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Complex structures adapted to smooth vector fields*

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Abstract. In this paper we study the flat geometry and real dynamics of meromorphic vector fields on compact Riemann surfaces. Necessary and sufficient conditions to assert the existence of meromorphic vector fields with prescribed singularities are given. A characterization of the real dynamics of meromorphic vector fields is also given. Several explicit examples of meromorphic vector fields, using singular flat metrics, are provided.

Introduction

Regarding holomorphic vector fields on a compact complex manifold is natural, since if they exist, they give rise to the Lie algebra of the group of holomorphic automorphisms. Unfortunately, holomorphic vector fields on compact complex manifolds are rare; however, meromorphic vector fields are abundant at least on projective manifolds. This paper studies meromorphic vector fields on compact Riemann surfaces from the differential geometry and dynamical systems point of view. In this paper we address the following questions:

A) Is the Poincaré–Hopf index formula a sufficient condition to assert the existence of meromorphic vector fields with prescribed singularities on compact Riemann surfaces ?

B) Let M be an orientable, compact, smooth two–manifold; remove $K \subset M$ a finite set of points and let F be a real smooth vector field on $M - K$. Under what dynamical conditions in F does a complex structure J exist such that F is the real part of a meromorphic vector field on the Riemann surface (M, J) ?

Problem (B) was communicated to the author by C. Bonatti and X. Gómez–Mont. These problems are obviously of a global nature. However to solve them we use the local invariant flat structure of meromorphic vector fields. Roughly speaking, our starting point is that on Riemann surfaces there is a natural one–to–one correspondence between

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i) meromorphic vector fields,
ii) meromorphic differential forms,
iii) orientable meromorphic quadratic differentials,
iv) some pairs of commuting real smooth vector fields, having suitable singularities,
v) and some singular flat structures with trivial holonomy, suitable singularities and a geodesible vector field.

The above is one aspect of the richness of Riemann surface theory; see Sect. 1 for a complete statement.

Since the study of meromorphic vector fields on Riemann surfaces lies in the intersection of many natural problems in geometry and dynamics, let us comment on some connections of (A) and (B) with other current topics before the statement of the results.

From a real point of point of view, a very natural problem in dynamical systems is the characterization of dynamical restrictions that impose a prescribed invariant geometric structure. In the case of C^k vector fields and their flows $(0 \leq k \leq \infty)$ on compact smooth two–manifolds, a complete description of the dynamical behavior follows mainly from the classical works of H. Poincaré, I. Bendixon, A. Denjoy, A. J. Schuartz, and recently C. Gutierrez [10]. We note in particular the following properties
any C^0 flow is topologically equivalent to some C^1 flow,
any C^2 flow is topologically equivalent to some C^k flow, here $3 \leq k \leq \infty$,
and any C^1 flow is topologically equivalent to some C^2 flow if and only if every minimal set of the C^1 flow is trivial.
The next obvious step in this program is understanding the dynamical behavior of real analytic vector fields over real analytic two–manifolds. Since very little is known in this case, here we are interested in an intermediate problem: understanding the real dynamics of meromorphic vector fields on Riemann surfaces.

Following (iv), a pair of smooth commuting real vector fields on a smooth two–manifold is equivalent to a probably local $(\mathbb{R}^2, +)$–action on the manifold, it is a classical subject in foliation theory. $(\mathbb{R}^2, +)$–actions appear in the seminal work of E. Lima on common zeros of commuting vector fields (see [18]) and in complete integrable Hamiltonian systems in four–dimensional symplectic manifolds. Our observation is that for a pair of commuting real vector fields (that are $\mathbb{R}$–linearly independent outside of their singularities), the vanishing of the commutator is equivalent to the Cauchy–Riemann equations (under suitable complex structure). See Sect. 1. In this paper we are interested in the description of $(\mathbb{R}^2, +)$–actions that arise from meromorphic vector fields.

Following (v), problem (B) concerns the geodesibility property of singular smooth vector fields under certain singular flat Riemannian metrics. Actually, there are many results on geodesible flows, for example [24]; however, the singular case remains comparatively unexplored.

Following (iii), quadratic differential theory is a highly developed and useful topic in geometry and dynamics as seen in [29], [22]. Related to problem (B), K. Strebel [28], J. Hubbard and H. Masur [13] have shown that given any measured foliation $\mathcal{F}$ on M and any complex structure J on M, there is a unique holomorphic quadratic differential on the Riemann surface (M, J) whose horizontal trajectory structure realizes $\mathcal{F}$. We note that in this case the area (or norm) of the quadratic differential is always finite. Here we consider different information: a real vector field F and the knowledge of its flow. Moreover, the existence of zeros for a meromorphic vector field implies that its associated meromorphic quadratic differential has infinite area, and this is the generic situation in the study of meromorphic vector fields.

The dynamical study of meromorphic vector fields on higher dimensional complex manifolds can be split into dynamics along the complex leaves and transverse dynamics. There are many results on transverse dynamics (for example [8]), and also on uniformization of the leaves of a meromorphic vector field, [12], [30]. In (B) we will focus on dynamics along the leaves with the natural uniformization given by the complex flow.

To provide us with explicit examples we study the topological realization problem (A) in Sect. 2, the analogous problem for quadratic differentials of finite area was solved in [23], and our result is as follows.

2.1. Theorem. *Let M be a compact oriented smooth two–manifold of genus* g. *There exist a complex structure J on M and a meromorphic vector field X on the Riemann surface (M, J) with exactly s zeros and k poles (counted with multiplicities) if and only if*
1) $s + k = 2 - 2\mathrm{g}$;
2) $s = 0$ *or* $s \geq 2$.

Condition (1) is the Poincaré–Hopf index formula for vector fields. Condition (2) comes from the fact that complex flows have an invariant area (or by the residue theorem for meromorphic differential forms). The proof of the result is given by finding suitable flat polyhedral metrics (as say (v) in the correspondence). As an application a folklore result is shown: the sufficiency of the Poincaré–Hopf index formula for the existence of smooth vector fields on compact oriented smooth two–manifolds (see 2.2).

Our main result describing the real dynamics is found below.

7.1 Theorem. *Let M be a compact oriented smooth two–manifold, consider a discrete set $K \subset M$ and a smooth vector field F on $M - K$ having smooth isolated singularities at $S \subset M - K$.*
There exists a complex structure J on M, making F the real part of a meromorphic vector field on (M, J) with simple zeros and poles if and only if
1) F has sources, sinks or centers at S.

2) F is up to diffeomorphism at points of K locally equivalent with

$$\frac{1}{x^2+y^2}\left(x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}\right).$$

3) The closure of non–trivial recurrences of F are two–manifolds with boundary $\overline{N}_i$ in M. On each $\overline{N}_i$ the vector field F has a smooth transverse invariant measure, and the flow has a smooth invariant transversal. If two of these recurrence two–manifolds are adjacent, then the smooth transverse invariant measure and the smooth invariant transversals exist on the union of both.
4) F has the isochronicity property for non–singular closed trajectories, saddle connections and recurrence connections.

Let us say a few words about the hypotheses. The result is stated only for simple zeros and poles. Following these ideas, it is easy to give conditions for vector fields having zeros and poles of higher order. Recall that a smooth vector field F on $M - K$ as above is the real part of a meromorphic vector field for some complex structure if and only if there exists a second smooth vector field G, transverse to F on $M-(K\cup S)$ and such that $[F, G]\equiv 0$. Hypotheses (1)–(4) assure the existence of such G. Isochronicity for centers is a classical concept: a center is isochronous if and only if the closed trajectories surrounding the center have the same period. For closed non–singular trajectories the definition is analogous, and other concepts in (4) will appear in Sect. 6.

As an application we study the realization of meromorphic vector fields having very simple trajectory structure. Holomorphic quadratic differentials having closed non–singular vertical trajectories form dense sub–spaces in the space of holomorphic quadratic differentials [28], [21], [13]. Recall that in the meromorphic case this density property is not true (see [26]). We consider families of vector fields whose real closed trajectories form isochronous centers. Recently different classes of isochronous centers have been studied [7], [9], [20]. However, the construction and recognition of smooth isochronous centers remain a difficult problem. In our case, isochronous centers appear in a natural way as zeros of meromorphic vector fields with pure imaginary linear parts. In Sect. 8, we describe families of holomorphic vector fields with isochronous centers on the plane.

Further applications are to study real Hamiltonian vector fields having an additional first integral [25], to characterize complete holomorphic vector fields on higher dimensional complex manifolds [19], and to describe meromorphic quadratic differentials with prescribed singularities [5].

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$\{[w_i, w_{i+1}]\}$. Up to holomorphic change of coordinates in $\mathbb{C}$, the above type of polynomials define a connected component in the space of complex polynomial vector fields as in corollary 8.3, which is diffeomorphic with $(\mathbb{R}^+)^{s-1}$. Conversely, given a complex polynomial vector field R having its zeros in a real straight line in $\mathbb{C}$, there are a complex number $e^{\sqrt{-1}\theta}$ and an affine change of coordinates $\phi(z) = az + b$ such that $e^{\sqrt{-1}\theta}\phi_*(R)$ is in the above family.
2. Another example is given by

$$P(z) = \sqrt{-1}z(z - w_1)(z - w_2)...(z - w_{s-1})\frac{\partial}{\partial z},$$

where $\{w_1, ...w_{s-1}\}$ are the $(s-1)$-th roots of unity. The associated graph $\Lambda(P)$ has s vertices $\{0, w_1, ..., w_{s-1}\} \subset \mathbb{C}$, and the arcs are given by the $s-1$ segments from 0 to w_i. However in this case it is difficult to effectively move the parameters of P in such way that isochronous centers persist and which result in a deformation of P in the families of corollary 8.3.
3. To show that hypothesis (3) in theorem 8.1 is necessary, we consider a vector field P on $\mathbb{C}P^1$ (as in the above examples). Take a finite length segment γ on a separatrix trajectory starting at $\infty \in \mathbb{C}P^1$, and remove this closed segment. The surface $\mathbb{C}P^1 - \{\gamma\}$ is diffeomorphic to $\mathbb{R}^2$ and supports the vector field $\Re e(P)$ satisfying hypotheses (1) and (2), but not (3).

Results 8.1 and 8.3 have easy extensions on other two–manifolds.

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