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# Topological and analytical classification of vector fields with only isochronous centres

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We study vector fields on the plane having only isochronous centres. The most familiar examples are *isochronous* vector fields, they are the real parts of complex polynomial vector fields on $\mathbb{C}$ having all their zeroes of centre type. We describe the number $N(s)$ of topologically inequivalent isochronous (singular) foliations that can appear for degree $s$, up to orientation preserving homeomorphisms. For each $s$, there exists a real analytic variety $\mathcal{I}(s)$ parametrizing the isochronous vector fields of degree $s$, the group of complex automorphisms of the plane Aut($\mathbb{C}$) acts on it. Furthermore, if $2 \leq s \leq 7$, then $\mathcal{I}(s)$ is a non-singular real analytic variety of dimension $s+3$, and their number of connected components is bounded by $2N(s)$. An explicit formula for the residues of the rational 1-form, canonically associated with a complex polynomial vector field with simple zeroes, is given. A collection of residues (i.e. periods) does not characterize an isochronous vector field, even up to complex automorphisms of $\mathbb{C}$. An exact bound for the number of isochronous vector fields, up to Aut($\mathbb{C}$), having the same collection of residues (periods) is given. We develop several descriptions of the quotient space $\mathcal{I}(s)/\text{Aut}(\mathbb{C})$ using residues, weighted $s$-trees and singular flat Riemannian metrics associated with isochronous vector fields.

**Keywords:** ordinary differential equations; isochronous centres; residues; complex polynomials

**AMS Subject Classification**: 37C10; 34C05; 58F23

## 1. Introduction

A real vector field on $\mathbb{R}^2$ has an isochronous centre when the periods of trajectories surrounding the singular point are constant. The simplest vector fields having isochronous centres are the real parts $\mathfrak{Re}(X)$ of complex polynomial vector fields $X$, with non-zero pure imaginary derivative at some zero.

Real and complex isochronous centres appear in the following situations:

In the problem of linearization of centres for real vector fields of class $C^{\omega}$, see [7,17]. They are good prospects to study the birth of limit cycles under perturbation of centres, this is the infinitesimal Hilbert's 16 problem [2,14]. In the topological classification of plane polynomial vector fields, the phase portraits of polynomial vector fields with only isochronous centres are between the simplest (see [1,4] or [23]) for the quadratic case. They are the most simple examples of Jenkins–Strebel quadratic differentials on the Riemann sphere [26, 27].

In all that follows, an *isochronous vector field*, in short, an *isochronous X* is a complex polynomial vector field on $\mathbb{C}$, having all their zeroes of centre type.

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An isochronous $X$ has associated a weighted $s$-tree $\Lambda(X)$ as follows, the $s$ vertices correspond to the zeroes, the edges are determined by adjacent centre basins and the weights are the periods. In particular, $\Lambda(X)$ determines an embedded $s$-tree in $\mathbb{C}$.

Following [18], we know the following three facts. Each embedded $s$-tree (without weights) in $\mathbb{C}$ is realized by an isochronous $X$. If two isochronous $X_1, X_2$ determine topologically equivalent isochronous foliations (up to preserving orientation homeomorphisms $\mathrm{Homeo}(\mathbb{C})^+$), then they have associated the same embedded $s$-tree. Furthermore, $X_1, X_2$ are biholomorphically equivalent (up to complex automorphisms $\mathrm{Aut}(\mathbb{C})$) if and only if they have associated isomorphic weighted $s$-trees.

Let $\mathbb{C}^{s+1}$ be the space of complex polynomial vector fields on $\mathbb{C}$ having degree at most $s$. The set of isochronous vector fields of degree $s$, denoted by $\mathcal{I}(s)$, forms a real analytic family. The complex Lie group $\mathrm{Aut}(\mathbb{C})$ acts holomorphically on $\mathbb{C}^{s+1}$ as changes of coordinates. We have the diagram

$$\mathbb{C}^{s+1} \supset \mathcal{I}(s) \rightarrow \frac{\mathcal{I}(s)}{\mathrm{Aut}(\mathbb{C})} \rightarrow \frac{\mathcal{I}(s)}{\mathrm{Homeo}(\mathbb{C})^+}. \tag{1}$$

Our goal is the computation of the above quotients from the point of view of combinatorics, topology, geometry and dynamics.

The first main result Theorem 5.3 answers a question in [2].

*The number of classes of topological isochronous foliations in* $\mathcal{I}(s)/\mathrm{Homeo}(\mathbb{C})^+$ *(i.e. without bearing in mind the orientation of the trajectories) is*

$$N(s) = \frac{1}{2(s-1)} \sum_{d|(s-1)} \phi\left(\frac{s-1}{d}\right)\binom{2d}{d} - \frac{1}{2}c_{s-1} + \frac{1}{2}\chi_{\mathrm{even}}(s)c_{(s/2)-1}, \tag{2}$$

*where* $c_s = (1/(s+1))\binom{2s}{s}$ *denotes the sth Catalan number,* $\phi$ *denotes the Euler's function and* $\chi_{\mathrm{even}}$ *is the characteristic function of even integers.*

*Furthermore, for* $1 \le s \le 15$ *the respective numbers are*

$$1, 1, 1, 2, 3, 6, 14, 34, 95, 280, 854, 2694, 8714, 28640 \text{ and } 95640. \tag{3}$$

The idea of enumeration (2) using $s$-trees was suggested Muciño-Raymundo in [18] Corollary 8.3, compared with the recent result of Rong [24] and see our diagram (16) for an accurate description. The analytic families are more subtle, we describe them in Theorem 6.1.

*Assume* $3 \le s \le 7$. *The set of isochronous vector fields* $\mathcal{I}(s) \subset \mathbb{C}^{s+1}$ *is a non-singular real analytic space of dimension* $s+3$.

*The quotient space* $\mathcal{I}(s)/\mathrm{Aut}(\mathbb{C})$ *is of dimension* $s-1$, *Hausdorff and admits a stratification by orbit types.*

*In particular for* $s \le 7$,

$$\text{The number of connected components of } \mathcal{I}(s) \text{ is } \begin{cases} = 2N(s) & \text{odd } s, \\ \le 2N(s)-1 & \text{even } s. \end{cases} \tag{4}$$

The low-dimensional quotients $\mathcal{I}(s)/\mathrm{Aut}(\mathbb{C})$, $s = 1, 2, 3$, are described in Proposition 6.3. The second assertion says (very roughly speaking) that $\mathcal{I}(s)/\mathrm{Aut}(\mathbb{C})$ is a locally finite union of manifolds, pieced together in a nice way. Equation (4) says that for odd $s$, the preimage of a point under $\mathcal{I}(s) \rightarrow \mathcal{I}(s)/\mathrm{Homeo}(\mathbb{C})^+$ in (1) has precisely two connected components of $\mathcal{I}(s)$, due to the change of sign $\pm X$. For even $s$, a more complex behaviour appears, some precise values for (4) are given in 6.1.3.

An isochronous vector field $X$ is characterized by their associated rational 1-form $\omega$, such that $\omega(X) \equiv 1$, having only one zero at infinity and simple poles with non-zero pure imaginary residues. Our main analytic tool is an explicit formula for these residues (see (8)). The hypothesis $s \leq 7$ in the above result depends on explicit computations with the residues. We conjecture that it remains valid for all $s$ (see Remark 2 in Section 7). The residues of complex polynomial vector fields are the most natural complex analytic invariants under the Aut($\mathbb{C}$)-action (see Proposition 2.3). A third result is as follows (see Theorem 7.8).

Aut($\mathbb{C}$)*-orbits fail to be separated by the residues. For* $s \geq 4$*, the number of different classes of isochronous vector fields in* $\mathcal{I}(s)/\text{Aut}(\mathbb{C})$ *having the same collection of residues varies between 1 and* $(s-2)!$

The determination of enough invariant quantities for the Aut($\mathbb{C}$)-action on $\mathcal{I}(s)$ is equivalent to the construction of the quotient space $\mathcal{I}(s)/\text{Aut}(\mathbb{C})$. This classical idea emerged from the work of Cayley and Hilbert (see [9], p. xiii for an illuminated explanation). Even more, we deal with real analytic spaces $\mathcal{I}(s)$, non-complex analytic. Hence, we require techniques from Lie group actions (see Chapter 2 in [10]).

Very roughly speaking, we describe $\mathcal{I}(s)/\text{Aut}(\mathbb{C})$, using the following points of view: realizable weighted $s$-trees, singular flat Riemannian metrics up to isometries, collections of residues and configurations of zeroes (see diagram (27)).

The similar problems (realization of analytic families, moduli spaces, use of weighted graphs, number of connected components, etc.) for holomorphic 1-forms on compact Riemann surfaces of genus $g \geq 1$ have received much attention (see [16,27]). However, the case of rational 1-forms having arbitrary zeroes and poles on the Riemann sphere is less studied. Our results for 'isochronous' rational 1-forms, describe new families on the Riemann sphere.

In Section 2, we construct complex manifolds which are parameter spaces for the complex polynomial vector fields with simple zeroes and their associated 1-forms. Also we recall the residues as Aut($\mathbb{C}$) invariants. In Section 3, we deal with real vector fields. A complete characterization of weighted $s$-trees coming from isochronous $X$ is given in Corollary 3.6. The residue theorem imposes a restriction, but also there are other restrictions (see Example 7.3). In Section 4, the flat metric associated with an isochronous $X$ is described as a gluing of flat cylinders. The main results are the topological classification and the construction of non-singular analytic families for $\mathcal{I}(s)$ (both are presented in Sections 5 and 6, respectively). The characterization of realizable residues by isochronous vector fields is studied in Section 7. In Section 8, explicit families of isochronous centres are described in terms of configurations of zeroes, inequalities for the residues and weighted graphs. Their dimensions and codimensions in $\mathcal{I}(s)$ are provided. As an application, Section 9 is devoted to results of bifurcations under rotation and the Hamiltonian nature of isochronous vector fields (see Sections 2 and 5). Conclusions and future directions are in Section 10.

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## 2. Different facets of complex polynomial vector fields

In all that follows we use the one-to-one correspondence between

(i) complex polynomial vector fields $X$ of degree $s$,
(ii) rational differential forms $\omega$ having only a zero at $\infty \in \hat{\mathbb{C}}$ of multiplicity $s-2$,
(iii) orientable rational quadratic differentials $\omega \otimes \omega$, having poles of even multiplicity in $\mathbb{C}$ and a zero at $\infty \in \hat{\mathbb{C}}$ of multiplicity $2s-4$,

(iv) singular flat Riemannian metrics $g$ on $\hat{\mathbb{C}}$, with suitable singularities and a geodesic unitary vector field; see [18] for a general explanation (the above multiplicities are for $s \geq 3$, the special cases $s = 1, 2$ are easy).

In the present case we recall some basic facts and notation.

Let $X$ be a complex polynomial vector field on $\mathbb{C}$ of degree $s \geq 1$, non-identically zero, as follows

$$X = (b_0 z^s + b_1 z^{s-1} + \cdots + b_{s-1} z + b_s)\frac{\partial}{\partial z} \doteq \frac{1}{\lambda}(z - p_1)\cdots(z - p_s)\frac{\partial}{\partial z}. \tag{5}$$

Its associated rational 1-form is

$$\omega = \frac{\mathrm{d}z}{(b_0 z^s + b_1 z^{s-1} + \cdots + b_{s-1} z + b_s)} \doteq \frac{\lambda\,\mathrm{d}z}{(z - p_1)\cdots(z - p_s)}. \tag{6}$$

On the Riemann sphere $\hat{\mathbb{C}}$, $\omega$ has a zero of muliplicity $s - 2$ at infinity. For (5), the associated orientable quadratic differential is $\omega \otimes \omega$. The flat Riemannian metric associated with $\omega \otimes \omega$ is

$$g = \begin{pmatrix} \frac{1}{u^2+v^2} & 0 \\ 0 & \frac{1}{u^2+v^2} \end{pmatrix}, \quad \text{on } (\mathbb{C} - \{\text{zeroes of } X\}),$$

here $X = (u + iv)(\partial/\partial z)$ is the polynomial expression and $i = \sqrt{-1}$. The metric $g$ is real analytic outside of the zeroes (we say that $g$ is singular at the zeroes and infinity).

By a *trajectory* of $X$, we understand a real trajectory of the polynomial vector field

$$\Re\mathrm{e}(X) \doteq u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}.$$

The trajectories of $X$ are geodesics for the flat metric $g$.

*Parameter spaces*. By (5), the space of complex polynomial vector fields having degree $s$ is isomorphic to the open set $\{b_0 \neq 0\}$ in the space of coefficients $\mathbb{C}^{s+1}_{\text{coef}} = \{(b_0, \ldots, b_s)\}$.

The condition of having simple zeroes is necessary in order to get isochronous vector fields. Let $\Delta = \{p_\iota = p_j | \iota \neq j, \iota, j \in 1, \ldots, s\}$ be the set of hyperplanes in $\mathbb{C}^s_{\text{roots}} = \{(p_1 \ldots, p_s)\}$, describing $p_\iota$ with non-simple multiplicity. Under the natural action of the symmetric group $\mathcal{S}(s)$ of order $s$, the quotient $(\mathbb{C}^s_{\text{roots}} - \Delta)/\mathcal{S}(s)$ is an open complex manifold.

The Viète map from roots to coefficients induces a holomorphic map

$$\begin{array}{rccc} \mathcal{V}_s: & \mathbb{C}^* \times \left(\dfrac{\mathbb{C}^s_{\text{roots}} - \Delta}{\mathcal{S}(s)}\right) & \longrightarrow & \mathbb{C}^{s+1}_{\text{coef}} \\ & (\lambda, [p_1, \ldots, p_s]) & \longmapsto & (b_0, \ldots, b_s) \\ & \dfrac{1}{\lambda}(z - p_1)\cdots(z - p_s)\dfrac{\partial}{\partial z} & \longmapsto & (b_0 z^s + \cdots + b_s)\dfrac{\partial}{\partial z}, \end{array} \tag{7}$$

described by the elementary symmetric polynomials $\lambda b_\alpha = \sigma_\alpha(p_1, \ldots, p_s)$.

Note that the bracket $[\ldots]$ means the unordered collection of roots.

The image of $\mathcal{V}_s$ is the complement of two algebraic sets: $\{b_0 = 0\}$ and the discriminant hypersurface $\{\mathcal{D} = 0\}$ comprised of polynomials with multiple roots. If we restrict the Viète map to the domain and image as above, $\mathcal{V}_s$ is a biholomorphism (see [6], p. 124).

COROLLARY 2.1. *The complex manifolds*

$$\mathbb{C}^* \times \left( \frac{\mathbb{C}^s_{\text{roots}} - \Delta}{\mathcal{S}(s)} \right) \quad \text{and} \quad \mathbb{C}^{s+1}_{\text{coef}} - (\{b_0 = 0\} \cup \{\mathcal{D} = 0\})$$

*are parameter spaces for the set of complex polynomial vector fields on* $\mathbb{C}$ *of degree* s, *with simple zeroes.*

We also recognize the above manifolds as parameter spaces for the set of rational 1-forms on $\mathbb{C}$ having $s$ simple poles and free of zeroes.

LEMMA 2.2.

1. *For* $s \geq 2$, *the residue of* $\omega$ *at* $p_j$ *is*

$$r_j(\lambda, [p_1, \ldots, p_s]) = \frac{1}{2\pi i} \int_{\gamma_j} \omega = \frac{\lambda}{(p_j - p_1)\cdots\widehat{(p_j - p_j)}\cdots(p_j - p_s)}, \tag{8}$$

*here the hat over the factor* $(p_j - p_j)$ *indicates that it is omitted.*

2. *For complex polynomial vector fields with simple zeroes, the residue map*

$$\mathcal{R} : \mathbb{C}^* \times \left( \frac{\mathbb{C}^s_{\text{roots}} - \Delta}{\mathcal{S}(s)} \right) \to \frac{(\mathbb{C}^*)^s}{\mathcal{S}(s)} \quad (\lambda, [p_1, \ldots, p_s]) \mapsto [r_1, \ldots, r_s] \tag{9}$$

*is rational (between complex algebraic varieties).*

*Proof.* Since all the poles of $\omega$ are simple, we can write $\omega = f(z)\mathrm{d}z/(z - p_j)$ at each pole, for $f(z)$ holomorphic on a neighbourhood of $z = p_j$. Thus

$$\text{Residue}\left( \frac{f(z)\mathrm{d}z}{z - p_j}, p_j \right) = f(p_j),$$

and (8) is done. For part 2, we recall that the symmetric product $(\mathbb{C}^*)^s/\mathcal{S}(s)$ is an algebraic variety, not a complex manifold (see [13], Example 10.23). □

If we express $X$ or $\omega$ in terms of $b_0, \ldots, b_s$ in (6), then a simple expression for $\mathcal{R}$ is impossible for $s \geq 5$, due to the Abel–Galois theorem on the non-existence of radical expressions for $\mathcal{V}_s^{-1}$.

Let $\mathrm{Aut}(\mathbb{C}) = \{T(z) = az + b\}$ be the group of complex automorphisms of $\mathbb{C}$. Each affine transformation $T$ acts over $X$ as change of coordinates. The language of 1-forms is more suitable.

PROPOSITION 2.3. *The holomorphic Lie group action* $\mathscr{A}(T, \omega) = T_* \omega$ *is*

$$\mathscr{A} : \mathrm{Aut}(\mathbb{C}) \times \left( \mathbb{C}^* \times \frac{\mathbb{C}^s_{\mathrm{roots}} - \Delta}{\mathcal{S}(s)} \right) \to \mathbb{C}^* \times \frac{\mathbb{C}^s_{\mathrm{roots}} - \Delta}{\mathcal{S}(s)}$$

$$(az + b), \frac{\lambda \, dz}{(z - p_1) \cdots (z - p_s)} \mapsto \frac{\lambda a^{s-1} dz}{(z - (ap_1 + b)) \cdots (z - (ap_s + b))}$$

*and the residues* $[r_1, \ldots, r_s]$ *are invariant.*

*Proof*. It is a routine calculation. □

The next result is well-known, compare with [15], pp. 10–11.

COROLLARY 2.4. *Let* X *be a complex polynomial vector field on* $\mathbb{C}$ *of degree* $s \geq 2$*, the following assertions are equivalent.*

1. X *has* s *isochronous centres.*
2. *Their derivatives satisfy*

$$X'(p_j) = \frac{1}{\lambda}(p_j - p_1) \cdots (\widehat{p_j - p_j}) \cdots (p_j - p_s) \in i\mathbb{R}^*, \quad \text{at } s-1 \text{ zeroes of } X.$$

3. *Their residues satisfy* $r_j(\omega) \in i\mathbb{R}^*$*, at* $s - 1$ *poles of* $\omega$*.*

Along this work, the residues of an isochronous $X \in \mathcal{I}(s)$, i.e. of its associated $\omega$ means the unordered collection

$$[r_1, \ldots, r_s], \quad \text{where } r_j \in i\mathbb{R}^*.$$

DEFINITION 2.5. *Consider two centres* $p_\iota$, $p_j$ *of an isochronous* X *having basins* $U_\iota, U_j \subset \mathbb{C}$ *with a common boundary. The respective* semi-residue *of* X (*i.e. of* $\omega$) *is*

$$S_{\iota j} = \int_{\lambda_{\iota j}} \omega \in \mathbb{R}^+. \tag{10}$$

*Here* $\lambda_{\iota j} \doteq \overline{U\iota} \cap \overline{Uj}$ *is the saddle connection* (*homoclinic*) *trajectory of* X *starting and ending at* $\infty \in \hat{\mathbb{C}}$*, describing the boundary of the basins. Obviously* $S_{\iota j} = S_{j\iota}$*.*

*Remark* 1. An isochronous centre $p_j$ of a complex polynomial vector field $X$ satisfies

$$\frac{1}{X'(p_j)} = r_j(\omega) = \frac{\{\text{period of } \Re\mathfrak{e}(X) \text{ at } p_j\}}{2\pi i} \doteq \frac{T_j}{2\pi i}. \tag{11}$$

In particular, the periods $T_j$ are 'oriented' as in (11) and $T_j \in \mathbb{R}^*$. In many places we will interchange the residues and the periods applying (11).

## 3. Real $C^r$ isochronous vector fields

We introduce vector fields $Y$ on $\mathbb{R}^2$ having differentiability class $C^r$, $r \in 1, \ldots, \infty, \omega$, and isochronous centres. A comparison with the complex polynomial case is useful, it increases the flexibility of our techniques.

The following local characterization of isochronous centres is classical in the real analytic case, it is due to Poincaré and Lyapunov (see [17]). We describe the differentiable case.

PROPOSITION 3.1. *For $r \in 1, \dots, \infty$, the following properties are equivalent.*

1. *The real $C^r$ vector field* Y *on* $\mathbb{R}^2$ *has an isochronous centre at* p *of period* $T \in \mathbb{R}^*$.
2. *Under a local,* $C^{r-1}$, *change of coordinates,* Y *is equivalent to*

$$\frac{2\pi}{T}\left(-y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}\right).$$

3. *Under a local,* $C^{r-1}$, *change of coordinates,* Y *is equivalent to the real part of the holomorphic vector field*

$$\frac{2\pi i z}{T}\frac{\partial}{\partial z}.$$

*Proof*. We show $1 \Rightarrow 2$. By a result of Sabatini [25], there exists a second vector field $W$, having class $C^r$, such that it is transversal with $Y$ in a punctured neighbourhood of $p$ and commutes with $Y$, this is $[Y, W] \equiv 0$. In polar coordinates $\mathcal{P}^{-1}(x, y) = (r(x, y), \theta(x, y))$, the corresponding vector fields $Y_1, W_1$, are of class $C^{r-1}$ at $\{r = 0\}$ (see [8], Proposition 1.135). Using that $Y_1$ and $W_1$ commute, there exists a $C^{r-1}$ diffeomorphism $\Psi(r, \theta)$ rectifying both vector fields. That is, $\Psi_* Y_1 = (2\pi/T)(\partial/\partial\theta)$ and $\Psi_* W_1 = \partial/\partial r$. The required change of coordinates is $\mathcal{P}\circ\Psi\circ\mathcal{P}^{-1}$. The other equivalences are immediate. □

We look at the global situation, consider in the plane the orientation coming from $i = \sqrt{-1}$ and the usual identification $\mathbb{R}^2 \cong \mathbb{C}$.

Assume that $Y$ is a real $C^r$ vector field on $\mathbb{R}^2$, $r \geq 1$, such that

(i) $Y$ has a finite number of singularities $\{p_1, \dots, p_s\} \subset \mathbb{R}^2$ which are isochronous centres with periods $\{T_1, \dots, T_s\}$,
(ii) the union of the closure of their basins $\{U_1, \dots, U_s\}$ is $\mathbb{R}^2$.

Let us give some topological facts. The boundary of $\overline{U}_\iota$ is a certain finite collection $\{\lambda_{\iota j}\}$ of trajectories of $Y$ as in (10), having $\alpha$ and $\omega$-limit at infinity of $\mathbb{R}^2 \cup \{\infty\}$.

If the intersection $\overline{U}_\iota \cap \overline{U}_j \subset \mathbb{R}^2$ is not empty, then $\lambda_{\iota j} \doteq U_\iota \cap U_j$.

Each vector field $Y$ as above have associated with a decomposition of $\mathbb{R}^2$ given by

$$\mathbb{R}^2 = U_1 \cup \cdots \cup U_s \cup_{\iota j} \lambda_{\iota j}.$$

The trajectory $\lambda_{\iota j}$ does not necessarily exist for each subindex pair $\iota j$.

By the isochronicity hypothesis, the flow of $Y$ is defined in $\lambda_{\iota j}$ for bounded time, say $S_{\iota j} \in \mathbb{R}^+$ similarly as in (10), since $\lambda_{\iota j}$ is oriented by $Y$.

We are attaching a sign to each period, $T_j > 0$ when $Y$ turns in the anticlockwise direction at $p_j$ and $T_\kappa < 0$ otherwise.

DEFINITION 3.2. *The* weighted s-tree of Y (*a vector field as above*) *is*

$$\Lambda(Y) = \left\{ \left( p_1, \frac{T_1}{2\pi i} \right), \ldots, \left( p_s, \frac{T_s}{2\pi i} \right), \Lambda_{\iota j} \right\},$$

*where the vertices are the centres $p_\iota$ of* Y, *their weights are $T_\iota/2\pi i \in i\mathbb{R}^*$ from the respective periods, the edges $\Lambda_{\iota j}$ (joining $p_\iota$ and $p_j$) are defined when $\overline{U}_\iota \cap \overline{U}_j \subset \mathbb{R}^2$ is not empty and $\Lambda(Y)$ has a natural embedding in $\mathbb{R}^2$.*

Following the notation of Harary and Palmer [11], pp. 66–67 (see also Bergeron et al. [3], p. 284, Example 4), we have the next concept.

DEFINITION 3.3. *A* plane s-tree $\Lambda \subset \mathbb{C}$ *is a class of embeddings of a* s-*tree to the plane up to orientation preserving homeomorphisms* $\text{Homeo}(\mathbb{C})^+$.

Recall that $\Lambda(Y)$ encodes an abstract *s*-tree and a plane *s*-tree.

If $Y = \Re\mathfrak{e}(X)$ for $X$ holomorphic, then the weights coincide with the residues (see (11)). We denote the associated plane *s*-tree simply by $\Lambda = \{v_1, \ldots, v_s, \Lambda_{\iota j}\}$.

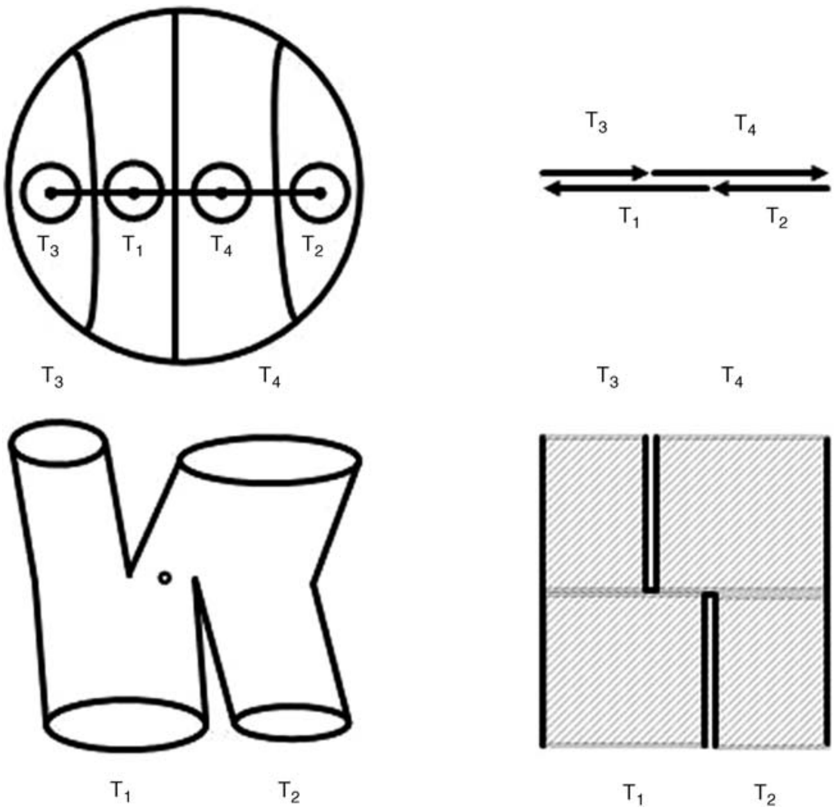


Figure 1. An isochronous $X$ of degree 4 as in Example 4.3 and four facets of it. The phase portrait in the Poincaré-Lyapunov disc with a plane 4-tree drawn on, having the periods $[T_1, \ldots, T_4]$ as weights. The arrangement of the periods in $\mathbb{R}$ is in the upper right drawing. The flat metric on the Riemann sphere with five punctures, the four zeroes and a point of cone angle $6\pi$, corresponding to the pole of $X$. The Riemann surface $\bar{\mathcal{B}} \subset \mathbb{C}$, from the gluing of four half bands (having bases in the arrangement) is drawing below right.

*Example* 3.4. In Figure 1 upper left drawing, the plane 4-tree $\Lambda$ corresponding to an isochronous $X \in \mathcal{I}(4)$ is shown ($\Lambda$ is placed on the phase portrait of $X$).

The next result appeared in [18] as 8.1.

THEOREM 3.5. *Let* Y *be a real* $C^r$ *vector field on* $\mathbb{R}^2$, $r \geq 1$, *as above. There exist a complex structure* J *on* $\mathbb{R}^2$, *a complex polynomial isochronous vector field* X *and a biholomorphic map* $\psi : (\mathbb{R}^2, J) \to \mathbb{C}$, *such that* $\psi_*(Y) = \Re\mathrm{e}(X)$ *if and only if*

(i) Y *has a finite number of singularities that are isochronous centres,*
(ii) *the union of the closure of their centre basins is* $\mathbb{R}^2$,
(iii) *the period* $T_\iota$ *of each isochronous centre satisfies*

$$|T_\iota| = \sum_j S_{\iota j}, \quad p_j \text{ adjacent with } p_\iota \text{ in } \Lambda(Y).$$

Let us remark that, in general, the complex structure $J$ depending on $Y$ is not the usual form $i = \sqrt{-1}$ on $\mathbb{C}$ (recall that there are plenty of complex structures on $\mathbb{R}^2$).

In the other direction, let $\Lambda_0$ be an 'abstract' weighted $s$-tree. We characterize under what conditions $\Lambda_0$ comes from an isochronous vector field.

COROLLARY 3.6. *Realizable weighted* s*-trees.*

*Let* $\Lambda_0 = \{(v_1, r_1), \ldots, (v_s, r_s), \Lambda_{\iota j}\}$ *be a weighted* s*-tree satisfying*

(i) $r_\iota \in i\mathbb{R}$ *and* $\Sigma r_\iota = 0$,
(ii) *each two adjacent vertices* $v_j$, $v_\iota$ *have weights* $r_j$ *and* $r_\iota$ *of opposite signs in* $i\mathbb{R}$,
(iii) *there exists a one-to-one correspondence* $\{\Lambda_{\iota j} \leftrightarrow S_{\iota j}\}$ *with suitable* $S_{\iota j} \in \mathbb{R}^+$, *such that*

$$|r_\iota| = \frac{1}{2\pi} \sum_j S_{\iota j}, v_j \text{ adjacent with } v_\iota \text{ in } \Lambda_0.$$

*Then* $\Lambda_0$ *determines an isochronous* $X \in \mathcal{I}(s)$, *unique up to* $\mathrm{Aut}(\mathbb{C})$.

*Proof.* By hypothesis, $\Lambda_0$ is embedded as abstract graph $\{v_1, \ldots, v_s, \Lambda_{\iota j}\}$ in $\mathbb{C}$. There exists a $C^1$ real vector field $Y$ having topological phase portrait and periods as the data in $\Lambda_0$. The factor $1/2\pi$ in condition (iii) fulfils the relation between periods and residues (11). By Theorem 3.5, there exists at least one $X \in \mathcal{I}(s)$ with associated weighted tree $\Lambda(X) = \Lambda_0$.

For the unicity, two isochronous $X_1, X_2$ coming from $\Lambda_0$ are biholomorphically equivalent to the vector field $(1/2)(Y - iJY)$ on $(\mathbb{R}^2, J)$ under $\psi_1^{-1}$, $\psi_2^{-1}$, respectively. Here again we are using 3.5 and [18] Remark 1.1. By transitivity, $X_1$ and $X_2$ are biholomorphically equivalent under $\psi_2 \circ \psi_1^{-1} \in \mathrm{Aut}(\mathbb{C})$. □

Let us summarize. Each realizable weighted $s$-tree $\Lambda(X)$ determines an equivalence class in $\mathcal{I}(s)/\mathrm{Aut}(\mathbb{C})$.

However, each plane $s$-tree $\Lambda$ a priori determines two equivalence classes $[X]$ and $[-X]$ in $\mathcal{I}(s)/\mathrm{Homeo}(\mathbb{C})^+$, we will study this in Lemma 6.7 (see diagram (27)).

## 4. Geometry of the flat metrics

We describe the metric associated with an isochronous $X$. This is instructive, provide us with simple intuition and will allow us avoid calculations in Sections 8 and 9.

LEMMA 4.1.

1. *A complex polynomial vector field* X *of degree* $s \geq 2$ *is isochronous if and only if the associated flat metric* g *on* $\mathbb{C} - \{\text{zeroes of } X\}$ *satisfies the following:*
   (i) g *is obtained by the gluing along the boundaries (using isometries) of* s *half flat cylinders*

$$\{S^1_{|T_\iota|} \times [0, \infty) | T_\iota > 0\}, \{S^1_{|T_j|} \times (-\infty, 0] | T_j < 0\},$$

   *here the subindex* $|T_\iota| = 2\pi |r_\iota|$ *means the perimeter and* $r_1, \ldots, r_s$ *are the residues of* X.
   (ii) *The gluing is performed in such way that only a singular point of cone angle* $2\pi(s-1)$ *appears corresponding to* $\infty \in \hat{\mathbb{C}}$*, the pole of* X.
   (iii) *The unitary closed geodesics in the cylinders and the trajectories of* $\Re\mathrm{e}(X)$ *coincide and have the same time orientation.*
2. *At level of sets there exists a one-to-one correspondence such that*

$$\frac{\mathcal{I}(s)}{\mathrm{Aut}(\mathbb{C}) \times \mathbb{Z}_2} = \left\{ \begin{array}{c} \text{flat metrics of} \\ \text{isochronous } X, \\ \text{up to isometries} \end{array} \right\}. \tag{12}$$

The metrics in (12) are real analytic on the sphere $\hat{\mathbb{C}}$ with $s+1$ punctures, the $s$ zeroes and the pole of the corresponding vector field $X$.

A more complete appreciation of (12) is given in diagram (27).

*Proof*. For (i), we use the ideas of Section 8 in [18]. Given an isochronous $X$, each centre basin $(U_j - \{p_j\}, g)$ as Riemannian surface is isometric to the interior of a half flat cylinder. The gluing of two half cylinders is by isometries along their boundaries, which are the trajectories $\lambda_{\iota j}$ in (10). This provided the global metric in $(\mathbb{C} - \{\text{zeroes of } X\})$, see examples 4.2–4.4.

For the converse, given the flat metric $g$ on $\hat{\mathbb{C}}$ coming from the gluing of $s$ half cylinders, $g$ has a unique point of cone of angle as in (ii) and $s$ punctures, which are the Riemannian ends of the cylinders.

On $(\mathbb{C} - \{\text{zeroes of } X\}, g)$, we define $\Re\mathrm{e}(X)$ as the geodesic vector field $F$ whose trajectories are unitary closed geodesics in the cylinders. From the real vector field we can recover the complex vector field as $X = (1/2)(F - iJF)$, here $J$ is the complex structure from $g$ (see [18], Remark 1.1). Using equation (11) we recognize the periods and the residues of $X$. It is isochronous.

For part 2, we consider $X \in \mathcal{I}(s)$ and $T \in \mathrm{Aut}(\mathbb{C})$. The vector fields $X$ and $T_* X$ determine the same metric $g$, i.e. $T$ is a Riemannian isometry. Moreover, the corresponding flat metrics of $X$ and $-X$ are isometric. Consider the action

$$\mathbb{Z}_2 \times \mathcal{I}(s) \rightarrow \mathcal{I}(s), \quad X \mapsto \pm X. \tag{13}$$

This action leaves invariant the isometry classes of the metrics. The correspondence between classes of vector fields and classes of metrics is done. □

*Example* 4.2. The metric $g$ of an isochronous $X \in \mathcal{I}(2)$ is a flat cylinder with a puncture.

The residues are $[r, -r] = [T/2\pi i, -T/2\pi i]$, $T > 0$. Consider the vertical band

$$B = \{x + iy | x \in [0, T]\} \subset \mathbb{C}.$$

We paste the opposite vertical sides of $B$ (each point $iy$ in the boundary is identified with $T + iy$) to get the cylinder $S^1_T \times \mathbb{R}$. The vector field $X$ is $\partial/\partial z$ on $B$. Finally, we remove one point $q = x + 0i$ in the cylinder, it corresponds to $\infty$ when we think $X$ on $\hat{\mathbb{C}}$.

*Example* 4.3. The metric $g$ of an isochronous $X \in \mathcal{I}(4)$, having residues

$$[r_1, r_2, r_3, r_4] = \left[\frac{T_1}{2\pi i}, \frac{T_2}{2\pi i}, \frac{T_3}{2\pi i}, \frac{T_4}{2\pi i}\right],$$

$$T_1 < T_2 < 0 < T_3, T_4, \qquad -T_1 - T_2 = T_3 + T_4$$

Note that $-T_3$ is different from $T_1$ or $T_2$. Without loss of generality assume $T_3 < -T_1$ see Figure 1, upper right drawing. Consider two lower half bands in $\mathbb{C}$ and two upper

$$\{x + iy | x \in [0, -T_1], y \leq 0\}, \quad \{x + iy | x \in [-T_1, -T_1 - T_2], y \leq 0\},$$

$$\{x + iy | x \in [0, T_3], y \geq 0\}, \quad \{x + iy | x \in [T_3, T_3 + T_4], y \geq 0\},$$

see Figure 1 lower right drawing. From each band we get a half cylinder. In addition, we paste the horizontal boundary of the half bands (respectively, the half cylinders) identifying each point $x \in \mathbb{R}$ in a lower band with the respective $x$ in the upper band, see the lower row drawings in Figure 1. The resulting surface fulfils the conditions in 4.1. $X$ is induced by $\partial/\partial z$. Let us remark that in this metric there are three trajectories

$$\{\lambda_{\iota j}\} = \{(0, T_3), (T_3, -T_1), (-T_1, T_3 + T_4)\},$$

which are the intersections of the centre basins $\bar{U}_\iota \cap \bar{U}_j$, determining the semi-residues.

Our choice of upper half bands for $0 < T_3, T_4$ is coherent with the orientation on $\mathbb{C}$, the definition of $r_3$, $r_4$ (as the integral along a path with the anticlockwise orientation) and equation (11).

*Example* 4.4. The metric $g$ of an isochronous $X \in \mathcal{I}(4)$, having residues

$$[r_1, r_2, r_3, r_4] = \left[\frac{T_1}{2\pi i}, \frac{T_2}{2\pi i}, \frac{T_3}{2\pi i}, \frac{T_4}{2\pi i}\right],$$

$$T_1 < 0 < T_2, T_3, T_4, \quad -T_1 = T_2 + T_3 + T_4.$$

Consider four half vertical bands in $\mathbb{C}$ one lower and three upper, which are as follows

$$\{x + iy | x \in [0, -T_1], y \leq 0\},$$

$$\{x + iy | x \in [0, T_2], y \geq 0\}, \quad \{x + iy | x \in [T_2, T_2 + T_3], y \geq 0\},$$

$$\{x + iy | x \in [T_2 + T_3, T_2 + T_3 + T_4], y \geq 0\}.$$

From each band we get a half cylinder. We paste the horizontal boundary of the half bands (respectively, the half cylinders) identifying each point $x \in \mathbb{R}$ in the lower band with the point $x$ in the respective upper band. The resulting surface fulfils the conditions in 4.1. $X$ is induced by $\partial / \partial z$.

Summing up, an isochronous $X \in \mathcal{I}(s)$ has associated with the following *data*, see (14), a rational 1-form, a weighted $s$-tree $\Lambda(X)$, a plane $s$-tree $\Lambda$ (e.g. Figures 2 and 3), a configuration of zeroes $[p_1, \ldots, p_s]$, an unordered collection of residues $[r_1, \ldots, r_s]$ and periods $[T_1, \ldots, T_s]$ and their semi-residues $\{S_{\iota j}\}$.

$$\begin{array}{ccccccc} & & (\lambda, [p_1, \ldots, p_s]) & & [p_1, \ldots, p_s] & & \\ & & \updownarrow & \nearrow & & & \\ \omega & \leftrightarrow & X & \rightarrow & \Lambda(X) & \rightarrow & \Lambda \\ & & \downarrow & \searrow & \downarrow & & \\ & & \{S_{\iota j}\} & & [r_1, \ldots, r_s] & & \\ & & & & \updownarrow & & \\ & & & & [T_1, \ldots, T_s] & & . \end{array} \tag{14}$$

In general, no further implications are possible. In Sections 6–9, we will derive additional conditions for the converse implications.

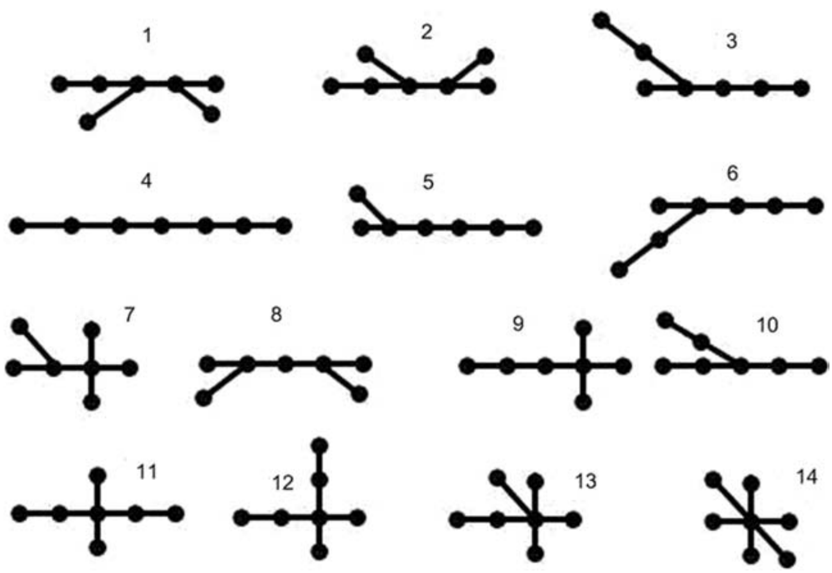


Figure 2. Plane 7-trees, i.e. embedded up to Homeo $(\mathbb{C})^+$.

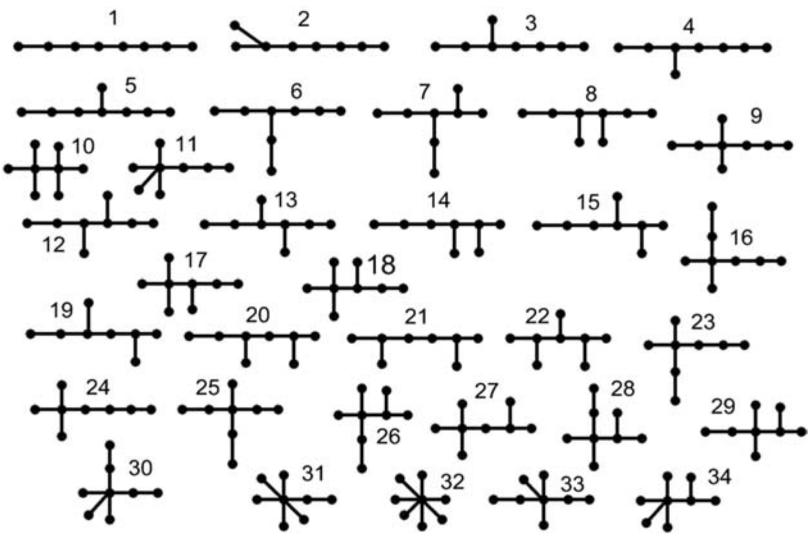


Figure 3. Plane 8-trees, i.e. embedded up to Homeo $(\mathbb{C})^{+}$.

## 5. Topological enumeration

We answer a question in [2] describing the topological classification of isochronous vector fields $X \in \mathcal{I}(s)$. In order to avoid confusions, we make explicit three notions of equivalence.

DEFINITION 5.1.

1. *Two isochronous vector fields* $X_1, X_2$ *determine* topologically equivalent isochronous foliations *if there exists a homeomorphism* $h \in \text{Homeo}(\mathbb{C})^{+}$ *which takes trajectories from* $X_1$ *to* $X_2$ (*independent of the time orientation*).
2. *Furthermore,* $X_1, X_2$ *are* topologically equivalent isochronous vector fields *if in addition the homeomorphism* h *as above preserves the time orientation along the trajectories. In particular the signs* (*in* $i\mathbb{R}$) *of their respective residues must coincide under* h.
3. $X_1$, $X_2$ *are* holomorphically equivalent isochronous vector fields *if there exists an affine map* $T \in \text{Aut}(\mathbb{C})$ *such that* $T_* X_1 = X_2$.

Our isochronous foliations are *singular*, by abuse of language we omit this. Clearly the above definitions satisfy $1 \Leftarrow 2 \Leftarrow 3$, however the converse implications fail.

*Example* 5.2. The implication $1 \Rightarrow 2$ depends on the degree.

For degree $s = 1$, the isochronous $iz(\partial/\partial z)$ and $-iz(\partial/\partial z)$ define topologically equivalent isochronous foliations. However they are topologically inequivalent isochronous vector fields, i.e. $h \in \text{Homeo}(\mathbb{C})^{+}$ cannot preserve the time orientation. In particular $N(1) = 1$ in equation (3).

For degree $s = 2$, every isochronous $X \in \mathcal{I}(2)$ is

$$X_1 = \left(\frac{r}{z+1} + \frac{-r}{z-1}\right)^{-1} \frac{\partial}{\partial z}, \quad r \in \mathbb{R}^{+},$$

under a suitable $h \in \mathrm{Aut}(\mathbb{C})$. Moreover, in this case, $X_1$ and $-X_1$ are topologically equivalent isochronous vector fields under $h(z) = -z$. In particular $N(2) = 1$ in equation (3).

Recall the $\mathbb{Z}_2$ action given by $X \mapsto \pm X$ (see (13)). A class of this action is the isochronous foliation of $\pm X$. Using the discussion in Section 3, Definitions 3.3 and 5.1.1 we have the following diagram:

$$\begin{array}{ccc} \mathcal{I}(s) & \longrightarrow & \frac{\mathcal{I}(s)}{\mathrm{Homeo}(\mathbb{C})^{+}} \\ & \searrow & \downarrow \end{array}$$

$$\frac{\mathcal{I}(s)}{\mathrm{Homeo}(\mathbb{C})^{+} \times \mathbb{Z}_2} = \left\{ \begin{array}{c} \text{topological} \\ \text{isochronous} \\ \text{foliations} \end{array} \right\} = \left\{ \begin{array}{c} \text{plane } s\text{-trees} \\ (\text{embedded } s\text{-trees} \\ \text{up to Homeo}(\mathbb{C})^{+}), \\ \Lambda \subset \mathbb{C} \end{array} \right\}. \tag{15}$$

Let us remark that $\{\Lambda\}$ are combinatorial objects. The proof of the right equality in (15) uses arguments in the phase portraits as in [2], we leave the details to the reader. A virtue of the concept of plane $s$-tree is its relation with the number of connected components in $\mathcal{I}(s)$ and $\mathcal{I}(s)/\mathrm{Aut}(\mathbb{C})$, as we see in Theorem 6.1.3. Now we introduce the enumeration of plane $s$-trees.

THEOREM 5.3. *The number of classes of topological isochronous foliations of degree* s *is*

$$N(s) = \frac{1}{2(s-1)} \sum_{d|(s-1)} \phi\left(\frac{s-1}{d}\right) \binom{2d}{d} - \frac{1}{2} c_{s-1} + \frac{1}{2} \chi_{\text{even}}(s) c_{(s/2)-1},$$

*where* $c_s = (1/(s+1)) \binom{2s}{s}$ *denotes the* s*th Catalan number,* $\phi$ *denotes the Euler's function and* $\chi_{\text{even}}$ *is the characteristic function of even integers.*

*Furthermore, for* $1 \le s \le 15$*, the respective numbers are*

$$1, 1, 1, 2, 3, 6, 14, 34, 95, 280, 854, 2694, 8714, 28640 \text{ and } 95640.$$

*Proof*. Let us recall two facts. The enumeration of tree-like structures started with Pólya [22]. The enumeration of $s$-trees as abstract graphs was achieved by Otter [21].

From the dynamical point of view, the enumeration of embeddings of $s$-trees in $\mathbb{C}$ can be consider in two ways:

(i) looking at the classes of embeddings up to homeomorphisms of the plane, $\mathrm{Homeo}(\mathbb{C})$, as was studied by Álvarez et al. [2] and Rong [24], or
(ii) looking as plane $s$-trees, i.e. classes of embeddings up to orientation preserving homeomorphisms of the plane, $\mathrm{Homeo}(\mathbb{C})^{+}$, see diagram (15).

Our choice is (ii).

In diagrams (1) and (15), we are looking at the canonical orientation on $\mathbb{C}$ (given by the complex structure $i = \sqrt{-1}$).

Summing up

$$\#\left\{\begin{array}{c}s\text{-trees}\\ \text{as abstract}\\ \text{graphs,}\\ \text{Otter [21]}\end{array}\right\} \leq \#\left\{\begin{array}{c}\text{embedded } s\text{-trees}\\ \text{up to Homeo}(\mathbb{C}),\\ \text{Rong [24]}\end{array}\right\} \leq \#\left\{\begin{array}{c}\text{plane } s\text{-trees}\\ \text{(embedded } s\text{-trees}\\ \text{up to Homeo}(\mathbb{C})^{+}),\\ (2)\end{array}\right\}. \qquad (16)$$

For $1 \leq s \leq 6$, the enumerations of Otter, Álvarez et al., Rong and our equation (2) coincide. For $7 \leq s$, the inequalities in (16) are strict, which are as follows. □

*Example* 5.4. The first case where the enumerations of Rong and equation (2) are different is the family of 7-trees. As abstract graphs there are eleven 7-trees, see [21]. In Figure 2, the plane $s$-trees labelled with 11 and 12 are equal as abstract graphs but different as classes of embeddings up to homeomorphisms of the plane, thus following Rong's enumeration we get 12. Moreover when we are considering the plane 7-trees, recall diagram (15) and graphs labelled with $\{1, 2\}$ in Figure 2 are different as plane 7-trees, and the same for graphs labelled $\{3, 6\}$, then our enumeration increases to being 14. Recall that Álvarez et al. [2] provided $\leq 12$, for $s = 7$.

*Example* 5.5. The next case where the enumerations are different is the family of 8-trees. As abstract graphs there are twenty three 8-trees, see [21]. In Figure 3, the trees labelled with 8 and 12 are equal as abstract graphs but different as classes of embeddings up to homeomorphisms of the plane. The same is true for the couples of graphs labelled $\{9, 16\}$, $\{26, 29\}$ and $\{30, 33\}$ in Figure 3. Thus following Rong's enumeration, we get 27. Moreover when we are considering the plane 8-trees, the couples of graphs labelled $\{3, 4\}$, $\{12, 13\}$, $\{14, 15\}$, $\{16, 23\}$, $\{17, 18\}$, $\{19, 20\}$ are different (see Figure 3). Then our enumeration increases to being 34. Recall that Álvarez et al. [2] provided $\leq 28$, for $s = 8$.

*The general case*. We started with Otter's enumeration. When a rooted tree is embedded in the plane a cyclic order is induced on the edges incident with the root, this is shown in Harary et al. [12]. Moreover they found an explicit formula for enumeration of these kinds of structures. In equations (3.3.23), (3.3.24), (3.3.25) and theorem in page 67 of Harary and Palmer [11], they resumed the computation of generating series and whence enumeration for planted, rooted and plane $s$-trees, respectively. Following the above ideas of Harary et al., in pages 284–285 and 291 of Bergeron et al. [3], a modern resume is presented, particularly (48) of [3] is our equation (2).

Following (2), for $1 \leq s \leq 15$, we get the numbers in (3).

## 6. Analytic structure for the space of isochronous vector fields

We leave the description of $\mathcal{I}(s)$ in the special cases $s = 1, 2$, for the reader.

THEOREM 6.1. *Assume* $3 \leq s \leq 7$.

1. *The set of isochronous centres* $\mathcal{I}(s) \subset \mathbb{C}^{s+1}_{\text{coef}}$ *is a non-singular real analytic space of dimension* $s + 3$. *Furthermore,* $\mathcal{I}(s)$ *has the structure of a real line bundle over a real analytic manifold, removing the zero section.*

2. *The quotient space* $\mathcal{I}(s)/\mathrm{Aut}(\mathbb{C})$ *is of dimension* $s-1$*, Hausdorff and admits a stratification by orbit types.*
3. *For* $s \leq 7$*, the number of connected components of* $\mathcal{I}(s)$ *is* $2N(s)$ *for odd* s*, and* $\leq 2N(s)-1$ *for even* s.

*The respective numbers are* $2, 1, 2, 3, 6, 10$and$14$.

*Proof of assertion* 6.1.1. The classes in $\mathcal{I}(s)/Aut(\mathbb{C})$ describe the holomorphically equivalent isochronous vector fields, recall Definition 5.1.3. Consider the diagram

$$\begin{array}{ccccc} \mathbb{C}^* \times \left(\frac{\mathbb{C}^s_{\text{roots}}-\Delta}{\mathcal{S}(s)}\right) & \supset & \mathcal{I}(s) & \longrightarrow & \frac{\mathcal{I}(s)}{\mathrm{Aut}(\mathbb{C})} \cong \left\{ \begin{array}{c} \text{realizable} \\ \text{weighted} \\ s\text{-trees } \Lambda(X) \end{array} \right\}, \\ & & \downarrow & & \\ & \mathcal{R} \searrow & \frac{(\mathbb{C}^*)^s}{\mathcal{S}(s)} & \swarrow & \end{array} \qquad (17)$$

where $\mathcal{R}$ maps each polynomial vector field to the unordered collection of their residues (see (9)). The other two down arrows are the restrictions of $\mathcal{R}$. The meaning of realizable weighted $s$-trees is in Corollary 3.6.

The map $\mathcal{R}$ is a sort of geometric transformation, sending each unordered configuration of zeroes to the respective unordered collection of residues. $\mathcal{R}$ is invariant under the Aut($\mathbb{C}$)-action, recall Proposition 2.3.

We define the *isochronicity equations* for $X = (\lambda, [p_1, \ldots, p_s])$ in $\mathbb{C}^* \times (\mathbb{C}^s_{\text{roots}} - \Delta/\mathcal{S}(s))$ as

$$\arg(r_1) = \cdots = \arg(r_{s-1}) = \arg(r_s). \qquad (18)$$

Thus, if the residues $[r_1, \ldots, r_s] = \mathcal{R}(\lambda, [p_1, \ldots, p_s])$ belong to the real line $\mathcal{L} = \{\rho e^{i\theta_0}\} \subset \mathbb{C}$ with $\theta_0 = \arg(r_\iota)(\text{mod } \pi)$, then $e^{i\theta_0}X$ is isochronous.

By the residue theorem, $(s-1)$ equalities in (18) imply the remaining.

Analytic line bundle structure of $\mathcal{I}(s)$. Let us define

$$C(s) = \{([p_1, \ldots, p_s]) | \arg(r_1) = \cdots = \arg(r_s)\} \subset \frac{\mathbb{C}^s_{\text{roots}} - \Delta}{\mathcal{S}(s)}$$

called the set of *isochronous configurations*. This is well defined, since $\lambda$ is a linear factor in every $r_\iota$ (see (8)).

The fact that $C(s)$ is a real analytic variety (possibly singular) is immediate from (18). We want to study whether $C(s)$ is non-singular and show that $(s-1)$ equations in (18) are non-redundant.

The property of being a non-singular space is local. We introduce *the ordered residue map* R as follows. Let $V_\alpha \subset \mathbb{C}^s - \Delta$ be a collection of open discs, such that $\cup_\alpha V_\alpha = \mathbb{C}^s - \Delta$. Note that each $(p_1, \ldots, p_s) \in V_\alpha$ is the usual ordered $s$-tuple. For each $V_\alpha$ we get an ordered residue map

$$\mathrm{R}(\lambda, p_1, \ldots, p_s) = (r_1, \ldots, r_s),$$

where both sides have order ($r_\iota$ is the obvious residue). R satisfies the following diagram

$$\begin{array}{ccccccc} \mathbb{C}^* \times V_\alpha & \overset{\mathrm{R}}{\rightarrow} & (\mathbb{C}^*)^s & \overset{1/r}{\rightarrow} & (\mathbb{C}^*)^{s-1} & \overset{\arg}{\rightarrow} & (S^1)^{s-1} \\ \downarrow & & \downarrow & & & & \\ \mathbb{C}^* \times \left(\frac{\mathbb{C}^s_{\text{roots}} - \Delta}{\mathcal{S}(s)}\right) & \overset{\mathcal{R}}{\rightarrow} & \frac{(\mathbb{C}^*)^s}{\mathcal{S}(s)} & & & & \end{array} . \tag{19}$$

By definition

$$\left(\frac{1}{r}\right)\circ \mathrm{R} : (\lambda, p_1, \dots, p_s) \mapsto \left(\frac{1}{r_1}, \dots, \frac{1}{r_{s-1}}\right),$$

$$\left(\frac{1}{r_j}\right)(\lambda, p_1, \dots, p_s) = \frac{1}{\lambda}(p_j - p_1)\cdots\widehat{(p_j - p_j)}\cdots(p_j - p_s)$$

are polynomial functions and we are considering $\lambda$ as a parameter, see (8). Moreover in (19), $S^1 = \mathbb{R}/2\pi\mathbb{Z}$, the down arrows are the inclusions and arg is the argument function on each coordinate. The maps R, $(1/r)\circ$R and arg depend on $V_\alpha$, by abuse of notation we omit this. □

Lemma 6.2. *For $s \leq 7$, the map $(1/r)\circ$R is a submersion.*

*Proof*. In order to introduce a manifold structure in $\mathcal{I}(s)$, only the computation up to $1/r_{s-1}$ is required, since the last residue can be expressed as $(1/r_s) = (1/(-r_1 - \cdots - r_{s-1}))$.

The differential of the map is

$$D\left(\left(\frac{1}{r}\right)\circ \mathrm{R}\right) = \left(\frac{\partial(1/r_j)}{\partial p_\iota}\right), \quad j \in \{1, \dots, s-1\}, \ \ \iota \in \{1, \dots, s\}.$$

Fixed $k \in \{1, \dots, s\}$, when we remove the row of partial derivatives $\partial/\partial p_k$, the determinant of the respective square matrix is

$$\det\left(\frac{\partial(1/r_j)}{\partial p_\iota}\right)_{\iota \neq k} = \frac{(s-1)!}{\lambda^{s-1}} \prod_{\alpha<\beta} (p_\alpha - p_\beta)^2, \quad \alpha, \beta \neq k. \tag{20}$$

For $s \leq 7$, we verify (20) using *Mathematica*. When $s \geq 4$, a theoretical proof of (20) seems to be very complex.

Furthermore, $\{(p_\alpha - p_\beta)^2 = 0\}$ are equations for $\Delta$, as sets. The determinant (20) does not vanish at the domain of R. Thus, $(1/r)\circ$R is a submersion. □

*Remark* 2. For $s \geq 8$, when we try to verify (20), the complexity of the calculations overcomes the memory of a common computer. In order to use a cluster, probably non-linear programmation will be required.

Conjecture 1. *Equation* (20) *is true for all $s \geq 8$.*

Since, equation (18) defines a submanifold of $(S^1)^{s-1}$, its inverse image under $(1/r)\circ\mathrm{R}$ is also a submanifold. Thus for $s \leq 7$, $C(s)$ is a real analytic submanifold of dimension $s+2$, having local coordinates

$$\{(p_1, \ldots, p_s) \in V_\alpha | \arg(r_1) = \cdots = \arg(r_{s-1})\}.$$

For each configuration $[p_1, \ldots, p_s] \in C(s)$, there exists a punctured real line $\mathcal{L} = \{\rho e^{i\theta_0} | \rho \in \mathbb{R}^*\} \subset \mathbb{C}^*$ parametrizing the isochronous vector fields with this configuration of zeroes. We have $\theta_0 + \arg(r_\iota) = \pm\pi/2$ and $\theta_0$ depends in a real analytic way of the configuration $[p_1, \ldots, p_s] \in C(s) \subset (\mathbb{C}^s_{\text{roots}} - \Delta)/\mathcal{S}(s)$.

*Summing up, there exists a $C^\omega$ real line bundle structure*

$$\pi : \mathcal{I}(s) \to C(s), \quad \pi^{-1}([p_1, \ldots, p_s]) = \{(\rho e^{i\theta_0}, [p_1, \ldots, p_s])\}.$$

Note that, the total space of the bundle $\mathcal{I}(s)$ does not contain the zero section, i.e. $\rho \neq 0$. This real line bundle parametrizes all the isochronous vector fields having s zeroes. The real dimension of $\mathcal{I}(s)$ is $s+3$. The proof of assertion 6.1.1 is done.

*Proof of assertion* 6.1.2. Dimension and stratification of $\mathcal{I}(s)/\mathrm{Aut}(\mathbb{C})$, $s \leq 7$. We compute the explicit quotients for $s \leq 3$.

PROPOSITION 6.3.

$$\frac{\mathcal{I}(1)}{\mathrm{Aut}(\mathbb{C})} = i\mathbb{R}^*, \quad \frac{\mathcal{I}(2)}{\mathrm{Aut}(\mathbb{C})} = i\mathbb{R}^+, \quad \frac{\mathcal{I}(3)}{\mathrm{Aut}(\mathbb{C})} = \frac{(i\mathbb{R}^+)^2}{\mathbb{Z}_2} \cup \frac{(i\mathbb{R}^-)^2}{\mathbb{Z}_2},$$

*in the last $\mathbb{Z}_2$-action is by the reflection $(r_1, r_2) \sim (r_2, r_1)$.*

*Proof.* The case $s=1$ follows from Example 5.2. For $s=2$, a representant in each equivalence class written as 1-form is

$$\omega = \left(\frac{r}{z+1} + \frac{-r}{z-1}\right) \mathrm{d}z, \quad r \in i\mathbb{R}^+.$$

Note that the vector fields, written as 1-forms,

$$\omega_1 = \left(\frac{r}{z+1} + \frac{-r}{z-1}\right) \mathrm{d}z, \quad \omega_2 = \left(\frac{-r}{z+1} + \frac{r}{z-1}\right) \mathrm{d}z$$

are in the same equivalence class, using the map $T(z) = -z$.

For $s=3$, in order to get explicit expressions, we use Lemma 7.1 and the residues as parameters. Since $r_1 + r_2 + r_3 = 0$, we can assume that $r_1$ and $r_2$ have the same sign in $i\mathbb{R}^*$. Whence a representant in each equivalence class is

$$\omega = \left(\frac{r_1}{z+1} + \frac{-r_1 - r_2}{z - ((r_2 - r_1)/(r_1 + r_2))} + \frac{r_2}{z-1}\right) \mathrm{d}z, \quad (r_1, r_2) \in (i\mathbb{R}^\pm)^2.$$

This quotient space has boundaries, they originate from classes $\{(r, r)\}$, which are the fixed points of the $\mathbb{Z}_2$-action induced by $(r_1, r_2) \mapsto (r_2, r_1)$. □

Since Aut($\mathbb{C}$) is connected, from the above quotients we get that the number of connected components of $\mathcal{I}(s)$ is 2, 1, 2, for $s = 1, 2, 3$, respectively, as (4) in 7.1.3 asserts.

We return to the study for $4 \leq s \leq 7$. The group Aut($\mathbb{C}$) is non-compact. In order to apply the well-known results of Lie group actions, e.g. [10], Chapter 2, we need to deal with a proper action having compact isotropy groups. Both properties are true.

LEMMA 6.4. *The push-forward action*

$$\mathscr{A} : \mathrm{Aut}(\mathbb{C}) \times \mathcal{I}(s) \longrightarrow \mathcal{I}(s), \quad (az+b), (\lambda, [p_1, \ldots, p_s]) \mapsto (\lambda a^{s-1}, [ap_1 + b, \ldots, ap_s + b])$$

*is proper.*

*Proof.* The assertion is for all $s$. $\mathscr{A}$ is induced on $\mathcal{I}(s)$ by the action in Proposition 2.3. We verify that $\mathscr{A}$ is proper on $\mathcal{I}(s)$ and proper at each $X_0 \in \mathcal{I}(s)$. Both conditions will simplify the application of the theory in [10].

The first property uses the definition in [10], p. 53. We must show that the map $\mathrm{Aut}(\mathbb{C}) \times \mathcal{I}(s) \longrightarrow \mathcal{I}(s) \times \mathcal{I}(s)$, given as $(T, X) \mapsto (T_*X, X)$, is proper. Note that $T : \mathbb{C} \longrightarrow \mathbb{C}$ is always a proper map. Whence $\mathscr{A}(T, X) = T_*X$ changes the position of the zeroes by $T$ and the coefficient $\lambda$ by $a^{s-1}$. The desired property follows.

Furthermore, the action $\mathscr{A}$ is proper at each $X_0 \in \mathcal{I}(s)$, see [10], pp. 98–99 for the definition. This is, for every pair of sequences $\{X_j\}$ of $\mathcal{I}(s)$ and $\{T_j\}$ of Aut($\mathbb{C}$); if $\lim_{j\to\infty} X_j = X_0$ and $\lim_{j\to\infty}(T_j)_*X_j = X_0$, then there is a subsequence $\{j(k)\}$ with convergent $\{T_{j(k)}\}$ in Aut($\mathbb{C}$), as $k \longrightarrow \infty$. We leave the details to the reader. □

Since $\mathcal{I}(s)$ is a $C^\omega$ manifold (for $s \leq 7$), by [10], Lemma 1.11.3 the quotient space $\mathcal{I}(s)/\mathrm{Aut}(\mathbb{C})$ is Hausdorff.

$\mathscr{A}$ is not a free action on $\mathcal{I}(s)$. The *isotropy group of* $X \in \mathcal{I}(s)$ is the subgroup $\{T \in \mathrm{Aut}(\mathbb{C}) | T_*X = X\}$. This statement is equivalent to the fact that $T$ sends the weighted graph $\Lambda(X)$ on itself as automorphism. As usual, the isotropy group of $X$ is called trivial if it is the identity.

Two examples. For $X \in \mathcal{I}(1)$ the isotropy group is $\mathbb{C}^*$, and for $X \in \mathcal{I}(2)$ the isotropy group is trivial, see the explicit descriptions in Proposition 6.3.

COROLLARY 6.5. *Assume* $s \geq 3$.

1. *The non-trivial isotropy groups of isochronous vector fields are cyclic subgroups of rotations* $\mathbb{Z}_p$, $2 \leq p \leq s-1$.
2. *Every* $\mathcal{I}(s)$ *admits vector fields with non-trivial isotropy group.*
3. *If a connected component* $\mathcal{C} \subset \mathcal{I}(s)$ *has associated an plane* s*-tree* $\Lambda$ *with a group of automorphisms* (*as abstract graph*) *trivial or different from* $\mathbb{Z}_p$, $p \geq 2$, *then each* $X \in \mathcal{C}$ *has trivial isotropy in* Aut($\mathbb{C}$).

*Proof.* To show (1), recall that the finite groups of Aut($\hat{\mathbb{C}}$) leaving invariant the point $\infty \in \hat{\mathbb{C}}$ are finite rotation groups $\mathbb{Z}_p$. Assertion (2) follows from the explicit construction of suitable isochronous vector fields, see Lemma 8.2 for the case $p = s-1$.

For the third assertion, since every continuous deformation of $X_t : [0,1] \longrightarrow \mathcal{C}$ determines the same phase portrait, the associated plane $s$-tree $\Lambda$ of $\mathcal{C}$ is well defined, diagrammatically

$$\mathcal{I}(s) \to \frac{\mathcal{I}(s)}{\mathrm{Homeo}\,(\mathbb{C})^{+} \times \mathbb{Z}_2} \quad \mathcal{C} \mapsto \Lambda.$$

By contradiction, if $X \in \mathcal{C}$ has a non-trivial isotropy group $\mathbb{Z}_p \subset \mathrm{Aut}(\mathbb{C})$, then $\mathbb{Z}_p$ induces a non-trivial automorphism of $\Lambda$ as abstract graph. □

*Example* 6.6. Consider the connected components $\mathcal{C}_\alpha$ of $\mathcal{I}(7)$ having associated 7-trees labelled with 1, 2, 3, 5, 6, 7, 9, 12 and 13 in Figure 2. Every isochronous $X \in \mathcal{C}_\alpha$ has trivial isotropy group.

Now, we introduce the set of associated *generic* isochronous vector fields $X \in \mathcal{G}(s) \subset \mathcal{C} \subset \mathcal{I}(s)$, i.e. the isochronous $X$ such that their weighted trees $\Lambda(X)$ have residues $[r_1, \ldots, r_s]$, with $r_\iota \neq r_j$, for each $\iota, j$, $\iota \neq j$. Here the plane $s$-tree $\Lambda$ is fixed.

For $s \leq 7$, $\mathcal{G}(s)$ determines an open and dense subset (a submanifold) of $\mathcal{I}(s)$ always with trivial isotropy groups in $\mathrm{Aut}(\mathbb{C})$.

The quotients $\mathcal{G}(s)/\mathrm{Aut}(\mathbb{C})$ or $\mathcal{C}/\mathrm{Aut}(\mathbb{C})$ (when $\mathcal{C}$ is a full connected component with trivial isotropy, see 6.5.3) are real $C^\omega$ manifolds. This follows by a classical result of Lie group actions on manifolds, see [10], p. 53, Theorem 1.11.4. The dimensions of these quotients are computed as

$$\dim[\mathcal{G}(s)] - \dim[\mathrm{Aut}(\mathbb{C})] = (s+3) - 4 = s - 1. \tag{21}$$

Moreover $\pi_1 : \mathcal{G}(s) \to \mathcal{G}(s)/\mathrm{Aut}(\mathbb{C})$ is a $C^\omega$ principal fibre bundle.

Assume $3 \leq s \leq 7$, in general a connected component $\mathcal{C} \subset \mathcal{I}(s)$ can behave in one of the following ways:

(i) all the vector fields in $\mathcal{C}$ have trivial isotropy as in 6.5.3,
(ii) there are vector fields with non-trivial isotropy, hence $\mathcal{C}$ splits in a finite number of subsets, depending in the isotropy groups that appear.

Moreover, the case (ii) means that $\mathcal{C}$ and hence $\mathcal{C}/\mathrm{Aut}(\mathbb{C})$ admit stratifications by orbit types. It follows from the fact that the action $\mathscr{A}$ is proper but non-free at several vector fields, see [10], Section 2.7 for a clear exposition of the theory.

The computation of $\dim[\mathcal{C}/\mathrm{Aut}(\mathbb{C})] = s - 1$ (by definition, the maximum of the dimensions of the strata in $\mathcal{C}$) is as follows.

In case (i), the computation follows directly as is (21).

In case (ii), the submanifold $\mathcal{G}(s) \subset \mathcal{C}$ is the subset of maximal dimension, $s+3$ in the stratification of $\mathcal{C}$. We regard that each submanifold $\mathcal{M}_p$ of $\mathcal{C} \subset \mathcal{I}(s)$, given by the stratification with non-trivial isotropy $\mathbb{Z}_p$, has dimension less that $s+3$. Hence the hypothesis of non-trivial isotropy imposes additional equalities for the residues $\{[r_1, \ldots, r_s]\}$ in (18). Now we apply the idea in (21) to each $M_p$, obtaining

$$\dim[M_p] - \dim\left[\frac{\mathrm{Aut}(\mathbb{C})}{\mathbb{Z}_p}\right] < s - 1.$$

The proof of assertion 6.1.2 is done.

*Proof of assertion* 6.1.3. We want to compute the number of connected components of $\mathcal{I}(s)$, for $1 \leq s \leq 7$. Our varieties $\mathcal{I}(s)$ are real affine (using the language of algebraic geometry), and their connectedness properties in general give origin to hard questions. However using that the ordered residue map R is a submersion, the connectedness problem becomes simple. The strategy is as follows.

Step 1. Given $X \in \mathcal{I}(s)$, there is a 'canonical' isochronous vector field, say $X^+_{\text{can}}$, such that $X$ and $X^+_{\text{can}}$ are in the same connected component of $\mathcal{I}(s)$.

Step 2. Compute how is the intersection of the Aut($\mathbb{C}$)-orbits of two canonical vector fields $X^+_{\text{can}}$ (this depends on the parity of $s$ and the plane $s$-tree $\Lambda$).

Step 3. Compute by inspection the number of connected components.

*Step* 1. Let $X$ be an isochronous vector field and its weighted $s$-tree $\Lambda(X) = \{(p_1, r_1), \ldots, (p_s, r_s), \Lambda_{\iota j}\}$. The collection of semi-residues $\{S_{\iota j}\} \in \mathbb{R}^+$ of $X$ is well determined (see Corollary 3.6). We consider the continuous paths

$$S_{\iota j,t} : [0,1] \longrightarrow \mathbb{R}^+, \quad t \longmapsto S_{\iota j,t} \doteq \frac{S_{\iota j}}{(1-t) + tS_{\iota j}},$$

note that $S_{\iota j,1} = 1$. There is a continuous path $t \mapsto \Lambda_t$ of weighted trees; starting at the weighted tree $\Lambda(X)$ and defined by the following two conditions.

(i) The weighted tree is

$$\Lambda_t = \{(v_{1,t}, r_{1,t}), \ldots, (v_{s,t}, r_{s,t}), \Lambda_{\iota j}\}$$

having residues

$$r_{\iota,t} = \frac{\pm 1}{2\pi i} \sum_j S_{\iota j,t} \in \frac{\mathbb{Z}}{2\pi i} \quad p_{j,t} \text{ are adjacent with } p_{\iota,t} \text{ in } \Lambda_t, \tag{22}$$

here we are using condition (iii) in Corollary 3.6, the choice of the sign $\pm$ is such that $r_\iota$ and $r_{\iota,t}$ are in the same component of $i\mathbb{R}^*$.

(ii) $\Lambda(X)$ and $\Lambda_t$ are the same plane $s$-trees in $\mathbb{C}$, up to $h \in \text{Homeo}(\mathbb{C})^+$.

By Corollary 3.6, for each $t$, there exists an isochronous $X_t$ that realizes the weighted tree $\Lambda_t$.

We must check that the realization of $X_t$ from $\Lambda_t$ is continuous in $\mathcal{I}(s)$.

The continuous path of unordered residues

$$t \longmapsto [r_{1,t}, \ldots, r_{s,t}] \in (\mathbb{C}^*)^s, \quad t \in [0,1]$$

is in the image $\text{R}(\mathcal{I}(s))$ of a suitable ordered residue map (see diagram (19)). For fixed $\text{t} \in [0,1]$, there exists a local continuous lift

$$\begin{array}{ccc} & (\lambda, p_{1,\tau}, \ldots, p_{s,\tau}) \in (\mathbb{C}^* \times V_\alpha) & \subset \mathbb{C}^* \times \frac{\mathbb{C}^s_{\text{roots}} - \Delta}{\mathcal{S}(s)} \\ \nearrow & \downarrow \text{R} & \\ \tau \quad \longrightarrow & (r_{1,\tau}, \ldots, r_{s,\tau}) \in (\mathbb{C}^*)^s & \end{array}$$

for $\tau \in (\text{t} - \varepsilon, \text{t} + \varepsilon)$, $\varepsilon > 0$, such that $\text{R}(\lambda, p_{1,\tau}, \ldots, p_{s,\tau}) = (r_{1,\tau}, \ldots r_{s,\tau})$. Here we are using suitable $\{V_\alpha\}$ and the fact that R is a submersion, in a similar way with Lemma 6.2. Hence it determines a local fibration over an open neighbourhood of each $(r_{1,\text{t}}, \ldots r_{s,\text{t}})$. The local lift is not unique, it can be changed by a continuous $T_t \in \text{Aut}(\mathbb{C})$. Using a finite cover of $[0, 1]$ and the above local lifts, we get the desired continuous path

$$t \mapsto X_t \in \mathcal{I}(s), \quad X_0 = X \quad \text{and} \quad X_1 \doteq X^+_{\text{can}}.$$

We define, the *canonical isochronous vector field* $X^+_{\text{can}} \in \mathcal{I}(s)$ *associated with* $X$ is an isochronous vector field determining the same plane $s$-tree that $X$, semi-residues 1 and residues of the same sign (in $i\mathbb{R}^*$) that $X$.

Note that, for all $T \in \text{Aut}(\mathbb{C})$, the vector field $T_* X^+_{\text{can}}$ is also canonical and associated under a continuous path with $X$. Hence the canonical vector field from $X$ is not unique. $X^+_{\text{can}}$ determines a *canonical* $\text{Aut}(\mathbb{C})$-orbit in $\mathcal{I}(s)$.

*Step* 2. We want to know how is the intersection of canonical $\text{Aut}(\mathbb{C})$-orbits in $\mathcal{I}(s)$. Let $X^+_{\text{can}}$ be a canonical isochronous vector field, we define

$$X^-_{\text{can}} \doteq -X^+_{\text{can}}.$$

Both have semi-residues 1, but their respective residues are of opposite sign in $i\mathbb{R}^*$. The key point is to characterize whether

$$X^+_{\text{can}} \quad \text{and} \quad X^-_{\text{can}}$$

are holomorphically equivalent isochronous vector fields. The characterization is as follows.

Lemma 6.7.

1. *If $s$ is odd or $X^+_{\text{can}}$ satisfies that the numbers of their residues of opposite sign in $i\mathbb{R}^*$ are different between them, then $X^+_{\text{can}}$ and $X^-_{\text{can}}$ determine different* $\text{Aut}(\mathbb{C})$*-orbits and different connected components of $\mathcal{I}(s)$.*
2. *If $s$ is even, then the following assertions are equivalent.*
   (i) *$X^+_{\text{can}}$ and $X^-_{\text{can}}$ determine the same* $\text{Aut}(\mathbb{C})$*-orbit (they are holomorphically equivalent).*
   (ii) *$X^+_{\text{can}}$ and $X^-_{\text{can}}$ determine the same connected component of $\mathcal{I}(s)$.*

Let us start with three illustrative objects.

*Example* 6.8. The action of $T(z) = -z$. Let $X = (\lambda, [p_1, \ldots, p_s])$ be a polynomial vector field written as in (7) a priori non-isochronous. Assume in addition that the collection of zeroes is invariant under $T \in \text{Aut}(\mathbb{C})$. Then $T_* X = (\lambda(-1)^{s-1}, [p_1, \ldots, p_s])$, see Lemma 6.4. Our assertions are the following.

$s$ is odd if and only if $T_* X = X$, i.e. $T$ is in the isotropy of $X$.
$s$ is even if and only if $T_* X = -X$, i.e. $X$ and $-X$ are in the same $\text{Aut}(\mathbb{C})$-orbit.

*Example* 6.9. *Case as in* 6.7.2 (*i*) with the same orbit. Assume even $s = 2\kappa$ and a canonical vector field having expression

$$X^+_{\text{can}} = i(z - x_1)\cdots(z - x_\kappa)(z + x_1)\cdots(z + x_\kappa)\frac{\partial}{\partial z} \in \mathcal{I}(s),$$

where $0 < x_1 < x_2 < \cdots < x_\kappa \in \mathbb{R}^+$. In Proposition 8.1 we will show the existence of a suitable collection of zeroes such that the semi-residues are 1 and their associated tree is a path graph. Then $X^+_{\text{can}}$ and $X^-_{\text{can}}$ are in the same Aut($\mathbb{C}$)-orbit, since $T(z) = -z$ makes true the equation $T_* X^+_{\text{can}} = X^-_{\text{can}}$.

Furthermore, $T$ induced the obvious permutation between the weighted vertices from these canonical vector fields, which are

$$\left\{\left(-x_\kappa, \frac{1}{2\pi i}\right), \left(-x_{\kappa-1}, \frac{-2}{2\pi i}\right), \ldots, \left(x_{\kappa-1}, \frac{2}{2\pi i}\right), \left(x_\kappa, \frac{-1}{2\pi i}\right)\right\} \quad \text{and}$$

$$\left\{\left(-x_\kappa, \frac{-1}{2\pi i}\right), \left(-x_{\kappa-1}, \frac{2}{2\pi i}\right), \ldots, \left(x_{\kappa-1}, \frac{-2}{2\pi i}\right), \left(x_\kappa, \frac{1}{2\pi i}\right)\right\}.$$

*Example* 6.10 *Case as in* 6.7.1. Assume odd $s = 2\kappa + 1$, the vector fields

$$X^+_{\text{can}} = i(z - x_1) \cdots (z - x_\kappa) z (z + x_1) \cdots (z + x_\kappa) \frac{\partial}{\partial z}, \quad X^-_{\text{can}}$$

are in different Aut($\mathbb{C}$)-orbits. In fact the collections of weighted vertices from these canonical vector fields are

$$\left\{\left(-x_\kappa, \frac{1}{2\pi i}\right), \left(-x_{\kappa-1}, \frac{-2}{2\pi i}\right), \ldots, \left(x_{\kappa-1}, \frac{-2}{2\pi i}\right), \left(x_\kappa, \frac{1}{2\pi i}\right)\right\} \quad \text{and}$$

$$\left\{\left(-x_\kappa, \frac{-1}{2\pi i}\right), \left(-x_{\kappa-1}, \frac{2}{2\pi i}\right), \ldots, \left(x_{\kappa-1}, \frac{2}{2\pi i}\right), \left(x_\kappa, \frac{-1}{2\pi i}\right)\right\},$$

a permutation induced by $T \in$ Aut($\mathbb{C}$) sending one collection in another is impossible.

*Example* 6.11 *Case as in* 6.7.2, *with different orbits*. Assume even $s$ and the canonical vector field having expression

$$X^+_{\text{can}} = i\rho z (z^{s-1} - 1) \frac{\partial}{\partial z} \in \mathcal{I}(s),$$

for suitable $\rho \in \mathbb{R}^+$. In Proposition 8.2 we will show their existence with the associated tree which is a star shape graph (with a vertex of degree $s - 1$ at $z = 0$) and having semi-residues 1. Thus $X^+_{\text{can}}$ and $X^-_{\text{can}}$ determine two different Aut($\mathbb{C}$)-orbits. In fact the collections of weighted vertices from these canonical vector fields are

$$\left\{\left(0, \frac{s-1}{2\pi i}\right), \left(1, \frac{-1}{2\pi i}\right), \ldots, \left(z_{s-1}, \frac{-1}{2\pi i}\right)\right\} \quad \text{and}$$

$$\left\{\left(0, \frac{-(s-1)}{2\pi i}\right), \left(1, \frac{1}{2\pi i}\right), \ldots, \left(z_{s-1}, \frac{1}{2\pi i}\right)\right\},$$

a permutation induced by $T \in$ Aut($\mathbb{C}$) sending one collection in another does not exist (here $1, \ldots, z_{s-1}$ are the $(s-1)$-th roots of the unit).

Now we return to Lemma 6.7.

*Proof*. For assertion 1, in order to show that $X^+_{\text{can}}$, $X^-_{\text{can}}$ are in different connected components, we proceed by contradiction. Assume there exists a continuous path $X_t : [0,1] \to \mathcal{I}(s)$ such that $X_0 = X^+_{\text{can}}$ and $X_1 = X^-_{\text{can}}$. We get a continuous family of homeomorphisms $h_t \in \text{Homeo}(\mathbb{C})^+$ such that $h_t$ makes $X_0 = X^+_{\text{can}}$ and $X_t$ topologically equivalent isochronous vector fields. $h_t$ is a topological isotopy.

$h_1$ induces a permutation $\sigma$ from the weighted vertices $\{(p_1, r_1), \ldots, (p_s, r_s)\}$ in $\Lambda(X^+_{\text{can}})$ of one canonical vector field to the other. The permutation must send the positive residues $r_\iota \in i\mathbb{R}^+$ to negative $r_j \in i\mathbb{R}^-$ and, *vice versa*.

However by hypothesis, the number of positive residues $r_\iota \in i\mathbb{R}^+$ is different from the number of negative residues. A permutation with this property does not exist. We have arrived to a contradiction. Assertion 1 is done.

For assertion 2, the Aut($\mathbb{C}$)-orbits are also connected, this and Step 1 shown (i) $\Rightarrow$ (ii).

For (i) $\Leftarrow$ (ii), by hypothesis there exists a continuous path $X_t : [0,1] \to \mathcal{I}(s)$ such that $X_0 = X^+_{\text{can}}$ and $X_1 = X^-_{\text{can}}$. Starting with $\{X_t\}$ we want to construct a suitable $T \in \text{Aut}(\mathbb{C})$, using the ideas in [18].

There exists a continuous family of homeomorphisms $h_t \in \text{Homeo}(\mathbb{C})^+$ such that $h_t$ makes $X_0 = X^+_{\text{can}}$ and $X_t$ topologically equivalent. $h_t$ is a topological isotopy.

Thus $h_1$ is a topological equivalence between $X^+_{\text{can}}$ and $X^-_{\text{can}}$. Using that the periods and the plane $s$-tree of both vector fields are the same, we can recognize that $h_1$ can be realized by an orientation preserving isometry

$$T : (\mathbb{C} - \{[p_1, \ldots, p_s]\}, g_0) \to (\mathbb{C} - \{[p_1, \ldots, p_s]\}, g_1),$$

here $g_0, g_1$ are the respective flat metrics. Moreover this isometry is a biholomorphic map between punctured Riemann surfaces. By the Riemann extension theorem, the biholomorphic map is well defined across the pole and zeroes of $X^+_{\text{can}}$. Furthermore by the Uniformization theorem, $T \in \text{Aut}(\mathbb{C})$ is the desired map such that $T_* X^+_{\text{can}} = X^-_{\text{can}}$. The respective Aut($\mathbb{C}$)-orbits coincide as (i) asserts. □

*Step 3*. For the third assertion in 7.1, given $s \leq 7$, we verify by inspection, which plane $s$-trees have the property that $X^+_{\text{can}}$ and $X^-_{\text{can}}$ belong to the same Aut($\mathbb{C}$)-orbit.

For $s = 1, 3, 5, 7$, using Lemma 7.6.1, the number of connected components of $\mathcal{I}(s)$ is $2N(s)$, where $N(s)$ is the number of topologically inequivalent isochronous foliations.

For $s = 2$, using Lemma 7.6.2 (ii), the number of connected components of $\mathcal{I}(2)$ is 1.

For $s = 4$, there are two 4-trees, a path graph and a star shape graph (i.e. having a vertex of degree 3). Using Lemma 6.7.2 (ii), the number of connected components of having the topology of the path graph is 1. Applying example 6.11, the number of components having the topology of the star shape tree is 2. The number of components of $\mathcal{I}(4)$ is 3.

For $s = 6$, $\mathcal{I}(6)$ has 10 connected components (we leave the details to the reader).

This end the proof of assertion 6.1.3.

Theorem 6.1 is done.

Conjecture 2. *For all* s $> 7$, *the total number of connected components* $\mathcal{I}(s)$ *is as follows.*

*If* s *is odd, then is* $2N(s)$.

*If* s *is even, then is bounded by* $2N(s) - 1$.

The odd case will follow by Conjecture 1 and Lemma 7.6.1. In the even case, we recall that the isochronous vector fields in Example 6.9 determine only a connected component. But, when $s$ increases there are more plane $s$-trees determining only one connected component. For example the 8-tree labelled 10 in Figure 3 determines only a connected component in $\mathcal{I}(8)$.

## 7. Combinatory of residues

Consider diagrams (14) and (17). An unordered collection $[r_1, \ldots, r_s]$ having $r_\iota \in i\mathbb{R}^*$ and $\Sigma r_\iota = 0$ is called *isochronous* if there exists an isochronous vector field $X$ realizing these periods.

Question 1: Given $[r_1, \ldots, r_s]$ as above, when is isochronous?

For rational vector fields, the similar problem has always positive answer. The 1-forms provide a more suitable language.

LEMMA 7.1. *Let $\{\eta\}$ be the family of rational 1-forms on $\hat{\mathbb{C}}$, having s simple poles and arbitrary zeroes. Every collection of poles and residues (satisfying the residue theorem) is realized by some $\eta$.*

*Proof.* We start with $s$ pairs $\{(p_\iota, r_\iota)\} \subset \mathbb{C} \times \mathbb{C}^*$, where $\{p_\iota\}$ are different points in the plane, since here we are working up to $PSL(2, \mathbb{C})$. The desired 1-form is

$$\eta = \left(\frac{r_1}{z - p_1} + \cdots + \frac{r_s}{z - p_s}\right) \mathrm{d}z = \left(\frac{a_1 z^{s-1} + a_2 z^{s-2} + a_3 z^{s-3} + \cdots + a_s}{(z - p_1) \cdots (z - p_s)}\right) \mathrm{d}z, \qquad (23)$$

where

$$a_1 = \sum_\iota r_\iota = 0, \quad a_2 = \sum_\iota r_\iota \left(\sum_{\alpha \neq \iota} p_\alpha\right),$$

$$a_3 = \sum_\iota r_\iota \left(\sum_{\alpha, j \neq \iota} p_\alpha p_j\right), \ldots, a_s = \sum_\iota r_\iota \big(p_1 \ldots \hat{p_\iota} \ldots p_s\big).$$

□

*Example* 7.2. The collection

$$[(s-1)r, \underbrace{-r, \ldots, -r}_{(s-1)}], \quad r \in i\mathbb{R}^*$$

is isochronous only for $X \in \mathcal{I}(s)$, $s \geq 3$, having a star shape $s$-tree $\Lambda$.

Thus, $\Lambda$ has a vertex with degree $s - 1$, for examples see trees labelled 14 and 32 in Figures 2 and 3. Up to $T \in \mathrm{Aut}(\mathbb{C})$, $X$ has zeroes at the origin and the $(s - 1)$-th roots of the unity, see Lemma 8.2 for an additional description.

Negative examples for Question 1 are as follow.

LEMMA 7.3. *The collection*

$$[r, r, -r, -r], \quad r \in i\mathbb{R}^*$$

*is not isochronous for any polynomial $X \in \mathcal{I}(4)$. Moreover, it is the unique forbidden collection of residues for $\mathcal{I}(4)$.*

*Proof*. We show several arguments. There are two 4-trees: one of order 3 (star shape) and another of order 2 (a path graph).

If we use the tree having order 3, then the residue $\pm r$ assigned at the vertex of order 3 cannot satisfy condition (iii) in Corollary 3.6, we get a contradiction. Similarly, when we study the tree having order 2.

Another argument uses the metric cylinders. Observe that the gluing of four half cylinders with these perimeters $2\pi|r_j| = |T_j|$ cannot be performed satisfying the conditions in Lemma 4.1. This is evident when we compare $[r, r, -r, -r]$ with Examples 4.4 and 4.3. Both examples shown that $[r, r, -r, -r]$ is the unique forbidden collection. □

*Remark* 3.

(i) The collection $[r, -r - i\varepsilon, r + i\varepsilon, -r]$, for enough small $\varepsilon \neq 0$, is isochronous. $\Lambda(X)$ is the path 4-tree in Example 4.3. Using (23), the explicit expression is

$$\omega = \left(\frac{r}{z} + \frac{-r - i\varepsilon}{z - 1} + \frac{r + i\varepsilon}{z - ((-i\varepsilon - 2r)/i\varepsilon)} + \frac{-r}{z - ((2(i\varepsilon + r))/i\varepsilon)}\right) \mathrm{d}z.$$

(ii) Moreover, using (23), $[r, r, -r, -r]$ is the collection of residues of a one parameter family of rational 1-forms $\{\eta\}$ on $\hat{\mathbb{C}}$ having four simple poles and two simple zeroes, hence the associated vector fields $X_\eta$ are not polynomial.

(iii) In simple words Lemma 7.3 says that, a real or complex polynomial $P(x)$ of degree 4 having derivatives $[r, -r, r, -r]$ at their zeroes, r $\in \mathbb{C}^*$, does not exist.

*Example* 7.4. The collection

$$[\underbrace{r, \ldots, r}_{k}, \underbrace{-r, \ldots, -r}_{k}], \quad r \in i\mathbb{R}^+, \ k \geq 2$$

is not isochronous for any $X \in \mathcal{I}(2k)$.

For the proof, we use the geometry of vertical bands and cylinders in Lemma 4.1. The obstructions are similar as in Lemma 7.3.

*Example* 7.5. The collection

$$[\underbrace{r, \ldots, r}_{k}, -\sigma_1 r, -\sigma_2 r, \ldots, -\sigma_\kappa r], \quad r \in i\mathbb{R}^+, \ \sigma_j \in \mathbb{N}, \ \kappa \geq 2, \ k = \Sigma\alpha_j,$$

is not isochronous for any $X \in \mathcal{I}(k + \kappa)$.

The proof uses Lemma 4.1.

COROLLARY 7.6. *For each $s \geq 4$, there are collections $[r_1, \ldots, r_s]$ having $r_\iota \in i\mathbb{R}^*$ and $\Sigma r_\iota = 0$, which are not isochronous for any $X \in \mathcal{I}(s)$.*

Now we study, how many times a collection of residues can appear for different classes $[X] \in \mathcal{I}(s)/\mathrm{Aut}(\mathbb{C})$.

*Example* 7.7. The collection

$$[\underbrace{2r, \ldots, 2r}_{k}, -kr, \underbrace{-r, \ldots, -r}_{k}], \quad r \in i\mathbb{R}^+, \ k \geq 1,$$

is isochronous for only one class in $\mathcal{I}(s)/\mathrm{Aut}(\mathbb{C})$ and its respective $s$-tree ($s = 2k + 1$).

In order to fix ideas, consider $s = 7$, $k = 3$ and the tree labelled 10 in Figure 2, say $\Lambda_7$. At the extreme vertices of $\Lambda_7$ we put residues $-r, -r, -r$, at the central vertex $-3r$, and at the others vertices $2r, 2r, 2r$. Following Corollary 3.6, there exists an isochronous $X$ realizing the data in $\Lambda_7$.

By simple inspection, any other 7-tree in Figure 2 cannot support these residues satisfying Corollary 3.6. Another argument avoiding large computations is as follows. We observe that the gluing of 7 half cylinders with these perimeters can be performed satisfying the conditions in Lemma 4.1 with a unique pattern.

The general case is left to the reader.

We summarize in the following result, the residues do not determine uniquely the orbits in $\mathcal{I}(s)/\mathrm{Aut}(\mathbb{C})$.

THEOREM 7.8. *For an unordered isochronous collection of residues $[r_1, \ldots, r_s]$ the number of its possible associated weighted* s*-trees, $\#\{\Lambda[r_1, \ldots, r_s]\}$, is the topological degree of the map*

$$\mathcal{R} : \frac{\mathcal{I}(s)}{\mathrm{Aut}(\mathbb{C})} \longrightarrow \frac{(\mathbb{C}^*)^s}{\mathcal{S}(s)}$$

*in* (17) *and satisfies*

$$1 \leq \#\{\Lambda[r_1, \ldots, r_s]\} \leq (s-2)! \tag{24}$$

*Proof*. We assume that $[r_1, \ldots, r_s]$ is isochronous. Since we are working in $\mathcal{I}(s)/\mathrm{Aut}(\mathbb{C})$ without loss of generality, two residues, say $r_1$ and $r_2$, correspond to zeroes at positions $p_1 = 0$ and $p_2 = 1$ in $\mathbb{C}$. It follows that the system of algebraic equations (23),

$$\{a_2(0, 1, p_3, \ldots, p_s) = 0\} \cap \cdots \cap \{a_{s-1}(0, 1, p_3, \ldots, p_s) = 0\},$$

admits solutions $\{(0, 1, p_{30}, \ldots, p_{s0})\}$.

The degree of the algebraic equation $a_\iota = 0$ with respect to the affine variables $\{(p_3, \ldots, p_s)\}$ is $(\iota - 1)$. By Bezout's theorem in these affine variables, the number of solutions is infinite or is at most the product of the degrees, that is $(s-2)!$

Note that infinite solutions are impossible, because we are working on $\mathcal{I}(s)/\mathrm{Aut}(\mathbb{C})$ (we imposed the conditions $p_1 = 0$, $p_2 = 1$).

The bound in equation (24) is sharp. Consider $\Lambda$ the $s$-tree having a vertex, say $v_1$, of order $s-1$ (this characterize the $s$-tree as plane graph), e.g. the 7-tree labelled 14 in Figure 2 and 8-tree labelled 32 in Figure 3.

Consider the isochronous periods

$$[-(s-1)i, (1+\varepsilon_2)i, \ldots, (1+\varepsilon_s)i],$$

here $0 < |\varepsilon_\iota| < 1$ and all the residues are different between them. In order to get a weighted tree, assume the following correspondence between periods and vertices

$$\{-(s-1)i, (1+\varepsilon_2)i, \ldots, (1+\varepsilon_s)i\} \leftrightarrow \{v_1, v_2, \ldots, v_s\}.$$

The above correspondence admits $(s-1)!$ permutations, given origin to non-equivalent weighted trees in $\mathbb{C}$. There are $(s-1)!$ classes in $\mathcal{I}(s)/\mathrm{Aut}(\mathbb{C})$ as equation (24) says. □

## 8. Isochronicity and configurations of zeroes

We return to diagrams (14) and (17). Let $X$ be a complex polynomial vector field as in (5), we assume the knowledge of the position of their zeroes.

Question 2: Under what conditions on the configuration of the zeroes of $X$, $[p_1, \ldots, p_s] \in ((\mathbb{C}^s_{\text{roots}} - \Delta)/\mathcal{S}(s))$, there exists a rotation, such that $e^{i\theta_0}X$ is isochronous?

In the affirmative case $[p_1, \ldots, p_s]$ belongs to the base $C(s)$ of the line bundle structure in $\mathcal{I}(s)$ and we say that $[p_1, \ldots, p_s]$ is an *isochronous configuration*, as in Section 6.

The isochronicity conditions (18) say that $[p_1, \ldots, p_s]$ is isochronous if and only if

$$\frac{r_j}{r_1} = -\frac{(\widehat{p_1 - p_1})(p_1 - p_2)\cdots(\widehat{p_1 - p_j})\cdots(p_1 - p_s)}{(\widehat{p_j - p_1})(p_j - p_2)\cdots(\widehat{p_j - p_j})\cdots(p_j - p_s)} \in \mathbb{R}^*, \quad j \in 2, \ldots, s. \tag{25}$$

Our task is the description of (25) from three points of view: elementary geometry of isochronous configurations, inequalities between the residues, and realizable weighted trees.

The following result is related with [1] (Theorem 1.1) (see also [18], Example 8.4.1).

PROPOSITION 8.1. *Isochronous vector fields whose zeroes are in a line.*

1. *For $s \geq 3$, the following assertions are equivalent.*
   (i) *The zeroes $\{p_j\}$ of* X *are in a line* L.
   (ii) *Up to suitable $T \in \mathrm{Aut}(\mathbb{C})$ and $\rho e^{i\theta_0}$, the vector field* X *satisfies*

   $$T_*(\rho e^{i\theta_0}X) = iz(z-1)(z-x_3)\cdots(z-x_s)\frac{\partial}{\partial z},$$

   *where $1 < x_3 < \cdots < x_s$.*
   (iii) *For $\theta_0 = -\arg(r_1)$, the rotated vector field $e^{i\theta_0}X$ is isochronous and its* s-*tree is a path graph.*
2. *These families of vector fields determine full connected components in each $\mathcal{I}(s)$.*

*Proof*. The equivalence (i) ⇔ (ii) is immediate.

For (ii) $\Rightarrow$ (iii), we want to describe the phase portrait of $X$. The conjugation $z \mapsto \bar{z}$ is a homeomorphism of $\mathbb{C}$, leaving invariant each trajectory (reversing the time along each trajectory) of $\mathfrak{Re}(X)$ and the zeroes are fixed points under $\bar{z}$. Both properties imply that the tree $\Lambda(X)$ is a path graph.

The easiest proof of (ii) $\Leftarrow$ (iii) uses conformal maps. Consider $X$ as in (iii), by Theorem 3.5 there exists a real $C^r$ vector field $Y$ on $\mathbb{R}^2$ having its weighted tree isomorphic with $\Lambda(X)$.

In addition to $C^r$ techniques, we can assume that $Y$ has all its zeroes in a real line $\tilde{L} \subset \mathbb{R}^2$ and the euclidean reflection R with respect to $\tilde{L}$ satisfies $\mathrm{R}_* Y = -Y$. In particular, the trajectories of $Y$ are invariant under R (independently of the time orientation).

Without loss of generality, the biholomorphic map $\psi : (\mathbb{R}^2, J) \to \mathbb{C}$ sending $Y$ to $\mathfrak{Re}(X)$ is such that $\psi(\tilde{L}) = \mathbb{R}$, here $\psi$ is as in 3.5. The composition $\psi \circ \mathrm{R} \circ \psi^{-1} : \mathbb{C} \to \mathbb{C}$ is an anti-conformal automorphism, that is $z \mapsto \bar{z}$ up to conjugation in Aut($\mathbb{C}$). All the zeroes of $X$ are in the line $\mathbb{R}$.

To show assertion 2, let us remark that these families have as effective number parameters $\{\rho, x_3, \dots, x_s\}$ (recalling that the ordered residue map is a submersion) plus the four in Aut($\mathbb{C}$). The total number of parameters is $s + 3$, that is the dimension of $\mathcal{I}(s)$. Then the families are open subsets of $\mathcal{I}(s)$. To show that the families in (1) fill completely the respective components of $\mathcal{I}(s)$, we can use a continuous path argument as in the proof of Theorem 6.1.3. We leave the details to the reader. □

The situation is more complicated for $s \geq 4$ as we describe below. We present an instructive second family, it is related with [1] (Theorem 1.1) (see also [18]), Example 8.4.2).

LEMMA 8.2. *Isochronous vector fields whose zeroes are at a regular polygon.*

1. *For $s \geq 4$, the following assertions are equivalent.*
   (i) *The zeroes of* X *are at the vertices $p_2, \dots, p_s$ and centre $p_1$ of a regular polygon.*
   (ii) *Up to suitable $T \in$* Aut($\mathbb{C}$) *and $\rho e^{i\theta_0}$, the vector field satisfies*
   $$T_*(\rho e^{i\theta_0} X) = iz(z^{s-1} - 1)\frac{\partial}{\partial z}.$$
   (iii) *The residues $\{r_j\}$ of* X *satisfy*
   $$r_1 = -(s-1)r_2, \quad r_2 = \cdots = r_s.$$
2. *These subfamilies $\{\rho e^{i\theta_0} X\}$ have* s*-trees with a vertex of degree $s - 1$ and dimension* 5 (*codimension $s - 2$*) *in the respective connected components of $\mathcal{I}(s)$.*

*Proof*. Equivalence (i)–(ii) is immediate. Assume (iii), by Corollary 3.6, there exists $X_1$ having residues $r_1, r_2 = \cdots = r_s$. It is easy to see that a vector field $X$ in (ii) provides this collection of residues. Hence, $X_1$ is $X$ up to a change of coordinates, (ii) is done. We leave assertion 3 to the reader. □

The last assertion says that the codimension of this subfamily is higher. An interesting problem is to find explicit polynomials for the above vector fields describing these full connected components in $\mathcal{I}(s)$.

*Case $s = 4$*. The next part was in [1,5] (Theorem 1.2).

LEMMA 8.3. *Isochronous vector fields whose zeroes are at a triangle.*

1. *For* s = 4, *the following assertions are equivalent.*
   (i) *The zeroes of* X *are at the vertices* $p_2, p_3, p_4$ *of a non-degenerate triangle and its orthocentre* $p_1$.
   (ii) *For* $\theta_0 = -\arg(r_1)$, *the residues of* $\mathrm{e}^{i\theta_0}X$ *satisfy*

$$[r_1, r_2, r_3, r_4] = \left[\frac{T_1}{2\pi i}, \frac{T_2}{2\pi i}, \frac{T_3}{2\pi i}, \frac{T_4}{2\pi i}\right], \quad \pm T_1 < 0 < \pm T_2, \quad \pm T_3, \pm T_4.$$

   (iii) *For some* $\theta_0$, *the rotated vector field* $\mathrm{e}^{i\theta_0}X$ *is isochronous and its tree has a vertex with degree* 3.
2. *These families* $\{\rho \mathrm{e}^{i\theta_0}X\}$ *determine two full connected components of* $\mathcal{I}(4)$.

*Proof.* Let $p_1, \ldots, p_4$ be as above, we have

$$\frac{r_2}{r_1} = -\frac{(p_1 - p_3)(p_1 - p_4)}{(p_2 - p_3)(p_2 - p_4)}.$$

Thus $(p_1 - p_3)$ and $(p_2 - p_4)$ are an altitude and a side of the triangle, they are orthonormal. Also $(p_1 - p_4)$ and $(p_2 - p_3)$ are orthonormal. Equation (25) holds, $[p_1, \ldots, p_4]$ is isochronous.

The collection of periods comes from Example 4.4. The associated tree is as in (iii). The assertion (2) follows from Example 7.9. □

COROLLARY 8.4.

1. *For* s = 4, *the families of isochronous vector fields having their zeroes in a line or in the vertices and the orthocentre of a non-degenerate triangle determine the three full connected components of* $\mathcal{I}(4)$.
2. *The unique forbidden collection of residues for* $\mathcal{I}(4)$ *is* $[r, r, -r, -r]$, $r \in i\mathbb{R}^*$.

*Proof.* Part (2) was done in Examples 4.4 and 4.3 and Lemma 7.3. □

*Case* $s = 5$. The isochronous configurations for $s \geq 5$ are very difficult to describe. There are three 5-trees. In the case of $\Lambda$, a path graph is completely described by (4). We present solutions for the other two classes of 5-trees. In order to find explicit solutions of (25), we assume some symmetry in the configuration of five zeroes.

Our symmetry hypothesis is that three zeroes $p_3, p_4, p_5$ are in the perpendicular bisector line of the segment defined by $p_1, p_2$. That is, up to suitable $T \in \mathrm{Aut}(\mathbb{C})$ and $\rho e^{i\theta_0}$, we deal with the vector fields

$$T_*(\rho \mathrm{e}^{i\theta_0}X) = i\left(z + \frac{1}{2}\right)\left(z - \frac{1}{2}\right)(z - iy_3)(z - iy_4)(z - iy_5)\frac{\partial}{\partial z},$$

where $y_3 < y_4 < y_5$.

Lemma 8.5. $[-(1/2),(1/2),iy_3,iy_4,iy_5]$ *is an isochronous configuration if and only if*

$$y_3+y_4+y_5=4y_3y_4y_5. \tag{26}$$

*Proof.* It is an explicit computation in order to verify (25). □

*Example* 8.6. Isochronous vector fields whose zeroes are at a rhombus.

1. For $s=5$, the following assertions are equivalent.
   (i) The zeroes of $X$ are at the vertices of a rhombus $p_2,\ldots,p_5$ and its centre $p_1$.
   (ii) The residues $[r_1,\ldots,r_5]$ of $X$ satisfy

$$r_1=-2(r_2+r_3),\quad r_2=r_4,\ \ r_3=r_5.$$

2. These isochronous subfamilies $\{\rho e^{i\theta_0}X\}$ have 5-trees with a vertex of degree 4, and dimension 6 (codimension 2) in the respective connected components of $\mathcal{I}(5)$.

In order to show it for the rhombus, we note that (26) reduces to $y_3+y_5=0$. Another proof uses a similar computation to 8.3.

The configurations in (26) determine the appearance of two classes of 5-trees. The first is when $p_4$ is in the convex hull of the other zeroes. This case contains as subfamilies the rhombus and the regular square in Lemma 8.2.

Lemma 8.7. *Isochronous vector fields with* 5 *zeroes symmetric respect to a line I.*

1. *For* $s=5$, *the following assertions are equivalent.*
   (i) *The zeroes of* X *satisfy that* $p_3,p_4,p_5$ *are in the bisector line of the segment by* $p_1$, $p_2$ *and moreover* $p_4$ *is in the convex hull of* $p_1,p_2,p_3,p_5$.
   (ii) *The residues* $[r_1,\ldots,r_5]$ *of* X *satisfy*

$$r_4=r_1+r_2+r_3+r_5 \quad\text{and}\quad |r_1|=|r_2|.$$

2. *These isochronous subfamilies* $\{\rho e^{i\theta_0}X\}$ *have* 5*-trees with a vertex of degree* 4*and dimension* 7 (*codimension* 1) *in the respective connected components of* $\mathcal{I}(5)$.

*Proof.* For (i) ⇔ (ii), use the fact that these phase portraits are invariant under a reflection $z\mapsto\bar{z}$ as in the proof of 8.1. □

The second case is when $p_3,p_4$ are in the convex hull of $p_1=(1/2),p_2=-(1/2),p_5$.

Lemma 8.8. *Isochronous vector fields with* 5 *zeroes symmetric respect to a line II.*

1. *For* $s=5$, *the following assertions are equivalent.*
   (i) *The zeroes of* X *satisfy that* $p_3,p_4,p_5$ *are in the bisector line of the segment by* $p_1$, $p_2$ *and moreover* $p_3,p_4$ *are in the convex hull of* $p_1,p_2,p_5$.
   (ii) *The residues* $[r_1,\ldots,r_5]$ *of* X *satisfy*

$$|r_3|<|r_1|+|r_2|+|r_4| \quad\text{and}\quad |r_4|<|r_5|.$$

2. *These isochronous subfamilies* $\{\rho e^{i\theta_0}X\}$ *have* 5*-trees with a vertex of degree* 3*and dimension* 7 (*codimension* 1) *in the respective connected components of* $\mathcal{I}(5)$.

We return to the case of isochronous $X$ having the star shape $s$-tree, extending the Lemma 8.2. □

LEMMA 8.9.

1. *For* $s \geq 3$, *the following assertions are equivalent.*
   (i) *The residues* $[r_1, \ldots, r_s] = [(T_1/2\pi i), \ldots, (T_s/2\pi i)]$ *of isochronous* $e^{i\theta_0}X$ *satisfy that only one period* $T_\iota$ *is positive* (*or negative*).
   (ii) *The isochronous* $\{\rho e^{i\theta_0}X\}$ *have* s*-trees with a vertex of degree* $s-1$.
2. *These isochronous subfamilies of vector fields determine two full connected components in each* $\mathcal{I}(s)$.
3. *Under the condition* (*i*) *every collection* $[r_1, \ldots, r_s]$ *having* $r_\iota \in i\mathbb{R}^*$ *and* $\Sigma r_\iota = 0$ *is isochronous.*

*Proof*. Assertion (i) ⇔ (ii) is direct from the fact of two adjacent poles have different sign.
For part 2, we use a continuous path argument as in the proof of Theorem 6.1.3.
Part 3 uses Lemma 4.1 and an explicit construction similar as in Example 4.4. □

Lemma 8.9 generalizes 8.2, however note the lack of explicit geometric descriptions for these isochronous configurations $[p_1, \ldots, p_s]$, when $s \geq 5$.

## 9. Bifurcations and Hamiltonian properties

Let $X$ be a complex polynomial vector field on $\mathbb{C}$, the family of their real-rotated vector fields is

$$\{e^{i\theta}\Re e(X) | \theta \in \mathbb{R}/2\pi\mathbb{Z}\}.$$

The trajectories of the family coincide (in the obvious sense) with the trajectories of the geodesic flow in the unitary tangent bundle of $(\mathbb{C} - \{\text{zeroes of } X\}, g)$.

COROLLARY 9.1. *Let* X *be a complex polynomial vector field on* $\mathbb{C}$ *with simple zeroes, the following assertions are equivalent.*

1. *For some* $\theta_0$, *the vector field* $e^{i\theta_0}X$ *is isochronous.*
2. *The family of the phase portraits* $\{e^{i\theta}\Re e(X)\}$ *has exactly two bifurcation values* $\{\theta_0, \theta_0 + \pi\}$.

*Proof*. Use that the closed geodesics on the half cylinders in Lemma 4.1 are trajectories of $\pm e^{i\theta_0}\Re e(X)$. □

The two bifurcation values in 9.1.2 are the dynamical expression of the $\mathbb{Z}_2$-action in (12). The problem of bifurcation under rotation was considered in [19]. Let $X$ be isochronous and let $e^{i\pi/2}\Re e(X) \doteq \Im m(X)$ be the rotated field. This vector field has sources or sinks at the zeroes of $X$. The residue theorem admits an interpretation as a conservation law for the $g$-area, which is as follows.

COROLLARY 9.2. *Let* $\Im\mathfrak{m}(X)$ *be as above. The* g*-area entering in* $(\mathbb{C} - \{\text{zeroes of } X\}, g)$ *under the flow of* $\Im\mathfrak{m}(X)$ *from a source* $p_j$ *in one unit of time is* $T_j$ *and the exiting area at the sink* $p_\iota$ *is* $T_\iota$. *Moreover*

$$\sum_{p_j \ \text{is a source}} T_j + \sum_{p_\iota \ \text{is a sink}} T_\iota = 0.$$

*Proof*. The flows of $\Re\mathfrak{e}(X)$ and $\Im\mathfrak{m}(X)$ leave invariant the $g$-area. □

The conservation law for the $g$-area of a meromorphic vector field on a compact Riemann surface is described in [20].

Question 3: Given a real polynomial vector field $Y$ with only isochronous centres, under what conditions $Y$ is Hamiltonian?

For a discussion, see [17,7] and references therein. In particular the problem of existence of (at least one) isochronous centre for a polynomial function $H(x, y)$ has received much attention. By Theorem 3.5, we can construct a family of vector fields with Hamiltonian structure.

For a complex polynomial vector field $X$, we look at the pair of real vector fields

$$\Re\mathfrak{e}(X) \doteq u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}, \quad \Im\mathfrak{m}(X) \doteq -v\frac{\partial}{\partial x} + u\frac{\partial}{\partial y}.$$

They commute, are linearly independent outside of the zeroes of $X$, and in the language of differential geometry determine a frame. The dual frame of 1-forms is

$$\omega_1 = \frac{u\,\mathrm{d}x + v\,\mathrm{d}y}{u^2 + v^2}, \quad \omega_2 = \frac{-v\,\mathrm{d}x + u\,\mathrm{d}y}{u^2 + v^2}.$$

Thus $\Omega = \omega_1 \wedge \omega_2 = ((\mathrm{d}x \wedge \mathrm{d}y)/(u^2 + v^2))$ determines a symplectic structure of $C^\omega$ class, outside of the zeroes of $X$.

PROPOSITION 9.3. *The isochronous vector field* $\Re\mathfrak{e}(X)$ *on* $(\mathbb{C} - \{\text{zeroes of } X\}, \Omega)$ *is Hamiltonian for the univalued,* $C^\omega$, *function*

$$H(x, y) = \int_{(x_0, y_0)}^{(x,y)} \omega_2.$$

*Proof*. The initial $(x_0, y_0)$ must be a regular point of $X$. We observe that $\omega_2$ is closed and $\omega_2(\Re\mathfrak{e}(X)) = 0$, thus $H$ is univalued. The value of $H$ at the zeroes of $X$ is $\pm\infty$. The Hamiltonian vector field of $H$ is $\Re\mathfrak{e}(X)$. □

## 10. Conclusions and future directions

We have enlarged the information of diagrams and equations (1), (12), (14), (15) and (17). Our new diagram is

$$
\begin{array}{ccccccc}
\mathcal{I}(s) & \rightarrow & \frac{\mathcal{I}(s)}{\mathrm{Aut}(\mathbb{C})} & \rightarrow & \frac{\mathcal{I}(s)}{\mathrm{Homeo}\,(\mathbb{C})^{+}} & & \\
 & & \updownarrow & \searrow & & \searrow & \\
 & & \left\{\begin{array}{c}\text{realizable}\\ \text{weighted}\\ s\text{-trees } \Lambda(X)\end{array}\right\} & & \frac{\mathcal{I}(s)}{\mathrm{Aut}(\mathbb{C})\times\mathbb{Z}_2} & \rightarrow & \frac{\mathcal{I}(s)}{\mathrm{Homeo}\,(\mathbb{C})^{+}\times\mathbb{Z}_2} \\
 & & \downarrow & \searrow & \updownarrow & & \updownarrow \\
 & & \left\{\begin{array}{c}\text{isochronous}\\ \text{residues;}\\ [r_1,\ldots,r_s]\end{array}\right\} & & \left\{\begin{array}{c}\text{flatmetrics of}\\ \text{isochronous } X,\\ \text{uptoisometries}\end{array}\right\} & \rightarrow & \left\{\begin{array}{c}\text{plane}\\ s\text{-trees}\\ \Lambda\subset\mathbb{C}\end{array}\right\}
\end{array}
\quad . \qquad (27)
$$

The $\mathbb{Z}_2$-action is induced by $X \mapsto \pm X$ and $\updownarrow$ means bijection. Having in mind the above results, we can state the following questions.

*Problem*. Characterize geometrically the isochronous configurations $[p_1, \ldots, p_s]$, for $s \geq 5$.

*Problem*. Given an isochronous configuration $[p_1, \ldots, p_s]$, construct an algorithm to determine the associated plane $s$-tree $\Lambda$.

In particular:

*Problem*. If for some isochronous configuration $[p_1, \ldots, p_s]$, with $s \geq 5$, only one point, say $p_1$, is in the open convex hull of the other $s - 1$ points; is it true that the associated $s$-tree has a vertex at $p_1$ with degree $s - 1$ (as in 8.9)?

*Conjecture*. For every isochronous residues $[r_1, \ldots, r_s]$, with $s \geq 5$:

the $\min_j\{|r_j|\}$ is attained at $p_\beta$ in the boundary of the convex hull of the zeroes,
the $\max_j\{|r_j|\}$ is attained at $p_\iota$ in the interior of the convex hull of the zeroes.

In other order of ideas. For $s \geq 2$, the symplectic structure making Hamiltonian an isochronous vector field in Proposition 9.3 has punctures at the zeroes.

*Problem*. Given an isochronous vector field, does exists a smooth symplectic structure on all $\mathbb{R}^2$ with the above property?

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