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JACOBIAN MATES FOR NON-SINGULAR POLYNOMIAL MAPS IN $\mathbb{C}^n$ WITH ONE–DIMENSIONAL FIBERS

ALVARO BUSTINDUY, LUIS GIRALDO, AND JESÚS MUCIÑO–RAYMUNDO

A nuestro maestro Xavier Gómez-Mont, con gratitud

ABSTRACT. Let $(F_2, \dots, F_n) : \mathbb{C}^n \to \mathbb{C}^{n-1}$ be a non-singular polynomial map. We introduce a non-singular polynomial vector field X tangent to the foliation $\mathcal{F}$ having as leaves the fibers of the map $(F_2, \dots, F_n)$. Suppose that the fibers of the map are irreducible in codimension ≥ 2, that the one forms of time associated to the vector field X are exact along the leaves, and that there is a finite set at the hyperplane at infinity containing all the points necessary to compactify the affine curves appearing as fibers of the map. Then, there is a polynomial F_1 (a Jacobian mate) such that the completed map $(F_1, F_2, \dots, F_n)$ is a local biholomorphism. Our proof extends the integration method beyond the known case of planar curves (introduced by Ilyashenko [Ily69]).

1. Introduction and Statement of Results

The topological or analytical classification of non-singular polynomial foliations in $\mathbb{C}^n$ is a very hard problem, even in the lowest dimensional case $n = 2$. See [ACL98], [BT06], [Fer05], [NN02], [Tib07] and references therein.

We study the (holomorphic) polynomial foliations by curves $\mathcal{F}$ in $\mathbb{C}^n$ which can be obtained from the fibers of complex polynomials $F_2, \dots, F_n \in \mathbb{C}[z_1, \dots, z_n]$, chosen in such a way that

$$(1) \qquad \begin{cases} (F_2, \dots, F_n) : \mathbb{C}^n \longrightarrow \mathbb{C}^{n-1} \quad \text{and} \\ dF_2 \wedge \cdots \wedge dF_n \quad \text{does not vanish at any } z \in \mathbb{C}^n. \end{cases}$$

The fibers of the map in (1) are nonsingular, but possibly reducible, affine curves that we denote by $\{\mathcal{A}_c\}$. The leaves of $\mathcal{F}$ are the connected components (a unique one generically) of those affine curves. We say that $\mathcal{F}$ is a *non-singular polynomial foliation having* $n-1$ *first integrals*.

As a first step toward a general classification a natural problem is to study topologically or analytically this family of foliations.

An interesting subfamily is as follows. The map $(F_2, \dots, F_n)$ has a *Jacobian mate* when there exists a polynomial $F_1 \in \mathbb{C}[z_1, \dots, z_n]$ such that

$$(2) \qquad \begin{cases} F = (F_1, F_2, \dots, F_n) : \mathbb{C}^n \longrightarrow \mathbb{C}^n \quad \text{and} \\ dF_1 \wedge dF_2 \wedge \cdots \wedge dF_n = dz_1 \wedge \cdots \wedge dz_n. \end{cases}$$

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Recall that the Jacobian Conjecture in $\mathbb{C}^n$ asserts the existence of the inverse map F^{-1} (which has to be also polynomial).

Given $\mathcal{F}$, where are the obstructions to the existence of F_1?

Note that the singularities of the extended foliation to projective space, still denoted by $\mathcal{F}$, are in the hyperplane at infinity of $\mathbb{C}^n$. In the classification problem one can study the singularities at infinity. Instead, our approach focus on the affine behavior and possible "jumps" in the geometry of the fibres $\{\mathcal{A}_c\}$. By a classical result of S. A. Broughton, see [Bro83], there exists an open Zariski set $U \subset \mathbb{C}^{n-1}$ such that the affine foliation $\mathcal{F}$ is a locally trivial fibration in $(F_2, \ldots, F_n)^{-1}(U)$.

Hence we must consider a priori the existence of atypical fibers (i.e. fibers outside U) of (1) and try to describe the behavior of $\mathcal{F}$. In particular, we point out that an example of (1) having atypical fibers and admitting a F_1, will provide a counterexample for the Jacobian Conjecture.

Another related problems with the existence of a Jacobian mate are the following. First, in the holomorphic category, on Stein manifolds the problem of the existence of F_1 is posed in [For03a] p. 146 and [For03b] p. 96, and it remains open (we thank Filippo Bracci for pointing this out to us). Second, the symmetric problem, i.e. given F_1 how to recognize the existence of $(F_2, \ldots, F_n)$ such that (2) is currently under study for $n \geq 3$, see [FR05] p. 3 or [Kal02].

The main tool that we introduce is a polynomial vector field X depending in an essential way of $\mathcal{F}$. Consider the Jacobian matrix of the map (1)

$$\left(\frac{\partial F_j}{\partial z_i}\right)_{2\leq j\leq n,\ 1\leq i\leq n},$$

and let $A_i(z_1, \ldots, z_n)$ be the determinant of the submatrix obtained after removing the i–th column, then

$$X := \sum_{i=1}^{n} (-1)^{i+1} A_i(z_1, \ldots, z_n) \frac{\partial}{\partial z_i}, \tag{3}$$

obviously X is nowhere zero. If there exists a Jacobian mate F_1, then

$$(F_1, \ldots, F_n)^* \frac{\partial}{\partial w_1} =.X. \tag{4}$$

X restricted to any fiber $\mathcal{A}_c$, $c \in \mathbb{C}^{n-1}$, of the map (1), gives a tangent vector field on $\mathcal{A}_c$, that we will denote by X_c. It determines a unique holomorphic one form ω_c on $\mathcal{A}_c$, when we require $\omega_c(X_c) = 1$. Thus, each map $(F_2, \ldots, F_n)$ produces a collection of pairs

$$\{(\mathcal{A}_c,\ X_c) \mid c \in \mathbb{C}^{n-1}\}, \qquad \text{equivalently } \{(\mathcal{A}_c,\ \omega_c)\}. \tag{5}$$

In Section 2, we briefly develop this ideas to make the argument more transparent.

Remark 1. *1. The vector field X defines a singular holomorphic foliation $\mathcal{F}$ by curves in $\mathbb{CP}^n$, such that its singular locus is contained in the hyperplane at infinity $\mathbb{CP}^{n-1}_\infty$.*
2. The polynomial vector field X has $n-1$ polynomial first integrals on $\mathbb{C}^n$, and the leaves of the foliation defined by X in $\mathbb{C}^n$ are given by the curves $\{\mathcal{A}_c \mid c \in \mathbb{C}^{n-1}\}$.
3. The hyperplane $\mathbb{CP}^{n-1}_\infty$ is saturated by leaves of $\mathcal{F}$.

In addition

Remark 2. *Up to multiplication by a non-zero constant, X is the unique non vanishing polynomial vector field giving a trivialization for the tangent line bundle of the non-singular holomorphic foliation $\mathcal{F}$ on $\mathbb{C}^n$.*

Indeed, if a second polynomial vector field Y (providing a trivialization of the tangent line bundle to the foliation) exists, then $X = \lambda Y$, for λ an entire function on $\mathbb{C}^n$, nowhere zero. But λ is clearly polynomial, hence it is necessarily a non-zero constant. Moreover, X is independent on the choice of any polynomial F_1 satisfying (2): it only depends on $(F_2, \ldots, F_n)$. Hence, we can use X to explore the existence of F_1.

The main result about affirmative conditions for the existence of F_1, is the following

Theorem 1. *Let $(F_2, \ldots, F_n) : \mathbb{C}^n \longrightarrow \mathbb{C}^{n-1}$ be a polynomial map such that $dF_2 \wedge \cdots \wedge dF_n$ does not vanish at any point of $\mathbb{C}^n$. Consider X as in (3), and suppose furthermore that:*
(i) The reducible fibers $\{\mathcal{A}_c\} \subset \mathbb{C}^n$ determine an algebraic subset of codimension at least 2.
(ii) For every $c \in \mathbb{C}^{n-1}$, the pairs $(\mathcal{A}_c,\ \omega_c)$ satisfy that

$$\int_\gamma \omega_c = 0, \quad \textit{for every } [\gamma] \in H_1(\mathcal{A}_c, \mathbb{Z}).$$

(iii) There is a finite set $Y \subset \mathbb{CP}^n_\infty$ such that each affine curve $\mathcal{A}_c$ is completed in $\mathbb{CP}^n$ by adding points in Y.
Then, there is a polynomial F_1 such that

$$dF_1 \wedge dF_2 \wedge \cdots \wedge dF_n = \ dz_1 \wedge \cdots \wedge dz_n.$$

Note that the second hypothesis is clearly necessary for ω_c to be an exact one form on the fibers $\mathcal{A}_c$. Concerning the first, it is in fact necessary for the integration method that we use: Example 1 shows a function with a reducible fiber (of codimension one), with zero periods, and such that the function constructed by integration as a candidate for Jacobian mate has a pole on that fiber (see Remark 6).

The third hypothesis, obviously satisfied in the case $n = 2$, is automatically satisfied in case that the map F is surjective. In this case, as F has no critical points, all the fibers are one dimensional, and according to [Ga99] p. 158, they share the same cone at infinity, i.e. all the affine curves are completed by adding the same points at infinity (a finite set). Note that this cone at infinity is defined by the vanishing of the polynomials in the ideal generated by the terms of highest degree of the elements of the ideal generated by the components of the function F. This cone at infinity is contained in, but not necessary equal to, the singular set of the foliation $\mathcal{F}$ extended to projective space.

After proving our result by integration method (see below), we realized that in case F is surjective, it is a consequence of a Theorem of Ph. Bonnet (Theorem 1.5 in [Bon03]). Nevertheless, even in that case, as his approach is algebraic, and our proof extends the integration method beyond the case of planar curves previously known (starting with Ilyashenko [Ily69]), we consider that it can be of interest for the people working in the field. Moreover, with this technique as a fundamental tool, together with some considerations on the degree of the map F and computations of the index of X restricted to the fibers of $(F_2, \ldots, F_n)$ (see the end of this Introduction), we have also obtained some new results on negative conditions for the existence of a jacobian mate. They will be presented in a future work, including the solution in a particular case (see Example 1) of a problem posed by L. Dũng Tráng and C. Weber in [DW94].

1.1. **Method and Structure of the proof.** The proof of Theorem 1 is given in several steps below. Note that, to avoid confusion we use $\mathbb{C}^n_z$ and $\mathbb{C}^{n-1}_w$ to denote the domain and the target in map (1).
Step 1. We construct a polynomial one form of time ω for X on $\mathbb{C}^n_z$. By integration of ω along the irreducible fibers of F, see equation (8), we get a candidate function $\widetilde{F}_1$.
Step 2. We verify that the candidate function is holomorphic on the whole $\mathbb{C}^n_z$, see Proposition

1.
Step 3. We estimate the growth of $\widetilde{F}_1$. This is the hardest step. We will study the growth of $\widetilde{F}_1$ at infinity. We recognize the growth of $|\widetilde{F}_1(z)|$ in a suitable set of complex lines in $\mathbb{C}^n_z$. This requires bounds for: the norm of the end points of the integration paths in (8), see Lemma 1, the norm of the ramification points in the fibers $\{\mathcal{A}_c\}$, see Lemma 2, and the length of the integration paths in (8), see Lemma 3. Thus, $|\widetilde{F}_1(z)|$ has polynomial growth in suitable lines, see Lemma 4.
Step 4. In order to show that $\widetilde{F}_1(z)$ is a polynomial, we make an argument by contradiction, using a property of the growth of entire non-polynomial functions, see Lemma 5 and Proposition 2. We show explicitly that F_1 satisfies $dF_1 \wedge dF_2 \wedge \ldots \wedge dF_n = dz_1 \wedge \ldots \wedge dz_n$.

Concerning the proof of Theorem 1, we point out that the powerful method of integration of one forms ω along the algebraic leaves of a polynomial foliation $\{\mathcal{A}_c\}$ in $\mathbb{C}^2$ to find F_1, was introduced by Yu. Ilyashenko, in his foundational work on the second part of the Hilbert's 16–th problem [Ily69]; see also Yakovenko's article [Yak94], that inspired us when searching for the estimates in Step 3 above. The higher dimensional method of integration of rational one forms ω along the leaves of singular codimension–one foliations in higher dimensional affine and projective manifolds appeared in the work [Muc95] of the third author of this article. In our Theorem 1, the bounds for the integration of one forms along the leaves of an one–dimensional foliation on $\mathbb{C}^n$ is more difficult.

2. Meromorphic maps and vector fields on Riemann surfaces

Let $\mathbb{CP}^1 = \mathbb{C}_w \cup \{\infty\}$ be the projective line, having affine coordinate w. The vector field $\partial/\partial w$ induces a holomorphic vector field in $\mathbb{CP}^1$ having double zero at $\infty \in \mathbb{CP}^1$. Let $\mathcal{L}$ be a compact Riemann surface.

Remark 3. *Let $f : \mathcal{L} \to \mathbb{CP}^1$ be a non-constant meromorphic function. The non-identically zero meromorphic vector field*

$$\frac{\partial}{\partial f} := f^*\left(\frac{\partial}{\partial w}\right)$$

is well defined on $\mathcal{L}$. Moreover, f has canonically associated a meromorphic one form ω, such that the diagram commutes

$$\begin{array}{ccc} & \nearrow & \{\omega\} \\ \{f : \mathcal{L} \to \mathbb{CP}^1\} & & \updownarrow \\ & \searrow & \{X = \frac{\partial}{\partial f}\}, \end{array} \tag{6}$$

X and ω are non-identically zero.

In fact, given f, we define $\omega = df$. The one to one correspondence between meromorphic vector fields and meromorphic one forms is given by the equation $\omega(X) \equiv 1$. This ω is called the one form of time for X, since for $p_0,\ p \in \mathcal{L}$ we have

$$f(p) - f(p_0) = \int_{p_0}^{p} \omega = \left\{ \begin{array}{l} \text{complex time to travel from} \\ p_0 \text{ to } p \text{ under the local flow of } \frac{\partial}{\partial f}. \end{array} \right.$$

The diagram (6) comes from the theory of quadratic differentials, see [Muc02]. The correspondence from ω to f in (6) is elementary.

In fact, in the righthand side the first and third integrals remain bounded when z goes to $1/a$. Hence, $|\widetilde{F}_1(a, z_2)|$ goes to infinity, when z_2 goes to $1/a$. $\widetilde{F}_1(z_1, z_2)$ is a rational function having a pole at the atypical fiber $\{1 - z_1 z_2 = 0\}$. □

Example 2. *A non-singular polynomial with non zero periods*

$$F_2(z_1, z_2) = z_1 - z_1^4 z_2^4.$$

This is also in the classification of polynomials with one critical value and no critical points in [Bod02]. The fiber over 0 is reducible, with a component which is topologically $\mathbb{C}$, and another one which is the Riemann sphere minus several points.

The level curve $\{F_2 = c\}$ corresponds to an octic in $\mathbb{CP}^2$ of equation:

$$z_0^7 z_1 - z_1^4 z_2^4 - c z_0^8 = 0.$$

The curve meets the line at infinity $z_0 = 0$ at the two points $[0 : 1 : 0]$ and $[0 : 0 : 1]$. It is singular at the two and if we look at the affine $\mathbb{C}^2 = \{z_1 \equiv 1\}$ of the first, we have the affine curve $z_0^7(1 - c z_0) - z_2^4 = 0$, that is singular (it has a cusp) at $(0, 0)$, with tangent line $z_2 = 0$. Furthermore, the contact of this tangent with the curve is $\dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}_0^2}}{(z_2, z_0^7(1-cz_0)-z_2^4)} = 8$.

On the other hand, if we look at the affine neighbourhood $\{z_2 \equiv 1\}$ of the second point, we see that the affine curve is given by $z_0^7 z_1 - z_1^4 - c z_0^8 = 0$. It is singular at $(0, 0)$ and the tangent is $z_1 = 0$. The contact of the curve and the tangent is $\dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}_0^2}}{(z_1, z_0^7 z_1 - z_1^4 - c z_0^8 = 0)} = 7$.

Hence, in order to parametrize we can consider the conics that pass through $(0 : 1 : 0)$, $(0 : 0 : 1)$ and have as tangents at them the lines $z_2 = 0$ and $z_1 = 0$, respectively. The conics fulfilling the conditions are those written as

$$s z_1 z_2 + \zeta z_0^2, \quad [s : \zeta] \in \mathbb{CP}^1.$$

They meet the octic at 16 points, 15 prescribed by the base conditions, and the remaining one giving the parametrization for the curve. Thus, we have

$$\Upsilon[s : \zeta] = [c s^8 + s^4 \zeta^4 : (c s^4 + \zeta^4)^2 : s^7 \zeta] : \mathbb{CP}^1 \to \mathcal{A}_c.$$

Note that $\Upsilon[0 : 1] = [0 : 1 : 0]$, while we have for points in $\mathbb{CP}^1$ (the roots of $cs^4 + \zeta^4 = 0$) whose image is $[0 : 0 : 1]$, there are four branches of the projective curve through that point.

Proceeding as before, we study the periods of the form ω such that $\omega(X) = 1$ on the level curve $\{F_2 = c\}$. Note that, topologically, this is $\mathbb{CP}^1$ with five points removed. As the affine parametrization is $\varphi(\zeta) = (\zeta^4 + c, \zeta/(\zeta^4 + c))$, we have that

$$X_c := X|_{\{F_2=c\}} = \varphi_* \left((\zeta^4 + c) \frac{\partial}{\partial \zeta} \right), \quad \text{hence} \quad \omega_c(\zeta) = \frac{d\zeta}{\zeta^4 + c}.$$

It is now easy to see that its periods around the finite poles are not zero.

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DEPARTAMENTO DE INGENIERÍA INDUSTRIAL
ESCUELA POLITÉCNICA SUPERIOR
UNIVERSIDAD ANTONIO DE NEBRIJA
C/ PIRINEOS 55, 28040 MADRID. SPAIN
E-mail address: abustind@nebrija.es

INSTITUTO DE MATEMÁTICA INTERDISCIPLINAR (IMI)
DEPARTAMENTO DE GEOMETRÍA Y TOPOLOGÍA
FACULTAD DE CIENCIAS MATEMÁTICAS
UNIVERSIDAD COMPLUTENSE DE MADRID
PLAZA DE CIENCIAS 3, 28040 MADRID. SPAIN
E-mail address: luis.giraldo@mat.ucm.es

CENTRO DE CIENCIAS MATEMÁTICAS
UNAM, CAMPUS MORELIA
A.P. 61–3 (XANGARI) 58089, MORELIA, MICHOACÁN, MÉXICO
E-mail address: muciray@matmor.unam.mx