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ORIGINAL ARTICLE

Geometry and dynamics of the Schur–Cohn stability algorithm for one variable polynomials

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Abstract

We provided a detailed study of the Schur–Cohn stability algorithm for Schur stable polynomials of one complex variable. Firstly, a real analytic principal $\mathbb{C} \times \mathbb{S}^1$-bundle structure in the family of Schur stable polynomials of degree n is constructed. Secondly, we consider holomorphic $\mathbb{C}$-actions $\mathscr{A}$ on the space of polynomials of degree n. For each orbit $\{s \cdot P(z) \mid s \in \mathbb{C}\}$ of $\mathscr{A}$, we study the dynamical problem of the existence of a complex rational vector field $\mathbb{X}(z)$ on $\mathbb{C}$ such that its holomorphic s-time describes the geometric change of the n-root configurations of the orbit $\{s \cdot P(z) = 0\}$. Regarding the above $\mathbb{C}$-action coming from the $\mathbb{C} \times \mathbb{S}^1$-bundle structure, we prove the existence of a complex rational vector field $\mathbb{X}(z)$ on $\mathbb{C}$, which describes the geometric change of the n-root configuration in the unitary disk $\mathbb{D}$ of a $\mathbb{C}$-orbit of Schur stable polynomials. We obtain parallel results in the framework of anti-Schur polynomials, which have all their roots in $\mathbb{C} \backslash \overline{\mathbb{D}}$, by constructing a principal $\mathbb{C}^* \times \mathbb{S}^1$-bundle structure in this family of polynomials. As an application for a cohort population model, a study of the Schur stability and a criterion of the loss of Schur stability are described.

Keywords Schur stable polynomials · Schur–Cohn stability algorithm · Principal G-bundles · Complex rational vector fields · Lie group actions

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1 Introduction

The space of complex polynomials $\mathbb{C}[z]_{=n} = \{P(z) = c_n z^n + \cdots + c_0 \mid c_n \neq 0\}$ of degree n admits two natural parameterizations: coefficient coordinates $\mathscr{C}_n = \{(c_n, \ldots, c_0)\}$ and root coordinates $\mathscr{R}_n = \{(c_n, [z_1, \ldots, z_n])\}$ determined by unordered n-root configurations $[z_1, \ldots, z_n] \in \mathbb{C}^n/\mathrm{Sym}(n)$ and the coefficient c_n. The Viète map $\mathscr{V}_n : \mathscr{R}_n \longrightarrow \mathscr{C}_n$ given by the elementary symmetric functions is a natural translator. The non-triviality of $\mathscr{V}_n^{-1}$ was performed by N. H. Abel and É. Galois.

A polynomial $P(z)$ is called *Schur stable* (respectively, *anti-Schur*) if all its roots lie in the unitary disk $\mathbb{D}$ (respectively, in $\mathbb{C}\backslash\overline{\mathbb{D}}$). A classical problem in control theory and algebraic/analytic theory of polynomials is the construction of algorithms that from information in $\mathscr{C}_n$ decide when a polynomial is Schur stable, which is a conclusion in $\mathscr{R}_n$.

We regard the seminal work of Schur [29] and Cohn [11], currently known as the *Schur–Cohn stability algorithm*; see [27], Theorem 11.5.3; [8], Sect. 1.4 for modern reviews and [15] for computational aspects. The algorithm allows us to determine the number of roots of a polynomial $P(z)$ of degree n in the unitary disk $\mathbb{D}$, the boundary $\partial\mathbb{D}$ and the complement $\mathbb{C}\backslash\overline{\mathbb{D}}$. In differential topology language, the algorithm depends on four real polynomial maps on $\mathscr{C}_n$, denoted as $\{R_{\alpha,n}\}_{\alpha=1}^{4}$. The first two maps

$$\begin{aligned} R_{1,n} : \mathscr{D}_{1,n} &= \{\, |c_n| < |c_0| \,\} \subset \mathbb{C}[z]_{=n} \longrightarrow \mathbb{C}[z]_{\leq n-1}, \\ R_{2,n} : \mathscr{D}_{2,n} &= \{\, |c_n| > |c_0| \,\} \subset \mathbb{C}[z]_{=n} \longrightarrow \mathbb{C}[z]_{=n-1} \end{aligned} \tag{1}$$

deal with anti-Schur and Schur stable polynomials, respectively, enjoying the following crucial properties (see [8,27]):
$P(z)$ is anti-Schur if and only if $R_{1,n}(P(z))$ is also anti-Schur, and
$P(z)$ is Schur stable if and only if $R_{2,n}(P(z))$ is also Schur stable.

A very remarkable/rare fact is the existence of maps $\{R_{\alpha,n}\}$ in coefficient coordinates $\mathscr{C}_n$ that enjoy the following two characteristics. Under $R_{\alpha,n}$ the degree of $P(z)$ decreases and the position (with respect to $\mathbb{D}$) of the roots of $R_{\alpha,n}(P(z))$ is preserved. Thus, the Schur–Cohn stability algorithm provides us with conclusions for the root coordinates $\mathscr{R}_n$.

Let us introduce $\mathbb{C}$, $\mathbb{C}^*$ and $\mathbb{S}^1$ the additive, multiplicative and circle Lie groups. In Aguirre-Hernández et al. [2,3], introduce a vector bundle structure on the space of monic Schur stable polynomials of degree n. Our starting result enlarged this geometric structure as follows (Theorem 2 in the text).

Theorem A *For $\alpha = 1, 2$, the maps $R_{\alpha,n}$ are real analytic submersions and determine trivial principal G-bundles*

$$\begin{tikzcd} \mathbb{C}^* \times \mathbb{S}^1 \arrow[r] & \mathscr{D}_{1,n} \arrow[d, "R_{1,n}"] \\ & \mathbb{C}^{n-1} \times \mathbb{R}^+ \end{tikzcd} \qquad \begin{tikzcd} \mathbb{C} \times \mathbb{S}^1 \arrow[r] & \mathscr{D}_{2,n} \arrow[d, "R_{2,n}"] \\ & \mathbb{R}^+ \times \mathbb{C}^{n-1} \end{tikzcd} \tag{2}$$

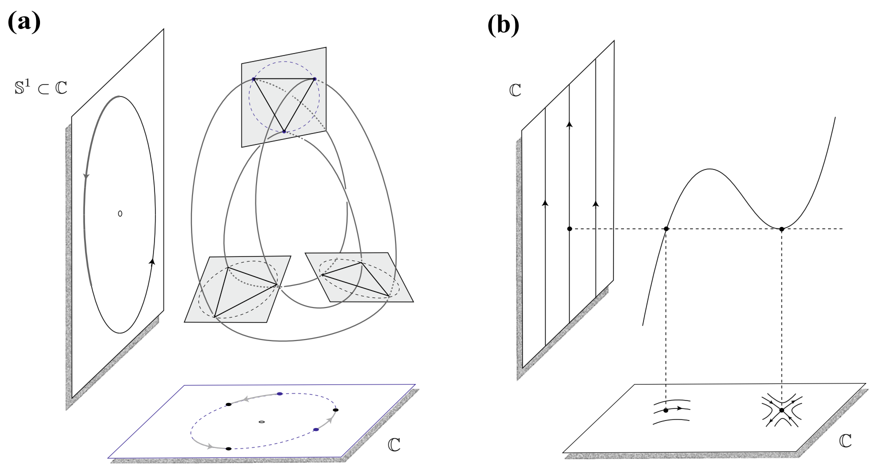

Fig. 1 Two families of polynomials: **a** a family over the circle $\mathbb{S}^1$ and **b** a family over the additive group $\mathbb{C}$. Note that a key fact is the use of s as the target variable for $P(z)$. The phase portraits sketched in $\mathbb{C}$ describe the n-root configuration dynamics, respectively

with structural Lie groups $G = \mathbb{C}^* \times \mathbb{S}^1$ *and* $\mathbb{C} \times \mathbb{S}^1$, *respectively.*

What is the meaning in root coordinates $\mathscr{R}_n$ of these principal G-bundle geometries?

We generalize the framework as follows. Let G be the Lie group $\mathbb{C}$ or $\mathbb{C}^*$, and consider

$$\mathscr{A} : G \times \mathbb{C}[z]_{=n} \longrightarrow \mathbb{C}[z]_{=n}, \quad \big(g, P(z)\big) \longmapsto g \cdot P(z)$$

any holomorphic (respectively, real analytic) G-action. Each orbit

$$\{g \cdot P(z) \mid g \in G\}$$

determines a holomorphic (respectively, real analytic) Weierstrass polynomial, in the sense of Hansen [19,20]; see Remark 5.

We consider the following *prototype dynamics*. Let $\mathscr{A}\big(s, P(z)\big) = P(z) - s$ be a holomorphic action by translations, here by simplicity, $P(z)$ is a monic polynomial of degree n, and consider $\{P(z) = 0\} = [z_1, \dots, z_n]$ its unordered n-root configuration. The dynamical problem is as follows.

$$\text{How can we describe the geometric change of the } n\text{-root configurations}$$
$$\{P(z) - s = 0\} = [z_1(s), \dots, z_n(s)], \ \text{ for } s \in \mathbb{C}?$$

See Fig. 1b. Two key points are as follows:

$\{P(z) - s \mid s \in \mathbb{C}\}$ is one orbit of the above Lie group $\mathbb{C}$-action $\mathscr{A}$, and each root $z_\iota(s)$ depends on the variable s of the additive Lie group $\mathbb{C}$.

Obviously, the topology of the n-root configurations changes when s crosses a critical value of $P(z)$. In a very rough analogy with the n-body problem we say that $[z_1(s), \ldots, z_n(s)]$ determines the n-root dynamics, where s plays the role of the complex time.

Our assertion is that the *complex analytic vector field* $\mathbb{X}(z) = (P(z)')^{-1}\frac{\partial}{\partial z}$ *describes the n-root configuration dynamics*. In this case, the diagram

$$\begin{array}{ccc} [z_1, \ldots, z_n] & \xrightarrow{\mathscr{V}_n} & P(z) = z^n + \cdots + c_1 z - c_0 \\ \varphi(s,\ )\Big\downarrow & & \Big\downarrow \mathscr{A}(s,\ ) \\ [\varphi(s, z_1), \ldots, \varphi(s, z_n)] & \xleftarrow[\mathscr{V}_n^{-1}]{} & P(z) - s = z^n + \cdots + c_1 z + (c_0 - s) \end{array} \tag{3}$$

commutes, whenever the holomorphic flow

$$\varphi(s, z) = z(s) \text{ of } \mathbb{X}(z)$$

is well defined. Diagram (3) is the prototype dynamics. See Definition 2 and Lemma 1 (where the commutativity property of (3) using the flow of $\mathbb{X}(z)$ has been proven).

Given a Schur polynomial $P(z)$, the dynamical problem is the description of the n-root configurations of the polynomials in its orbit $\{s \cdot P(z) \mid s \in \mathbb{C}\}$, which originates from the real analytic Lie group $\mathbb{C}$-action in the bundle omitting the $\mathbb{S}^1$-action; see the right diagram in (2). In an analogous way, given an anti-Schur polynomial $P(z)$, the dynamical problem makes sense. In other words, we are asking for new versions of diagram (3).

Since the Lie group $\mathbb{C}$ is simply connected, the Schur case is simpler than the anti-Schur case requiring $\mathbb{C}^*$. Our main dynamical result for Schur and anti-Schur polynomials is as follows (Theorems 3 and 4 in the text).

Theorem B (n-root configuration dynamics of Schur $\mathbb{C}$-orbits and anti-Schur $\mathbb{C}^*$-orbits)

1. *Let $P(z)$ be a Schur stable polynomial and consider $\{s \cdot P(s) \mid s \in \mathbb{C}\}$ its orbit in the corresponding principal $\mathbb{C}$-bundle from (2). There exists a complex rational vector field $\mathbb{X}(z)$ on $\mathbb{C}$ describing the n-root configuration dynamics of*

$$\{s \cdot P(z) = 0\},$$

up to a suitable reparameterization of its complex time.

2. *Let $P(z)$ be an anti-Schur polynomial and consider $\{w \cdot P(s) \mid w \in \mathbb{C}^*\}$ its orbit in the corresponding principal $\mathbb{C}^*$-bundle from (2). There exists a complex rational vector field $\mathbb{Y}(z)$ on $\mathbb{C}$ describing the n-root configuration dynamics of*

$$\{w \cdot P(z) = 0\},$$

up to a suitable reparametrization of its complex time.

For accurate statements and proofs see Theorems 3 and 4 in Sects. 8 and 9, respectively. Also in Sect. 8, Fig. 5a–d provides numerical examples of phase portraits of suitable vector fields $\mathbb{X}(z)$ on $\mathbb{D}$; Fig. 5e and f provide sketches of phase portraits of $\mathbb{Y}(z)$ on $\mathbb{C}\backslash\mathbb{D}$. A complete explanation of the suitable reparametrization of complex time is in Eqs. (51) and (59) for Schur stable and anti-Schur, respectively.

The analogous versions of Theorems A and B remain true for Hurwitz polynomials, considering the Möbius transformation T that sends $\mathbb{D}$ on to the left half-plane $\mathbb{H} = \{\mathfrak{Im}(z) < 0\}$.

In order to describe the content of the work, the following diagram should provide a logical guideline for the major themes in it:

$$\begin{array}{ccccccc} G\text{-action} & & \text{a } G\text{-orbit or} & & \text{its } n\text{-root configurations} & & \text{a vector field describing} \\ \text{on } \mathbb{C}[z]_{=n} & & \text{a Weierstrass polynomial} & & & & \text{these configuration dynamics} \\ \mathscr{A} & \dashrightarrow & g \cdot P(z) = \mathscr{P}(g, z) & \dashrightarrow & \{\mathscr{P}(g, z) = 0\} & \dashrightarrow & \mathbb{X}(z)\ . \end{array} \tag{4}$$

Note that the G-actions and orbits are given in coefficient coordinates $\mathscr{C}_n$, whereas the configurations and vectors fields belong to the root coefficient $\mathscr{R}_n$ realm.

Section 2 describes the Schur, anti-Schur and other useful families of degree n polynomials, according to the position of their roots. In Sect. 3, the Schur–Cohn stability algorithm is reviewed. The proof of Theorem A is given in Sect. 4, this provide us with the natural actions (orbits) of the Lie groups $G = \mathbb{C}^* \times \mathbb{S}^1$ and $\mathbb{C} \times \mathbb{S}^1$ on $\mathbb{C}[z]_{=n}$. The general notion of Lie group actions on the space of polynomials of degree n, G-orbits, their associated Weierstrass polynomials (see Definition 1) and the proof of the prototype dynamics (Lemma 1) are done in section Sect. 5. The complex dynamics of singular complex analytic vector fields $\mathbb{X}(z)$ and their application to our dynamical problem are given in Sect. 6. A dictionary between singular points of vector fields and n-root configuration dynamics is in Definition 3, and several simple examples are provided. Section 7 explores complex rational vector fields that arose from $\mathbb{C}$ and $\mathbb{C}^*$ Lie group actions on $\mathbb{C}[z]_{\leq n}$. In Sect. 8, Theorem B for the case of Schur stable polynomials has been proven; see Theorem 3. Furthermore, a certain criterion of the loss of Schur stability is given in Proposition 4. The assertion of Theorem B for the anti-Schur case is done in Sect. 9; see Theorem 4. Finally in Sect. 10, we apply Theorems A and B to study the Schur stability of a cohort population model; Theorem B and the criterion of the loss of Schur stability are computed for precise parameter values, see Example 16.

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2 Families of polynomials

Let $\mathbb{C}[z]_{\leq n} = \{P(z) = c_n z^n + \cdots + c_0\}$ be the space of complex polynomials of degree at most n. Let $\mathbb{C}[z]_{=n} = \{P(z) = c_n z^n + \cdots + c_0 \mid c_n \neq 0\}$ be the subset of complex polynomials of degree n. We introduce root coordinates, $\mathscr{R}_n$, on $\mathbb{C}[z]_{=n}$ using the coefficient $\{c_n\}$ and the unordered roots $\{[z_1, \ldots z_n]\}$ for each polynomial $P(z)$, as follows:

$$
\begin{array}{ccccc}
\mathbb{C}^* \times \mathbb{C}^n & \longrightarrow & \mathbb{C}^* \times \frac{\mathbb{C}^n}{\mathrm{Sym}(n)} & \underset{\mathscr{V}_n^{-1}}{\overset{\mathscr{V}_n}{\rightleftarrows}} & \mathbb{C}[z]_{=n} \\
(c_n, z_1, \ldots, z_n) & \longmapsto & (c_n, [z_1, \ldots, z_n]) & \longmapsto & (c_n, c_{n-1}, \ldots, c_0) \\
 & & & & = \Big(c_n, -c_n(z_1 + \cdots + z_n), \ldots, (-1)^n c_n (z_1 \cdots z_n)\Big),
\end{array}
\tag{5}
$$

where the symmetric group of order n, $\mathrm{Sym}(n)$, acts on the roots and $[\ldots]$ means a class under this action.

$\mathscr{V}_n$ is the n degree Viète map and the inverse map essentially $\mathscr{V}_n^{-1}$ sends a polynomial to its roots; see [1,23].

Remark 1 By abusing notation in several places, we identify the polynomial $P(z) = c_n z^n + \cdots + c_0$ with the vector $(c_n, \ldots, c_0)$, whence also the identification between $\mathbb{C}[z]_{\leq n}$ and $\mathbb{C}^{n+1}$ is used.

Let $\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$ be the unitary disk, having boundary $\partial\mathbb{D} := \{z \in \mathbb{C} \mid |z| = 1\}$. It will be useful to consider families of polynomials of degree n, depending on the numbers of roots $\mathtt{p}$, $\mathtt{s}$, $\mathtt{q}$ in $\mathbb{D}$, $\partial\mathbb{D}$ and $\mathbb{C}\backslash\overline{\mathbb{D}}$, respectively ($n = \mathtt{p} + \mathtt{s} + \mathtt{q}$).

The natural motivation is as follows. Let F be a holomorphic germ map on $(\mathbb{C}^n, \overline{0})$ and $\det(DF(\overline{0})) = P(z)$ be its Jacobian polynomial having eigenvalues $[z_1, \ldots, z_n]$. It is well known that generically under the iteration of F, there appear stable, central and unstable local manifolds at the origin $\overline{0} \in \mathbb{C}^n$ of dimensions $\mathtt{p}$, $\mathtt{s}$ and $\mathtt{q}$, respectively. Other extensive references that include interesting information about Schur stable polynomials are [4,10,14,22].

We describe the families for $\mathbb{C}[z]_{=n}$ in the Table 1, considering on three attributes: the position of the roots, some inequality on the coefficients and the Schur–Cohn map required by the stability algorithm. Note that $|c_n| > 0$.

- *Schur*[1] is the family of polynomials that have all their roots in the open unitary disk.
- *Anti-Schur* is the family of polynomials that have all their roots in the exterior of the unitary disk (following the usual name [8], p. 115, [21], p. 341).
- *Semi-Schur* is the topological closure of the Schur family, i.e., the family of polynomials that have all their roots in the closed unitary disk $\overline{\mathbb{D}}$.
- *Semianti-Schur* is the topological closure of the anti-Schur family.
- *Saddle polynomials, the* The term comes from the existence of non-empty local stable and unstable manifolds under the iteration of F, as we remark above. Type 1 or 2 depends on the respective Schur–Cohn map that will be required.
- We define that $P(z)$ is *self-inverse* when $P(z) = \sigma P^*(z)$, for some point in $\{|\sigma| = 1\} = \mathbb{S}^1$, and

$$
\begin{array}{rcl}
(\ )^* : \mathbb{C}[z]_{=n} & \longrightarrow & \mathbb{C}[z]_{=n} \\
P(z) = (c_n, c_{n-1}, \ldots, c_0) & \longmapsto & P(z)^* = z^n \overline{P\left(\frac{1}{\overline{z}}\right)} = (\overline{c}_0, \ldots, \overline{c}_{n-1}, \overline{c}_n),
\end{array}
\tag{6}
$$

here $\overline{z}$ denotes the conjugate. Each root $z_\ell \neq 0$ of $P(z)$ comes with its reciprocal conjugate $1/\overline{z_\ell}$; see [27], p. 375, [31], p. 109, [6,9].

[1] We simplify Schur stable to say only Schur.

Table 1 Families of polynomials according to roots $\{\mathtt{p}, \mathtt{s}, \mathtt{q}\}$ and coefficients $\{|c_n|, |c_0|\}$

		$\mathbb{D}$ $\mathtt{p}$	$\partial\mathbb{D}$ $\mathtt{s}$	$\mathbb{C}\backslash\overline{\mathbb{D}}$ $\mathtt{q}$		Required Schur–Cohn map
Schur	$\mathscr{S}_n$	$\mathtt{p} = n$	0	0	$\lvert c_n\rvert > \lvert c_0\rvert$	$R_{2,n}$
Semi-Schur	$s\mathscr{S}_n$	$\mathtt{p} \geq 1$	$\mathtt{s} \geq 1$	0		
Saddle polynomials type 2	$\mathrm{us}_2\mathscr{S}_n$	$\mathtt{p} \geq 1$	$\mathtt{s}$	$\mathtt{q} \geq 1$		
Anti-Schur	$a\mathscr{S}_n$	0	0	$\mathtt{q} = n$	$\lvert c_n\rvert < \lvert c_0\rvert$	$R_{1,n}$
Semianti-Schur	$\mathrm{sa}\mathscr{S}_n$	0	$\mathtt{s} \geq 1$	$\mathtt{q} \geq 1$		
Saddle polynomials type 1	$\mathrm{us}_1\mathscr{S}_n$	$\mathtt{p} \geq 1$	$\mathtt{s}$	$\mathtt{q} \geq 1$		
Balanced	$\mathscr{B}_n$	$\mathtt{p} \geq 1$	$\mathtt{s}$	$\mathtt{q} \geq 1$	$\lvert c_n\rvert = \lvert c_0\rvert$	$R_{3,n}$
Self-inverse	$\mathscr{C}_n$	$\mathtt{p} \geq 0$	$\mathtt{s} \geq 0$	$\mathtt{q} = \mathtt{p}$	$\lvert c_n\rvert = \lvert c_0\rvert$	$R_{4,n}$

Remark 2 1. The operator $(\ )^*$ in (6) is an involution.
2. The polynomial $P(z)^*$ is Schur if and only if $P(z)$ is anti-Schur.

- We define that $P(z)$ is *balanced* when $|c_n| = |c_0|$ and it is not self-inverse.

Looking at the eight families in Table 1, the intersection of two of them is empty.

Recall that $\{|c_n| > |c_0|\}$ is a necessary but no sufficient condition in order to characterize Schur polynomials. Following the fifth column in Table 1, let

$$\Sigma_n = \{P(z) \mid |c_n| = |c_0|\} \subset \mathbb{C}[z]_{=n} \tag{7}$$

be a real hypersurface. Σ_n determines two open and connected domains

$$\mathscr{D}_{1,n} = \{P(z) \mid |c_n| < |c_0|\}, \qquad \mathscr{D}_{2,n} = \{P(z) \mid |c_n| > |c_0|\} \tag{8}$$

of $\mathbb{C}[z]_{=n}$. In addition, we define

$$\mathscr{D}_{4,n} = \{P(z) \mid |c_n| = |c_0|\}, \quad P(z) = \sigma P^*(z) \ \text{ for } \sigma \in \mathbb{S}^1\}, \qquad \mathscr{D}_{3,n} = \Sigma_n \backslash \mathscr{D}_{4,n}, \tag{9}$$

whence

$$\mathbb{C}[z]_{=n} = \mathscr{D}_{1,n} \cup \cdots \cup \mathscr{D}_{4,n}.$$

Moreover, $\{\mathscr{D}_{\alpha,n}\}_{\alpha=1}^4$ will be the domain of the Schur–Cohn maps $R_{\alpha,n}$, as we will show in the next section.

3 The Schur–Cohn maps

We recall the Schur–Cohn maps for polynomial map families of degree n. In order to avoid double subindexes at the target, which is the space of polynomials of degree at most $n - 1$, we use the notation

$$\mathbb{C}[z]_{\leq n-1} = \{b_{n-1}z^{n-1} + \cdots + b_0\} = \{(b_{n-1}, \ldots, b_0)\}$$

and the contention $\mathbb{R}^+ \subset \mathbb{C}$ as usual. Moreover, we follow the enumeration of Schur–Cohn maps as in [27], p. 375.

When $P(z) \in \mathscr{D}_{1,n} \cup \mathscr{D}_{2,n}$, which is an open and dense set of the polynomials of degree n, the maps are defined as follows.
The *Schur–Cohn map 1* is

$$\begin{aligned} R_{1,n} : \mathscr{D}_{1,n} \subset \mathbb{C}[z]_{=n} &\longrightarrow (\mathbb{C}^{n-1} \times \mathbb{R}^+) \subset \mathbb{C}[z]_{\leq n-1} \\ (c_n, \ldots, c_0) &\longmapsto (c_{n-1}\overline{c}_0 - \overline{c}_1 c_n,\ c_{n-2}\overline{c}_0 - \overline{c}_2 c_n,\ \ldots,\ c_1\overline{c}_0 \\ &\qquad -\overline{c}_{n-1}c_n,\ |c_0|^2 - |c_n|^2) = (b_{n-1}, \ldots, b_0). \end{aligned} \tag{10}$$

The *Schur–Cohn map 2* is

$$\begin{aligned} R_{2,n} : \mathscr{D}_{2,n} \subset \mathbb{C}[z]_{=n} &\longrightarrow (\mathbb{R}^+ \times \mathbb{C}^{n-1}) \subset \mathbb{C}[z]_{=n-1} \\ (c_n, \ldots, c_0) &\longmapsto (|c_n|^2 - |c_0|^2,\ c_{n-1}\overline{c}_n - c_0\overline{c}_1,\ \ldots,\ c_2\overline{c}_n \\ &\qquad -c_0\overline{c}_{n-2},\ c_1\overline{c}_n - c_0\overline{c}_{n-1}) = (b_{n-1}, \ldots, b_0). \end{aligned} \tag{11}$$

Now consider the nongeneric case $P(z) \in \Sigma_n = \{|c_0| = |c_n|\}$. In fact, $|c_n| = |c_0|$ if and only if $c_n - \sigma c_0 = 0$ for some $\sigma \in \mathbb{S}^1 = \{|\sigma| = 1\}$. *Case i*. If $\sigma\overline{c}_{n-k} - c_k = 0$ for every $k \in 1, \ldots, n$, then we apply Schur–Cohn rule 4 below, in particular $|c_0|^2 - |c_n|^2 = 0$ holds. *Case ii*. If there exists k_0 such that $\sigma\overline{c}_{n-k_0} - c_{k_0} \neq 0$, let $k_1 = \min\{k_0\}$ be for some $0 < k_0 < n$, then we have the following.
The *Schur–Cohn map 3* is

$$\begin{aligned} R_{3,n} : \mathscr{D}_{3,n} \subset \mathbb{C}[z]_{=n} &\longrightarrow \mathbb{C}[z]_{=n-1} \\ P(z) &\longmapsto P_1(z) = \tfrac{1}{z}\left[g_1^*(0)g_1(z) - g_1(0)g_1^*(z)\right], \end{aligned} \tag{12}$$

with

$$g(z) = \left(z^k + \frac{2b}{b}\right)P(z), \qquad g_1(z) = \overline{g}(0)g(z) - \overline{g}^*(0)g^*(z), \qquad b = \frac{c_{n-k} - \sigma\overline{c}_k}{c_n}.$$

Example 1 For degree two, the map is

$$\begin{aligned} R_{3,2} : \mathscr{D}_{3,2} \subset \mathbb{C}[z]_{=2} &\longrightarrow \mathbb{C}[z]_{=1} \\ P(z) \longmapsto P_1(z) = &\left(\left|\frac{2\overline{c}_0\overline{b}c_1}{|b|} + 4\overline{c}_0c_2 - c_2\overline{c}_0 - 2c_2\overline{c}_1\frac{\overline{b}}{|b_0|}\right|^2 \right. \\ &\left. - \left|4|c_0|^2 - |c_2|^2\right|^2\right) z \\ &+ \left(\frac{2|c_0|^2\overline{b}}{|b|} + 4\overline{c}_0c_1 - c_2\overline{c}_1 - \frac{2|c_2|^2\overline{b}}{|b|}\right) \end{aligned}$$

$$\left(\frac{2c_0 b\bar{c}_1}{|b|} + 4c_0\bar{c}_2 - \bar{c}_2 c_0 - \frac{2\bar{c}_2 c_1 b}{|b|}\right)$$
$$-\left(4|c_0|^2 - |c_2|^2\right)\left(\frac{2|c_0|^2 b}{|b|} + 4c_0\bar{c}_1\right.$$
$$\left.-\bar{c}_2 c_1 - \frac{2|c_2|^2 b}{|b|}\right),$$

where $b = \dfrac{c_1\bar{c}_2 - c_0\bar{c}_1}{|c_2|^2} \neq 0$.

The *Schur–Cohn map 4* is

$$\begin{aligned} R_{4,n} : \mathscr{D}_{4,n} \subset \mathbb{C}[z]_{=n} &\longrightarrow \mathbb{C}[z]_{\leq n-1} \\ (c_n, \ldots, c_0) &\longmapsto (c_{n-1}, 2c_{n-2}, 3c_{n-3}, \ldots, (n-1)c_1, nc_0) \\ &\quad= (b_{n-1}, \ldots, b_0). \end{aligned} \tag{13}$$

The precise statement for the Schur–Cohn algorithm is stated in Theorem 1. The original sources are [11,29], modern versions can be found in [8], p. 55, [21], pp. 355, 368 and [27], pp. 375, 395, our redaction follows verbatim from the latter.

Theorem 1 (Schur–Cohn stability algorithm) *Let $P(z)$ be a polynomial of degree n, having zeros $\mathtt{p}, \mathtt{s}, \mathtt{q}$ as in Table 1, and consider its image under the Schur–Cohn stability algorithm; thus*

$$P_1(z) = R_{\alpha,n}(P(z)),$$

where $\alpha \in \{1, \ldots, 4\}$ is determined by $P(z) \in \mathscr{D}_{\alpha,n}$, as in Eqs. (8) and (9).

Let $\mathtt{p}_1, \mathtt{s}_1, \mathtt{q}_1$ be the corresponding zeros of $P_1(z)$ in $\mathbb{D}$, $\partial\mathbb{D}$, $\mathbb{C}\backslash\overline{\mathbb{D}}$.

1. *If $|c_0| > |c_n|$, then $P_1(z) = \bar{c}_0 P(z) - c_n P^*(z)$ is not identically zero and we have $\deg P_1 < \deg P$. In this case*

$$\mathtt{p}_1 = \mathtt{p}, \mathtt{s}_1 = \mathtt{s} \text{ and } 0 \leq \mathtt{q}_1 < \mathtt{q}.$$

2. *If $|c_0| < |c_n|$, then $P_1(z) = (\bar{c}_n P(z) - c_0 P^*(z))/z$ is of degree $n-1$. In this case*

$$\mathtt{p}_1 = \mathtt{p} - 1, \mathtt{s}_1 = \mathtt{s} \text{ and } \mathtt{q}_1 = \mathtt{q}.$$

3. *If there is an nonnegative integer $k \leq n/2$ such that*

$$c_0 = \sigma\bar{c}_n,\ \ c_1 = \sigma\bar{c}_{n-1}, \ldots, c_{k-1} = \sigma\bar{c}_{n-k+1},\ \ c_k \neq \sigma\bar{c}_{n-k} \quad \text{with } |\sigma| = 1$$

(equivalently $P(z) \in \mathscr{D}_{3,n}$, since $|\sigma| = 1$ imply $|c_n| = |c_0|$), then define $b = (c_{n-k} - \sigma\bar{c}_k)/c_n$,

$$g(z) = \left(z^k + \frac{2b}{b}\right)P(z), \quad g_1(z) = \overline{g}(0)g(z) - \overline{g}^*(0)g^*(z)$$

and

$$P_1(z) = \frac{1}{z}\left[g_1^*(0)g_1(z) - g_1(0)g_1^*(z)\right],$$

which yields that P_1 *is of degree* $n-1$*. In detail,*

$$\mathtt{p}_1 = \mathtt{p} - 1, \mathtt{s}_1 = \mathtt{s}, \ \textit{and}\ \mathtt{q}_1 = \mathtt{q}.$$

4. *If* $P(z)$ *is self-inversive (equivalently* $P(z) \in \mathscr{D}_{4,n}$*), then* $P_1(z) = nP(z) - zP'(z)$ *is not identically zero, and we have* $\deg P_1 < \deg P$ *and*

$$\mathtt{p}_1 = \mathtt{p}, \mathtt{s} = n - 2\mathtt{p}, \mathtt{s}_1 + \mathtt{q}_1 < \mathtt{s} + \mathtt{q}.$$

□

Remark 3 A qualitative root description of the Schur–Cohn maps is as follows.

- If $P(z) \in \mathscr{D}_{1,n}$ has $\mathtt{p}, \mathtt{s}, \mathtt{q}$ roots in $\mathbb{D}$, $\partial\mathbb{D}$, $\mathbb{C}\backslash\overline{\mathbb{D}}$, then the map $R_{1,n}$ removes $\ell \geq 1$ from the original $\mathtt{q}$ roots and relocates $\mathtt{p}$ roots in $\mathbb{D}$, $\mathtt{s}$ roots in $\partial\mathbb{D}$, $\mathtt{q} - \ell$ roots in $\mathbb{C}\backslash\overline{\mathbb{D}}$.
- If $P(z) \in \mathscr{D}_{2,n}$ has $\mathtt{p}, \mathtt{s}, \mathtt{q}$ roots, then the map $R_{2,n}$ removes one of the original $\mathtt{p}$ roots and relocates $\mathtt{p} - 1$ roots in $\mathbb{D}$, $\mathtt{s}$ roots in $\partial\mathbb{D}$, $\mathtt{q}$ roots in $\mathbb{C}\backslash\overline{\mathbb{D}}$.
- If $P(z) \in \mathscr{D}_{3,n}$ has $\mathtt{p}, \mathtt{s}, \mathtt{q}$ roots, then $R_{3,n}$ removes one of the $\mathtt{q}$ roots and relocates $\mathtt{p}$ roots in $\mathbb{D}$, $\mathtt{s}$ roots in $\partial\mathbb{D}$, $\mathtt{q} - 1$ roots in $\mathbb{C}\backslash\overline{\mathbb{D}}$.
- $R_{4,n}$ acts on the roots as in the fourth assertion, and it is the restriction of a linear submersion.

Remark 4 1. $R_{\alpha,n}$ are real analytic maps in coefficient coordinates $\{(c_n, \ldots, c_0)\}$, for $\alpha = 1, \ldots, 4$.
2. For $n \geq 2$, the Schur polynomials $\mathscr{S}_n \subset \mathscr{D}_{2,n} = \{|c_n| > |c_0|\}$ determine an open but not dense subset in this component.
   Similar properties are fulfilled by the anti-Schur $a\mathscr{S}_n$, polynomials in $\mathscr{D}_{2,n} = \{|c_n| > |c_0|\}$.
   The set $us\mathscr{S}$ is open and has two connected components coming from $\{|c_n| > |c_0|\}$ and $\{|c_n| < |c_0|\}$.
3. The *topological boundary* $\partial\mathscr{S}_n$ of $\mathscr{S}_n$ has the following property; the intersection $\partial\mathscr{S}_n \cap \{c_n = 0\}$ is only the polynomial $P(z) \equiv 0$. On the whole $\mathbb{C}[z]_{=n}$, we have

$$\partial\mathscr{S}_n = s\mathscr{S}_n \cup \{\text{polynomials with all roots in the unitary circle}\}.$$

4 Geometry of Schur–Cohn maps in coefficient coordinates

Our first goal is the study of $R_{\alpha,n},\ \alpha \in 1, 2$, in the domain where they are nonsingular maps. This domain of regular points corresponds to $\mathscr{D}_{\alpha,n}$ in $\mathbb{C}[z]_{=n}\backslash\Sigma_n$, where we get associated G-bundle structures.

Let us consider $\mathbb{C}$, $\mathbb{C}^*$ and $\mathbb{S}^1$ the additive, multiplicative and circle Lie groups, respectively.

In Aguirre-Hernández et al. [2,3], introduce a vector bundle structure on the space of monic Schur polynomials of n degree. This geometric structure is enlarged on $\mathbb{C}[z]_{=n}\backslash\Sigma_n$ as follows.

Theorem 2 1. *The map*

$$R_{1,n} : \mathscr{D}_{1,n} \subset \mathbb{C}[z]_{=n} \longrightarrow \mathbb{C}^{n-1} \times \mathbb{R}^+ \tag{14}$$

is a submersion and determines a trivial principal $\mathbb{C}^* \times \mathbb{S}^1$*-bundle.*

2. *The map*

$$R_{2,n} : \mathscr{D}_{2,n} \subset \mathbb{C}[z]_{=n} \longrightarrow \mathbb{R}^+ \times \mathbb{C}^{n-1} \tag{15}$$

is a submersion and determines a trivial principal $(\mathbb{C} \times \mathbb{S}^1)$*-bundle.*

These submersions and G-bundle structures are in the real analytic category.

An interesting point is that the submersion $R_{2,n}$ in (15) is well defined in the whole component $\mathscr{D}_{2,n}$, even when Schur polynomials $\mathscr{S}_n$ are strictly proper subsets for $n \geq 3$, similarly for anti-Schur polynomials $a\mathscr{S}_n \subsetneq \mathscr{D}_{1,n}$ in (14).

Proof Firstly, we study the map $R_{2,n}$ in full detail.
Step 1 $R_{2,n}$ is a submersion in the domain $\mathscr{D}_{2,n}$. By simple inspection, using (11) we note that

$$R_{2,n}(\mathscr{D}_{2,n}) = \mathbb{R}^+ \times \mathbb{C}^{n-1} \subsetneq \mathscr{D}_{2,n-1}.$$

For a description of polynomials of type $P(z) \in \mathscr{D}_{2,2}\backslash\mathscr{S}_2$, see Example 13. We introduce the following map:

$$\begin{aligned}
\Xi_{2,n} : \mathbb{C} \times \mathbb{S}^1 \times \mathbb{R}^+ \times \mathbb{C}^{n-1} &\longrightarrow \mathscr{D}_{2,n} \\
(s, \mathrm{e}^{i\theta}, b_{n-1}, b_{n-2}, \ldots, b_0) &\longmapsto (c_n, \ldots, c_0) \\
&= \Bigg(\sqrt{b_{n-1} + |s|^2}\mathrm{e}^{i\theta}, \frac{\sqrt{b_{n-1}+|s|^2}\mathrm{e}^{i\theta} b_{n-2}}{b_{n-1}} \\
&\quad + \frac{\overline{b}_0 s}{b_{n-1}}, \ldots, \frac{\sqrt{b_{n-1}+|s|^2}\mathrm{e}^{i\theta} b_0}{b_{n-1}} + \frac{\overline{b}_{n-2} s}{b_{n-1}}, s \Bigg) \\
&= \frac{\mathrm{e}^{i\theta}}{b_{n-1}} \sqrt{b_{n-1} + |s|^2} \Big(b_{n-1}, b_{n-2}, \ldots, b_0, 0 \Big) \\
&\quad + \frac{s}{b_{n-1}} \Big(0, \overline{b}_0, \ldots, \overline{b}_{n-2}, \overline{b}_{n-1} \Big)
\end{aligned} \tag{16}$$

by assumption $b_{n-1} \in \mathbb{R}^+$. Note that the condition $|c_n| > |c_0|$ matches in the last line above.

Thus, $\Xi_{2,n}$ is a real analytic map, having inverse

$$\begin{aligned}
\Xi_{2,n}^{-1} : \mathscr{D}_{2,n} &\longrightarrow \mathbb{C} \times \mathbb{S}^1 \times \mathbb{R}^+ \times \mathbb{C}^{n-1} \\
(c_n, \ldots, c_0) &\longmapsto \big(c_0, \mathrm{e}^{i\theta_n}, b_{n-1}, \ldots, b_1, b_0 \big) \\
&= \big(c_0, \mathrm{e}^{i\theta_n}, |c_n|^2 - |c_0|^2, c_{n-1}\overline{c}_n - c_0\overline{c}_1, \ldots, c_2\overline{c}_n \\
&\quad - c_0\overline{c}_{n-2}, c_1\overline{c}_n - c_0\overline{c}_{n-1} \big),
\end{aligned} \tag{17}$$

here we use the retraction

$$\rho : \mathbb{C}\backslash\{0\} \longrightarrow \mathbb{S}^1, \quad c_n \longmapsto \frac{c_n}{|c_n|} = \mathrm{e}^{i\theta_n}. \tag{18}$$

Observe that $\Xi_{2,n}^{-1}$ restricted to the last n coordinates coincides with $R_{2,n}$. In fact,

$$\Xi_{2,n}^{-1} \circ \Xi_{2,n} = Id, \qquad \Xi_{2,n} \circ \Xi_{2,n}^{-1} = Id. \tag{19}$$

We compute the first composition as follows:

$$\begin{aligned}
&\Xi_{2,n}^{-1} \circ \Xi_{2,n}(s, \mathrm{e}^{i\theta}, b_{n-1}, b_{n-2}, \ldots, b_1, b_0) \\
&\quad = \Xi_{2,n}^{-1}\Bigg(\sqrt{b_{n-1}+|s|^2}\mathrm{e}^{i\theta}, \frac{\sqrt{b_{n-1}+|s|^2}\mathrm{e}^{i\theta} b_{n-2}}{b_{n-1}} \\
&\qquad + \frac{\overline{b}_0}{b_{n-1}} s, \ldots, \frac{\sqrt{b_{n-1}+|s|^2}\mathrm{e}^{i\theta} b_0}{b_{n-1}} + \frac{\overline{b}_{n-2}}{b_{n-1}} s, s\Bigg) \\
&\quad = \Bigg(s, \mathrm{e}^{i\theta}, b_{n-1}, \left[\frac{\sqrt{b_{n-1}+|s|^2}\mathrm{e}^{i\theta} b_{n-2}}{b_{n-1}} + \frac{\overline{b}_0}{b_{n-1}} s\right]\sqrt{b_{n-1}+|s|^2}\mathrm{e}^{-i\theta} \\
&\qquad - s\left[\frac{\sqrt{b_{n-1}+|s|^2}\mathrm{e}^{-i\theta}\overline{b}_0}{b_{n-1}} + \frac{b_{n-2}}{b_{n-1}}\overline{s}\right], \ldots, \\
&\qquad \left[\frac{\sqrt{b_{n-1}+|s|^2}\mathrm{e}^{i\theta} b_0}{b_{n-1}} + \frac{\overline{b}_{n-2}}{b_{n-1}} s\right]\sqrt{b_{n-1}+|s|^2}\mathrm{e}^{-i\theta} \\
&\qquad - s\left[\frac{\sqrt{b_{n-1}+|s|^2}\mathrm{e}^{-i\theta}\overline{b}_{n-2}}{b_{n-1}} + \frac{b_0}{b_{n-1}}\overline{s}\right]\Bigg) \\
&\quad = \left(s, \mathrm{e}^{i\theta}, b_{n-1}, \frac{b_{n-2}}{b_{n-1}}(b_{n-1}|s|^2)\frac{b_{n-2}}{b_{n-1}}|s|^2, \ldots, \frac{b_0}{b_{n-1}}(b_{n-1}+|s|^2) - \frac{b_0}{b_{n-1}}|s|^2\right) \\
&\quad = (s, \mathrm{e}^{i\theta}, b_{n-1}, b_{n-2}, \ldots, b_1, b_0).
\end{aligned}$$

Looking at the other composition, we observe

$$\begin{aligned}
&\Xi_{2,n} \circ \Xi_{2,n}^{-1}(c_n, c_{n-1}, \ldots, c_0) \\
&\quad = \Xi_n(s, \mathrm{e}^{i\theta_n}, |c_n|^2 - |c_0|^2, c_{n-1}\overline{c}_n - c_0\overline{c}_1, c_{n-2}\overline{c}_n - c_0\overline{c}_2, \ldots, c_1\overline{c}_n - c_0\overline{c}_{n-1}) \\
&\quad = \Bigg(\sqrt{|c_n|^2 - |c_0|^2 + |s|^2}\mathrm{e}^{i\theta_n}, \frac{\sqrt{|c_n|^2 - |c_0|^2 + |s|^2}\mathrm{e}^{i\theta_n}(c_{n-1}\overline{c}_n - c_0\overline{c}_1)}{|c_n|^2 - |c_0|^2} \\
&\qquad + \frac{\overline{c_1\overline{c}_n - c_0\overline{c}_{n-1}}}{|c_n|^2 - |c_0|^2} s, \\
&\qquad \frac{\sqrt{|c_n|^2 - |c_0|^2 + |s|^2}\mathrm{e}^{i\theta_n}(c_{n-2}\overline{c_n} - c_0\overline{c}_2)}{|c_n|^2 - |c_0|^2} + \frac{\overline{c_2\overline{c}_n - c_0\overline{c}_{n-2}}}{|c_n|^2 - |c_0|^2} s, \ldots, s\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&= \left(|c_n|\mathrm{e}^{i\theta}, \frac{|c_n|\mathrm{e}^{i\theta_n}(c_{n-1}\overline{c}_n - c_0\overline{c}_1)}{|c_n|^2 - |c_0|^2} + \frac{(\overline{c}_1 c_n - \overline{c}_0 c_{n-1})c_0}{|c_n|^2 - |c_0|^2}, \dots, s \right) \\
&= \left(c_n, \frac{c_{n-1}|c_n|^2 - |c_0|^2 c_{n-1}}{|c_n|^2 - |c_0|^2}, \frac{c_{n-2}|c_n|^2 - |c_0|^2 c_{n-2}}{|c_n|^2 - |c_0|^2}, \dots, s \right) \\
&= (c_n, c_{n-1}, \dots, c_0).
\end{aligned}
$$

Step 2 Each preimage $R_{2,n}^{-1}(b_{n-1}, \dots, b_0)$ of (11) is diffeomorphic to the Lie group $\mathbb{C} \times \mathbb{S}^1$ and it is provided with a natural action. Let $P(z) \in \mathscr{D}_{2,n}$ be a polynomial, we recall that by definition $P_1(z) = R_{2,n}(P(z))$ and $b_{n-1} = |c_n|^2 - |c_0|^2$. By using (6), (18), (19) and the identification in Remark 1, we get the following decomposition:

$$
\begin{aligned}
P(z) &= c_n z^n + \dots + c_1 z + c_0 = (c_n, \dots, c_0) \\
&= \frac{\mathrm{e}^{i\theta_n}}{b_{n-1}} \sqrt{b_{n-1} + |c_0|^2} \left(|c_n|^2 - |c_0|^2, c_{n-1}\overline{c}_n - c_0\overline{c}_1, \dots, c_1\overline{c}_n - c_0\overline{c}_{n-1}, 0 \right) \\
&\quad + \frac{c_0}{b_{n-1}} \left(0, \overline{c}_1 c_n - \overline{c}_0 c_{n-1}, \dots, \overline{c}_{n-1} c_n - \overline{c}_0, c_1, |c_n|^2 - |c_0|^2 \right) \\
&= \frac{\mathrm{e}^{i\theta_n}}{b_{n-1}} \sqrt{b_{n-1} + |c_0|^2} z P_1(z) + \frac{c_0}{b_{n-1}} P_1^*(z),
\end{aligned}
$$

here $(\,)^*$ is the operator in Eq. (6). The inverse image of $P_1(z) = b_{n-1}z^{n-1} + \dots + b_0 \in \mathbb{R}^+ \times \mathbb{C}^{n-1}$ is

$$
R_{2,n}^{-1}(b_{n-1}, \dots, b_0) = \left\{ \frac{\mathrm{e}^{i\theta_n}}{b_{n-1}} \sqrt{b_{n-1} + |s|^2} z P_1(z) + \frac{s}{b_{n-1}} P_1^*(z) \,\middle|\, s \in \mathbb{C},\ \mathrm{e}^{i\theta_n} \in \mathbb{S}^1 \right\}. \tag{20}
$$

Moreover, new coordinates on $\mathscr{D}_{2,n}$ are as follows:

$$
\begin{aligned}
\mathscr{D}_{2,n} = \Bigg\{ P(z) = \frac{\mathrm{e}^{i\theta_n}}{b_{n-1}} \sqrt{b_{n-1} + |c_0|^2} z P_1(z) + \frac{c_0}{b_{n-1}} P_1^*(z) \,\Bigg|\, c_0 \in \mathbb{C},\ \ \mathrm{e}^{\theta_n} \in \mathbb{S}^1, \\
P_1(z) \in \mathbb{R}^+ \times \mathbb{C}^{n-1} \Bigg\}.
\end{aligned} \tag{21}
$$

On these coordinates, the Lie group action admits a plain expression

$$
\begin{aligned}
\mathscr{A}_{2,n} : \mathbb{C} \times \mathbb{S}^1 \times \mathscr{D}_{2,n} &\longrightarrow \mathscr{D}_{2,n} \\
\left(s, \mathrm{e}^{i\theta}, \tfrac{\mathrm{e}^{i\theta_n}}{b_{n-1}} \sqrt{b_{n-1} + |c_0|^2} z P_1(z) + \tfrac{c_0}{b_{n-1}} P_1^*(z) \right) &\longmapsto \tfrac{\mathrm{e}^{i(\theta_n + \theta)}}{b_{n-1}} \sqrt{b_{n-1} + |c_0 + s c_n|^2} z P_1(z) \\
&\qquad + \tfrac{c_0 + s c_n}{b_{n-1}} P_1^*(z),
\end{aligned} \tag{22}
$$

where $\mathrm{e}^{i\theta_n} = c_n/|c_n|$. Recalling that $P_1(z) = R_{2,n}(P(z))$, for any $s_1,\ s_2 \in \mathbb{C}$ and $\mathrm{e}^{i\theta_1},\ \mathrm{e}^{i\theta_2} \in \mathbb{S}^1$, the associative rule of $\mathscr{A}_{2,n}$ is as follows:

$$
\left(s_1 + s_2, \mathrm{e}^{i\theta_1 + i\theta_2}, \frac{\mathrm{e}^{i\theta_n}}{b_{n-1}} \sqrt{b_{n-1} + |c_0|^2} z P_1(z) + \frac{c_0}{b_{n-1}} P_1^*(z) \right)
$$

$$\longmapsto \frac{\mathrm{e}^{i(\theta_n+\theta_1+\theta_2)}}{b_{n-1}}\sqrt{b_{n-1}+|c_0+(s_1+s_2)c_n|^2}zP_1(z)+\frac{c_0+(s_1+s_2)c_n}{b_{n-1}}P_1^*(z)$$
$$=\left(s_2, \mathrm{e}^{i\theta_2}, \frac{\mathrm{e}^{i(\theta_n+\theta_1)}}{b_{n-1}}\sqrt{b_{n-1}+|c_0+s_1c_n|^2}zP_1(z)+\frac{c_0+s_1c_n}{b_{n-1}}P_1^*(z)\right).$$

There exists a global section

$$\begin{aligned}&\varXi_{2,n}:\mathbb{R}^+\times\mathbb{C}^{n-1}\longrightarrow\mathscr{D}_{2,n},\\ &\varXi_{2,n}(0,1,b_{n-1},\ldots,b_0)=\left(\sqrt{b_{n-1}}, \frac{b_{n-2}}{\sqrt{b_{n-1}}},\ldots,\frac{b_0}{\sqrt{b_{n-1}}},0\right).\end{aligned}$$

By using the identification of Remark 1, the map $\varXi_{2,n}(0,1,\ldots,b_0)$ determines a section of the bundle. Hence, the principal $\mathbb{C}\times\mathbb{S}^1$-bundle in (15) is trivial.

In coefficient coordinates of $\mathscr{D}_{2,n}\subsetneq\mathbb{C}[z]_{=n}$, the action $\mathscr{A}_{2,n}$ is

$$\begin{aligned}(s,\mathrm{e}^{i\theta},c_n,c_{n-1},\ldots,c_0)=\Bigg(&\sqrt{|c_n|^2-|c_0|^2+|(c_0+sc_n)|^2}\frac{c_n}{|c_n|}\mathrm{e}^{i\theta},\\
&\frac{c_n}{|c_n|}\frac{\sqrt{|c_n|^2-|c_0|^2+|(c_0+sc_n)|^2}\mathrm{e}^{i\theta}(c_{n-1}\bar{c}_n-c_0\bar{c}_1)}{|c_n|^2-|c_0|^2}\\
&+\overline{\frac{c_1\bar{c}_n-c_0\bar{c}_{n-1}}{|c_n|^2-|c_0|^2}}(c_0+sc_n),\\
&\frac{c_n}{|c_n|}\frac{\sqrt{|c_n|^2-|c_0|^2+|(c_0+sc_n)|^2}\mathrm{e}^{i\theta}(c_{n-2}\bar{c}_n-c_0\bar{c}_2)}{|c_n|^2-|c_0|^2}\\
&+\overline{\frac{c_2\bar{c}_n-c_0\bar{c}_{n-2}}{|c_n|^2-|c_0|^2}}(c_0+sc_n),\\
&\times\frac{c_n}{|c_n|}\frac{\sqrt{|c_n|^2-|c_0|^2+|(c_0+sc_n)|^2}\mathrm{e}^{i\theta}(c_1\bar{c}_n-c_0\bar{c}_{n-1})}{|c_n|^2-|c_0|^2}\\
&+\overline{\frac{c_{n-1}\bar{c}_n-c_0\bar{c}_1}{|c_n|^2-|c_0|^2}}(c_0+sc_n),(c_0+sc_n)\Bigg).\end{aligned}$$

Step 3 The description of $R_{1,n}$ in the first assertion 1 of Theorem 2 follows the analogous steps. In order to show that $R_{1,n}$ is a submersion on the domain $\mathscr{D}_{1,n}$, we introduce the following map:

$$\begin{aligned}\varXi_{1,n}:\mathbb{C}^*\times\mathbb{S}^1\times\mathbb{R}^+\times\mathbb{C}^{n-1}&\longrightarrow\mathscr{D}_{1,n}\\
(w,\mathrm{e}^{i\theta},b_{n-1},\ldots,b_0)&\longmapsto(c_n,\ldots,c_0)\\
&=\Bigg(w,\frac{w\bar{b}_1}{b_0}+\frac{\sqrt{b_0+|w|^2}\mathrm{e}^{i\theta}b_{n-1}}{b_0},\ldots,\frac{w\bar{b}_\ell}{b_0}\\
&\quad+\frac{\sqrt{b_0+|w|^2}\mathrm{e}^{i\theta}b_{n-\ell}}{b_0},\ldots,\sqrt{b_0+|w|^2}\mathrm{e}^{i\theta}\Bigg),\end{aligned}\tag{23}$$

Two observations are in order; w coincides with the leader term coefficient $c_n z^n$ of the resulting polynomial $P(z) \in \mathscr{D}_{1,n}$, and by assumption $(b_0 + |w|^2) \in \mathbb{R}^+$. It follows that $\Xi_{1,n}$ is a real analytic map.

Recall that by definition $P_1(z) = R_{1,n}(P(z))$. The inverse image of $P_1(z) = b_{n-1}z^{n-1} + \cdots + b_0 \in \mathbb{C}^{n-1} \times \mathbb{R}^+$ is

$$R_{1,n}^{-1}(b_{n-1}, \ldots, b_0) = \left\{ \frac{w}{b_0} z P_1^*(z) + \frac{\mathrm{e}^{i\theta_0}}{b_0} \sqrt{b_0 + |w|^2} P_1(z) \;\middle|\; w \in \mathbb{C}^*,\ \mathrm{e}^{i\theta_0} \in \mathbb{S}^1 \right\}. \tag{24}$$

Moreover, new coordinates on $\mathscr{D}_{1,n}$ are as follows:

$$\mathscr{D}_{1,n} = \left\{ P(z) = \frac{c_n}{b_0} z P_1^*(z) + \frac{\mathrm{e}^{i\theta_0}\sqrt{b_0 + |c_n|^2}}{b_0} P_1(z) \;\middle|\; c_n \in \mathbb{C}^*,\ \ \mathrm{e}^{i\theta_0} \in \mathbb{S}^1, \right.$$
$$\left. P_1(z) \in \mathbb{C}^{n-1} \times \mathbb{R}^+ \right\}. \tag{25}$$

A straightforward computation shows that the coefficient of z^n in $P(z)$ is c_n.

By requiring Lie group variables w and $\mathrm{e}^{i\theta}$, the $\mathbb{C}^* \times \mathbb{S}^1$-action can be recognized as

$$\mathscr{A}_{1,n} : \mathbb{C}^* \times \mathbb{S}^1 \times \mathscr{D}_{1,n} \longrightarrow \mathscr{D}_{1,n}$$
$$\left(w, \mathrm{e}^{i\theta}, \tfrac{c_n}{b_0} z P_1^*(z) + \tfrac{\mathrm{e}^{i\theta_0}\sqrt{b_0+|c_n|^2}}{b_0} P_1(z) \right) \longmapsto \tfrac{wc_n}{b_0} z P_1^*(z) + \tfrac{\mathrm{e}^{i(\theta_0+\theta)}\sqrt{b_0+|wc_n|^2}}{b_0} P_1(z), \tag{26}$$

the computation follows as above. The trivial principal $\mathbb{C}^* \times \mathbb{S}^1$-bundle in Eq. (14) is done. □

Corollary 1 1. *Let $P(z) \in \mathscr{D}_{2,n}$ be a polynomial, the following assertions are equivalent.*

(i) *The $\mathbb{C} \times \mathbb{S}^1$-orbit of $P(z)$*

$$\left\{ \frac{e^{i(\theta_n+\theta)}}{b_{n-1}} \sqrt{b_{n-1} + |c_0 + sc_n|^2} z P_1(z) + \frac{c_0 + sc_n}{b_{n-1}} P_1^*(z) \;\middle|\; s \in \mathbb{C},\ e^{i\theta} \in \mathbb{S}^1 \right\} \tag{27}$$

is contained in the Schur stable polynomials.

(ii) *The polynomial $P(z)$ is Schur stable.*

(iii) *The polynomial $P_1(z)$ is Schur stable.*

2. *The analogous assertions are true for anti-Schur polynomials considering $P(z) \in \mathscr{D}_{1,n}$ and its $\mathbb{C}^* \times \mathbb{S}^1$-orbit*

$$\left\{ \frac{wc_n}{b_0} z P_1^*(z) + \frac{e^{i(\theta_0+\theta)}\sqrt{b_0 + |wc_n|^2}}{b_0} P_1(z) \;\middle|\; w \in \mathbb{C}^*,\ e^{i\theta} \in \mathbb{S}^1 \right\}. \tag{28}$$

Proof The orbit (27) comes from the action $\mathscr{A}_{2,n}$ in (22). Similarly, the orbit (28) originates from the action $\mathscr{A}_{1,n}$ in (26). □

The geometrical meaning of the orbits (27) shows a certain weak notion of convexity, since the Schur stable set $\mathscr{S}_n$ and $\mathscr{D}_{2,n}$ are filled by $\mathbb{C} \times \mathbb{S}^1$-orbits.

5 Lie group actions and *n*-root configurations: a prototype

Our goal in this section is to explore Lie group actions on the space of complex polynomials of n degree, regarding the respective n-root configuration dynamics. We consider two ingredients in order to construct our main Definition 2.

5.1 Lie group actions and *n*-root configurations

Let G be a real (respectively, complex) Lie group; we are mainly interested when G is $\mathbb{S}^1$, $\mathbb{C}$ and $\mathbb{C}^*$. Let

$$\begin{aligned} \mathscr{A} : G \times \mathbb{C}[z]_{\leq n} &\longrightarrow \mathbb{C}[z]_{\leq n} \\ (g, P) &\longmapsto g \cdot P(z) \end{aligned} \tag{29}$$

be a real (respectively, complex) analytic action of G on the complex manifold of polynomials of degree at most n. The associated principal G-bundle is

$$\begin{array}{ccc} G & \xrightarrow{\ \mathtt{i}\ } & \mathscr{D}^0 \subseteq \mathbb{C}[z]_{\leq n} \\ & & \big\downarrow \Pi \\ & & \mathscr{D}^0/G, \end{array}$$

here $\mathscr{D}^0$ is the open manifold where $\mathscr{A}$ is proper and free from fixed points, see [13], Theorem 1.11.4, Chapter 1. Now we look at the orbits of the G-bundle

$$\{g \cdot P(z) \mid g \in G\}.$$

The Weierstrass preparation theorem [18], p. 8 and the work of Hansen [19,20] give origin to the following concept for each G-orbit.

Definition 1 1. A *Weierstrass polynomial of degree $n \geq 1$ over a Lie group G* is a map

$$\mathscr{P}(g, z) = c_n(g)z^n + \cdots + c_0(g) : G \times \mathbb{C} \longrightarrow \mathbb{C}, \tag{30}$$

where $c_\iota(g) : G \to \mathbb{C}$ are real analytic functions, $\iota \in \{0, \ldots, n\}$, and $c_n(g)$ is nonidentically zero.[2] Furthermore, if G is a complex Lie group, then we require $c_\iota(g)$ to be complex analytic.

2. For a fixed $g_0 \in G$, the *n-root configuration of $\mathscr{P}(g_0, z)$* is

$$\{\mathscr{P}(g_0, z) = 0\} = [z_1(g_0), \ldots, z_n(g_0)] \in \frac{\mathbb{C}^n}{\mathrm{Sym}(n)}. \tag{31}$$

[2] For simplicity, here we use degree n; however, the case $\leq n$ is also useful.

3. The set of *all the n-root configurations of* $\mathscr{P}(g, z)$ is

$$\mathscr{Z} = \{\mathscr{P}(g, z) = 0 \mid \forall\, (g, z)\} \subset G \times \mathbb{C}. \tag{32}$$

Example 2 (a) Let

$$\mathscr{P}(\theta, z) = z^3 - \mathrm{e}^{i\theta} : \mathbb{S}^1 \times \mathbb{C} \longrightarrow \mathbb{C}$$

be a Weierstrass polynomial of degree 3 over a Lie group $\mathbb{S}^1$. If $\mathrm{e}^{i\theta}$ varies over the whole circle, then the 3-root configurations of $\{\mathscr{P}(\theta, z) = 0 \mid \mathrm{e}^{i\theta} \in \mathbb{S}^1\}$ describe three arcs of angle $2\pi/3$ in $\mathbb{C}$. Considering the embedding of $\{\mathrm{e}^{i\theta}\}$ into $\mathbb{C}$ the target of the polynomial, we get Fig. 1a. See [28] for a general study of these kinds of knots and braids in $\mathbb{C} \times \mathbb{C}$.

(b) Consider $P(z)$ a complex polynomial of degree $n \geq 2$, and let

$$\{\mathscr{P}(s, z) = P(z) - s : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}\} \tag{33}$$

be a Weierstrass polynomial of degree $n \geq 1$ over the additive Lie group $\mathbb{C}$. Considering the embedding of s in the target $\mathbb{C}$ of $P(z)$, we get the following behavior. If s_1 varies in a neighborhood of regular values of $P(z)$, then the n-root configurations of $\{P(z) - s_1 = 0\}$ do not change topologically. If s_1 varies in a neighborhood of a critical value s_0 of $P(z)$, then the n-root configuration of $\{P(z) - s_1 = 0\}$ changes topologically at s_0. See Fig. 1b. for an sketch. We will describe both situations accurately in Lemma 1 and Fig. 4a.

Remark 5 Throughout this work, we identify each orbit of a G-action $\mathscr{A}$ to a Weierstrass polynomial over G

$$\{g \cdot P(z) \mid g \in G\} = \{\mathscr{P}(g, z) = c_n(g) z^n + \cdots + c_0(g) : G \times \mathbb{C} \longrightarrow \mathbb{C}\}, \tag{34}$$

in the sense of (30). Clearly the converse is not true; a Weierstrass polynomial over G as in (30) does not always come from the orbit of a suitable G-action.

As in [19,20], the induced n-fold branched polynomial covering map Π is

$$\begin{tikzcd} \mathscr{Z} \arrow[r, hook, "I"] \arrow[dr, "\Pi"'] & G \times \mathbb{C} \arrow[r, "\mathrm{proj}_2"] \arrow[d, "\mathrm{proj}_1"] & \mathbb{C} \\ & G & \end{tikzcd} \tag{35}$$

here I is the natural embedding and proj_1, proj_2 are the two natural projections.

Example 3 Let $P(z)$ be a polynomial of degree $n \geq 1$. The set of all the n-root configurations

$$\mathscr{Z} = \{\mathscr{P}(s, z) = P(z) - s = 0\} \subset \mathbb{C} \times \mathbb{C}$$

is essentially the graph of $P(z)$.

The second ingredient is the dynamics of vector fields. Let

$$\mathbb{X}(z) = f(z)\frac{\partial}{\partial z}$$

be a *singular complex analytic vector field* on the plane $\mathbb{C}$ or the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, here the adjective *singular* means that $f(z)$ can admits as singularities zeros, poles and isolated essential singularities; denoted as $\mathrm{Sing}(\mathbb{X}(z))$, see [5]. In the domain where $\mathbb{X}(z)$ is holomorphic, the vector field $\mathbb{X}(z)$ determines an a priori local holomorphic flow:

$$\varphi : \mathbb{D}^2\big((s_0, z_0), \epsilon\big) \subset \big(\mathbb{C} \times \mathbb{C}\backslash(\mathrm{Sing}(\mathbb{X}(z)))\big) \longrightarrow \mathbb{C}, \quad (s, z) \longmapsto \varphi(s, z), \tag{36}$$

where s is the complex time, $\mathbb{D}^2\big((s_0, z_0), \varepsilon\big)$ is an open two-dimensional disk centered at (s_0, z_0) with radius $\varepsilon > 0$.

Given a continuous time path $\gamma(\tau) : [0, 1] \longrightarrow \mathbb{C}$ such that $\varphi(\gamma(0), z_0) = z_0 \in \mathbb{C}\backslash\{\mathrm{Sing}(\mathbb{X}(z))\}$, we consider the analytic continuation of the flow $\varphi(\gamma(\tau), z_0)$ starting at the initial condition z_0. Depending on the path and the initial condition, the analytic continuation of $\varphi(\ , z_0)$ can be well defined or not.

In summary, $\varphi(s, z)$ is called the *flow*[3] of $\mathbb{X}(z)$ over the maximal domain under the analytic continuation process.

Now we explain the guideline diagram (4).

The coupling between the G-action $\mathscr{A}$, their G-orbits as Weierstrass polynomials $\mathscr{P}(s, z) : G \times \mathbb{C} \longrightarrow \mathbb{C}$ and the flow $\varphi(s,\ )$ of a vector field $\mathbb{X}(z)$ is given essentially by the Viète map as follows.

Definition 2 Consider $\mathscr{A} : G \times \mathbb{C}[z]_{=n} \longrightarrow \mathbb{C}[z]_{=n}$ an additive $\mathbb{C}$-action. Let $\mathscr{P}(g, z)$ be a Weierstrass polynomial coming from $\mathscr{A}$ and having $\big\{\mathscr{P}(0, z) = 0\big\} = [z_1, \ldots, z_n]$ as n-root configuration. A complex analytic vector field $\mathbb{X}(z)$ on $\mathbb{C}$ *describes the n-root configuration dynamics* of $\mathscr{P}(s, z)$ if

$$\mathscr{P}(s, z) = c_n(s)\big(z - \varphi(s, z_1)\big) \cdots \big(z - \varphi(s, z_n)\big) \tag{37}$$

holds whenever the flow $\varphi(s,\ )$ of $\mathbb{X}(z)$ is well defined for a suitable function $c_n(s)$. For the case of a multiplicative $\mathbb{C}^*$-action, in (37) use the variable $w = \mathrm{e}^s \in \mathbb{C}^*$.

Remark 6 The use of s, the variable of the Lie group $\mathbb{C}$, as the time of the flow $\varphi(s,\ )$ of $\mathbb{X}(z)$ is a key trick in (37).

[3] As a matter of record, a Lie group *action* on a manifold, $G \times M \longrightarrow M$ is well defined for all the pairs $\{(g, p)\}$, whereas a *local action* is defined only for a certain open subset of pairs $\{(g, p)\}$; here we agree that *flows* can be local or global $\mathbb{C}$-actions.

In the case of $\mathbb{C}$-actions, Definition 2 is equivalent to the fact that the diagram

$$\begin{tikzcd}
[z_1, \dots, z_n] \arrow[r, "\mathscr{V}_n"] \arrow[d, "\varphi(s,\ )"'] & \mathscr{P}(0, z) = c_n(0)z^n + \cdots + c_0(0) \arrow[d, "\mathscr{A}(s,\ )"] \\
{[\varphi(s, z_1), \dots, \varphi(s, z_n)]} & \mathscr{P}(s, z) = c_n(s)z^n + \cdots + c_0(s) \arrow[l, "\mathscr{V}_n^{-1}"]
\end{tikzcd} \tag{38}$$

commutes. Here, $\mathscr{V}_n$ is the Viète map and the n-root configuration $\{\mathscr{P}(0, z) = 0\} = [z_1, \dots, z_n]$ is considered as the initial condition for $\varphi(s,\ )$ (or $\{\mathscr{P}(1, z) = 0\} = [z_1, \dots, z_n]$ in the case of $\mathbb{C}^*$-actions). The diagram (3) is a particular case.

Recall that in the domain where $\mathbb{X}(z) = (u(z) + iv(z))\frac{\partial}{\partial z}$ is holomorphic, $\mathbb{X}(z)$ is equivalent to two real analytic commuting vector fields

$$\mathfrak{Re}\,(\mathbb{X}(z)) = u(z)\frac{\partial}{\partial x} + v(z)\frac{\partial}{\partial y} \quad \text{and} \quad \mathfrak{Im}(\mathbb{X}(z)) = -v(z)\frac{\partial}{\partial x} + u(z)\frac{\partial}{\partial y},$$

see [25], p. 1213 and [26], p. 234. Hence, the complex flow $\varphi(g,\ )$ in diagram (38) is equivalent to the two real commuting flows (of the above pair of vector fields); one for pure real time and the second for pure imaginary time.

5.2 The prototype

By reviewing Example 2b and Definition 2, we want to show that each arbitrary n-point configuration $[z_1, \dots, z_n] \subset \mathbb{C}$ determines the n-root configuration dynamics of a suitable Weierstrass polynomial $\mathscr{P}(s, z)$.

Lemma 1 (The prototype) *Let*

$$\begin{aligned} \mathscr{A} : \mathbb{C} \times \mathbb{C}[z]_{=n} &\longrightarrow \mathbb{C}[z]_{=n} \\ (s,\ c_n z^n + \cdots + c_1 z + c_0) &\longmapsto c_n z^n + \cdots + c_1 z + (c_0 - s) \end{aligned} \tag{39}$$

be a holomorphic action of the additive group $\mathbb{C}$. For each orbit

$$\mathscr{P}(s, z) = c_n z^n + \cdots + c_1 z + (c_0 - s) : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C},$$

the rational vector field

$$\mathbb{X}(z) = \frac{1}{P(z)'}\frac{\partial}{\partial z} \quad \textit{on } \mathbb{C}$$

describes the n-root configuration dynamics of $\mathscr{P}(s, z)$.

Proof The key idea is to look at any polynomial

$$P(z) = c_n \Pi_{\iota=1}^{n}(z - z_\iota) : \mathbb{C} \longrightarrow \mathbb{C}$$

determined by a configuration of simple roots $[z_1, \dots, z_n]$ and $c_n \in \mathbb{C}^*$, by construction 0 is a regular value of $P(z)$.

Let $\{\mathscr{P}(s, z) = 0\} = \{P(z) + s = 0\}$ be the n-root configurations of the orbit of $P(z)$ under $\mathscr{A}$.

We associate with $P(z)$ a polynomial 1-form and a complex rational vector field, as follows:

$$P(z) \longleftrightarrow \quad \omega(z) = P'(z)\mathrm{d}z \quad \longleftrightarrow \quad \mathbb{X}(z) = \frac{1}{P'(z)} \frac{\partial}{\partial z}.$$

In addition, $\omega(z)$ and $\mathbb{X}(z)$ enjoy the two (equivalent) properties

$$\omega(z)\big(\mathbb{X}(z)\big) \equiv 1 \quad \text{and} \quad P_*\big(\mathbb{X}(z)\big) = \frac{\partial}{\partial s},$$

where P_* denotes the pushforward and s is the time of the associated ordinary differential equation $\frac{\mathrm{d}z}{\mathrm{d}s} = (P'(z))^{-1}$.

Consider n open disks $\mathbb{D}(z_\iota, \epsilon) \subset \mathbb{C}$ with center at z_ι and a small enough radius $\epsilon > 0$, $\iota \in \{1, \dots, n\}$ such that

$$P(z) : \mathbb{D}(z_\iota, \epsilon) \subset \mathbb{C} \longrightarrow \mathbb{C}$$

are local biholomorphisms. The local flows $\{\varphi(s, z_\iota) \mid \iota \in 1, \dots, n\}$ of $\mathbb{X}(z)$, for complex time $\{s \bigm| |s| < \epsilon\}$, are well defined. Using $\omega(\mathbb{X}) \equiv 1$, it follows that

$$P(\varphi(s, z_\iota)) = s \quad \text{for all } \iota \in \{1, \dots, n\}.$$

The diagram (38) commutes whenever the analytic continuation of $\varphi(s, \ \ )$ is well defined. □

Example 4 Figure 2 shows phase portraits of vector fields $\mathbb{X}(z) = (P'(z))^{-1}\frac{\partial}{\partial z}$ describing the 3-root dynamics of Schur polynomials in $\mathbb{C}$ of $\mathscr{P}(s, z) = z(z-1/2)(z-(1+i)/2) - s$ and $\mathscr{Q}(s, z) = z(2z-1)^2 + 2(z-2)^2 - s$. In Fig. 2, the lines of flow with arrows come from the real part of the vector field $\mathfrak{Re}\,(\mathbb{X}(z))$ and the orthogonal lines originate from $\mathfrak{Im}\,(\mathbb{X}(z))$.

Let us recall a principal $\mathbb{C}$-bundle interpretation of the prototype. Given the Lie group action $\mathscr{A}$ in (39), the associated holomorphic trivial principal $\mathbb{C}$-bundle is

$$\begin{tikzcd} \mathbb{C} \arrow[r, "I"] & \mathbb{C}[z]_{=n} \arrow[d, "\varPi"] \\ & \mathbb{C}^n = \{(c_n, \dots, c_1)\}, \end{tikzcd}$$

where $\varPi : (c_n, \dots, c_0) \longmapsto (c_n, \dots, c_1)$ and I denotes the embedding of an orbit in the total space.

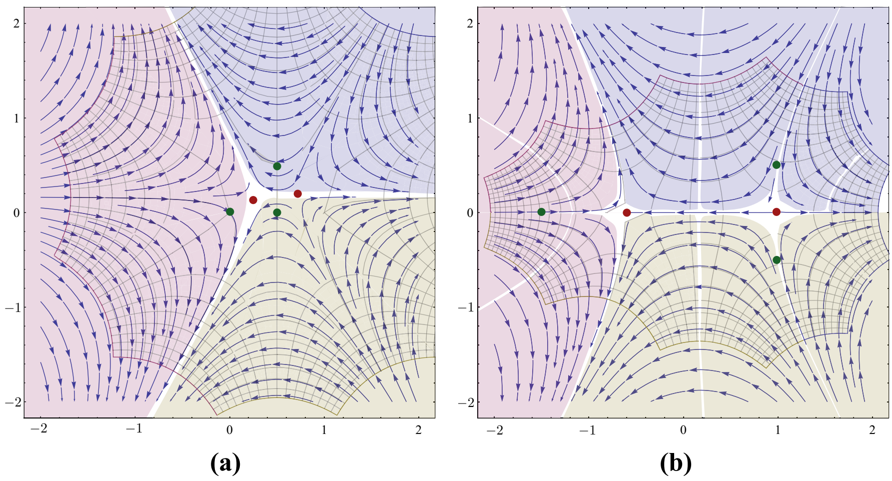

Fig. 2 Phase portraits **a** and **b** of vector fields $\mathbb{X}(z)$ in $\mathbb{C}$ describing 3-root configuration dynamics correspond to $\mathscr{P}(s, z)$ and $\mathscr{Q}(s, z)$ in Example 4. The green points correspond to $\{\mathscr{P}(0, z) = 0\}$ and $\{\mathscr{Q}(s, z) = 0\}$; the red points show the poles of the respective vector fields. Each color island originates from the complex flow of $\mathbb{X}(z)$ using a root (one green point) as an initial condition. The color islands show that any point z_0 can appear as a root of $\{\mathscr{P}(s_0, z) = 0\}$, for suitable complex time s_0

6 Rational vector fields and *n*-root configuration dynamics

6.1 Complex rational vector fields on the Riemann sphere $\widehat{\mathbb{C}}$

We enlarge the description of $\mathbb{X}(z)$ in the prototype in order to consider rational vector fields. Let

$$\mathbb{X}(z) = \frac{P(z)}{Q(z)}\frac{\partial}{\partial z} = \frac{\lambda(z-p_1)^{\mu_1}\cdots(z-p_\nu)^{\mu_\nu}}{(z-q_1)^{\kappa_1}\cdots(z-q_m)^{\kappa_m}}\frac{\partial}{\partial z} \quad \text{on} \quad \widehat{\mathbb{C}} = \mathbb{C}\cup\{\infty\} \tag{40}$$

be a rational vector field; here, $P(z)$, $Q(z)$ are polynomials without common factors and $\kappa,\ \mu \leq 0$ their respective degrees. The vector field $\mathbb{X}(z)$ extends rationally to the whole Riemann sphere $\widehat{\mathbb{C}}$. Moreover, a simple calculation shows that

$$\mu_1 + \cdots + \mu_\nu - \kappa_1 - \cdots - \kappa_m = \mu - \kappa = 2$$

on $\widehat{\mathbb{C}}$. This is equivalent to the fact that the point $\infty \in \widehat{\mathbb{C}}$ is of multiplicity $2-\kappa+\mu \in \mathbb{Z}$ for $\mathbb{X}(z)$; by definition the multiplicity of poles of $\mathbb{X}(z)$ is negative.

Following [7], p. 579, a singular complex analytic (probably multivalued) function $\psi : \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ is called *additively automorphic* if $d\psi$ is an univalued singular complex analytic 1-form on $\mathbb{C}$: i.e., for ψ_α, ψ_β any two branches of ψ,

$$\psi_\alpha(z) = \psi_\beta(z) + a_{\alpha\beta}, \quad a_{\alpha\beta} \in \mathbb{C}.$$

A concrete example is $\psi(z) = \ln(z)$.

Lemma 2 [25,26,30] *On the Riemann sphere* $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, *there are canonical correspondences between the following:*

1. *Complex analytic vector fields* $\mathbb{X}(z)$.
2. *Complex analytic 1-forms* $\omega(z)$.
3. *Multivalued additively automorphic complex analytic functions* $\Psi(z)$ *with a rational derivative.*
4. *Weierstrass polynomials* $\mathscr{P}(s, z) = \psi(z) - s$.

Proof By using (40), we define $\Psi(z) = \int_{z_0}^{z} \omega : \widehat{\mathbb{C}} \backslash \{p_\iota\}_{\iota=1}^{\nu} \longrightarrow \mathbb{C}$. Diagrammatically, we have

$$
\begin{tikzcd}
 & \mathbb{X}(z) = \frac{P(z)}{Q(z)}\frac{\partial}{\partial z} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & & \\
\omega(z) = \frac{P(z)}{Q(z)}\mathrm{d}z \arrow[rr, leftrightarrow] & & \Psi(z) = \int^{z} \frac{P(\zeta)}{Q(\zeta)} d\zeta \arrow[r, leftrightarrow] & \{\Psi(z) = s\}.
\end{tikzcd}
\tag{41}
$$

Obviously, s is the time parameter of $X(z)$. □

Recall that $\Psi(z) = \ln(z)$ provides an example with rational $\mathbb{X}(z) = z\frac{\partial}{\partial z}$. Hence, the enlarged hypothesis within multivalued additively automorphic complex analytic functions for $\Psi(z)$ is useful.

A singular complex analytic vector field $\mathbb{X}(z)$ on $\mathbb{C}$ is *complete* when its flow is well defined for all initial conditions and for all complex time s. It is a very restrictive condition; as is well known, $\mathbb{X}(z)$ is complete on $\mathbb{C}$ if and only if $\mathbb{X}(z) = (bz + c)\frac{\partial}{\partial z}$; see [24]. Moreover, the comprehension of rational vector fields (and their flows) at poles is required.

Proposition 1 (Complex analytic normal forms at poles and zeros of vector fields) *Let* $\mathbb{X}(z)$ *be a rational vector field germ on* $(\mathbb{C}, 0)$.

1. *If* 0 *is a pole of order* $-\kappa \leq -1$ *for* $\mathbb{X}(z)$, *then it is holomorphically equivalent to* $z^{-\kappa}\frac{\partial}{\partial z}$.
2. *If* 0 *is a zero of order one for* $\mathbb{X}(z)$, *then it is holomorphically equivalent to* $\lambda z\frac{\partial}{\partial z}$, *where* $\lambda = \mathbb{X}'(0)$.
3. *If* 0 *is a zero of order* $\mu \geq 2$ *for* $\mathbb{X}(z)$, *then it is holomorphically equivalent to* $\frac{z^{\mu}}{1+\lambda z^{\mu-1}}\frac{\partial}{\partial z}$, *here* $\lambda \in \mathbb{C}$ *is the residue of the associated rational differential 1-form* $\omega(z)$ *at 0.*

Proof The result is well known see [16,17,30]. The phase portraits are in Fig. 3. □

Now we introduce how the flow singularities of a rational vector field $\mathbb{X}(z)$ on $\widehat{\mathbb{C}}$ can be related to the n-root configuration dynamics of a Weierstrass polynomial $\mathscr{P}(s, z)$.

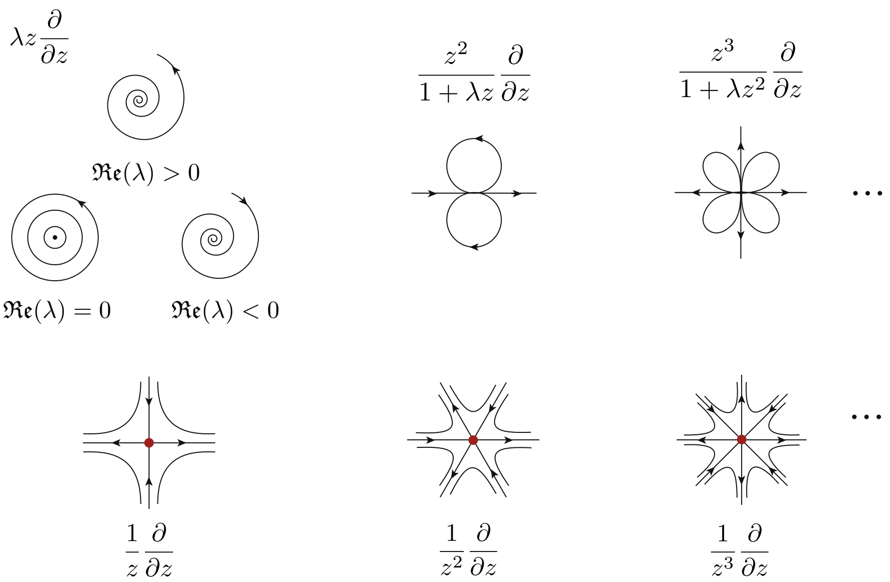

Fig. 3 Topological phase portraits of $\mathfrak{Re}\,(\mathbb{X}(z)) = u(z)\frac{\partial}{\partial x} + v(z)\frac{\partial}{\partial y}$ for the complex analytic normal forms of poles and zeros of $\mathbb{X}(z) = \big(u(z) + iv(z)\frac{\partial}{\partial x}\big)$

6.2 A dictionary between singular points of vector fields and *n*-root configuration dynamics

Definition 3 Let $\mathbb{X}(z)$ be a complex analytic vector field on the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ describing the n-root configuration dynamics of a Weierstrass polynomial $\mathscr{P}(s, z)$ of degree at most n.

1. A *fixed root* of $\{\mathscr{P}(s, z) = 0\}$ is a finite zero $p \in \mathbb{C}$ of $\mathbb{X}(z)$ such that it is root of $\{\mathscr{P}(s_0, p) = 0\}$ for a s_0.
2. An *unattainable root* of $\{\mathscr{P}(s, z) = 0\}$ is a finite zero $p \in \mathbb{C}$ of $\mathbb{X}(z)$ such that $\mathscr{P}(s_0, p) \neq 0$ for all $s_0 \in \mathbb{C}$.
3. A *collision of* $(\kappa + 1)$*-roots* of $\{\mathscr{P}(s, z) = 0\}$ is a finite pole $q \in \mathbb{C}$ of $\mathbb{X}(z)$ having order $-\kappa \leq -1$.
4. $\infty \in \widehat{\mathbb{C}}$ a regular point ($\kappa = 0$) or a pole of order $-\kappa \leq -1$ of $\mathbb{X}(z)$ determines *a* $1 + \kappa$ *reduction of the degree of* $\{\mathscr{P}(s, z) = 0\}$.

The four concepts in Definition 3 are justified by using the normal forms of $\mathbb{X}(z)$ as follows.

Let $[z_1(s_0), \ldots, z_n(s_0)] \subset \mathbb{C}$ be the zeros of $\{\mathscr{P}(s_0, z) = 0\}$ for a value s_0. Assume that p is a zero of $\mathbb{X}(z)$: if it is a zero of $\{\mathscr{P}(s_0, z) = 0\}$, then p is *fixed* under the flow of $\mathbb{X}(z)$.

Let $[z_1(s_0), \ldots, z_n(s_0)] \subset \mathbb{C}$ be the zeros of $\{\mathscr{P}(s_0, z) = 0\}$ for a value s_0. Assume that p is a zero of $\mathbb{X}(z)$ and does not a zero of $\{\mathscr{P}(s_0, z) = 0\}$. Clearly for all initial conditions $z_0 \notin \{z_1, \ldots, z_n\}$, we have that

$$\varphi(s_1, z_0) \neq p$$

for every analytic continuation of the flow of $\mathbb{X}(z)$ along a continuous time path $\gamma(\tau) : [0, 1] \longrightarrow \mathbb{C}$ with $\gamma(0) = z$.

Hence, p_ℓ is *an unattainable root* of $\{\mathscr{P}(s, z) = 0\}$.

The last two items in Definition 3 require an accurate theoretical study. Let $\mathscr{P}(g, z) : G \times \mathbb{C} \longrightarrow \mathbb{C}$ be Weierstrass polynomial.

When does a complex analytic vector field $\mathbb{X}(z)$ exists on $\mathbb{C}$ describing the n-root configurations of $\mathscr{P}(g, z)$?

The following result gives an answer.

Corollary 2 *Let $\mathscr{P}(s, z) : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}$ be a holomorphic Weierstrass polynomial over the complex additive Lie group $\mathbb{C}$. If the set all the n-roots configurations $\mathscr{Z} = \{\mathscr{P}(s, z) = 0\} \subset \mathbb{C} \times \mathbb{C}$, Eqs. (32) and (35), can be described as the graph of a rational function*

$$s = \Psi(z) : \mathbb{C} \longrightarrow \mathbb{C},$$

then there exists a rational vector field $\mathbb{X}(z)$ on $\mathbb{C}$ that describes the n-root configuration dynamics of $\mathscr{P}(s, z)$, even if the flow of $\mathbb{X}(z)$ is non holomorphic.

Proof Starting with $s = \Psi(z)$, the correspondence (41) provides a vector field $\mathbb{X}(z)$ that is not necessarily holomorphic.

For the study at the poles of $\mathbb{X}(z)$ using local holomorphic coordinates $(\mathbb{C}, 0)$ provided by the local normal forms, we can assume that locally the Weierstrass polynomial is $\mathscr{P}(s, z) = z^n - s$. By using the correspondence (41) (which is well defined up to a change of coordinates), we have

$$\mathbb{X}(z) = \frac{1}{nz^{n-1}} \frac{\partial}{\partial z} \quad \longleftrightarrow \quad \omega(z) = nz^{n-1} \mathrm{d}z \quad \longleftrightarrow \quad \Psi(z) = z^n. \tag{42}$$

The n-th roots of the unity $[\mathrm{e}^{2\pi i/n}, \mathrm{e}^{4\pi i/n}, \ldots, \mathrm{e}^{2n\pi i/n} = 1]$ collide at z_0 for complex time $s_0 = \int_1^0 n\zeta^{n-1} d\zeta$.

In fact, despite the non-holomorphicity of the vector field $\mathbb{X}(z)$ and its flow in (42), the Weierstrass polynomial $\mathscr{P}(s, z)$ is holomorphic on both variables. It follows that the n-root dynamics is well described by $X(z)$. See Fig. 4. □

Recall that the study of collisions in the n-body problem is a hard subject; in flat billiards, the trajectories arriving to the singularities are usually removed, since they have zero measure. In our case, root collisions are natural and easy to describe.

Example 5 Let

$$\mathscr{A} : \mathbb{C} \times \mathbb{C}[z]_{=2} \longrightarrow \mathbb{C}[z]_{=2}, \quad (s, c_2 z^2 + c_1 z + c_0) \longmapsto c_2 z^2 + (c_1 + s)z + c_0$$

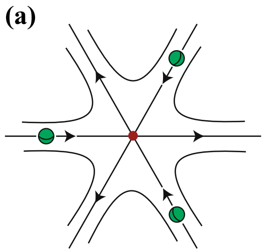

Fig. 4 Let $\varphi(s,z)$ be the holomorphic flow of $\mathbb{X}(z) = \frac{1}{3z^2}\frac{\partial}{\partial z}$. **a** The 3-root configuration $[-1, \mathrm{e}^{5\pi i/3}, \mathrm{e}^{2\pi i/6}]$ from $\{z^3 - 1 = 0\}$ collides at $z_0 = 0$ for real positive time $s_0 = 1$ under $\varphi(s,\ )$ of $\mathbb{X}(z)$, and the polynomial $\mathscr{P}(s,z) = \prod(z - \varphi(s, z_i))$ is holomorphic for all complex time $s \in \mathbb{C}$. **b** For only one starting point $z_1 = -1$, the flow $\varphi(s,\ )$ and the respective polynomial $(z - \varphi(s, z_1)$ do not make sense for real time $s > 1$. c) For three starting points in arbitrary positions (different from 0), the resulting polynomial $\prod(z - \varphi(s, z_i))$ is not well defined for certain time values $\{s_j\} \subset \mathbb{C}$

be a holomorphic action. Consider the orbit $\{\mathscr{P}(s,z) = z^2 - sz + 1\}$. We apply the correspondence (41);

$$\mathbb{X}(z) = \frac{z^2}{z^2-1}\frac{\partial}{\partial z} \quad \longleftrightarrow \quad \omega(z) = \frac{z^2-1}{z^2}\mathrm{d}z \quad \longleftrightarrow \quad \Psi(z) = \frac{z^2+1}{z} = s.$$

This vector field describes the 2-root configuration dynamics of $\{\mathscr{P}(s,z) = 0\}$.

The point $z_1 = 0$ is an unattainable root of $\{\mathscr{P}(s,z) = 0\}$.

Two collisions of 2-root appear at $z = 1, -1$; they correspond to the poles of $\mathbb{X}(z)$.

The point ∞ is a double zero of $\mathbb{X}(z)$, the degree of $\{\mathscr{P}(s,z) = 0\}$ remains 2 for all complex time s.

On the other hand, if we consider the orbit $\{\mathscr{P}(s,z) = z^2 - sz\}$. Then, the vector field $\mathbb{X}(z) = \frac{\partial}{\partial z}$ describes the 2-root configuration dynamics of it; $z_1 = 0$ is a fixed root and the second root $z_2 = s_0$ moves linearly with respect to the s-time flow of $\mathbb{X}(z)$.

Let $\mathbb{X}$ be a vector field describing the n-root configuration dynamics of a Weierstrass polynomial. Recall Fig. 3, in a small enough neighborhood of a pole q of order $-\kappa$ of X, there are $\{z_1, \ldots, z_{\kappa+1}\}$ roots (i.e., initial conditions) such that

$$\varphi(s_0, z_1) = \cdots = \varphi(s_0, z_{k+1}) = q \quad \text{for a suitable time } s_0 \in \mathbb{C},$$

whence the pole determines the collision of $(\kappa+1)$-roots of $\{\mathscr{P}(s,z) = 0\}$. In other words, $\kappa + 1$ simple roots give an origin to a root of multiplicity $\kappa + 1$.

Example 6 A Weierstrass polynomial (coming from a holomorphic Lie group action) that does not possess a complex analytic vector field $\mathbb{X}(z)$ describing its roots. The Weierstrass polynomial $\{\mathrm{e}^s z + s = 0\}$ is an orbit of the additive Lie group action

$$\begin{aligned} \mathscr{A} : \mathbb{C} \times \mathbb{C}[z]_{=1} &\longrightarrow \mathbb{C}[z]_{=1} \\ (s, c_1, c_0) &\longmapsto (\mathrm{e}^{-s}c_1, c_0 - s). \end{aligned}$$

The set of all the n-root configurations of orbit of $(c_1, c_0) = (1, 0)$ is

$$\mathscr{Z} = \{\mathrm{e}^{-s}z - s = 0\} = \{s\mathrm{e}^{s} - z = 0\},$$

which is the graph of the multivalued nonadditively automorphic Lambert W function $s = W(z)$. We would want to settle $W(z) = \Psi(z)$; see Corless et al. [12]. The hypothesis in Proposition 2 does not hold, since

$$\frac{\mathrm{d}W}{\mathrm{d}z} = \frac{W(z)}{z(1 + W(z))} \quad \text{for } z \notin 0, \ -1/\mathrm{e}.$$

The diagram (41) produces a multivalued complex analytic 1-form $\omega(z) = \mathrm{d}\psi(z)$, whence the vector field $\mathbb{X}(z)$ on $\mathbb{C}$ shares the mutivaluedness.

Example 7 (Transcendental functions $s = \Psi(z)$).

1. The Weierstrass polynomial $\mathscr{P}(s, z) = \mathrm{e}^{s}z + \mathrm{e}^{-s} = 0 : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}$ comes from an additive action on $\mathbb{C}[z]_{=1}$. Moreover, by using correspondence (41), there exists a complex analytic vector field $\mathbb{X}(z) = -2z\frac{\partial}{\partial z}$ on $\mathbb{C}$, which describes its roots dynamics of $\{\mathrm{e}^{s}z + \mathrm{e}^{-s} = 0\}$.
2. Let $s = \Psi(z) = \ln(z)$ be a multivalued additively complex analytic function. By using the correspondence (41), the rational vector field is $\mathbb{X}(z) = z\frac{\partial}{\partial z}$ and the Weierstrass polynomial is $\mathscr{P}(s, z) = \mathrm{e}^{s} - z : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}$.

See Proposition 3 for a generalization of both examples.

7 Vector fields describing *n*-root dynamics from Lie group actions of $\mathbb{C}$ and $\mathbb{C}^*$

Suitable tools for the study of Schur and anti-Schur polynomials are given in the following two subsections.

7.1 Actions on $\mathbb{C}[z]_{\leq n}$ by translations

Proposition 2 (Action on $\mathbb{C}[z]_{\leq n}$ by translations) *Let $Q \in \mathbb{C}[z]_{\leq n}$ be a polynomial. There exists a correspondence between the following objects.*

1. *A holomorphic Lie group action*

$$\begin{aligned} \mathscr{A} : \mathbb{C} \times \mathbb{C}[z]_{\leq n} &\longrightarrow \mathbb{C}[z]_{\leq n} \\ \big(s, P(z)\big) &\longmapsto P(z) - sQ(z). \end{aligned}$$

2. *A family of Weierstrass polynomials (obits of $\mathscr{A}$)*

$$\mathscr{P}(s, z) = P(z) - sQ(z) : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C} \tag{43}$$

over the complex additive Lie group.

3. *A family of rational maps*

$$\left\{ \Psi(z) = \frac{P}{Q}(z) - s : \mathbb{C} \longrightarrow \mathbb{C} \;\middle|\; P(z) \in \mathbb{C}[z]_{=n},\ s \in \mathbb{C} \right\}.$$

4. *A family of rational vector fields*

$$\left\{ \mathbb{X}(z) = \frac{Q^2(z)}{P'(z)Q(z) - Q'(z)P(z)} \frac{\partial}{\partial z} \;\middle|\; P(z) \in \mathbb{C}[z]_{=n} \right\} \quad on \quad \widehat{\mathbb{C}}_z. \tag{44}$$

Proof Each polynomial $P(z) \in \mathbb{C}[z]_{\leq n}$ must be considered as the initial condition for an orbit $\{P(z) - sQ(z) \mid s \in \mathbb{C}\}$ of the action. The key point is the definition of the rational functions $\Psi(z) = \frac{P}{Q}(z)$; it follows that $\{\frac{P}{Q}(z) - s = 0 \mid s \in \mathbb{C}\}$ and the diagram (41) provides the correspondence (2)–(4). □

Note that a rational vector field $\mathbb{X}(z)$ on $\widehat{\mathbb{C}}$ in (44) has a flow $\varphi : \Omega \times \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ such that

$$\frac{\Pi_\iota(z - \varphi(s, z_\iota(s)))}{Q(z)} = \frac{P(z)}{Q(z)} = s.$$

Recall Definition 3, the zeros $\{Q(z) = 0\}$ are fixed points under the flow of $\mathbb{X}(z)$.

Example 8 Given the polynomial $Q(z) = z^2$, let us consider the action

$$\mathscr{A} : \mathbb{C} \times \mathbb{C}[z]_{\leq 2} \longrightarrow \mathbb{C}[z]_{\leq} 2, \quad (s, c_2 z^2 + c_1 z + c_0) \longmapsto (c_2 + s)z^2 + c_1 z + c_0.$$

For each polynomial $P(z) = c_2 z^2 + c_1 z$ as initial condition, the action $\mathscr{A}$ gives origin to the Weierstrass polynomial $\mathscr{P}(s, z) = P(z) + sQ(z) = (c_2 + s)z^2 + c_1 z$. The associated vector field is

$$\mathbb{X} = \frac{z^2}{c_1} \frac{\partial}{\partial z}.$$

Note that $z_1 = 0$ is a fixed root of all $\{\mathscr{P}(s, z) = 0\}$. For $s = -c_2$, the second root $z_2 = -c_1/(c_2 + s)$ escapes to infinity.

7.2 Linear actions on $\mathbb{C}[z]_{\leq n}$

Let us consider a linear action $\mathscr{A}$ of $\mathbb{C}$ on $\mathbb{C}[z]_{\leq n} = \{(c_n, \ldots, c_1, c_0)\}$, with $\{z^n, \ldots, z, 1\}$ as a basis such that its infinitesimal generator is the linear holomorphic vector field

$$\sum_{j=0}^{n} \lambda_j c_j \frac{\partial}{\partial c_j}, \qquad \lambda_j \in \{\lambda_\alpha, \lambda_\beta\} \subset \mathbb{C}, \tag{45}$$

which has two different eigenvalues, at least one of them nonzero. Each polynomial $P(z) \in \mathbb{C}[z]_{\leq n}$ can be written as

$$P(z) = \sum_{j=0}^{n} c_j z^j = \sum_j c_{j(\alpha)} z^{j(\alpha)} - \sum_j c_{j(\beta)} z^{j(\beta)} = R(z) - Q(z),$$

by definition $R(z)$, $Q(z)$ belong to the eigenspaces of λ_1 and λ_2. Under (45), each $\mathbb{C}$-orbit is $\{e^{\lambda_\alpha s} R(z) - e^{\lambda_\beta s} Q(z) \mid s \in \mathbb{C}\}$. Moreover, if $\lambda_\alpha = \lambda_\beta$ or one of them are zero, then we get a $\mathbb{C}^*$-orbit.

Proposition 3 (Linear actions with two different nonzero eigenvalues) *The following objects are equivalent.*

1. *A holomorphic Lie group action*

$$\begin{aligned} \mathscr{A} : \mathbb{C} \times \mathbb{C}[z]_{\leq n} &\longrightarrow \mathbb{C}[z]_{\leq n} \\ \big(s, P(z)\big) &\longmapsto e^{\lambda_\alpha s} R(z) - e^{\lambda_\beta s} Q(z), \end{aligned}$$

   with an infinitesimal generator as in (45).
2. *Weierstrass polynomials*

$$\mathscr{P}_{R,Q}(s, z) = e^{\lambda_\alpha s} R(z) - e^{\lambda_\beta s} Q(z) : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}$$

   over the complex additive Lie group.
3. *Families of multivalued additively automorphic functions*

$$\left\{ \Psi_{R,Q}(z) = \frac{1}{\lambda_\alpha - \lambda_\beta} \ln\left(\frac{R(z)}{Q(z)}\right) \;\middle|\; s \in \mathbb{C},\ Q(z), R(z) \in \mathbb{C}[z]_{\leq n} \right\}.$$

4. *A family of rational vector fields*

$$\left\{ \mathbb{Y}(z) = \frac{R(z)Q(z)}{Q'(z)R(z) - R'(z)Q(z)} \frac{\partial}{\partial z} \quad \text{on } \mathbb{C} \;\middle|\; R(z),\ Q(z) \in \mathbb{C}[z]_{\leq n} \right\}.$$

Proof For each multivalued additively automorphic function $\Psi_{R,Q}(z)$, we have the diagram

$$\begin{tikzcd} & \mathbb{Y}(z) = \frac{R(z)Q(z)}{Q'(z)R(z)-R'(z)Q(z)} \frac{\partial}{\partial z} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\ \omega(z) = \left(\frac{1}{\lambda_\alpha-\lambda_\beta} ln\left(\frac{R(z)}{Q(z)}\right)\right)' dz \arrow[r, leftrightarrow] & \Psi(z) = \int^z \left(\frac{1}{\lambda_\alpha-\lambda_\beta} ln\left(\frac{R(z)}{Q(z)}\right)\right)' dz \arrow[r, leftrightarrow] & \left\{\frac{1}{\lambda_\alpha-\lambda_\beta} ln\left(\frac{R(z)}{Q(z)}\right) - s = 0\right\}. \end{tikzcd} \tag{46}$$

□

Example 9 Let $\lambda_\alpha = 1$ and $\lambda_\beta = 0$ be eigenvalues in (45). The families of Weierstrass polynomials are

$$\mathscr{P}(s,z) = \mathrm{e}^s R(z) + Q(z) : \mathbb{C}^* \times \mathbb{C} \longrightarrow \mathbb{C},$$

where $\mathrm{e}^s = w \in \mathbb{C}^*$, and the family of vector fields is

$$\mathbb{Y}(z) = \frac{R(z)Q(z)}{Q'(z)R(z) - R'(z)Q(z)} \frac{\partial}{\partial z} \quad \text{on} \quad \mathbb{C}. \tag{47}$$

Assume that there are points in $\{R(z) = 0\} \cap \{Q(z) = 0\} \subset \mathbb{C}$, they are then fixed under the flow of $\mathbb{Y}(z)$ and roots of the Weierstrass polynomial. The points $\{R(z) = 0\} \cup \{Q(z) = 0\} \backslash \big(\{R(z) = 0\} \cap \{Q(z) = 0\}\big) \subset \mathbb{C}$ are invariant under the flow of $\mathbb{Y}(z)$. They are unattainable roots under the analytic extension of the flow of $\mathbb{Y}(z)$.

8 Schur–Cohn map 2 in root coordinates

Recall the additive $\mathbb{C}$-action $\mathscr{A}_{2,n}(0, 1, P(z)) = P(z)$ in (22), where $s = 0$ and $\mathrm{e}^{i\theta} = 1$. We obtain a reduced version

$$\begin{aligned} \mathscr{A}_{2,n} : \mathbb{C} \times \mathscr{D}_{2,n} &\longrightarrow \mathscr{D}_{2,n} \\ (s, P(z)) &\longmapsto \tfrac{\mathrm{e}^{i\theta_n}}{b_{n-1}} \sqrt{b_{n-1} + |c_0 + sc_n|^2} z P_1(z) + \tfrac{c_0 + sc_n}{b_{n-1}} P_1^*(z), \end{aligned} \tag{48}$$

here $\mathrm{e}^{i\theta_n} = c_n/|c_n|$ and $R_{2,n}(P(z)) = P_1(z)$. Note that by abuse of notation, the above $\mathscr{A}_{2,n}$ is denoted as in (22). The associated principal $\mathbb{C}$-bundle is

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\ \mathtt{i}\ } & \mathscr{D}_{2,n} \subset \mathbb{C}[z]_{=n} \\ & & \Big\downarrow R_{2,n} \\ & & \mathbb{R}^+ \times \mathbb{C}^{n-1} \subsetneq \mathscr{D}_{2,n-1}. \end{array}$$

For each polynomial $P(z) \in \mathscr{D}_{2,n}$, the action $\mathscr{A}_{2,n}$ in (48)

$$\mathscr{P}(s,z) = \Xi_{2,n}(s, P_1(z)) : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}$$

provides a $\mathbb{C}$-orbit

$$\begin{aligned} \{s \cdot P(z) \mid s \in \mathbb{C}\} = \Bigg\{ \mathscr{P}(s,z) = \frac{\mathrm{e}^{i\theta_n}}{b_{n-1}} \sqrt{b_{n-1} + |c_0 + sc_n|^2} z P_1(z) \\ + \frac{c_0 + sc_n}{b_{n-1}} P_1^*(z) \,\Big|\, s \in \mathbb{C} \Bigg\}. \end{aligned} \tag{49}$$

In fact, for the identity element $s = 0 \in \mathbb{C}$ in (48), we recover $P(z) = \mathscr{P}(0, z)$.

A very useful expression for zeros of the $\mathbb{C}$-orbit is given by

$$\mathscr{Z} = \left\{ \mathscr{P}(s,z) = \frac{e^{i\theta_n}}{b_{n-1}} \sqrt{b_{n-1} + |c_0 + sc_n|^2} z P_1(z) \right.$$
$$\left. + \frac{c_0 + sc_n}{b_{n-1}} P_1^*(z) = 0 \,\middle|\, s \in \mathbb{C} \right\} \subset \mathbb{C} \times \mathbb{D}. \quad (50)$$

For each $\mathscr{P}(s,z)$, we require the change of time parameter

$$\varepsilon : \mathbb{C} \longrightarrow \mathbb{D} \subset \mathbb{C}, \quad s \longmapsto \frac{c_0 + sc_n}{\frac{c_n}{|c_n|}\sqrt{b_{n-1} + |c_0 + sc_n|^2}} = t, \quad (51)$$

which is a real analytic diffeomorphism. Consider Eq. (50); if we divide by the coefficient of $zP_1(z)$, then the equation of all the n-roots configurations $\mathscr{Z}$ assumes the form

$$\mathscr{Z} = \{\mathscr{P}(s,z) = 0 \mid s \in \mathbb{C}\} = \left\{ \widehat{\mathscr{P}}(t,z) = zP_1(z) + tP_1^*(z) = 0 \,\middle|\, t \in \mathbb{D} \right\}. \quad (52)$$

By Proposition (2), we recognize $\widehat{\mathscr{P}}(t,z)$ as a Weierstrass polynomial over $\mathbb{C}$ (a priori only for $t \in \mathbb{D}$). As an advantage, we have constructed a new $\{\widehat{\mathscr{P}}(t,z)\}$, which is a holomorhic $\mathbb{C}$-orbit with respect to t. It enjoys the following property; for each $s_0 \in \mathbb{C}$, there exists a time $t_0 \in \mathbb{D}$ such that

$$\{\mathscr{P}(s_0,z)\} = \{\widehat{\mathscr{P}}(t_0,z) = 0\}.$$

If we add as a hypothesis that $P(z)$ is Schur stable, then we distinguish three cases.

Proposition 4 (The loss of Schur stability for t-time) *Let $P(z) = \widehat{\mathscr{P}}(0,z)$ be a Schur stable polynomial and consider the associated zeros of the $\mathbb{C}$-orbit $\{\widehat{\mathscr{P}}(t,z) \mid t \in \mathbb{C}\}$ as in (52).*

1. *For $|t_0| < 1$, the polynomial $\widehat{\mathscr{P}}(t_0,z)$ is Schur stable.*
2. *For $|t_0| = 1$, the polynomial $\widehat{\mathscr{P}}(t,z)$ has all its zeros in the unitary circle $\partial\mathbb{D}$.*
3. *For $|t_0| > 1$, the polynomial $\widehat{\mathscr{P}}(t_0,z)$ is anti-Schur.*

Proof For $|t_0| < 1$, we recall that $\{\widehat{\mathscr{P}}(t_0,z) = 0\}$ already appeared in Eq. (21), so the assertion follows.

For $|t_0| > 1$, note that by Eq. (52), $\mathscr{P}(t,z) = zP_1(z) + tP_1^*(z)$ is Schur for $t \in \mathbb{D}$. We use coefficient coordinates, if $P_1(z) = b_{n-1}z^{n-1} + b_{n-2}z^{n-2} + \ldots b_1 z + b_0$ then

$$\begin{aligned}(zP_1(z) + tP_1^*(z))^* &= \big(z(b_{n-1}z^{n-1} + b_{n-2}z^{n-2} + \ldots b_1 z + b_0)\\ &\quad + t(\overline{b}_0 z^{n-1} + \overline{b}_1 z^{n-2} + \ldots \overline{b}_{n-2} z + \overline{b}_{n-1})\big)^*\\ &= \big((\overline{b}_{n-1} z + \overline{b}_{n-2} z + \ldots \overline{b}_1 z^{n-2} + \overline{b}_0 z^{n-1})\\ &\quad + \overline{t} z(b_0 + b_1 z + \ldots b_{n-2} z^{n-2} + b_{n-1} z^{n-1})\big)\\ &= \overline{t} z P_1(z) + P_1^*(z).\end{aligned}$$

Since $zP_1(z) + tP_1^*(z)$ is Schur for $t \in \mathbb{D}$, by Remark 2 we have that $(zP_1(z) + tP_1^*(z))^* = \bar{t}zP_1(z) + P_1^*(z)$ is anti-Schur for $t \in \mathbb{D}$ and $zP_1(z) + (1/\bar{t})P_1^*(z)$ is anti-Schur for $t \in \mathbb{D}\backslash\{0\}$.

Finally $zP_1(z) + tP_1^*(z)$ is anti-Schur for suitable $t \in \mathbb{C}\backslash\overline{\mathbb{D}}$, we have assertion 3.

For $|t_0| = 1$, the assertion follows by a continuity argument and the assertions 1 and 3. □

Summing up, we have obtained the following.

Theorem 3 (n-root configuration dynamics of Schur stable polynomials) *Let*

$$\mathscr{A}_{2,n} : \mathbb{C} \times \mathscr{D}_{2,n} \longrightarrow \mathscr{D}_{2,n}$$

be the real analytic action (48) *from the principal* $\mathbb{C}$*-bundle defined by the Schur–Cohn map* $R_{2,n}$*, and let* $P(z) \in \mathscr{A}_{2,n}$ *be a Schur stable polynomial. For the respective orbit*

$$\left\{ \mathscr{P}(s, z) = \frac{e^{i\theta_n}}{b_{n-1}}\sqrt{b_{n-1} + |c_0 + sc_n|^2}\, zP_1(z) + \frac{c_0 + sc_n}{b_{n-1}} P_1^*(z) \;\middle|\; s \in \mathbb{C} \right\}, \quad (53)$$

the rational vector field

$$\mathbb{X}(z) = \frac{P_1^{*2}(z)}{zP_1(z)P_1^{*'}(z) - (zP_1'(z) + P_1(z))P_1^*(z)} \frac{\partial}{\partial z} \quad (54)$$

describes the n-root configuration dynamics of $\mathscr{P}(s, z)$ *on* $\mathbb{D}$*. In particular*

$$\varphi(t, z_1), \ldots, \varphi(t, z_n) \in \mathbb{D} \quad \textit{for } t \in \mathbb{D}, \quad (55)$$

by using $\{P(z) = \mathscr{P}(0, z) = 0\} = [z_1, \ldots, z_n]$ *as initial conditions and t as in* (51).

Equation (55) is a geometrical expression of Eqs. (35) and (52).

Proof Let $P(z)$ be a Schur stable polynomial.

We recognize that the Weierstrass polynomial $\mathscr{P}(s, z)$, in (53), produces a second Weierstrass polynomial $\widehat{\mathscr{P}}(t, z)$ that fills up the conditions in Proposition 2, Eqs. (43) and (52), where t plays the role of the time in Eq. (43). Hence, following Eq. (44), the vector field $\mathbb{X}(z)$ in (54) is well defined.

By Proposition 2, the flow of the vector field $\mathbb{X}(z)$ for time $t \in \mathbb{D}$ describes the n-root configuration dynamics of $\{\mathscr{P}(0, z) = 0\}$. In fact, we can interchange the set of all the n-root configurations $\{\mathscr{P}(t, z) = 0\}$ by $\{\widehat{\mathscr{P}}(s, z) = 0\}$, hence the equation

$$\widehat{\mathscr{P}}(t, z) = c_n(t)\big(z - \varphi(t, z_1)\big) \cdots \big(z - \varphi(t, z_n)\big)$$

holds true whenever the flow $\varphi(t, \ )$ is well defined, as in Definition 2. □

An advantage of $\mathbb{X}(z)$ in (54) is that it is rational having an explicit expression in term of $P_1(z)$ and $P_1^*(z)$. The next result describes the dynamics of $\mathbb{X}(z)$ in the sense of Definition 3.

Corollary 3 1. *The unattainable points of* $\{\mathscr{P}(s,z)=0\}$ *are* $\{P_1^*(z)=0\}\subset\mathbb{C}\backslash\overline{\mathbb{D}}$.

2. *The poles of* $\mathbb{X}(z)$ *are* $2n-2$ *counted with multiplicity, and there are at most* $n-1$ *inside* $\mathbb{D}$*, which produce collisions of the roots of* $\{\mathscr{P}(s,z)=0\}$.

Proof For assertion 1, we note that the zeros of $\mathbb{X}(z)$ are in $\mathbb{C}\backslash\overline{\mathbb{D}}$ and by Lemma 4, it follows that $\{\mathscr{P}(t,z)=0\}$ has its unattainable points outside of the closed unitary disk.

For 2, Recall that the vector field (54) has an anti-Schur numerator and a self-inverse denominator. In fact,

$$
\begin{aligned}
&zP_1(z)P_1^{*'}(z)-(zP_1'(z)+P_1(z))P_1^*(z)\\
&\quad=\overline{b}_0b_{n-1}z^{2n-2}+\cdots+\left(\sum_{k=0}^{n-3}(n-2-2k)\overline{b}_{n-(k+3)}b_{n-(k+1)}\right)z^{n+1}\\
&\qquad+\left(\sum_{k=0}^{n-2}(n-1-2k)\overline{b}_{n-(k+2)}b_{n-(k+1)}\right)z^{n}\\
&\qquad+\left(\sum_{k=0}^{n-1}(n-2k)|b_{n-(k+1)}|\right)z^{n-1}+\left(\sum_{k=0}^{n-2}(n-1-2k)b_{n-(k+2)}\overline{b}_{n-(k+1)}\right)z^{n-2}\\
&\qquad+\left(\sum_{k=0}^{n-3}(n-2-2k)b_{n-(k+3)}\overline{b}_{n-(k+1)}\right)z^{n+1}+\cdots+b_0b_{n-1}
\end{aligned}
$$

is self-inverse by using [27] Definition 11.5.1, pp. 375. The vector field $\mathbb{X}(z)$ has at most $n-1$ poles inside of $\mathbb{D}$. □

The following examples and Fig. 5 illustrate these facts.

Example 10 Let

$$P(z)=z^3$$

be a Schur polynomial. Its associated polynomials are $P_1(z)=R_{2,n}(P(z))=z^2$ and $P_1^*(z)=1$. By Theorem A (equivalently Theorem 2) and Eq. (49), the respective $\mathbb{C}$-orbit is

$$\left\{\mathscr{P}(s,z)=\sqrt{1+|s|^2}z^3+s\;\middle|\;s\in\mathbb{C}\right\}.$$

The change of time parameter (51) determines the set of all the 3-root configurations

$$\left\{\widehat{\mathscr{P}}(t,z)=t+z^3=0\;\middle|\;t\in\mathbb{D}\right\}.$$

By Theorem B (equivalently Theorem 3) and Eq. (54), the vector field

$$\mathbb{X}(z)=\frac{-1}{3z^2}\frac{\partial}{\partial z}$$

describes the 3-root configuration dynamics of $\{\mathscr{P}(s,z)=0\}$ over $\mathbb{C}$. The 3-root configurations are inside $\mathbb{D}$ and there is a triple collision in $z=0$; see Fig. 5a. Hence, a triple pole appears at $z=0$.

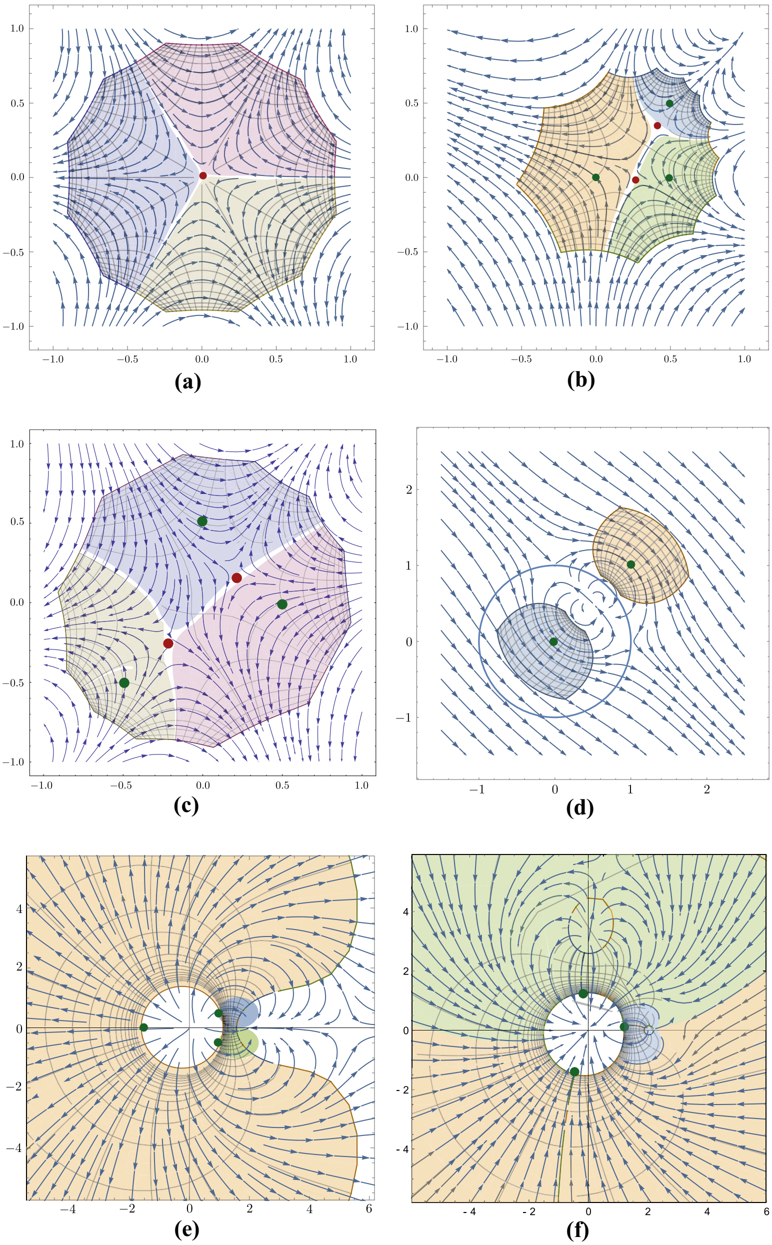

Fig. 5 Phase portraits **a**, **b** and **c** of suitable vector fields $\mathbb{X}(z)$ describing the 3-root dynamics of Schur polynomials in $\mathbb{D}$ for time $\{s \mid |s| < 1\}$ correspond to Examples 10, 11 and 12; they illustrate Eq. (55). **d** Phase portrait of a vector field $\mathbb{X}(z)$ describing the 2-root dynamics of a polynomial in $\mathscr{D}_{2,2}$, not Schur or anti-Schur, see Example 13. Phase portraits **e**–**f** of vector fields $\mathbb{Y}(z)$ describing the 3-root dynamics of anti-Schur polynomals in $\mathbb{C}\backslash\mathbb{D}$ for time $\{w \mid 0.1 \leq |w| \leq 0.8,\ 0 \leq Arg(w) \leq 2\pi\}$ correspond to Examples 14 and 15; they illustrate Eq. (63). Each color island originates from the complex flow of $\mathbb{X}(z)$ or $\mathbb{Y}(Z)$ using a root of $\{P(z) = 0\}$ (one green point) as initial condition

Example 11 Let

$$P(z) = z(z - 1/2)(z - (1 + i)/2)$$

be a Schur polynomial. Its associated polynomial is $P_1(z) = R_{2,3}(P(z)) = (z - 1/2)(z - (1 + i)/2)$ and the respective anti-Schur polynomial is $P_1^*(z) = \frac{1-i}{4}z^2 - (1 - i/2)z + 1$. By Theorem A (equivalently Theorem 2) and Eq. (49), the respective $\mathbb{C}$-orbit is

$$\left\{ \mathscr{P}(s, z) = \sqrt{1 + |s|^2} z\Big(z - \frac{1}{2}\Big)\Big(z - \frac{1+i}{2}\Big) \right. \\ \left. +s\Big(\frac{1-i}{4}(z - 2)(z - (1 + i))\Big) \,\Big|\, s \in \mathbb{C} \right\}.$$

The change of the time parameter (51) determines the set of all the 3-root configurations

$$\left\{ \widehat{\mathscr{P}}(t, z) = z\Big(z - \frac{1}{2}\Big)\Big(z - \frac{1+i}{2}\Big) + t\Big(\frac{1-i}{4}(z - 2)(z - (1 + i))\Big) = 0 \,\Big|\, t \in \mathbb{D} \right\}.$$

By Theorem B (equivalently Theorem 3) and Eq. (54), the vector field

$$\mathbb{X}(z) = -\frac{\big((2 + 2i) - (3 + i)z + z^2\big)^2}{(-2 + 2i) + (8 - 16i)z + 33iz^2 - (8 + 16i)z^3 + (2 + 2i)z^4}\frac{\partial}{\partial z}$$

describes the 3-root configuration dynamics of $\{\mathscr{P}(s, z) = 0\}$ over $\mathbb{C}$. The number of poles of $\mathbb{X}$ is four. The root collisions appear in $z_1 = 0.25+0.03i$, $z_2 = 0.42+0.32i$, $z_3 = 1.48 + 1.14i$ and $z_4 = 3.83 + 0.49i$, but only z_1 and z_2 are in the unitary disk, so the other collisions appear in $\mathbb{C}\backslash\overline{\mathbb{D}}$. See Fig. 5b.

Furthermore, we verify numerically Proposition 4.3; for $t \in \overline{\mathbb{D}}$: if $|t| \to 1$ then the orbit $\widehat{\mathscr{P}}(t, z)$ loss of Schur stability.

Example 12 Let

$$P(z) = (-1 + i) - 2iz + 8z^3$$

be a Schur polynomial. Its associated polynomial is $P_1(z) = R_{2,n}(P(z)) = -16i + (2 + 2i)z + 62z^2$ and the respective anti-Schur polynomial is $P_1^*(z) = 62 + (2 - 2i)z + 16iz^2$. By Theorem A (equivalently Theorem 2) and Eq. (49), the respective $\mathbb{C}$-orbit is

$$\left\{ \mathscr{P}(s, z) = \frac{1}{62}((-1 + i) + 8s)\left(62 + (2 - 2i)z + 16iz^2\right) \right. \\ \left. +\frac{1}{62}z\left(-16i + (2 + 2i)z + 62z^2\right)\sqrt{62 + |(-1 + i) + 8s|^2} \,\Big|\, s \in \mathbb{C} \right\}.$$

The change of the time parameter (51) determines the set of all the 3-root configurations

$$\left\{\widehat{\mathscr{P}}(t,z)=t(\frac{1-i}{4}(z-2)(z-(1+i)))+z(z-1/2)(z-(1+i)/2)=0 \,\Big|\, t\in\mathbb{D}\right\}.$$

By Theorem B (equivalently Theorem 3) and Eq. (54), the vector field

$$\mathbb{X}(z)=-\frac{i\left(31i+(1+i)z-8z^2\right)^2}{31\left(-8+(2-2i)z-91iz^2-(2+2i)z^3+8z^4\right)}\frac{\partial}{\partial z}$$

describes the 3-root configuration dynamics of $\{\mathscr{P}(s,z)=0\}$ over $\mathbb{C}$. The number of poles of $\mathbb{X}$ is four. The root collisions appear in $z_1=0.2+0.2i$, $z_2=-0.22-0.22i$, $z_3=2.24-2.24i$ and $z_4=2.51+2.51i$, but only z_1 and z_2 are in the unitary disk, so the other collisions appear in $\mathbb{C}\backslash\mathbb{D}$. See Fig. 5c.

Example 13 Let

$$P(z)=z(z-(1+i)) \quad \in \mathscr{D}_{2,2}\backslash\mathscr{S}_2 \tag{56}$$

be a non-Schur polynomial, but in the domain $\mathscr{D}_{2,2}$ of $R_{2,2}$. Its associated polynomials are

$$P_1(z)=R_{2,2}(P(z))=z-(1+i) \quad \notin \mathscr{D}_{2,1} \quad \text{and} \quad P_1^*(z)=(-1+i)z+1.$$

By Theorem A (equivalently Theorem 2) and Eq. (49), the respective $\mathbb{C}$-orbit is

$$\left\{\mathscr{P}(s,z)=\sqrt{1+|s|^2}z(z-(1+i))+s((-1+i)z+1) \,\Big|\, s\in\mathbb{C}\right\}.$$

The change of the time parameter (51) determines the set of all the 3-root configurations

$$\left\{\widehat{\mathscr{P}}(t,z)=z(z-(1+i))+t((-1+i)z+1)=0 \,\Big|\, t\in\mathbb{C}\right\}.$$

By Theorem B (equivalently Theorem 3) and Eq. (54), the vector field

$$\mathbb{X}(z)=\frac{(2-2i)(z-\frac{1+i}{2})^2}{2(z-1)(z-i)}\frac{\partial}{\partial z}$$

describes the 2-root configuration dynamics of $\{\mathscr{P}(s,z)=0\}$ over $\mathbb{C}$. Since the initial $P(z)$ is not Schur, for $|t|=1$ the roots $\{\widehat{\mathscr{P}}(t,z)=0\}$ do not belong necessarily to the unitary circle $\partial\mathbb{D}$. See Fig. 5d.

9 Schur–Cohn map 1 in root coordinates

Recall the multiplicative $\mathbb{C}^*$-action $\mathscr{A}_{1,n}\big(1,0,P(z)\big)=P(z)$ in (26), where $w=1$ and $e^{i\theta}=1$. We obtain a reduced version

$$\begin{gathered}\mathscr{A}_{1,n} : \mathbb{C} \times \mathscr{D}_{1,n} \longrightarrow \mathscr{D}_{1,n} \\ \left(w,\ \tfrac{c_n}{b_0} z P_1^*(z) + \tfrac{e^{i\theta_0}\sqrt{b_0+|c_n|^2}}{b_0} P_1(z)\right) \longmapsto \tfrac{wc_n}{b_0} z P_1^*(z) + \tfrac{e^{i\theta_0}\sqrt{b_0+|c_n w|^2}}{b_0} P_1(z),\end{gathered} \tag{57}$$

where $e^{i\theta_0} = c_0/|c_0|$ and $R_{1,n}(P(z)) = P_1(z)$. Note that by abuse of notation, the above $\mathscr{A}_{1,n}$ is denoted as in (26). The associated principal $\mathbb{C}^*$-bundle is

$$\begin{array}{ccl} \mathbb{C}^* & \xrightarrow{\ \mathtt{i}\ } & \mathscr{D}_{1,n} \subset \mathbb{C}[z]_{=n} \\ & & \Big\downarrow R_{1,n} \\ & & \mathbb{C}^{n-1} \times \mathbb{R}^+ \subsetneq \mathscr{D}_{1,n-1}. \end{array}$$

For each polynomial $P(z) \in \mathscr{D}_{1,n}$, the action $\mathscr{A}_{1,n}$ provides a Weierstrass polynomial of degree at most n:

$$\mathscr{P}(w, z) = \Xi_{1,n}(w, P_1(z)) : \mathbb{C}^* \times \mathbb{C} \longrightarrow \mathbb{C},$$

over the multiplicative Lie group $\mathbb{C}^*$. Note that since we are interested in the dynamics of the roots without loss of generality we can start with $P_1(z) = R_{1,n}(P(z))$ having $b_0 \in \mathbb{R}^+$.

A very useful expression for the $\mathbb{C}^*$-orbit of the bundle defined by $R_{1,n}$ and its zeros is given by

$$\mathscr{Z} = \left\{ \mathscr{P}(w, z) = \frac{wc_n}{b_0} z P_1^*(z) + \frac{e^{i\theta_0}\sqrt{b_0 + |wc_n|^2}}{b_0} P_1(z) = 0 \ \Big|\ w \in \mathbb{C}^* \right\} \subset \mathbb{C}\backslash\overline{\mathbb{D}}, \tag{58}$$

where $\theta_0 = c_0/|c_0|$ is given as in (18).

For each $\mathscr{P}(s, z)$, we require the change of time parameter

$$\epsilon : \mathbb{C}^* \longrightarrow \mathbb{C}\backslash\overline{\mathbb{D}}, \qquad w \longmapsto \frac{e^{i\theta_0}\sqrt{b_0 + |wc_n|^2}}{wc_n} = e^t, \tag{59}$$

where $b_0 \in \mathbb{R}^+$ and $\mathfrak{Re}\,(t) > 0$; the change is a real analytic diffeomorphism.

Consider Eq. (58), if we divide by the coefficient of $\big(wc_n\big)/b_0$, then the equation of all the n-roots configurations $\mathscr{Z}$ assumes the form

$$\mathscr{Z} = \{\mathscr{P}(w, z) = 0 \mid w \in \mathbb{C}^*\} = \left\{\widehat{\mathscr{P}}(e^t, z) = zP_1^*(z) + e^t P_1(z) = 0 \mid e^t \in \mathbb{C}\backslash\overline{\mathbb{D}}\right\}. \tag{60}$$

As in Proposition 3, we recognize the right side as a the set of all the n-roots configurations of the Weierstrass polynomial $\widehat{\mathscr{P}}(e^t, z)$ over $\mathbb{C}^*$.

Recall the two properties below:

By using the identity element $1 \in \mathbb{C}^*$ by (57) and (58), we recover $P(z) = \mathscr{P}(1, z)$.

For each $w_0 \in \mathbb{C}^*$, there exists $t_0 \in \mathbb{D}$ such that $\{\mathscr{P}(w_0, z) = 0\} = \{\widehat{\mathscr{P}}(e^{t_0}, z) = 0\}$.

We assume that $P(z)$ is anti-Schur.

Lemma 3 *Let $P(z)$ be an anti-Schur polynomial and consider the associated Weierstrass polynomial, $\widehat{\mathscr{P}}(t, z)$ as in (60).*

1. *For $e^{t_0} \in \mathbb{C}\backslash\overline{\mathbb{D}}$, the polynomial $\widehat{\mathscr{P}}(t_0, z)$ is anti-Schur.*
2. *For $e^{t_0} \in \mathbb{S}^1$, the polynomial $\widehat{\mathscr{P}}(t_0, z)$ has all its zeros in the unitary circle $\partial\mathbb{D}$.*
3. *For $e^{t_0} \in \mathbb{D}$, the polynomial $\widehat{\mathscr{P}}(t_0, z)$ is Schur.*

Proof It is analogous to the proof in Lemma 4. □

By using Proposition 3, we have an auxiliary rational vector field $\mathbb{Y}(z)$ on $\widehat{\mathbb{C}}$, associated with the family (60).

In summary

Theorem 4 (*n*-root configuration dynamics of anti-Schur stable polynomials) *Let*

$$\mathscr{A}_{1,n} : \mathbb{C}^* \times \mathscr{D}_{1,n} \longrightarrow \mathscr{D}_{1,n}$$

be the real analytic action (57) from the principal $\mathbb{C}^*$-bundle defined by the Schur-Cohn map $R_{1,n}$, and let $P(z)$ be an anti-Schur polynomial. For the respective orbit

$$\left\{ \mathscr{P}(w, z) = \frac{wc_n}{b_0} z P_1^*(z) + \frac{e^{i\theta_0}\sqrt{b_0 + |wc_n|^2}}{b_0} P_1(z) \;\middle|\; w \in \mathbb{C}^* \right\}, \tag{61}$$

the rational vector field

$$\mathbb{Y}(z) = -\frac{z P_1(z) P_1^*(z)}{(P_1^*(z) + z P_1^{*'}(z)) P_1(z) - z P_1^*(z) P_1'(z)} \frac{\partial}{\partial z} \quad on \ \widehat{\mathbb{C}} \tag{62}$$

describes the n-root configuration dynamics of $\mathscr{P}(w, z)$ in $\mathbb{C}\backslash\overline{\mathbb{D}}$. In particular

$$\varphi(t, z_1), \ldots, \varphi(t, z_n) \in \mathbb{C}\backslash\overline{\mathbb{D}} \quad for\ all\ e^t \in \mathbb{C}\backslash\overline{\mathbb{D}}, \tag{63}$$

by using $\{P(z) = \mathscr{P}(1, z) = 0\} = [z_1, \ldots, z_n]$ as initial conditions. □

The next result describes the dynamics of $\mathbb{Y}(z)$ in the sense of Definition 3.

Corollary 4 *Consider the vector field $\mathbb{Y}(z)$ in (62).*

1. *The unattainable points are the roots of $z P_1^*(z) P_1(z)$, and there are $n-1$ unattainable points in $\mathbb{C}\backslash\overline{\mathbb{D}}$.*
2. *The poles of $\mathbb{Y}(z)$ are at most $2n-2$ counted with multiplicity, and there are at most $n-1$ in $\mathbb{C}\backslash\overline{\mathbb{D}}$, which produce root collisions of $\mathscr{P}(w, z)$.*

Proof It is analogous to the proof for Corollary 3. □

Example 14 Let

$$P(z) = z(2z-1)^2 + 2(z-2)^2$$

be an anti-Schur polynomial. Its associated polynomials are $P_1(z) = R_{1,n}(P(z)) = 12(z-2)^2$ and $P_1^*(z) = 12(2z-1)^2$. By Theorem A (equivalently Theorem 2) and Eq. (61), the respective $\mathbb{C}^*$-orbit is

$$\left\{ \mathscr{P}(w, z) = \frac{4w}{48} 12z(2z-1)^2 + \frac{\sqrt{48 + |4w|^2}}{48} 12(z-2)^2 \;\Big|\; w \in \mathbb{C}^* \right\}.$$

The change of the time parameter (59) determines the set of all the n-root configurations

$$\left\{ \widehat{\mathscr{P}}(e^t, z) = z(2z-1)^2 + e^t 12(z-2)^2 \;\Big|\; e^t \in \mathbb{C}\backslash\overline{\mathbb{D}} \right\}.$$

By Theorem B (equivalently Theorem 4) and Eq. (62), the vector field

$$\mathbb{Y}(z) = -\frac{(-2+z)z(-1+2z)}{2 - 11z + 2z^2} \frac{\partial}{\partial z}$$

describes the 4-root configuration dynamics of $\{\mathscr{P}(w, z) = 0\}$ over $\mathbb{C}^*$. In $\mathbb{C}\backslash\overline{\mathbb{D}}$, there is a collision of 2-roots at $z = 5.31$, and $z = 2$ is an unattainable root. See Fig. 5e.

Example 15 Let

$$P(z) = -6iz^3 + 3iz^2 - (3+6i)z + 12i$$

be an anti-Schur polynomial. Its associated polynomial is $P_1(z) = -18i(z-2)(z-3i)$, which is an anti-Schur polynomial, the associated Schur polynomial is $P_1^*(z) = 108z^2 - (54+36i)z + 18i$. By Theorem A (equivalently Theorem 2) and Eq. (61), the respective $\mathbb{C}$-orbit is

$$\left\{ \mathscr{P}(w, z) = \left\{ \frac{-6iw}{108} z(108z^2 - (54+36i)z + 18i) \right.\right.$$
$$\left.\left. + \frac{i\sqrt{108 + |-6iw|^2}}{108} (-18i)(z-2)(z-3i) \;\Big|\; w \in \mathbb{C}^* \right\}\right. .$$

The change of the time parameter (59) determines the set of all the n-root configurations

$$\left\{ \widehat{\mathscr{P}}(e^t, z) = \{z(108z^2 - (54+36i)z + 18i) + e^t(-18i)(z-2)(z-3i) \;\Big|\; e^t \in \mathbb{C}\backslash\overline{\mathbb{D}} \right\}.$$

By Theorem B (equivalently Theorem 4) and Eq. (62), the vector field

$$\mathbb{Y}(z) = \frac{z\left(6i - (2+3i)z + z^2\right)\left(i - (3+2i)z + 6z^2\right)}{6\left(-1 + (4-6i)z + 20iz^2 - (4+6i)z^3 + z^4\right)} \frac{\partial}{\partial z}$$

describes the root dynamics of $\{\mathscr{P}(w, z) = 0\}$. The roots belong to $\mathbb{C}\backslash\overline{\mathbb{D}}$, and there is a collision of 2-roots in $z_1 = 0.55 + 5.47i$, $z_2 = 3.11 + 0.31i$. The Weierstrass polynomial $\mathscr{P}(w, z)$ has unattainable roots in 0 and $3i$. See Fig. 5f.

10 Theorems A and B and loss of Schur stability of a cohort population model

In this section, we use the ideas of Theorems A and B, trying to attract the interest of specialists in control theory to use them in Schur stability problems. Let us consider a cohort population model presented in [21], p. 356, as follows

$$x(T+1) = \begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} & \alpha_n \\ \beta_0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \beta_1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \beta_{n-1} & 0 \end{bmatrix} x(T),$$

having characteristic polynomial

$$P(z) = z^{n+1} - \alpha_0 z^n - \alpha_1\beta_0 z^{n-1} - \alpha_2\beta_0\beta_1 z^{n-2} - \cdots - \alpha_n\beta_0\beta_1\cdots\beta_{n-1}, \quad (64)$$

where $\alpha_i \geq 0$ for $i = 0, 1, \ldots, n$ and $0 \leq \beta_i \leq 1$ for $i = 0, 1, \ldots, n-1$.

The model describes the evolution of the age distribution of a given population depending on the time. The coordinate $x_i(T)$ represents the number of the i-th age group at time period T.

The number of the youngest age group at time $T+1$ is given by $x_0(T+1) = \alpha_0 x_0(T) + \cdots + \alpha_n x_n(T)$, where α_i is the constant birth rate of the i-th age group with $i = 0, \ldots, n$.

The $(i+1)$-th and i-th age groups are related to the equation $x_{i+1}(T+1) = \beta_i x_i(T)$, $i = 0, 1, \ldots, n-1$, $T \in \mathbb{N}$, where β_i is the constant survival rate of the i-th age group.

For illustrating the application of Theorem B, we consider $n = 2$. The system

$$x(T+1) = \begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 \\ \beta_0 & 0 & 0 \\ 0 & \beta_1 & 0 \end{bmatrix} x(T)$$

has characteristic polynomial $P(z) = z^3 - \alpha_0 z^2 - \alpha_1\beta_0 z - \alpha_2\beta_0\beta_1$.

Example 16 For $\alpha_0 = 1/4$, $\alpha_1 = 3/4$, $\alpha_2 = 0$, $\beta_0 = 1/2$, we get the characteristic Schur stable polynomial

$$P(z) = z^3 - \frac{1}{4}z^2 - \frac{3}{8}z = z\Big(z - \frac{3}{4}\Big)\Big(z + \frac{1}{2}\Big).$$

Its associated polynomials are

$$P_1(z) = R_{2,3}(P(z)) = z^2 - \frac{1}{4}z - \frac{3}{8} \quad \text{and} \quad P_1^*(z) = -\frac{3}{8}z^2 - \frac{1}{4}z + 1.$$

By Theorem A (equivalently Theorem 2) and Eq. (49), the respective $\mathbb{C}$-orbit is

$$\left\{\mathscr{P}(s,z)=\sqrt{1+|s|^2}z\Big(z^2-\frac{1}{4}z-\frac{3}{8}\Big)+s\Big(-\frac{3}{8}z^2-\frac{1}{4}z+1\Big)\;\Big|\;s\in\mathbb{C}\right\}.$$

The change of the time parameter (51) determines the set of all the 3-root configurations

$$\left\{\widehat{\mathscr{P}}(t,z)=z\Big(z^2-\frac{1}{4}z-\frac{3}{8}\Big)+t\Big(-\frac{3}{8}z^2-\frac{1}{4}z+1\Big)=0\;\Big|\;t\in\mathbb{D}\right\}.$$

By Theorem B (equivalently Theorem 3) and Eq. (54), the vector field

$$\mathbb{X}(z)=\frac{(2+z)^2(-4+3z)^2}{24+32z-187z^2+32z^3+24z^4}\frac{\partial}{\partial z}$$

describes the 3-root configuration dynamics of $\mathscr{P}(t,z)$ over $\mathbb{C}$.

The implications of Theorem B are shown in Fig. 6a.

For $|t|<1$ the resulting polynomial $\widehat{\mathscr{P}}(t,z)$ remains Schur stable.

The loss of stability is shown in Fig. 6b.

For $|t|\geq 1$ the resulting polynomial $\widehat{\mathscr{P}}(t,z)$ lose Schur stability, since the 3-root configurations belong to $\mathbb{C}\backslash\overline{\mathbb{D}}$. Figure 6b shows the behavior of the three original roots $\{-1/2, 0, 3/4\}$ of $P(z)$, under the complex flow $\varphi(t,\ )$ of $\mathbb{X}(z)$ for the choice of two segments of complex time

$$t_0\in\{\tau+i\tau\mid -1<\tau<1\}\cup\{\tau-i\tau\mid -1<\tau<1\}\subset\mathbb{C},$$

where τ is real. Each segment of time gives origin to a trajectory of roots $\{\widehat{\mathscr{P}}(t_0,z)=0\}$ starting at one of the original roots. Note that the loss of Schur stability $\widehat{\mathscr{P}}(t,z)$ occurs at $|t|=1$, equivalently to $|\tau|=\sqrt{2}/2$.

Example 17 For $\alpha_0=1$, $\alpha_1=92$, $\alpha_2=640$, $\beta_0=1/2$, $\beta_1=1/4$, we get the characteristic anti-Schur polynomial

$$P(z)=z^3-z^2-46z-80=(z+5)(z+2)(z-8).$$

Its associated polynomials are

$$P_1(z)=R_{1,3}(P(z))=\frac{3}{8}z^2+\frac{1}{4}z-1\ \text{ and }\ P_1^*(z)=-z^2+\frac{1}{4}z+\frac{3}{8}.$$

By Theorem A (equivalently Theorem 2) and Eq. (61), the respective $\mathbb{C}^*$-orbit is

$$\left\{\mathscr{P}(w,z)=-wz\Big(-z^2+\frac{1}{4}z+\frac{3}{8}\Big)+\sqrt{-1+|w|^2}\Big(\frac{3}{8}z^2+\frac{1}{4}z-1\Big)\;\Big|\;w\in\mathbb{C}^*\right\}.$$

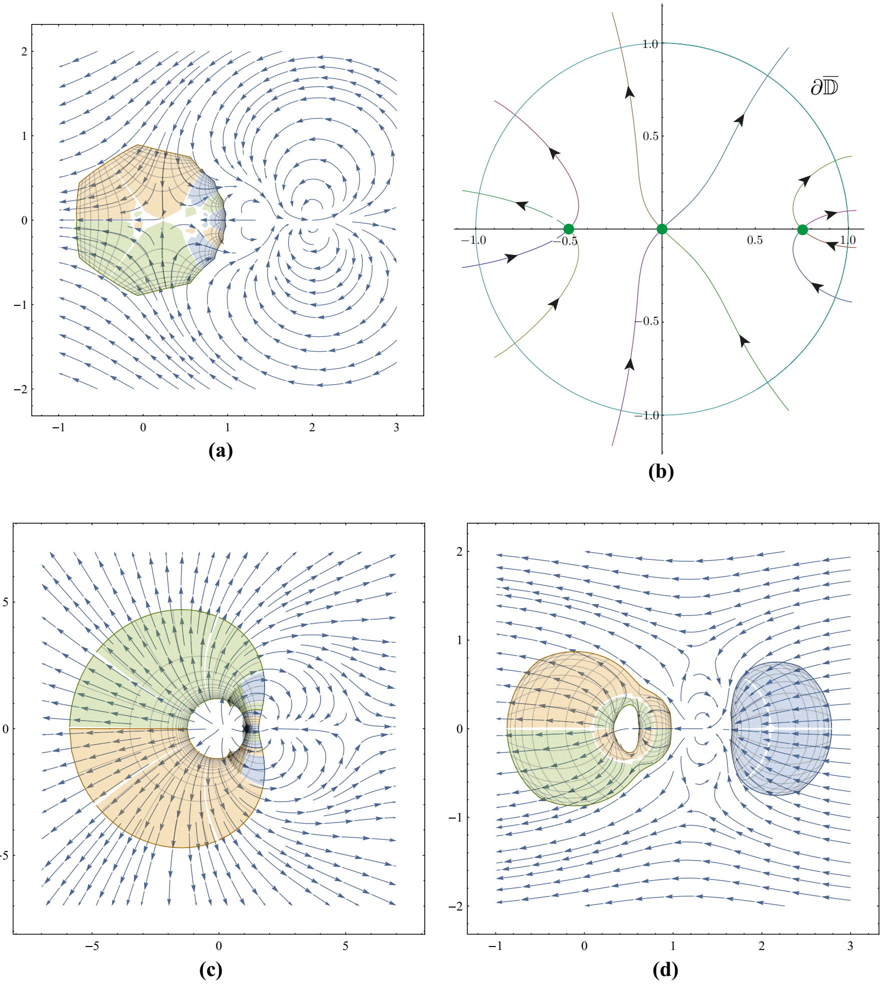

Fig. 6 Phase portraits **a**, **c**, **d** describing the 3-root configuration dynamics of a cohort population model, correspond to Examples 16, 17 and 18, respectively. Furthermore, **b** describes the behavior of the initial roots $[0, 3/4, -1/2]$ under the complex flow of $\mathbb{X}(z)$ corresponding to Example 16 for two segments of complex time; when the segments leave the unitary disk, then the polynomials $\widehat{\mathscr{P}}(t, z) = z\big(z^2 - \frac{1}{4}z - \frac{3}{8}\big) + t\big(-\frac{3}{8}z^2 - \frac{1}{4}z + 1\big)$ lose Schur stability (the trajectories of roots scape from the unitary disk $\mathbb{D}$)

The change of the time parameter (59) determines all the 3-root configurations

$$\left\{ \widehat{\mathscr{P}}(\mathrm{e}^t, z) = z\Big(-z^2 + \frac{1}{4}z + \frac{3}{8}\Big) + \mathrm{e}^t\Big(\frac{3}{8}z^2 + \frac{1}{4}z - 1\Big) \,\Big|\, \mathrm{e}^t \in \mathbb{C}\backslash\overline{\mathbb{D}} \right\}.$$

By Theorem B (equivalently Theorem 4) and Eq. (62), the vector field is

$$\mathbb{Y}(z) = \frac{(-z^2 + \frac{1}{4}z + \frac{3}{8})(\frac{-3}{8}z^3 - \frac{1}{4}z^2 + z)}{\frac{3}{8}z^4 + \frac{13}{16}z^3 - \frac{17}{4}z^2 - \frac{3}{16}z + \frac{3}{8}}.$$

The implications of Theorem B are shown in Fig. 6c.

Example 18 For $\alpha_0 = 5/4$, $\alpha_1 = 3$, $\alpha_2 = 0$, $\beta_0 = 1/2$, we get the characteristic polynomial

$$P(z) = z^3 - \frac{5}{4}z^2 - \frac{3}{2}z = z\Big(z + \frac{3}{4}\Big)\Big(z - 2\Big).$$

Its associated polynomials are

$$P_1(z) = R_{2,3}(P(z)) = z^2 - \frac{5}{4}z - \frac{3}{2} \quad \text{and} \quad P_1^*(z) = -\frac{3}{2}z^2 - \frac{5}{4}z + 1.$$

By Theorem A (equivalently Theorem 2) and Eq. (49), the respective $\mathbb{C}$-orbit is

$$\left\{ \mathscr{P}(s, z) = \sqrt{1 + |s|^2} z\Big(z^2 - \frac{5}{4}z - \frac{3}{2}\Big) + s\Big(- \frac{3}{2}z^2 - \frac{5}{4}z + 1\Big) \,\Big|\, s \in \mathbb{C} \right\}.$$

The change of the time parameter (51) determines all the 3-root configurations

$$\left\{ \widehat{\mathscr{P}}(t, z) = z\Big(z^2 - \frac{5}{4}z - \frac{3}{2}\Big) + t\Big(- \frac{3}{2}z^2 - \frac{5}{4}z + 1\Big) = 0 \,\Big|\, t \in \mathbb{D} \right\}.$$

By Theorem B (equivalently Theorem 3) and Eq. (54), the vector field

$$\mathbb{X}(z) = \frac{(-1 + 2z)^2(4 + 3z)^2}{24 + 40z - 37z^2 + 40z^3 + 24z^4} \frac{\partial}{\partial z},$$

which is neither Schur polynomial nor anti-Schur polynomial. The implications of Theorem B are shown in Fig. 6d.

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