+f^2)^2$$
$$+\frac{2}{\rho_1^2}\left[\alpha_8\left[4-\frac{1}{\epsilon}\right]^2+2\alpha_9\left[4-\frac{1}{\epsilon^2}\right]\right]=0 . \tag{3.37}$$

The remaining set of equations comes from taking the free indices as *internal* coordinates; after substituting from (3.19)–(3.31) they can be shown to reduce to the (final) algebraic conditions

$$-\frac{\kappa}{\epsilon}+\frac{7}{12}\frac{\lambda}{\epsilon}(d^2+f^2)+\tfrac{1}{2}\kappa\rho_1\Lambda+\left[\frac{a^2}{\rho_1}(1-\tfrac{1}{2}ag)^2(1+2\epsilon)+\frac{c^2}{\rho_1}+\tfrac{1}{4}(d^2+f^2)[2(1-\tfrac{1}{2}ag)^2+\epsilon(1-\tfrac{1}{2}ag+\tfrac{1}{2}cg')^2]-\tfrac{1}{4}m^2\rho_1(d^2+f^2)+\tfrac{1}{8}\lambda\rho_1(d^2+f^2)^2\right]$$
$$-\frac{2}{\rho_1}a^2(1-\tfrac{1}{2}ag)^2(1+\epsilon)-\frac{2c^2}{\rho_1}-\tfrac{1}{2}(1-\tfrac{1}{2}ag)^2(d^2+f^2)-\frac{2}{\rho_1}\left[\alpha_8\left[4-\frac{1}{\epsilon}\right]^2+2\alpha_9\left[4-\frac{1}{\epsilon^2}\right]\right]$$
$$+\frac{8}{\rho_1}\left[4-\frac{1}{\epsilon}\right]\left[1-\frac{1}{2\epsilon}\right](2\alpha_8+\tfrac{3}{2}\alpha_9)-\frac{16}{\rho_1}\alpha_9\left[1-\frac{1}{2\epsilon}\right]^2=0\ , \tag{3.38}$$

$$-\kappa(4\epsilon-3)+\tfrac{7}{12}\lambda(4\epsilon-3)(d^2+f^2)+\tfrac{1}{2}\kappa\Lambda\rho_1\epsilon$$
$$+\rho_1\epsilon\left[\frac{a^2}{\rho_1^2}(1-\tfrac{1}{2}ag)^2(1+2\epsilon)+\frac{c^2}{\rho_1^2}+\frac{1}{4\rho_1}(d^2+f^2)[2(1-\tfrac{1}{2}ag)^2+\epsilon(1-\tfrac{1}{2}ag+\tfrac{1}{2}cg')^2]-\tfrac{1}{4}m^2(d^2+f^2)+\tfrac{1}{8}\lambda(d^2+f^2)^2\right]-\frac{4}{\rho_1}a^2\epsilon^2(1-\tfrac{1}{2}ag)^2-\tfrac{1}{2}\epsilon^2(1-\tfrac{1}{2}ag+\tfrac{1}{2}cg')^2(d^2+f^2)$$
$$-\frac{2}{\rho_1}\epsilon\left[\alpha_8\left[4-\frac{1}{\epsilon}\right]^2+2\alpha_9\left[4-\frac{1}{\epsilon^2}\right]\right]+\frac{4}{\rho_1}\left[4-\frac{1}{\epsilon}\right](2\alpha_8+\tfrac{3}{2}\alpha_9)-\frac{4}{\rho_1}\alpha_9\frac{1}{\epsilon}=0\ . \tag{3.39}$$

The Einstein-Yang-Mills-Higgs system of field equations has thus been reduced to seven algebraic conditions which relate the 13 parameters of the theory. At first sight, this apparently large number of free parameters would suggest an infinite set of possible solutions; however, as will be shown in the next section, because of the high nonlinearity of the system the admissible solutions are in fact extremely restricted, and this fact allows for some very specific predictions on the values of some of these parameters, including the coupling constants.

## IV. SPONTANEOUS COMPACTIFICATION AND COUPLING CONSTANTS

In the previous section the Euler-Lagrange equations of our model were solved in terms of a system of seven highly nonlinear coupled algebraic equations. Here we wish to show that the system indeed admits solutions for which the dimensions of the internal space spontaneously compactify in a satisfactory manner.

Finding solutions for the algebraic system (3.32)–(3.39) is a much more difficult procedure than those required in Refs. 7 and 20, due to the fact that the symmetry group considered here on the base manifold is more complicated than the ones used in the above-mentioned papers, resulting in a greater number of parameters and equations. Furthermore, as will be noted later on, quadratic terms in the Lagrangian are essential for the existence of nontrivial solutions, and thus must be kept. However, even though he system of seven coupled equations appears to be analytically untractable, one can unfold it sufficiently to arrive at expressions which lend themselves to numerical analysis, from where explicit values for physically interesting parameters such as $g/g'$ can be predicted at energy scales of the order of the Planck energy.

To this end note first that Eq. (3.34) may be written as

$$f^2+d^2=\frac{4}{g^2\rho_1(1+\epsilon)}\{1-[1-ag(1+\epsilon)]^2\}\ . \tag{4.1}$$

This immediately leads to the condition

$$|1-ag(1+\epsilon)|\le 1\ , \tag{4.2}$$

from where it clearly follows that $a\ge 0$, and (3.34) then implies

$$0\le ag\le\frac{2}{1+\epsilon}\ . \tag{4.3}$$

On the other hand, note that assuming $c>0$ in (3.36) gives $(eg'/8)(1-\frac{1}{2}ag+\frac{1}{2}cg')<0$, which contradicts (4.3). Hence,

$$0\ge\tfrac{1}{2}cg'\ge -1\ . \tag{4.4}$$

Next substitute (3.36) into the right-hand side (RHS) of (3.35) to get

$$a(1-\tfrac{1}{2}ag)[1-2\epsilon(1-\tfrac{1}{2}ag)]=c\frac{g}{g'}\ ; \tag{4.5}$$

alternatively, substituting (3.34) into (3.35) yields

$$a(1-\tfrac{1}{2}ag)(1+\tfrac{1}{2}\epsilon ag)(1-\epsilon)=-\tfrac{1}{16}\epsilon\rho_1 cgg'(f^2+d^2)\ . \tag{4.6}$$

Equations (4.5) and (4.3) together with $c\le 0$ imply

$1 \le 2\epsilon(1-\frac{1}{2}ag) \le 2\epsilon$, with (4.6) implies $\epsilon \le 1$. Consequently,

$$\tfrac{1}{2} \le \epsilon \le 1 \ . \tag{4.7}$$

So far we have only made use of the Yang-Mills equations to obtain constraints on the range of some of our parameters. We can make further progress by resorting to the Einstein equations. Multiplying (3.37) by $\rho_1$ and adding to (3.38) results in

$$\kappa\left[2-\frac{1}{\epsilon}\right]-\tfrac{7}{12}\lambda\left[2-\frac{1}{\epsilon}\right](d^2+f^2)-\frac{a^2}{\rho_1}(1-\tfrac{1}{2}ag)^2(1+\epsilon)-\frac{c^2}{\rho_1}-\tfrac{1}{4}(1-\tfrac{1}{2}ag)^2(f^2+d^2)$$
$$+\frac{2}{\rho_1}\left[4-\frac{1}{\epsilon}\right]\left[2-\frac{1}{\epsilon}\right](2\alpha_8+\tfrac{3}{2}\alpha_9)-\frac{2}{\rho_1}\alpha_9\left[2-\frac{1}{\epsilon}\right]^2=0 \ . \tag{4.8}$$

Similarly, multiplying (3.37) by $\epsilon\rho_1$ and adding to (3.39) gives

$$\kappa-\tfrac{7}{12}\lambda(d^2+f^2)-\frac{2\epsilon^2a^2}{\rho_1}(1-\tfrac{1}{2}ag)^2-\tfrac{1}{4}\epsilon^2(1-\tfrac{1}{2}ag+\tfrac{1}{2}cg')^2(f^2+d^2)+\frac{2}{\rho_1}\left[4-\frac{1}{\epsilon}\right](2\alpha_8+\tfrac{3}{2}\alpha_9)-\frac{2}{\rho_1}\frac{\alpha_9}{\epsilon}=0 \ . \tag{4.9}$$

The occurrence of the cosmological constant has been eliminated in (4.8) and (4.9). Moreover, by multiplying (4.9) by $(2-1/\epsilon)$, substracting from (4.8), and making use of (3.34), (3.36), and (4.5), we can also remove the dependence on $\kappa$, $\lambda$, and $(f^2+d^2)$ and arrive at

$$\frac{8a}{g}(1-\tfrac{1}{2}ag)^2(1-\epsilon)+2c^2-4\alpha_9\frac{1}{\epsilon}\left[2-\frac{1}{\epsilon}\right]\left[1-\frac{1}{\epsilon}\right]=0 \ . \tag{4.10}$$

One more relation comes from combining (3.34) and (3.36):

$$|c|\left[\frac{g}{g'}+\tfrac{1}{2}\epsilon ag'[1-\tfrac{1}{2}ag(1+\epsilon)]\right]-\epsilon a(1-\tfrac{1}{2}ag)[1-\tfrac{1}{2}ag(1+\epsilon)]=0 \ . \tag{4.11}$$

Defining

$$A \overset{\text{def}}{=} ag, \quad C \overset{\text{def}}{=} |c|g, \quad G \overset{\text{def}}{=} \frac{g}{g'} \ , \tag{4.12}$$

Eqs. (4.5), (4.10), and (4.11) may be rewritten as

$$-2\epsilon A(1-\tfrac{1}{2}A)^2+A(1-\tfrac{1}{2}A)+CG=0 \ , \tag{4.13}$$

$$8A(1-\tfrac{1}{2}A)^2(1-\epsilon)+2C^2-4(\alpha_9g^2)\frac{1}{\epsilon}\left[2-\frac{1}{\epsilon}\right]\left[1-\frac{1}{\epsilon}\right]=0 \ , \tag{4.14}$$

$$CG+\frac{\epsilon C}{2G}A[1-\tfrac{1}{2}A(1+\epsilon)]-\epsilon A(1-\tfrac{1}{2}A)[1-\tfrac{1}{2}A(1+\epsilon)]=0 \ , \tag{4.15}$$

and substituting $C$ from (4.13) into (4.14)–(4.15) yields finally

$$A(1-\tfrac{1}{2}A)^2\{8(1-\epsilon)+\frac{2A}{G^2}[2\epsilon(1-\tfrac{1}{2}A)-1]^2\}-4\alpha_9g^2\frac{1}{\epsilon}\left[2-\frac{1}{\epsilon}\right]\left[\frac{1}{\epsilon}\right]=0 \ , \tag{4.16}$$

and

$$-A(1-\epsilon)(1+\tfrac{1}{2}\epsilon A)+\frac{\epsilon}{2G^2}A^2[2\epsilon(1-\tfrac{1}{2}A)-1][1-\tfrac{1}{2}A(1+\epsilon)]=0 \ . \tag{4.17}$$

So far we have treated $\epsilon$ and $G$ as independent parameters. However, in the original five-dimensional Kaluza-Klein theory the coupling constant is quantized according to

$$e=\frac{1}{R}\sqrt{16\pi G_N} \ , \tag{4.18}$$

where $G_N$ is here the universal gravitational constant, and $2\pi R$ is the circumference of the compact fifth dimension. The dynamical mechanism responsible for spontaneous compactification thus imposes a relation between the characteristic length of the internal space and the electric charge. One should expect that in $n$-dimensional Kaluza-Klein theories spontaneous compactification lead also to similar relations between the coupling constants and the characteristic lengths of the internal space. In fact, Weinberg[21] has shown that for semisimple groups the coupling constants are given by

$$g_i=\frac{2\pi\sqrt{16\pi G_N}}{N_i}\frac{1}{\sqrt{s_i^2}} \ , \tag{4.19}$$

where $s_i$ is the circumference of the compactified dimenson and $N_i$ is the winding number associated to the symmetry group. When U(1) subgroups of the symmetry group are present, $g_i$ is obtained via a similar calculation but considering a scalar field in the $4+n$ dimensions. The important fact here is that the coupling constant goes as the inverse of the radius of compactification, so that if we want the relative strengths between the SU(2) and U(1) forces to be reflected in the topology of the compactified sector of the base manifold in our model, it seems natural to set

$$\epsilon=\frac{1}{G^2} \tag{4.20}$$

in agreement with the philosophy behind Kaluza-Klein theories. [Recall that $\epsilon=\rho_1/\rho_2$, and $\rho_i$ $(i=1,2)$ represent the square of the radii of compactification. Winding numbers and other numerical constants may be absorbed in the definition of $\rho_i$]. With this ansatz not only do the Yang-Mills and Higgs fields have a geometric origin in our theory, but so does the relative strength $G$ of the gauge fields, as it is determined, via (4.20), by the "eccentricity" $\epsilon$ of the hyperellipsoid into which the internal space compactifies. Equations (4.16)–(4.17) now read

$$A(1-\tfrac{1}{2}A)^2\{8(1-\epsilon)+2A\epsilon[2\epsilon(1-\tfrac{1}{2}A)-1]^2\} -4\alpha_9 g^2\frac{1}{\epsilon}\left[2-\frac{1}{\epsilon}\right]\left[1-\frac{1}{\epsilon}\right]=0\,, \tag{4.21}$$

and

$$-A(1-\epsilon)(1+\tfrac{1}{2}\epsilon A) +\tfrac{1}{2}\epsilon^2A^2[2\epsilon(1-\tfrac{1}{2}A)-1][1-\tfrac{1}{2}A(1+\epsilon)]=0\,, \tag{4.22}$$

where the range of $\epsilon$ is constrained to [cf. Eq. (4.7)] $\frac{1}{2}\le\epsilon\le1$, while the domain of $A$ is [cf. Eq. (4.3)] $0\le A\le 2/(1+\epsilon)$.

There are two obvious solutions to the system (4.21) and (4.22): (i) $\epsilon=\frac{1}{2}$, $A=0$; and (ii) $\epsilon=1$, $A=0$, or $A=1$. From Eqs. (4.5), (4.3), (4.6), and (4.1), it follows however that

$$\epsilon=\tfrac{1}{2}\Longrightarrow a=c=f^2+d^2=0 \quad (g\neq0,g'\neq0)\,. \tag{4.23}$$

Clearly this solution is trivial, since both the gauge and Higgs fields vanish. We therefore discard it. The subcase $\epsilon=1$, $A=0$ implies by (4.1), (4.5), (3.34), and (3.35), that either $g=g'=0$, i.e., no coupling of the gauge fields, or $a=c=d^2+f^2=0$, i.e., no gauge nor Higgs fields present. This solution is therefore also trivial, and we likewise discard it. Finally, the subcase $\epsilon=1$, $A=1$ implies, also by (4.1), (4.5), (3.34), and (3.35), that $c=d^2+f^2=0$. The gauge fields for U(1) and the Higgs fields both disappear in this case, so that this solution corresponds to a pure SU(2)-gauge field with the spherical symmetry of the three-dimensional internal space hypersphere $(\epsilon=1)$. Consequently, if we want to find nontrivial, nonlimiting solutions to (4.21) and (4.22), we need to eliminate the end points for $\epsilon$ and $A$, i.e., we need to restrict our solutions to the open intervals $\frac{1}{2}<\epsilon<1$ and $0<A<2/(1+\epsilon)$.

With the above constraints on $\epsilon$ and $A$, numerical analysis shows that solutions to the system *only* exist for

$$G=\frac{1}{\sqrt{\epsilon}}\in(1,1.027)\,. \tag{4.24}$$

[Within this range for $G$, $A$ decreases monotonically from 0.920 (for $G=1.001$) to 0.300 (for $G=1.0266$), while $\alpha_9 g^2$ varies nonmonotonically from $-0.925$ $(G=1.001)$ to $-0.935$ $(G=1.01)$ to $-0.627$ $(G=1.0266)$.]

*Spontaneous compactification.* We next use the range of values obtained above for $G$, $\alpha_9 g^2$, and $A$, to show that spontaneous compactification of our base manifold can actually occur, and that the two radii of the hyperellipsoid can indeed both be of the order of the Planck length.

To this purpose, substitute (3.34) into (4.8) and make use of (4.14) to get

$$\rho_1=\frac{2}{g^2\kappa}\left[\tfrac{7}{3}\lambda A[1-\tfrac{1}{2}A(1+\epsilon)]+\epsilon A(1-\tfrac{1}{2}A)^2 -g^2\alpha_9\left[4-\frac{2}{\epsilon}+\frac{1}{\epsilon^2}\right]-2g^2\alpha_8\left[4-\frac{1}{\epsilon}\right]\right]\,. \tag{4.25}$$

For the range of values of $A$, $\epsilon$, and $g^2\alpha_9$ in our solutions, this expression is positive provided $g^2\alpha_8\le0.084\lambda+0.354\,46$, which can clearly be satisfied. In this case,

$$\rho_1\sim\frac{1}{\kappa g^2}\,. \tag{4.26}$$

We cannot, however, as yet identify $\kappa$ with the observable gravitational constant $\kappa_4$ which appears in the four-dimensional effective Lagrangian. To establish this relationship we examine perturbations to our solutions induced by the metric

$$ds^2=g_{\mu\nu}(x)dx^\mu dx^\nu-\frac{\rho_1}{4}(\tilde\sigma^1\otimes\tilde\sigma^1)-\frac{\rho_1}{4}(\tilde\sigma^2\otimes\tilde\sigma^2) -\frac{\rho_2}{4}(\tilde\sigma^3\otimes\tilde\sigma^3)\,. \tag{4.27}$$

This implies

$$\underline{R}=\underline{R}_s-\frac{2}{\rho_1}\left[4-\frac{1}{\epsilon}\right] \tag{4.28}$$

where $\underline{R}_s$ is the ordinary space-time Ricci scalar, and the only contributions to the perturbed action, linear in $\underline{R}_s$, are

$$\delta I_g=\left[-\kappa-\frac{4\alpha_1}{\rho_1}\left[4-\frac{1}{\epsilon}\right]+\tfrac{7}{12}\lambda(d^2+f^2)\right] \times\int\sqrt{|g|}\underline{R}_s d^4x\,. \tag{4.29}$$

Hence, the quantity to be identified with the inverse square of the Planck length $\kappa_4$ $(\equiv c^3/16\pi\hbar G_N=0.762\times10^{64}\ \mathrm{cm}^{-2})$ is

$$\kappa_4=\kappa+\frac{4\alpha_1}{\rho_1}\left[4-\frac{1}{\epsilon}\right]-\tfrac{7}{12}\lambda(d^2+f^2)\,, \tag{4.30}$$

which from (3.34), (4.25), and our previous results, is of the same order as $\kappa$. Therefore, Eq. (4.26) just says that $\rho_1$ is of the order of the square of the Planck length; and since $\epsilon\approx1$, so is $\rho_2$.

It is remarkable that this model not only predicts a value for the ratio of the SU(2)- and U(1)-coupling constants at the energy where both compactification and the unification of gravity with electroweak interactions would occur, but that this is so close to 1, in agreement with that predicted by the standard model when applying the renormalization group to it.

## V. CONCLUSIONS

Based on a principal fiber-bundle approach to a Kaluza-Klein-type theory for the unification of gravitation with the bosonic sector of the standard electroweak model, we have shown that there exist $SU(2)\times U(1)$-invariant connections which induce spontaneous compactification of the original base space $M$ to $\mathcal{M}^4\times S^3$. We have also shown that if the resulting gauge field components in the internal space are required to be linear in the Euclidean coordinates $x_1,x_2,x_3,x_4$ of the $\mathcal{R}^4$ in which $S^3$ is immersed, then the existence of compactified solutions requires that quadratic terms in the curvature occur in the Lagrangian. On the other hand, this restriction on the gauge fields together with the need for $R^2$-type terms (i.e., quadratic in the curvature) in the Lagrangian, lead to very definite predictions for the value of the ratio of the SU(2)- and U(1)-coupling constants at the compactification energies ($\sim$Planck mass), which are in close agreement with those resulting from applying the renormalization group to the standard model. Furthermore, the dimensions of the characteristic parameters in the metric for $S^3$ turn out to be of the order of the Planck length, so in this respect our formalism does not present the inconsistencies which occur in other approaches in the literature.

It is important to observe, however, that although having quadratic Lagrangians is also an advantage from the point of view of superstring theories, as these terms arise naturally in the low-energy limit of such theories, the appearance of $R^2$-type terms may pose some problems when carrying our results over to the quantum realm. Indeed, it is well known that unless the $R^2$-type terms are of the Gauss-Bonnet form: $R^{\mu\nu\sigma\tau}R_{\mu\nu\sigma\tau}-4R^{\mu\nu}R_{\mu\nu}+R^2$, quantum perturbation theory leads inevitably to the appearance of ghosts.[22] Even though it could be argued that we need not worry about this problem at this stage, since quantum gravity is nonrenormalizable anyhow, there might be alternative ways within our formalism to circumvent the problem by seeking solutions compatible with Gauss-Bonnet-type Lagrangians. This latter possibility clearly exists, since we still have at our disposal the more general solutions for the $SU(2)\times U(1)$-invariant connections found in Sec. II, as well as other options in the choice of the form of the action of the symmetry group on the fibers of our bundle space [cf. Eq. (2.26)], or investigating other admissible topologies such as case (3) in (2.1), or, lastly, a combination of these options. Work along these lines is in progress.

As a final remark on the occurrence of Gauss-Bonnet terms in the Lagrangian, we recall that for spaces of dimension $\leq 4$, such terms contribute at most with a total derivative to the Lagrangian density, and may therefore be dropped out in a variational principle. In our case, where $M=\mathcal{M}^4\times S^3$, requiring a Gauss-Bonnet condition in the Lagrangian is tantamount to having only Einstein gravity coupled to Yang-Mills and Higgs fields. Moreover, since we already know that no compactification is possible in this case with gauge field solutions linear in the Euclidean coordinates, it becomes necessary to turn to the alternatives described above to search for compactified solutions. The situation is quite different when considering internal spaces with dimension $>4$, such as is the case of Kaluza-Klein theories with grand unification, where the smallest dimension of the compact homogeneous space is 7. There the Gauss-Bonnet terms in the Lagrangian are no longer a total derivative, and may be determinant to the existence of solutions leading to spontaneous compactification of an eleven-dimensional base manifold.

To end this section, we shall make some remarks concerning the cosmological constant. First note that multiplying (3.32) by $\frac{1}{4}(d^2+f^2)$ and substituting into (3.37), yields

$$\frac{\kappa}{\rho_1}\left[4-\frac{1}{\epsilon}\right]-\frac{1}{2}\kappa\Lambda-\frac{1}{\rho_1^2}a^2\left[1-\frac{1}{2}ag\right]^2(1+2\epsilon)-\frac{c^2}{\rho_1^2}+\frac{\lambda}{8}(d^2+f^2)^2+\frac{2}{\rho_1^2}\left[\alpha_8\left[4-\frac{1}{\epsilon}\right]^2+2\alpha_9\left[4-\frac{1}{\epsilon^2}\right]\right]=0\;. \tag{5.1}$$

Also, dividing (4.9) by $\epsilon$ and adding to (3.38) gives

$$\tfrac{1}{2}\kappa\rho_1\Lambda-\frac{1}{\rho_1}a^2(1-\tfrac{1}{2}ag)^2(1+2\epsilon)-\frac{c^2}{\rho_1}-\tfrac{1}{4}\rho_1 m^2(d^2+f^2)+\frac{\lambda}{8}\rho_1(d^2+f^2)^2+\frac{2}{\rho_1}\left[4-\frac{1}{\epsilon}\right]^2(\alpha_8+\tfrac{1}{2}\alpha_9)=0\;. \tag{5.2}$$

We can now combine these last two expressions as follows: multiply (5.1) by $\rho_1$ and subtract (5.2). The result is

$$\Lambda=\frac{1}{\rho_1}\left[4-\frac{1}{\epsilon}\right]+\frac{1}{4}\frac{m^2}{\kappa}(d^2+f^2)+\frac{\alpha_9}{\epsilon\kappa\rho_1^2}\left[8-\frac{5}{\epsilon}\right]\;. \tag{5.3}$$

It may be clearly seen from (5.3) and our previous results that $\Lambda=O(\kappa)$. Note, furthermore, that this $\Lambda$ appears in the Lagrangian with negative sign and thus may be used to cancel the positive contributions to the cosmological constant originating in the same Lagrangian from changes in the vacuum energy due to phase transitions from symmetry breaking at the different energy scales, as well as other contributions from quantum effects. This procedure, albeit conceptually unsatisfactory since it requires extreme fine-tuning of the available parameters in (5.3), offers at least a possibility to achieve the small values required by observation for the "physical" $\Lambda$.

Related to the same subject, but as a different approach to it, note that the nonminimal coupling term $\sim\underline{R}(\Phi_A\Phi^A)$, which appears in a natural way in the Lagrangian (1.1) as a result of the requirement of semisymmetric torsion on the fibers of the bundle, is of the same form as the one used on compensating field models for the damping of the cosmological constant. Moreover, by modifying the condition of semisymmetric torsion, our formalism could also yield terms of the form $R_{\mu\nu}U^\mu U^\nu\Phi_A\Phi^A$, which are also being considered for the same purpose. Although these models also pose some so-far unresolved problems, it remains a suggestive fact that the required terms in the Lagrangian appear naturally in our theory, rather than in the *ad hoc* fashion by means of which they are introduced in other approaches.[15–17]

In the Introduction we made some general remarks regarding the problems associated with the incorporation of fermions in Kaluza-Klein-type theories. Though this is beyond the scope of the present work, we nevertheless wish to outline the general procedure that is currently being investigated by our group.

First, massless gravitinos are obtained from a locally supersymmetrized version ($N=1$) of the ground state of the model described in paper I. These constitute a Rarita-Schwinger spin-$\frac{3}{2}$ field $\Psi_{\mu\alpha}$ (where $\mu=0,1,\ldots,n-1$) is a vector index while $\alpha$ is a spinor index). Upon compactification of the extra dimensions of the base space, the $\mu\geq 4$ vector components would become indices of an internal symmetry and would thus carry spin zero, that is, from the point of view of the four-dimensional spacetime the gravitino components $\Psi_{\mu\alpha}$ with $\mu=0,\ldots,3$ are spin-$\frac{3}{2}$ fields, while the components with $\mu=4,\ldots,n-1$ are spin-$\frac{1}{2}$ fields. One would then go on to dimensionally reduce the theory by harmonically expanding the four-dimensional spin components on the extra dimensions, i.e., by letting[23]

$$\Psi=\sum_{\sigma k}\Psi^{(\sigma\pm)k}(x)U^{(\sigma\pm)k}(y)\ , \tag{5.4}$$

where $U^{(\sigma\pm)k}(y)$ are spinor harmonics in the internal dimensions, while the coefficients $\Psi^{(\sigma\pm)k}(x)$ represent the fermion fields in four-dimensional spacetime. The labels in (5.4) are such that $\sigma$ specifies the representation of the gauge group, $k$ labels the components of the representation $\sigma$, and the superscripts ($\pm$) denote the chirality of the states.

Be regarding the compactifying gauge fields, though geometric in origin, as elementary fields in the base space, one is free to consider small oscillations resulting from their interaction with the dimensionally reduced fermionic fields, and in particular search for the presence of zero modes of the mass operator $\not{D}^{(n-4)}$ in a complex representation, following an approach similar to that used by Horvath *et al.*[8]

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