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EXISTENCE OF AN ADDITIONAL FIRST INTEGRAL AND COMPLETENESS OF THE FLOW FOR HAMILTONIAN VECTOR FIELDS

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Pairs of real analytic Hamiltonian vector fields X_h, X_g in Poisson involution over (not necessarily compact) symplectic manifolds are considered. We address the following problem: describe how a two–dimensional orbit $\mathcal{L}$ of the induced $(\mathbb{R}^2,+)$–action falls to an isolated common zero of X_h and X_g. A generalization of the Poincaré–Hopf index is introduced to describe the dynamics of X_h on $\mathcal{L}$. Poincaré–Hopf index at least three on some $\mathcal{L}$, implies that X_h has incomplete flow (i.e. is not well defined for all time). Completeness of the X_h–flow implies Poincaré–Hopf index one or two on $\mathcal{L}$, and a full description of $\mathcal{L}$ and X_h is provided. Explicit examples of the index computation are given.

1 Introduction

Let (M,ω) be a $2n$–dimensional real analytic symplectic manifold, not necessarily compact, $2n \geq 4$. Given a real analytic Hamiltonian function $h : M \to \mathbb{R}$, two desirable and nice properties of the associated Hamiltonian vector field X_h are:

Existence of an additional first integral g for X_h; i.e. the existence of a real analytic function $g : M \to \mathbb{R}$, functionally independent with h (almost everywhere) and in Poisson involution $\{h,g\} = 0$.

Completeness of X_h; i.e. for all initial condition the corresponding trajectory of X_h is well defined for all real time.

It is elementary that if M is compact then X_h is complete. However many interesting phase spaces M are non–compact and the problem of deciding whether a Hamiltonian vector field is complete, remains nontrivial and interesting. Recall, for example the study of escapes and non–collision singularities on higher dimensional manifolds in celestial mechanics [14]. On the other hand, under the assumption of complete integrability inside nonsingular compact energy levels $\{h^{-1}(c)\}$ many dynamical aspects of X_h are very well understood. This study began with the celebrated Liouville–Arnold theory, but we remark the more recent work of Fomenko for $dim(M) = 4$ and $\{h^{-1}(c)\}$ a three–dimensional smooth manifold, see [6] chapter 2.

The goal of this paper is to show the existence of a link between completeness and existence of a second first integral near the singular points of X_h in M.

We assume the following hypothesis (H1), (H2).

(H1) The *existence of an additional first integral* g for X_h, with $p \in M$ a common zero of X_h and X_g.

Since h and g Poisson commute they define a local $(\mathbb{R}^2, +)$–action, giving rise to a singular analytic foliation $\mathcal{F}$ on M, having as leaves two–dimensional orbits of the action (the orbits can be zero, one or two–dimensional). Let $\mathcal{L}^2$ be a two–dimensional orbit, where by definition the orbits are connected. We address the following questions:

How can a two–dimensional orbit $\mathcal{L}^2$ "fall" to a singularity p ?

How is the dynamics of X_h inside of $\mathcal{L}^2$?

Recall that an *analytic set* in M is the common zero locus of a finite collection of real analytic functions.

(H2) $\mathcal{F}$ has a *separatix* $\mathcal{L}$ by p, i.e. there exists a two–dimensional connected analytic set $\mathcal{L} \subset M$ such that;
$p \in \mathcal{L}$ is an isolated singularity of $\mathcal{L}$, and
$\mathcal{L} = \cup_j \mathcal{L}_j^2$ is the closure of two–dimensional orbits, containing zero–dimensional orbits but not one–dimensional orbits.

The last condition says that on $\mathcal{L}$ the vector fields X_h, X_g are $\mathbb{R}$–linearly independent (whenever one of them is non zero), and otherwise they have common isolated zeros. For example, in a four–dimensional M, the ansatz for a separatrix is

$$\mathcal{L} = \{h^{-1}(0)\} \cap \{g^{-1}(0)\}$$

(where we may assume without loss of generality that $h(p) = 0 = g(p)$), whenever we can verify that X_h and X_g are $\mathbb{R}$–linearly independent as above.

Now we describe the shape of a separatrix $\mathcal{L}$ near a singular point p.

We intersect the separatrix with a small ball $B_\epsilon(p) \subset M$, centered at p and of radius $\epsilon > 0$ in some Riemannian metric in M. It is known that for small enough ball, every $\mathcal{L} \cap B_\epsilon(p)$ is homeomorphic to a finite union of copies of a *cone*, where the vertex correspond to p, see Section 2 and [4]. Here by a cone $\mathcal{L}_i$ we mean any surface homeomorphic to $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 = 0,\ z \geq 0\}$. Note that since the $(\mathbb{R}^2, +)$–action is not necessarily complete, $\mathcal{L}$ can assume intricate topological patterns, in particular there may exist several cones $\{\mathcal{L}_i\} \subset \mathcal{L}$.

Fixing a cone $\mathcal{L}_i \subset \mathcal{L}$, there exists a *resolution* of $\mathcal{L}_i$ at p, by this we mean a continuous map

$$\psi_i : B_{\epsilon^2}(0) \subset \mathbb{R}^2 \to \mathcal{L}_i\ ,$$

(for $B_{\epsilon^2}(0)$ an open ball centered at 0 of radius ϵ^2), which is a smooth embedding of the punctured ball $B_{\epsilon^2}(0) - \{0\}$ over its image in $\mathcal{L}_i$ and $\psi_i(0) = p$. The existence of resolutions is shown in Section 2. Note that at p the cone $\mathcal{L}_i$ can be smoothly embedded in M or be singular (see examples on Section 7).

The dynamics of X_h inside $\mathcal{L}_i$ is described by the pulled back smooth non–singular vector field $\psi_i^* X_h$ on the punctured ball $B_{\epsilon^2}(0) - \{0\}$. In section 2 we will show that (H2) implies that the Poincaré–Hopf index of $\psi_i^* X_h$ at 0;

$$PH(\psi_i^* X_h, 0)$$

is a well defined positive integer number. For example, if $\mathcal{L}_i$ is smoothly embedded at p, then $PH(\psi_i^* X_h, 0)$ is the usual Poincaré–Hopf index of X_h restricted to the submanifold $\mathcal{L}_i$.

Our main result is as follows:

1.1 Theorem. *Let $\mathcal{L} \subset M$ be a separatrix by $p \in M$ coming from the closure of a two–dimensional orbits of the $(\mathbb{R}^2, +)$–action induced by X_h, X_g, satisfying (H1) and (H2). Given a cone $\mathcal{L}_i \subset \mathcal{L}$ and ψ_i its resolution.*

1.- If $PH(\psi_i^* X_h, 0) \geq 3$, then X_h (or X_g) is incomplete.

2.- If $PH(\psi_i^* X_h, 0) = 2$ and X_h, X_g are complete, then $\mathcal{L}$ is homeomorphic to a two–sphere (having p as unique singularity of X_h in $\mathcal{L}$).

3.- If $PH(\psi_i^* X_h, 0) = 1$ and X_h, X_g are complete, then $\mathcal{L}$ is homeomorphic to a union of pieces of the following types:

a plane (having p as a unique singularity of X_h in $\mathcal{L}$), or

a two–dimensional sphere (having two singularities of X_h in $\mathcal{L}$), or

a singular surface obtained from a two–dimensional sphere identifying two different points to one (this last one will correspond to p, the unique singularity of X_h in $\mathcal{L}$).

Note that parts (2) and (3) are closely related with Liouville–Arnold Theorem in four–dimensional manifolds. Completeness is a hypothesis in Liouville-Arnold theory, see [2] p. 6 (the section called: What the Liouville–Arnold does not say). If completeness is assumed as in (2) and (3), the two–dimensional orbits falling to a singularity are; a plane or a cylinder. Hence (2) and (3) describe how they are immersed in M.

If we pick exactly one two-dimensional orbit in M a new version of 1.1 parts (2) and (3) is as follows.

1.2 Corollary. *Let $\mathcal{L}$ be a separatrix that is the closure of a two-dimensional orbit $\mathcal{L}^2$ of the $(\mathbb{R}^2,+)$-action induced by X_h, X_g, whose singular points $\{p_j\} \subset \mathcal{L}$ satisfy (H1) and (H2). If X_h, X_g are complete then*

$$\sum PH(\phi_i^* X_h, 0) = 1 \text{ or } 2 ,$$

here the sum runs over all the cones $\{\mathcal{L}_i\} \subset \mathcal{L}$ having resolutions $\{\phi_i\}$.

Note that compactness in M or $\mathcal{L}$ is not required, only completeness for X_h and X_g on $\mathcal{L}^2$ is essential. For incompleteness we have:

1.3 Corollary. *The existence of separatrix that is the closure of a two-dimensional orbit and with three or more singular points and hence three or more cones, implies that X_h (or X_g) is incomplete on this orbit.*

The basic ideas of this paper come from holomorphic dynamical systems. We follow the seminal ideas of J. C. Rebelo on the study of complete holomorphic $(\mathbb{C},+)$-actions [13]. Roughly speaking our main arguments are:
* The two-dimensional orbits, that are leaves of the foliation $\mathcal{F}$ (associated to the Hamiltonian vector fields X_h and X_g), have natural Riemann surface structures (i.e. are one-dimensional complex manifolds).
* Moreover, the real vector fields X_h and X_g are recognized as the "real" and "imaginary" parts of suitable holomorphic vector fields over these Riemann surfaces.

The reason is very simple: the Cauchy–Riemann equations for a real two-dimensional vector field F are equivalent with the existence of an additional $\mathbb{R}$-linearly independent real vector field G, such that they commute $[F,G] = 0$, see Section 4.

The $(\mathbb{R}^2,+)$-actions satisfying (H2) are very close to holomorphic $(\mathbb{C},+)$-actions on $\mathcal{L}$, and some exchange of ideas between Hamiltonian vector fields having an additional first integral and holomorphic dynamics, is possible.

Note that the analyticity assumption for (M,ω), h and g allow us to apply the classical theory of analytic singularities in the description of $\mathcal{L}$ near p. See 6.1 for necessary assumptions in the smooth case.

In this paper smooth means C^∞, and manifold means C^∞-manifold.

2 Conic structure, resolutions and Hamiltonian vector fields on separatrices

Let X_h be a real analytic Hamiltonian vector field on (M,ω) satisfying (H1) and (H2). We start by reviewing the description of the cone structure for a separatix $\mathcal{L}$ at singular points p as in Section 1. This is due to H. Whitney, J. Milnor et al.

Consider some Riemannian metric in M and let d be the induced distance function, $B_r(p) \subset M$ denotes the open ball centered at p with radius r.

By (H2), the point p is an isolated singularity of $\mathcal{L}$ as a real analytic set. We work locally in some $B_r(p)$ such that $B_r(p) \cap \mathcal{L}$ has p as unique singular point, and where obviously $B_r(p) \cap (\mathcal{L} - \{p\})$ is a smooth submanifold.

A key fact is:

2.1 Lemma. *The distance function $d : B_r(p) \cap (\mathcal{L} - \{p\}) \to \mathbb{R}^+$ given by $q \mapsto d(p,q)$, can have at most a finite number of critical values.*

Proof. See [9] p. 16 in the case when $\mathcal{L}$ is an algebraic set and [4] p. 58–59 for the analytic case. □

Hence we can find a number $\epsilon \in (0,r)$ such that for all $q \in B_\delta(p) \cap (\mathcal{L} - \{p\})$, where $\delta \in (0,\epsilon)$, the distance function d is free of critical values. In consequence the tangent plane $T_q\mathcal{L} \not\subset T_q S_\delta$, where $\delta = d(p,q)$ and S_δ is the sphere of d-radius δ centered at p.

It follows that every sufficiently small sphere S_δ intersects $\mathcal{L}$ in a one-dimensional smooth compact manifold (not necessarily connected), i.e. a finite collection of disjoint circles. For example if $S_\delta \cap \mathcal{L}$ is only one circle, then $\mathcal{L}$ must be a topological manifold near p.

Fixing a circle in $S_\delta \cap \mathcal{L}$, when ϵ goes to 0, the associated family of circles describes a topological cone $\mathcal{L}_i \subset \mathcal{L}$. We want to show that $\mathcal{L}_i$ can be parametrized, the classical ideas are as follows (see also [9] p. 16–22 and [4] p. 58–59).

There exists a smooth vector field V on the punctured ball $(B_\epsilon(p) - \{p\}) \subset M$ with the following properties:

* The vector $V(q)$ will point away from p for all $q \in B_\epsilon(p) - \{p\}$; assume without loss of generality the existence of normal Riemannian coordinates in $B_\epsilon(p)$, let $W(q) \in T_qM$ be the tangent vector of the unique geodesic between p and q in the normal Riemannian coordinates, then the Riemannian inner product $< V(q), W(q) >$ will be strictly positive.

* The vector $V(q)$ will be tangent to $\mathcal{L}_i$ in all $q \in B_\epsilon(p) \cap (\mathcal{L}_i - \{p\})$.

* The local trajectories $q(t)$ of V are such that

$$d(q(t)) = d(q(t), p) = t + \text{constant} ,$$

where d is the Riemannian distance. In particular for each initial condition $a \in S_\epsilon \cap \mathcal{L}_i$ the corresponding trajectory solution $q(t) : (0, \epsilon^2] \to M$ with initial condition $q(\epsilon^2) = a$ are well defined, and $q(t)$ tends uniformly to p as t goes to zero.

Hence, the restriction of the flow of V to the invariant submanifold $B_\epsilon(p) \cap (\mathcal{L}_i - \{p\})$ maps the product $(S_\epsilon \cap \mathcal{L}_i) \times (0, \epsilon^2]$ diffeomorphically onto the two–dimensional punctured ball $B_{\epsilon^2}(0) - \{0\} \subset \mathbb{R}^2$.

We summarize all the above in the:

2.2 Corollary. *[Existence of resolutions.] Let $\mathcal{L}_i \subset \mathcal{L}$ be a cone in a separatrix, then there exists a resolution of $\mathcal{L}_i$ at p. This is a continuous map*

$$\psi_i : B_{\epsilon^2}(0) \subset \mathbb{R}^2 \to \mathcal{L}_i ,$$

(for an open ball $B_{\epsilon^2}(0)$ centered at 0 of radius $\epsilon^2 > 0$), which is a smooth embedding of the punctured ball $B_{\epsilon^2}(0) - \{0\}$ over its image in $\mathcal{L}_i$ and $\psi_i(0) = p$.

□

Now, we want to study the induced Hamiltonian vector fields.

2.3 Lemma. *The vector fields $\psi_i^* X_h$ and $\psi_i^* X_g$ on the punctured ball $(B_{\epsilon^2}(0) - \{0\}) \subset \mathbb{R}^2$, have the following properties.*

1.- They are smooth vector fields.

2.- They are free of zeros, everywhere $\mathbb{R}$–linearly independent and commute.

3.- Any trajectory of $\psi_i^* X_h$ or $\psi_i^* X_g$ having the origin $0 \in B_{\epsilon^2}(0)$ as α–limit (or ω–limit) is well defined for all values of time in some interval $(-\infty, a)$ (respectively $(a, +\infty)$).

Proof. Part (1) follows since ψ_i is an embedding outside of the origin and part (2) follows from (H1), (H2). For (3), a trajectory of $\psi_i^* X_h$ having the origin as α–limit corresponds to a trajectory of X_h in M having $p \in M$ as α–limit. Since X_h is real analytic at $p \in M$, the existence of trajectories for time $(-\infty, a)$ follows. For trajectories having 0 as ω–limit the idea is similar. □

To get a classification of pairs $\psi_i^* X_h$, $\psi_i^* X_g$ on the punctured ball $B_{\epsilon^2}(0) - \{0\}$, we introduce in it a smooth Riemannian metric g defined by

$$\mathrm{g}(\psi_i^* X_h, \psi_i^* X_h) = 1 = \mathrm{g}(\psi_i^* X_g, \psi_i^* X_g) , \ \mathrm{g}(\psi_i^* X_h, \psi_i^* X_g) = 0 .$$

Some main features of this metric are:

i) Since the vector fields commute and they are orthonormal the metric g is flat (see [15] p. 261).

ii) Condition (3) in Lemma 2.3 says that the metric g is geodesically complete around the origin $0 \in B_{\epsilon^2}(0)$.

2.4 Definition. A *flat punctured ball with a frame* $\{(B_r(0) - \{0\},\ \mathrm{g}),\ F,\ G\}$ is an oriented smooth punctured ball $(B_r(0) - \{0\}) \subset \mathbb{R}^2$ provided with a flat Riemannian metric g and a positively oriented orthonormal frame F, G of smooth vector fields, satisfying (i) and (ii) above.

Obviously pairs of Hamiltonian vector fields X_h and X_g, satisfying (H1) and (H2) on each cone $\mathcal{L}_i$ give origin to flat punctured balls with frame

$$\{(B_{\epsilon^2}(0) - \{0\},\ \mathrm{g}),\ \psi_i^* X_h,\ \psi_i^* X_g\} .$$

We say that two flat punctured balls with a frame are *isometric* if there exists a Riemannian isometry between them sending an orthonormal frame on the other. We want to classify the flat punctured balls with a frame up to isometry. To perform this, we introduce two isometry invariants; the Poincaré–Hopf index and the residue (for simplicity to describe them we continue with the above notation using Hamiltonian vector fields).

The Poincaré–Hopf index:
Consider $\psi_i^* X_h$ as a smooth vector field on $B_{\epsilon^2}(0) - \{0\}$. Let $\gamma \subset B_{\epsilon^2}(0)$ be a clockwise circle (in the usual metric of $\mathbb{R}^2$) of radius $\epsilon^2/2$ centered at 0. The topological degree of the map

$$\begin{aligned} PH : \gamma &\to S^1 \\ q &\mapsto \frac{\psi_i^* X_h(q)}{\|\psi_i^* X_h(q)\|} , \end{aligned}$$

is well defined, where $\| \ \|$ is the usual Euclidean norm in $\mathbb{R}^2$, and $S^1 \subset \mathbb{R}^2$ is the clockwise unitary circle. This degree is called the *Poincaré–Hopf index* and we denote it by $PH(\psi_i^* X_h, 0) \in \mathbb{Z}$.

Note that in our case $\psi_i^* X_h$ is not necessarily well defined at 0, hence a priori the above number is not the classical Poincaré–Hopf index of an isolated smooth zero. However, let us recall explicitly that:

The number $PH(\psi_i^* X_h, 0)$ is independent on the choice of the closed path γ.

Let γ_1 be any path homotopic to γ in $B_{\epsilon^2}(0) - \{0\}$. The difference of the index computed with γ or γ_1 is the usual Poincaré–Hopf index of $\psi_i^* X_h$ in the interior of the region bounded by $\gamma \cup \gamma_1$, that is always zero since $\psi_i^* X_h$ is free of zeros in $B_{\epsilon^2}(0) - \{0\}$, and the assertion follows.

$PH(\psi_i^* X_h, 0) = PH(\psi_i^* X_g, 0)$.
Note that on each tangent plane of the Riemannian surface $(B_{\epsilon^2}(0) - \{0\}, g)$, the vector fields $\psi_i^* X_h$ and $\psi_i^* X_g$ differ by a $(\pi/2)$–rotation (with respect to g). It is follows that corresponding PH maps differ by a $(\pi/2)$–translation in S^1, and they have the same degree.

The residue:
Define in $B_{\epsilon^2}(0) - \{0\}$ two closed smooth one–forms α and β as follows:

$$\alpha(\psi_i^* X_h) = 1 \ , \quad \alpha(\psi_i^* X_g) = 0 \ ,$$

$$\beta(\psi_i^* X_h) = 0 \ , \quad \beta(\psi_i^* X_g) = 1 \ .$$

Let δ be the usual flat Riemannian metric in $\mathbb{R}^2$. Using the flatness of the metric g, there exist local isometry maps $\phi : V \subset (\mathbb{R}^2, \delta) \to (B_{\epsilon^2}(0) - \{0\}, g)$, where $\phi^*\alpha = dx$ and $\phi^*\beta = dy$, hence in fact they are closed forms.

Given a punctured flat ball we define its *residue* at 0 by

$$\left(\int_\gamma \alpha \, , \int_\gamma \beta \right) \in \mathbb{R}^2 \ ,$$

where γ is a trajectory as above. It measures the $\mathbb{R}^2$–time required to travel along γ under the $(\mathbb{R}^2, +)$–action defined by $\psi_i^* X_h$ and $\psi_i^* X_g$.

A pictorial description is as follows. Consider a closed path γ_1 in the punctured ball, constructed using small pieces of $\psi_i^* X_h$ and/or $\psi_i^* X_g$ trajectories, such that is homotopic to γ in the punctured ball $B_\epsilon^2(0) - \{0\}$. The integration of α along γ_1 measures the time required in the dynamical system defined by $\psi_i^* X_h$ to travel along its $\psi_i^* X_h$ trajectory pieces in γ_1. The integration of β measures the same thing for $\psi_i^* X_g$. Note that since α and β are closed one–forms, the residue is the same for γ_1 as for γ (since γ_1 is homotopic to the original γ).

The Poincaré–Hopf index and the residue are well defined for any flat punctured balls with frame and are isometry invariants. To show that they characterize up to isometry flat punctured balls with a frame, we require complex analysis.

3 Holomorphic vector fields; flat structures and classification of zeros

We start by reviewing how a holomorphic vector field gives rise to a flat Riemannian metric. This is a basic idea coming from the theory of meromorphic quadratic differentials, see [16] for the general theory and [10], [11] for the relation with meromorphic vector fields.

Let $f = u + \sqrt{-1}v : \Omega \subset \mathbb{C} \to \mathbb{C}$ be a holomorphic function, in some domain Ω. We have the following associated objects:
* A holomorphic vector field

$$X = f(z)\frac{\partial}{\partial z} \ .$$

* A meromorphic differential form

$$\omega = \frac{dz}{f(z)} \ .$$

* A pair of real smooth vector fields on Ω

$$\Re e(X) \doteq (X + \overline{X}) = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} \quad , \quad \Im m(X) \doteq J(X - \overline{X}) = -v\frac{\partial}{\partial x} + u\frac{\partial}{\partial y} \ ,$$

here $\overline{X}$ means the conjugate vector field and $J : T\mathbb{R}^2 \to T\mathbb{R}^2$ is the usual complex structure in $\mathbb{R}^2 \cong \mathbb{C}$.
* A smooth flat Riemannian metric in $\Omega - \{$ zeros of $f\}$ is given by

$$g_X = \begin{pmatrix} \frac{1}{u^2+v^2} & 0 \\ 0 & \frac{1}{u^2+v^2} \end{pmatrix} \ .$$

Some features and relations between the above objects are as follows:

The vector field X has ω as time–form, namely $\omega(X) \equiv 1$, and for any smooth trajectory γ from z_0 to z_1 in $\Omega - \{$ zeros of $f\}$, the definite integral

$$\int_\gamma \frac{dz}{f(z)} \in \mathbb{C}$$

is the complex time required to travel from z_0 to z_1 along γ under the field X.

For $z_0 \in \Omega - \{$ zeros of $f\}$ the function

$$F(z) = \int_{z_0}^{z} \frac{dw}{f(w)} : B(z_0) \subset \Omega \to \mathbb{C} \ ,$$

here z in some ball $B(z_0)$ around z_0 where $f \neq 0$, is well defined, holomorphic and its differential F_* satisfies:

$$F_*(X) = \frac{\partial}{\partial z} \ , \ F_*(\Re e(X)) = \frac{\partial}{\partial x} \ , \ F_*(\Im m(X)) = \frac{\partial}{\partial y} \ .$$

The first equality says that F is a holomorphic flow box for X, and the last two describe smooth flow boxes for $\Re e(X)$ and $\Im m(X)$.

From the expressions above the commutator gives

$$[\Re e(X), \Im m(X)] = [u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} \ , \ -v\frac{\partial}{\partial x} + u\frac{\partial}{\partial y}] \equiv 0 \ .$$

Since the Riemannian metric g_X has $\Re e(X), \Im m(X)$ as orthonormal frame, it is well known that the curvature of g_X is identically zero, [15] p. 261. Moreover, every map

$$F(z) : B(z_0) \subset (\Omega - \{ \text{ zeros of } f\}, g_X) \to (\mathbb{C}, \delta)$$

is a local isometry, where δ is the usual flat metric.

Let us give some elementary examples of the above situation.

3.1 Example. Consider $c = a + \sqrt{-1}b \in \mathbb{C}^*$, and let $X = c\frac{\partial}{\partial z}$ be the complex vector field on $\Omega = \mathbb{C}$. The space $(\mathbb{C}, g_X)$ is isometric with the usual flat plane $\mathbb{R}^2$, where the isometry is given by the map $x + \sqrt{-1}y \mapsto \frac{1}{\|c\|}(x, y)$. The associated real vector field is

$$\Re e(c\frac{\partial}{\partial z}) = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} \ ,$$

having as trajectories straight lines of slope $arg(c)$.

3.2 Example. For $X = z\frac{\partial}{\partial z}$, let $\omega = dz/z$ be the meromorphic form in $\Omega = \mathbb{C}$. The metric space $(\mathbb{C} - \{0\}, g_X)$ is isometric to the cylinder $S^1_{2\pi} \times \mathbb{R}$, where the subindex 2π means the length of the closed geodesics. The associated real vector field is

$$\Re e(z\frac{\partial}{\partial z}) = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \ ,$$

having as trajectories straight lines through 0 in $\mathbb{C}$. The associated imaginary vector field is

$$\Im m(z\frac{\partial}{\partial z}) = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \ ,$$

having as trajectories closed circles around zero. They correspond to closed geodesics in the cylinder $S^1_{2\pi} \times \mathbb{R}$.

3.3 Proposition. *[Analytic classification of zeros.] Let $X = f(z)\frac{\partial}{\partial z}$ be a holomorphic vector field on a ball $B_r(0) \subset \mathbb{C}$ centered at 0. Up to a holomorphic change of coordinates it has the following form.*
1.- If X has a zero of order one in 0, then it is

$$\lambda z \frac{\partial}{\partial z} \ ,$$

where $\lambda = f'(0)$.
2.- If X has a zero of order $s \geq 2$ in 0, then it is

$$\frac{z^s}{1 + \lambda z^{s-1}} \frac{\partial}{\partial z} \ ,$$

for $\lambda = \int_\gamma \frac{dz}{f(z)}$ the complex residue of the associated differential form where γ is a simple clockwise path enclosing 0.

Proof. See [3] p. 312. □

Given a holomorphic vector field X on a ball $B_r(0) \subset \mathbb{C}$, having a unique zero at 0, obviously it defines a flat punctured ball with a frame $\{(B_r(0) - \{0\}, g_X), \Re e(X), \Im m(X)\}$.

A key fact is:

3.4 Corollary. *[Equivalence of the analytic and geometric classifications.] Let X_1, X_2 be two holomorphic vector fields on balls $B_{r_i}(0) \subset \mathbb{C}$ centered at 0, having unique zeros at 0. Then X_1 and X_2 are holomorphically equivalent if and only if their associated flat punctured balls with frame are isometrically equivalent.*

Proof. Assume that there exists a smooth isometry

$$\nu : (B_{r_1}(0) - \{0\}, g_{X_1}) \longrightarrow (B_{r_2}(0) - \{0\}, g_{X_2})$$

such that

$$\nu_* \Re e(X_1) = \Re e(X_2) \ , \quad \nu_* \Im m(X_1) = \Im m(X_2)$$

holds in the punctured ball $B_{r_1}(0) - \{0\}$, we want to show that ν is holomorphic everywhere in $B_{r_1}(0)$.

Since g_{X_1}, g_{X_2}, and δ (the usual Riemannian metric in $\mathbb{C}$) are conformal and 0 is a conformal puncture in the usual Riemann surface structures of $B_{r_i}(0)$ (see [1] p. 40 for the concept of conformal puncture). It follows that ν is a conformal map from $B_{r_1}(0) - \{0\}$ onto $B_{r_2}(0) - \{0\}$, hence ν is holomorphic and bounded. Finally by the Riemann continuation Theorem, ν is holomorphically extendable as $\nu(0) = 0$.

The converse is immediate.

$\square$

Now we describe explicitly the geometry of the metric g_X and the singular foliation by $\Re e(X)$ trajectories at any zero.

Consider the Riemann sphere $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$ provided with the natural flat metric in $\mathbb{C}$, and where ∞ is a "singular point" of this metric. Also introduce in $\mathbb{C}P^1$ the singular real foliation by geodesic trajectories of $\frac{\partial}{\partial x}$ in $\mathbb{C}$, this foliation is singular at ∞ (this type of singularity is called a *dipole*). Define a half sphere as the subset

$$\mathcal{H} = \{z \in \mathbb{C} \mid Im(z) \geq 0\} \cup \{\infty\} \subset \mathbb{C}P^1 .$$

A *flat elliptic sector* is an open neighborhood of $\infty \in \mathcal{H}$ (which does not contains 0).

The above sector is a Riemannian surface with boundary, having a foliation by unitary geodesics.

3.5 Lemma. *[Geometry and dynamics of zeros.] Let X be a holomorphic vector field on a ball $B(0) \subset \mathbb{C}$ centered at 0.*

1.- If X has a zero of order $s = 1$ at 0, then $(B(0) - \{0\}, g_X)$ is locally isometric to an Euclidean cylinder $S^1_T \times (0, \infty)$ (where $0 \in B(0)$ corresponds to the extreme having ∞ as second coordinate). The trajectories of the vector field $\Re e(X)$ assume one of the following models: center, source or sink.

2.- If X has a zero of order $s \geq 2$ at 0, then $(B(0) - \{0\}, g_X)$ is locally isometric to a suitable glue of $2s - 2$ flat elliptic sectors (see the proof for full details). The trajectories of the vector field $\Re e(X)$ define $2s - 2$ elliptic sectors.

3.- The real associated vector fields satisfy

$$PH(\Re e(X), 0) = PH(\Im m(X), 0) = \textit{ order of } X \textit{ at } 0 .$$

Proof. For case (1), using Proposition 3.3 (1) it follows that the vector field is $X = \lambda z \frac{\partial}{\partial z}$. We consider two sub cases.

If $\Re e(\lambda) = 0$, then the linear part of $\Re e(X)$ has pure imaginary eigenvalues. Its trajectories are circles, i.e. closed geodesics in the metric g_X, giving rise to a center. The flow of the orthonormal vector field $\Im m(X)$, sends closed geodesics to closed geodesics. Moving a fixed closed geodesic with the flow of $\Im m(X)$ in the direction of the zero of X, we get the description of a cylinder.

If $\Re e(\lambda) \neq 0$, then we consider the rotated vector field $e^{\sqrt{-1}\theta} X$, where $\theta \in [0, 2\pi)$, such that $\Re e(e^{\sqrt{-1}\theta}\lambda) = 0$ as in the above sub case. The Riemannian metrics coming from X and $e^{\sqrt{-1}\theta} X$ are isometric (see [11] for rotated vector fields). Finally note that the trajectories of $\Re e(X)$ correspond to open geodesics in the cylinder, describing a source or a sink.

For case (2), if the order of the zero is $s \geq 2$, there are two isometric invariants of $(B(0) - \{0\}, g_X)$: the order $s \in \mathbb{N}$, and $\lambda \in \mathbb{C}$ the residue of $dz/f(z)$ at $0 \in \mathbb{C}$.

Applying Corollary 3.4 we can make the description of the metric by cut and paste methods.

In the sub case $\lambda = 0$, the glue of $2s - 2$ flat elliptic sectors produce the metric space $(B(0) - \{0\}, g_X)$.

Assume $\lambda = a + \sqrt{-1}b \neq 0$ and $s = 2$. Consider the above global model $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$ with $\lambda = 0$. It is necessary to consider two bands

$$A = \{x + \sqrt{-1}y \in \mathbb{C} \mid a \geq x \geq 0,\ y \geq 0\} ,$$

$$B = \{x + \sqrt{-1}y \in \mathbb{C} \mid b \geq y \geq 0,\ x \geq 0\} .$$

Remove the bands from $\mathbb{C}P^1$. Now we glue the boundaries, using isometrics:
In A, glue X to $x + \sqrt{-1}b$ for $x \geq a$.
In B, glue $\sqrt{-1}y$ to $a + \sqrt{-1}y$ for $y \geq b$.
Then, an open neighborhood of the point coming from $\infty \in \mathbb{C}P^1$, in the new flat surface, is the local model for $[z^2/(1 + \lambda z)]\frac{\partial}{\partial z}$ having $s = 2$ and $\lambda = a + \sqrt{-1}b$.

The case $\lambda = a + \sqrt{-1}b \neq 0$ and $s \geq 3$ is now easy, following the same ideas.

Assertion (3) follows by simple inspection.

$\square$

Note that the complex residue $\lambda \in \mathbb{C}$ in 3.3 and in the proof of 3.5, is the same that the residue defined in Section 2, when we consider the associated flat punctured ball of X and the natural identification $\mathbb{R}^2 \cong \mathbb{C}$.

4 Hamiltonian and holomorphic vector fields

Now we are ready to describe how a pair of Hamiltonian vector fields in Poisson involution produce holomorphic vector fields on Riemann surfaces.

4.1 Proposition. *[Real description of holomorphic vector fields.] Let L be an oriented paracompact smooth two–dimensional manifold. There exists a correspondence between:*

1.- Pairs of non vanishing smooth vector fields F and G in L, everywhere $\mathbb{R}$–linearly independent and commuting $[F,G]=0$.

2.- Pairs g, F, where g is a flat Riemannian metric and F is a unitary geodesic vector field on (L, g).

3.- Pairs J, X, where $J : TL \to TL$, $J^2 = -Id$, is a smooth complex structure making (L, J) a Riemann surface and X is a holomorphic non vanishing holomorphic vector field.

Proof. Given (1), define g using F and G as orthonormal frame. In the other direction given (2), define G as the rotated vector field of F by a positive oriented $(\pi/2)$–angle (using the metric g).

Given (3), consider g coming from X as in Section 2, and define $F = \Re e(X)$. For the converse given g, define the complex structure $J : TL \to TL$ as the rotation by a positive oriented $(\pi/2)$–angle. The explicit formula for the holomorphic vector field is $X = F + \sqrt{-1}JF$, see [12] p. 116 for full details. □

A key observation in the paper is the following:

4.2 Corollary. *Given two Hamiltonian vector fields X_h, X_g on a symplectic manifold (M, ω) in Poisson involution. For a two–dimensional orbit $\mathcal{L}^2 \subset M$ of their associated $(\mathbb{R}^2, +)$–action, there exists a Riemann surface structure $L = (\mathcal{L}^2, J)$ and a holomorphic vector field X on L satisfying*

$$X_h = \Re e(X) \quad \text{and} \quad X_g = \Im m(X) \ .$$

Proof. Recall that by definition $\mathcal{L}^2$ is a connected smooth manifold where X_h and X_g are always $\mathbb{R}$–linearly independent (and in particular never zero). □

A very useful result is the extension of 4.1 to cover F and G with isolated common zeros, as follows.

4.3 Corollary *[A compactification procedure.] Given $\{(B_r(0) - \{0\}, \mathrm{g}), F, G\}$ any flat punctured ball with a frame (here F, G is a frame of real smooth vector fields). There exists a Riemann surface structure in the full ball $B_r(0) = (B_r(0), J)$ coming from the flat structure of g, and a holomorphic vector field X on $B_r(0)$ with a zero in 0, satisfying*

$$F = \Re e(X) \quad \text{and} \quad G = \Im m(X) \ .$$

Proof. Applying results 4.1 and 4.2, the punctured ball $B_r(0) - \{0\}$ has a Riemann surface structure and a holomorphic vector field X, such that $F = \Re e(X)$ and $G = \Im m(X)$.

By results 3.4 and 3.5 the flat structure (arising from F and G) can be recognized as the conformal structure of the holomorphic vector field X. Then the Riemann surface structure J extends to 0 in a unique way, since it is a conformal puncture. Moreover, the holomorphic vector field extends to 0 (applying Riemann continuation Theorem). □

We summarize the work of Sections 2–4 in the following scheme:

$$\left\{ \begin{array}{c} \text{Pairs of Hamiltonian vector fields } \psi_i^* X_h \text{ and } \psi_i^* X_g \text{ coming from} \\ \text{the resolution of a separatrix, satisfying (H1) and (H2).} \end{array} \right\}$$
$$\cap$$
$$\left\{ \begin{array}{c} \text{Holomorphic vector fields } X = f(z)\frac{\partial}{\partial z} \text{ on} \\ \text{some ball } B(0) \subset \mathbb{C}, \text{ with a zero at } 0. \end{array} \right\}$$
$$\|$$
$$\{ \text{Flat punctured balls with a frame.} \}$$

It is an open question for us, if every holomorphic vector field in a ball as above comes from a pair of Hamiltonian vector fields, we believe that the answer is affirmative.

5 Complete holomorphic vector fields on Riemann surfaces

By Section 4, the classification of complete holomorphic vector fields on Riemann surfaces is equivalent to the classification of pairs of smooth complete and commuting vector fields F and G on two–dimensional manifolds (where F and G are $\mathbb{R}$–linearly independent or have common isolated zeros). A geometric version is the following:

5.1 Corollary. *Equivalence between flow completeness and geodesic completeness. Given X a holomorphic vector field on a Riemann surface L the following assertions are equivalent.*

1.- The Riemannian surface $(L - \{\text{ zeros of } X\}, \mathrm{g}_X)$ is geodesically complete.

2.- The real vector fields $\Re e(X)$ and $\Im m(X)$ on $L - \{\text{ zeros of } X\}$ are complete (i.e. for all initial conditions the corresponding trajectories are well defined for all real time).

□

5.2 Lemma. *Let $L = (L, J)$ be a connected Riemann surface, and a non–identically zero complete holomorphic vector field X on L. Then, up to bi-holomorphism, X and L are as follow:*

Case:	*Vector field X:*	*Topology of the real vector field $\Re e(X)$:*
1	$\lambda z\frac{\partial}{\partial z}$ *in* $\mathbb{C}P^1$	*a source–sink in the sphere, or two centers in the sphere*
2	$z^2\frac{\partial}{\partial z}$ *in* $\mathbb{C}P^1$	*a dipole in the sphere*
3	$\frac{\partial}{\partial z}$ *in* $\mathbb{C}$	*parallel lines in the plane*
4	$\lambda z\frac{\partial}{\partial z}$ *in* $\mathbb{C}$	*a linear; center, source or sink in the plane*
5	$\lambda z\frac{\partial}{\partial z}$ *in* $\mathbb{C}^*$	*parallel geodesics in the cylinder*
6	$\lambda\frac{\partial}{\partial z}$ *in* $\mathbb{C}/\Lambda$	*parallel geodesics in the torus*

Here $\lambda \in \mathbb{C}^*$, and Λ is a rank two lattice.

Proof. Consider in $L - \{$ zeros of $X\}$ the flat metric g_X.

Applying the classical classification of geodesically complete flat surfaces with a frame (as in Liouville–Arnold theory), we know that $L - \{$ zeros of $X\}$ assume one of the following possibilities:
a plane,
a cylinder,
a torus.

If the original vector field X is free of zeros; for L a plane we have case (3), for L a cylinder we have case (5), and when L is a torus we get (6).

Now we assume X having at least one zero.

If $L - \{$ zeros of $X\}$ is a plane, we can compactify it by adding a point, and get a holomorphic vector field with a zero at the new point (apply Corollary 4.3) Moreover, L is the Riemann sphere $\mathbb{C}P^1$. By Poincaré–Hopf index theory the zero of the vector field has order 2, giving case (2).

Moreover, if $L - \{$ zeros of $X\}$ is a cylinder we can compactify with 1 or 2 points having two different possibilities for L and the corresponding holomorphic vector fields.

If we add one point then L is a plane $\mathbb{C}$ and we have case (4).

If we compactify with two points then L is the Riemann sphere $\mathbb{C}P^1$. By Poincaré–Hopf index theory the two zeros of the vector field have order 1, giving case (1).

□

6 Proof of the main results

Proof of Theorem 1.1.
Let $\mathcal{L} \subset M$ be a separatrix by $p \in M$ coming from the closure of a two–dimensional orbit of the $(\mathbb{R}^2, +)$–action induced by X_h, X_g, satisfying (H1) and (H2).

Basically the problem is that $\mathcal{L} \subset M$ is not necessarily a topological manifold. Remove from the separatrix its singular points $\mathcal{L} - \{p_j\}$ obtaining a two–dimensional manifold $\mathcal{L}^0$. Thinking in $\mathcal{L}^0$ as an abstract manifold, we can compactify the punctured cones $(\mathcal{L}_i - \{p_j\}) \subset \mathcal{L}^0$ by adding one point in each punctured cone (see Corollary 4.3). Obtaining a new two–dimensional manifold L (probably disconnected), having smooth commuting vector fields, moreover by results 4.1 and 4.2 we get a Riemann surface L with a holomorphic vector field X having as real and imaginary parts the original Hamiltonian vector fields X_h and X_g.

The key point is: completeness hypothesis for X_h on $\mathcal{L} \subset M$ is equivalent to completeness for X on L, by Lemma 5.1, hence the third column in Lemma 5.2 describes real complete Hamiltonian vector fields.

To show 1.1 part (1) we note that Poincaré–Hopf indices coming from complete holomorphic vector fields in Lemma 5.2 are 1 or 2, since index 3 or more in a cone of $\mathcal{L}$ implies incompleteness.

For 1.1 part (2), note that the existence of a singularity of Poincaré–Hopf index 2 corresponds to case (2) in Lemma 5.2.

For 1.1 part (3), the existence of a cone having Poincaré–Hopf index 1 corresponds to cases (1) or (4) in Lemma 5.2. Hence if $\mathcal{L}$ a two–dimensional manifold, then is homeomorphic to a plane or a sphere. In particular if $\mathcal{L}$ is compact but not a manifold, it has two cones by the same singular point p, we get that $\mathcal{L}$ is homeomorphic to the singular surface obtained from the sphere identifying two different points to one. □

Proof of Corollary 1.2.
In presence of zeros of X_h on $\mathcal{L}$ we are speaking of cases (1), (2) or (4) in Lemma 5.2. Where the sum of the Poincaré–Hopf indices is 1 or 2. □

6.1 Remark. On the analyticity assumption:
Note the results 1.1 and 1.2 can be stated for smooth Hamiltonian vector fields X_h, X_g on smooth symplectic manifolds (M, ω) satisfying (H1) and replacing (H2) as follows.

(H2') the singular foliation $\mathcal{F}$ associated to X_h, X_g has a *smooth separatrix* $\mathcal{L}$ by p, i.e.

$\mathcal{L}$ is the common zero locus of a finite collection of smooth functions, $\mathcal{L} = \cup_j \overline{\mathcal{L}_j^2}$ is the closure of two–dimensional orbits, containing zero–dimensional orbits but not one–dimensional orbits.
conclusion in Lemma 2.1 remains true for $\mathcal{L}$ as above.

7 Examples of the index computation

Our first example uses normal forms theory for four–dimensional integrable Hamiltonian systems at simple singular points, due to J. Moser, H. Rüssmann (for the analytic case), and L. H. Eliasson (for the smooth case).

Assume that X_h and X_g form an integrable Hamiltonian system in a four–dimensional manifold M. Let $p \in M$ be a simple singular point of the Hamiltonian vector field X_h. Locally we can identify p with $0 \in \mathbb{R}^4$ by a symplectic change of coordinates, where $\mathbb{R}^4 = \{(x_1, x_2, y_1, y_2)\}$ and $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$ describe the canonical symplectic four–dimensional manifold.

7.1 Theorem. *In a neighborhood of* $0 \in (\mathbb{R}^4, \omega)$ *a simple singular point of a real analytic integrable Hamiltonian vector field* X_h *with two degrees of freedom there exist real analytic (smooth) coordinates in which*

$$h = \lambda_1 \xi_1 + \lambda_2 \xi_2 + \widetilde{h}(\xi_1, \xi_2) \ , \ g = \mu_1 \xi_1 + \mu_2 \xi_2 + \widetilde{g}(\xi_1, \xi_2) \ ,$$

where the functions $\widetilde{h}$, $\widetilde{g}$ *are real analytic (smooth) and their initial terms are of second degree at* 0. *The quadratic functions* ξ_1 *and* ξ_2 *depend on the type of the point and have the form:*

1.- $\xi_1 = \frac{1}{2}(x_1^2 + y_1^2)$, $\xi_2 = \frac{1}{2}(x_2^2 + y_2^2)$ *(elliptic point).*

2.- $\xi_1 = \frac{1}{2}(x_1^2 + y_1^2)$, $\xi_2 = x_2 y_2$ *(saddle–center point).*

3.- $\xi_1 = x_1 y_1$, $\xi_2 = x_2 y_2$ *(saddle point).*

4.- $\xi_1 = x_1 y_1 + x_2 y_2$, $\xi_2 = x_1 y_2 - x_2 y_1$ *(saddle–focus point).*

Where $\lambda_1 \mu_2 - \lambda_2 \mu_1 \neq 0$.

Proof. See L. M. Lerman and Ya. L. Umanskiy expositions in [7] p. 515 and [8] p. 27. □

Our assertion is:

7.2 Corollary. *For saddle–focus points there are two smooth separatrices by*

$0 \in \mathbb{R}^4$, *and the Hamiltonian vector field* X_h *satisfies*

$$PH(\psi_i^* X_h, 0) = 1$$

on both separatrices.

Proof. Let us begin by sketch the computation in the simplest case. Consider the Hamiltonian functions:

$$h = \xi_1 = x_1 y_1 + x_2 y_2 \ , \quad g = \xi_2 = x_1 y_2 - x_2 y_1 \ ,$$

that is the particular case $\lambda_1 = \mu_2 = 1$, $\lambda_2 = \mu_1 = 0$ and $\widetilde{h} \equiv 0 \equiv \widetilde{g}$ in Theorem 7.1. The Hamiltonian vector fields are:

$$X_h = -x_1 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2} + y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2} \ ,$$

$$X_g = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} + y_2 \frac{\partial}{\partial y_1} - y_1 \frac{\partial}{\partial y_2} \ .$$

There are two separatrices by $0 \in \mathbb{R}^4$ given by

$$\mathcal{L}_1 = \{y_1 = 0 = y_2\} \ \text{ and } \ \mathcal{L}_2 = \{x_1 = 0 = x_2\} \ .$$

The Hamiltonian vector fields assume the expressions

$$\psi_1^* X_h = -x_1 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2} \quad , \quad \psi_1^* X_g = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2}$$

in $\mathcal{L}_1$, and

$$\psi_2^* X_h = y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2} \quad , \quad \psi_2^* X_h = y_2 \frac{\partial}{\partial y_1} - y_1 \frac{\partial}{\partial y_2}$$

in $\mathcal{L}_2$. They have Poincaré–Hopf index 1. In particular, each separatrix $\mathcal{L}_i - \{0\} \subset \mathbb{R}^4$ is isometric to a flat cylinder.

Note that $\psi_1^* X_h$ and $\psi_1^* X_g$ on $\mathcal{L}_1$, are the real and imaginary vector fields associated to the holomorphic vector field

$$X = -z \frac{\partial}{\partial z} \ ,$$

making $z = x_1 + \sqrt{-1} x_2$ see Example 3.2, and similarly for $\mathcal{L}_2$.

Assume $\lambda_1\mu_2 - \lambda_2\mu_1 \neq 0$ and the vanishing of the higher order terms $\widetilde{h}$, $\widetilde{g}$ for the next case in the proof. Use that the above separatrices persist, and note that $\psi_i^* X_h$, $\psi_i^* X_g$ are linear combinations of the linear Hamiltonian vector fields in the above case. Hence Poincaré–Hopf indices are also 1.

We leave the general case (when the higher order terms $\widetilde{h}$ and $\widetilde{g}$ are non zero), as an exercise for the interested reader.

□

7.3 Example. *An incomplete Hamiltonian vector field X_h having a singular separatrix $\mathcal{L}$ and*

$$PH(\psi^* X_h, 0) = 2 \,.$$

Let be the Hamiltonian functions

$$h = x_1^2 - x_2^2 - y_1^3 + 3y_1y_2^2 \quad \text{and} \quad g = -2x_1x_2 - 3y_1^2y_2 + y_2^3$$

in $(\mathbb{R}^4, \omega)$. The Hamiltonian vector field is

$$X_h = (3y_1^2 - 3y_2^2)\frac{\partial}{\partial x_1} - 6y_1y_2\frac{\partial}{\partial x_2} + 2x_1\frac{\partial}{\partial y_1} - 2x_2\frac{\partial}{\partial y_2} \,.$$

The vanishing of the Poisson bracket is an easy computation

$$\begin{aligned} 0 = \{h, g\} = & -(-6y_1y_2)(2x_1) + (-3y_1^2 + 3y_2^2)(-2x_2) \\ & -(-3y_1^2 + 3y_2^2)(-2x_2) + (6y_1y_2)(-2x_1) \,. \end{aligned}$$

There exists a separatrix by $0 \in \mathbb{R}^4$, given by

$$\mathcal{L} = \{h^{-1}(0)\} \cap \{g^{-1}(0)\} \,.$$

The verification of (H1) and (H2) for the singular point $0 \in \mathcal{L}$ is very simple.

Find an explicit resolution for a singular analytic set, usually is a very difficult (and unpleasant) task, fortunately for this separatrix the resolution is given by

$$\begin{aligned} \psi : \mathbb{R}^2 \to \quad & \mathcal{L} \subset \mathbb{R}^4 \\ (t, s) \mapsto \ & (-t^3 + 3ts^2, 3t^2s - s^3, t^2 - s^2, 2ts) \,. \end{aligned}$$

Note that ψ is an embedding of $\mathbb{R}^2 - \{0\}$ over $\mathcal{L} - \{0\}$.

Consider an auxiliary vector field $F = A(t,s)\frac{\partial}{\partial t} + B(t,s)\frac{\partial}{\partial s}$ in $\mathbb{R}^2$, satisfying $\psi_* F = X_h$ on $\mathcal{L}$, straightforward computation shows that

$$F \doteq \psi^* X_h = (-t^2 + s^2)\frac{\partial}{\partial t} - 2ts\frac{\partial}{\partial s} \,.$$

This vector field can be recognized as the real vector field $\Re e(X)$ associated to the holomorphic vector field

$$X = -z^2\frac{\partial}{\partial z} \,,$$

introducing coordinates $z = t + \sqrt{-1}s$. Hence $\psi^* X_h$ has Poincaré–Hopf index 2 at $0 \in \mathbb{R}^2$.

If we note that $\mathcal{L}$ is unbounded in $\mathbb{R}^4$, it follows using Theorem 1.1 part (2) that X_h is an incomplete Hamiltonian vector field on $\mathbb{R}^4$.

7.4 Examples. *Six–dimensional systems.*

From 7.2 higher dimensional examples are easy to see. Let $(\mathbb{R}^6, \omega)$ be the canonical sympléctic six–dimensional manifold, consider the Hamiltonian functions:

$$h = x_1y_1 + x_2y_2 + x_3\widetilde{h}(x_3, y_3) \,, \quad g = x_1y_2 - x_2y_1 + y_3\widetilde{g}(x_3, y_3) \,,$$

where $\{x_3\widetilde{h}(x_3, y_3), y_3\widetilde{g}(x_3, y_3)\} = 0$.

There are two analytic separatrices by $0 \in \mathbb{R}^6$ given by

$$\mathcal{L}_1 = \{y_1 = y_2 = x_3 = y_3 = 0\} \quad \text{and} \quad \mathcal{L}_2 = \{x_1 = x_2 = x_3 = y_3 = 0\} \,.$$

The hypothesis (H1) and (H2) hold on both separatrices and the Poincaré–Hopf index is 1 (see proof of 7.2).

A more interesting example is in the work of L. H. Eliasson [5] p. 33, let us follow him word for word. Consider the Lagrangian spinning top, having principal moments of inertia $I_1 = I_2 \neq I_3$, it is rotational invariant around the third principal axis of inertia and the gravitational field is invariant around the vertical. It can be described by a Hamiltonian system on $T^*SO(3)$, having two additional first integrals Q_3^S and Q_3^B from the rotational invariance. The vertical positions is a circle T in the configuration space. In local coordinates $(\mathbb{R}^6, \omega)$ the Hamiltonian functions are:

$$\begin{aligned} H(x_1, ..., y_3) = \frac{1}{2I_1}(y_1^2 + y_2^2) + (\frac{1}{2I_1}y_3^2 - \frac{m}{2})x_1^2 - \frac{m}{2}x_2^2 + \frac{1}{I_1}y_3x_1y_2 \\ + \frac{1}{2I_3}y_3^2 + O^3(x_1, x_2, y_1, y_2) \,, \end{aligned}$$

$$Q_3^S(x_1, ..., y_3) = x_1y_2 - x_2y_1 + \frac{1}{2}y_3(x_1^2 - x_2^2) + y_3 + O^3(x_1, x_2, y_1, y_2) \,,$$

$$Q_3^B(x_1, ..., y_3) = y_3 \,.$$

If we fix the value $Q_3^B(x_1,...,y_3) = y_3 = c_2$, then H and Q_3^S become functions only on the (x_1,x_2,y_1,y_2)–space. For the point $(x_1,x_2,x_3,y_1,y_2,y_3) = (0,0,c_1,0,0,c_2)$, a separatrix is given by the equations

$$\mathcal{L} = \{x_3 = c_1,\ y_3 = c_2,\ H^{-1}(c_3),\ (Q_3^S)^{-1}(c_4)\}\ ,$$

where c_3, c_4 are suitable constants. If $y_3^2 > 4mI_1$, then L. H. Eliasson asserts that under suitable symplectic coordinates the Hamiltonians are generated by

$$h = x_1y_1 + x_2y_2\ ,\quad g = x_1y_2 - x_2y_1\ .$$

From 7.2 the Poincaré–Hopf index on the separatrix is 1.

Acknowledgments

The author thanks Professors D. del Castillo, J. Seade and the referee by very useful comments. Partially supported by DGAPA–UNAM and Conacyt 28492-E.

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