Operator Theory:
Advances and Applications, Vol. 114

On the problem of deciding whether a holomorphic vector field is complete

Jorge L. López and Jesús Muciño-Raymundo

1. Introduction.

Let M be a complex manifold provided with a holomorphic vector field X. We say that X is *complete* if its flow $\Phi : \mathbb{C} \times M \to M$ is well defined for all complex values of the time, otherwise X is incomplete.

Complete holomorphic vector fields are interesting from several points of view:

* In differential geometry, complete holomorphic vector fields describe monoparametric groups of holomorphic automorphisms. In particular if M is compact they form the Lie algebra of the Lie group of holomorphic automorphisms of the manifold, [18] p. 77. Over non compact manifolds the situation is more complicated. For $\mathbb{C}^n$, $n \geq 2$, the group of complex automorphisms is infinite dimensional, see [2], [18] p. 77, however very few automorphisms are the time–1 map of complete holomorphic vector fields [4]. On the other hand, a bounded domain has a real Lie group of complex automorphisms, but not a non–identically zero complete holomorphic vector field, a result due to H. Cartan, [1], [18] p. 78, and Section 3.

* The iteration dynamics of complex automorphisms of M is easier to describe if the iterated functions are the time–1 map of complete holomorphic vector fields, [4], [9] p. 44.

* Algebraic or semi–algebraic $\mathbb{C}$ or $\mathbb{C}^*$–actions, in affine and projective complex manifolds, produce complete holomorphic vector fields. See [19] for a list of results and problems in this area, and [8], [34].

* In singular holomorphic foliation theory, foliations coming from complete holomorphic vector fields are rare. In fact, they are singular holomorphic foliations by complex curves having the simplest intrinsic types of leaves: planes $\mathbb{C}$, cylinders $\mathbb{C}^*$, or tori $\mathbb{C}/\Lambda$, [6], [10]. In particular foliations by complex curves having hyperbolic leaves, which is the generic case on projective manifolds [12], never support complete holomorphic vector fields.

1991 *Mathematics Subject Classification.* 34A20.
Key words and phrases. holomorphic vector fields, complete flows.
Partially suported by DGAPA-UNAM and CONACYT 28492-E.

* The problem of completeness (or incompleteness) for real analytic vector fields on open manifolds, is also very interesting. For example, in the n–body problem, it is related to the problem of finding non–collision singularities [29]. See also [15] for results about completeness of real vector fields and relations with the completeness problem for partial differential equations. Hence, we can see the solution of the more rigid problem of completeness for holomorphic vector fields, as a first step to understand the real analytic case.

It is elementary that holomorphic vector fields on compact manifolds are always complete. In this case the main problem is to determine whether a non-identically zero holomorphic vector field exists, see for example [1], [5], [11], [18], [22], [23].

Moreover, the construction and/or recognizing of complete holomorphic vector fields on open manifolds, remains as a very interesting problem [4], [10], [11], [19], [28], [34], even in the simplest non compact complex manifold $M = \mathbb{C}^n$, $n \geq 2$, or in the case of quasi–projective manifolds, where holomorphic vector fields always exist.

The inspiration for this paper is the recent works of J. C. Rebelo [28], F. Forsterneric [10], G. T. Buzzard and J. E. Fornæss [4].

Our aim is to survey some well known facts on complete holomorphic vector fields, giving some simple ideas on the following problem.

Given an open complex manifold M and a holomorphic vector field X, recognize in an effective way whether X is complete.

Obviously this is a very difficult task, because it is almost always impossible to compute the flow directly from X (that is, to solve explicitly the associated system of holomorphic differential equations). We split the original problem in the following more geometric subproblems:

i) To classify all complete holomorphic vector fields on Riemann surfaces.

ii) To determine where the original vector field X assumes the above one-dimensional models, in trajectories across the zeros of X.

iii) To study how the complex trajectories of X escape to infinity in M, and to compute the flow along these escapes (for this we only consider the case $M = \mathbb{C}^2$ and X polynomial).

Here is the outline of the paper.

In Section 2 we study flat structures induced by a meromorphic vector field in a Riemann surface, as our main tool for the problem. Recall that natural one to one correspondences exist between: meromorphic vector fields, meromorphic forms, and orientable meromorphic quadratic differentials. Using this idea we identify the flat metric coming from the $(\mathbb{C}, +)$–action of a meromorphic vector field with the metric of the associated quadratic differential. From the classical description of zeros and poles in quadratic differential theory, we describe normal forms for poles and zeros of meromorphic vector fields.

The classification of complete holomorphic vector fields on arbitrary Riemann surfaces is given in Section 3, solving subproblem (i). Six families of vector fields appear. We also remark that only zeros of order one or two appear.

For subproblem (ii), by a *separatrix* we understand a complex analytic curve $\mathcal{L} \subset M$ (probably with singularities) such that it is invariant under the local flow of X, having a discrete and non empty intersection with the zeros of X. Following J. C. Rebelo [28], we show that a complete vector field X with a separatrix by a zero $p \in M$, is such that the order of X at p is at most two. Basically, we follow the seminal idea of J. C. Rebelo; however, our proof uses the explicit classification of complete holomorphic vector fields on Riemann surfaces. Several simple facts follow easily, for example, if the manifold M is Stein and has a separatrix, the order of a complete vector field X at the corresponding zero is exactly one.

We show several simple examples of complete (or incomplete) vector fields in Section 5.

In Section 6, we consider basic properties of holomorphic automorphisms that are the time–1 map of some holomorphic vector field. As a first step, we show that the time–1 map of a complete holomorphic vector field, having zeros of order one and separatrices across these zeros, has cylinders $\mathbb{C}^* \subset M$ of periodic points. In particular, hyperbolic holomorphic automorphisms are never the time–1 map of complete vector fields.

In Section 7, we address subproblem (iii) for polynomial vector fields in $\mathbb{C}^2$. One basic fact is that a polynomial vector field extends to a rational vector field in the compactification given by the complex projective plane $\mathbb{C}P^2 = \mathbb{C}^2 \cup \mathbb{C}P^1_\infty$. We introduce simple elements from singular holomorphic foliation theory in $\mathbb{C}P^2$. Polynomial vector fields of degree at least two which are (almost everywhere) transverse to the line at infinity $\mathbb{C}P^1$ are incomplete. On the other hand, vector fields having the line at infinity as a leaf can be either complete or incomplete. The problem can be localized on:

i) The type of the flow at the singular points of the associated foliation in the line at infinity.

ii) The existence of non-trivial recurrences of some leaves around the line at infinty.

Here we only study (i). We generalize the concept of indetermination point for rational functions to rational vector fields in $\mathbb{C}P^2$. Then we study how the flow is of a polynomial vector field restricted to the separatrices by the singularities of its foliation in the line at infinity. Some results are:

A polynomial vector field in $\mathbb{C}^2$ having a separatrix $\mathcal{L}$ by some point p in the line at infinity, such that the vector field induces a regular point or a pole at p is incomplete in $\mathbb{C}^2$, see 7.6.

A polynomial vector field in $\mathbb{C}^2$ having a polynomial first integral, such that on every separatrix $\mathcal{L}$ by points $\{p\}$ in the line at infinity, induces zeros at $p \in \mathcal{L}$, is complete in $\mathbb{C}^2$, see 7.8.

A polynomial Hamiltonian vector field in $\mathbb{C}^2$, such that one of its leaves intersects the line at infinity in at least three points (set theoretically) is incomplete in $\mathbb{C}^2$, see 7.9.

For the convenience of the freshman reader, we include several simple examples, complete proofs and extensive bibliography.

2. Singular flat metrics from meromorphic vector fields.

2.1. Flat geometry at regular points.

To reduce the background on quadratic differential theory, we describe some simple ideas from it. Let $f = u + \sqrt{-1}v : \Omega \subset \mathbb{C} \to \mathbb{C}P^1$ be a meromorphic function in some domain Ω. We have the following associated objects:

* A meromorphic vector field

$$X = f(z)\frac{\partial}{\partial z}.$$

* A meromorphic differential form

$$\omega = \frac{dz}{f(z)} .$$

* An orientable meromorphic quadratic differential

$$\omega \otimes \omega = \frac{dz}{f(z)} \otimes \frac{dz}{f(z)} = \frac{dz^2}{f(z)^2} .$$

* A pair of smooth vector fields in $\Omega - \{\text{poles and zeros of } f\}$

$$\Re e(X) = (X + \overline{X}) = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} \quad , \quad \Im m(X) = J(X - \overline{X}) = -v\frac{\partial}{\partial x} + u\frac{\partial}{\partial y} ,$$

here $\overline{X}$ means the conjugate vector field and $J : T\mathbb{R}^2 \to T\mathbb{R}^2$ is the usual complex structure on $\mathbb{R}^2 \cong \mathbb{C}$.

* A smooth flat Riemannian metric in $\Omega - \{\text{poles and zeros of } f\}$ given by

$$g_f = \begin{pmatrix} \frac{1}{u^2+v^2} & 0 \\ 0 & \frac{1}{u^2+v^2} \end{pmatrix} .$$

Some features and relations between the above objects are as follow:

The vector field X has ω as time–form, namely $w(X) \equiv 1$, and for all smooth trajectories γ from z_0 to z_1 in $\Omega - \{\text{poles and zeros of } f\}$, the number

$$\int_\gamma \frac{dz}{f(z)}$$

is the complex time required to travel from z_0 to z_1 along γ under the field X.

For $z_0 \in \Omega - \{\text{poles and zeros of } f\}$ the holomorphic function

$$F(z) = \int_{z_0}^{z} \frac{dw}{f(w)} : B(z_0) \subset \Omega \to \mathbb{C} ,$$

for z in some disk $B(z_0)$ around z_0 free of poles and zeros, is called a local parameter for the quadratic differential $\omega \otimes \omega$. If $f(z_0) \neq 0, \infty$, then the differential F_* maps:

$$F_*(X) = \frac{\partial}{\partial z} \ , \ F_*(\Re e(X)) = \frac{\partial}{\partial x} \ , \ F_*(\Im m(X)) = \frac{\partial}{\partial y} .$$

The first equality says that F is a holomorphic flow box for X. The last two say that the real trajectories of $\Re e(X)$ are (by definition) the horizontal trajectories of the quadratic differential $\omega \otimes \omega$, and the real trajectories of $\Im m(X)$ are the vertical ones.

From the above it is easy to compute the commutator

$$[\Re e(X), \Im m(X)] = [u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\ ,\ -v\frac{\partial}{\partial x} + u\frac{\partial}{\partial y}] \equiv 0\ .$$

Since the Riemannian metric g_f has $\Re e(X)$, $\Im m(X)$ as orthonormal frame, it is well known that the curvature of g_f is identically zero, [31] p. 261. Moreover, every map

$$F(z) : B(z_0) \subsetneq (\Omega - \{\text{poles and zeros of } f\}, g_f) \to (\mathbb{C}, \delta)$$

is a local isometry, where δ is the usual flat metric.

Let us give some elementary examples of the above.

2.1 Example. Consider $c = a + \sqrt{-1}b \in \mathbb{C}^*$, and let $X = c\frac{\partial}{\partial z}$ be the complex vector field on $\Omega = \mathbb{C}$. The space $(\mathbb{C}, g_f)$ is isometric with the usual flat plane $\mathbb{R}^2$, where the isometry is given by the map $x + \sqrt{-1}y \mapsto \frac{1}{|c|}(x, y)$. The real associated vector field is

$$\Re e(c\frac{\partial}{\partial z}) = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\ ,$$

having as trajectories straight lines of slope $arg(c)$.

2.2 Example. For $f(z) = z$, let $\omega = dz/z$ be the meromorphic form in $\Omega = \mathbb{C}$. The metric space $(\mathbb{C} - \{0\}, g_f)$ is isometric to the cylinder $S^1_{2\pi} \times \mathbb{R}$, where the subindex 2π means the length of the closed geodesics. The associated real vector field is

$$\Re e(z\frac{\partial}{\partial z}) = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\ ,$$

having as trajectories straight lines through 0 in $\mathbb{C}$. The associated imaginary vector field is

$$\Im m(z\frac{\partial}{\partial z}) = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\ ,$$

having as trajectories closed circles around zero; they correspond to closed geodesics in the cylinder $S^1_{2\pi} \times \mathbb{R}$.

2.3 Example. For $f(z) = 1/z$, let $\omega = zdz$ on $\Omega = \mathbb{C}$. The metric space $(\mathbb{C} - \{0\}, g_f)$ is isometric to the glue of four copies of the half flat plane $\{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$. We glue the positive x–axis of one copy with the negative part of the x–axis in another, using isometries. The real vector field is

$$\Re e(\frac{1}{z}\frac{\partial}{\partial z}) = \frac{x}{x^2 + y^2}\frac{\partial}{\partial x} - \frac{y}{x^2 + y^2}\frac{\partial}{\partial y}$$

in $\mathbb{C}$, having as trajectories hyperbolas. Is easy to see that the real flow is incomplete along the trajectories in the x and y axes.

2.4 Example. Consider $f(z) = e^z$, and let $X = e^z\frac{\partial}{\partial z}$ be the complex vector field in $\Omega = \mathbb{C}$. The space $(\mathbb{C}, g_f)$ is obtained gluing an infinite number of copies of the usual half flat plane. Here we glue the positive x–axis of one copy with the negative part of the x–axis in another. The real associated vector field is

$$\Re e(e^z \frac{\partial}{\partial z}) = e^x(\cos y \frac{\partial}{\partial x} + \sin y \frac{\partial}{\partial y}) \ .$$

An elementary exercise shows that the associated real vector field is incomplete on the real trajectories $\{y = k\pi\}$, for $k \in \mathbb{Z}$.

2.2. Flat geometry at poles and zeros.

We start by studying the normal forms for meromorphic vector fields at poles and zeros.

2.5 Lemma. *Let $f(z)\frac{\partial}{\partial z}$ be a meromorphic vector field in a neighborhood $B(0) \subset \mathbb{C}$ of* 0*, up to holomorphic change of coordinates it is as follows:*
1.- If $f(z)\frac{\partial}{\partial z}$ has a zero of order one in 0*, then it is*

$$\lambda z \frac{\partial}{\partial z} \ ,$$

for $\lambda = f'(0)$.
2.- If $f(z)\frac{\partial}{\partial z}$ has a zero of order $s \geq 2$ in 0*, then it is*

$$\frac{z^s}{1 + \lambda z^{s-1}} \frac{\partial}{\partial z} \ ,$$

for λ the residue of the associated differential form at 0*.*
3.- If $f(z)\frac{\partial}{\partial z}$ has a pole of order $-k \leq -1$ in 0*, then it is*

$$\frac{1}{z^k} \frac{\partial}{\partial z} \ .$$

Proof. We use the obvious idea, consider a local holomorphic change of coordinates, compute how the vector field changes in the new coordinates. For cases (1) and (2), see [3]. Moreover, also for cases (1)–(3) we can consider the associated quadratic differential $dz^2/f(z)^2$, and apply the corresponding theory, see [32] p. 29–31. □

To describe in an explicit way the geometry of the metric g_f and the singular foliation by horizontal trajectories at poles and zeros, we need some preliminary definitions.

Consider the Riemann sphere $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$ provided with the natural flat metric on $\mathbb{C}$, and where ∞ is a "singular point" of this metric. Also introduce in $\mathbb{C}P^1$ the singular real foliation by trajectories of $\frac{\partial}{\partial x}$ in $\mathbb{C}$, this foliation is singular at ∞. Define a half sphere as the subset $\mathcal{H} = \{z \in \mathbb{C} \mid Im(z) \geq 0\} \cup \{\infty\} \subset \mathbb{C}P^1$.
A *flat hyperbolic sector* is an open neighborhood of $0 \in \mathcal{H}$ (which does not contains ∞).
A *flat elliptic sector* is an open neighborhood of $\infty \in \mathcal{H}$ (which does not contains 0).
Both types of sectors are Riemannian surfaces with boundary, and having a foliation by unitary geodesics.

2.6 Lemma. *Let $f(z)\frac{\partial}{\partial z}$ be a meromorphic vector field on a neighborhood $B(0) \subset \mathbb{C}$ of 0.*
1.- If $f(z)\frac{\partial}{\partial z}$ has a zero of order one at 0, then $(B(0) - \{0\}, g_f)$ is isometric to the end of an infinite euclidean cylinder $S^1_T \times (0, \infty)$. The real trajectories of the vector field assume one of the following models: center, source or sink.
2.- If $f(z)\frac{\partial}{\partial z}$ has a zero of order $s \geq 2$ at 0, then $(B(0) - \{0\}, g_f)$ is isometric to a suitable glue of $2s - 2$ flat elliptic sectors (see the proof for full details).
3.- If $f(z)\frac{\partial}{\partial z}$ has a pole of order $-k \leq -1$ at 0, then $(B(0) - \{0\}, g_f)$ is isometric to the glue of $2k + 2$ flat hyperbolic sectors (the metric has a point of cone angle $(2k + 2)\pi$).
4.- In each case the Poincaré–Hopf index for the real singular foliation is equal to the order of $f(z)$ at $0 \in \mathbb{C}$.

Proof. Cases (1) and (3) follow from the classical theory of quadratic differentials [32].

Let us explain (2) in more detail. When the order of the zero is $s \geq 2$, there are two isometric invariants of $(B(0) - \{0\}, g_f)$: the order $s \in \mathbb{N}$, and $\lambda \in \mathbb{C}$ the residue of $dz/f(z)$ at $0 \in \mathbb{C}$. We make the description of the metric by cut and paste methods.

Assume $\lambda = 0$, the glue of $2s - 2$ flat elliptic sectors produce the metric space $(B(0) - \{0\}, g_f)$.

Now assume $\lambda = a + \sqrt{-1}b \neq 0$ and $s = 2$. Consider the above global model $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$ with $\lambda = 0$. It is necessary to consider two bands

$$A = \{x+\sqrt{-1}y \in \mathbb{C} \mid a \geq x \geq 0,\ y \geq 0\}\,,\ B = \{x+\sqrt{-1}y \in \mathbb{C} \mid b \geq y \geq 0,\ x \geq 0\}\,.$$

Remove the bands from $\mathbb{C}P^1$. Now we glue the boundaries, using isometries:
In A, glue x to $x + \sqrt{-1}b$ for $x \geq a$.
In B, glue $\sqrt{-1}y$ to $a + \sqrt{-1}y$ for $y \geq b$.
Then an open neighborhood of the point coming from $\infty \in \mathbb{C}P^1$, in the new flat surface, is the local model for $[z^2/(1 + \lambda z)]\frac{\partial}{\partial z}$ having $s = 2$ and $\lambda = a + \sqrt{-1}b$.

The case $\lambda = a + \sqrt{-1}b \neq 0$ and $s \geq 3$ is now easy, following the same ideas. □

2.3. Global description.

The following is well known for the specialist:

2.7 Lemma. *Let $\mathcal{L}$ be a Riemann surface. One to one correspondences exist between: meromorphic vector fields, meromorphic forms and orientable meromorphic quadratic differentials, given locally as:*

$$f(z)\frac{\partial}{\partial z} \quad \leftrightarrow \quad \frac{dz}{f(z)} \quad \leftrightarrow \quad \frac{dz^2}{f(z)^2}\,.$$

Proof. It is an easy computation with local charts. Let $\{z\}$ and $\{w\}$ be local holomorphic charts for $\mathcal{L}$, with transition function $w = T(z)$. Two local meromorphic vector fields $f(z)\frac{\partial}{\partial z}$ and $g(w)\frac{\partial}{\partial w}$ define the same meromorphic vector field in $\mathcal{L}$ iff

$$T'(z)f(z) = g(T(z)) = g(w) \ .$$

The associated one forms $dz/f(z)$ and $dw/g(w)$ define a differential form in $\mathcal{L}$ iff

$$(T^{-1})'(w)\frac{1}{f(z)} = \frac{1}{g(T(z))} = \frac{1}{g(w)} \ .$$

This is equivalent with the above equality by the inverse function theorem. □

As elementary application, since every Riemann surface admits meromorphic functions then it also admits meromorphic one forms and meromorphic vector fields. It is easy to see that the above correspondence (between vector fields and one forms) is well defined on manifolds modeled on a field. For its applications to real one–dimensional manifolds see [17].

Another explicit description of the associated flat structure is:

2.8 Corollary. *Let $\mathcal{L}$ be a Riemann surface provided with a meromorphic vector field X, or equivalently with a meromorphic form ω. There exists in $\mathcal{L}$ −* {poles and zeros of X} *a flat holomorphic atlas $\{(B(p_i), F_i)\}$ with coordinate functions given by*

$$F_i(p) = \int_{p_i}^{p} \omega : B(p_i) \subset \mathcal{L} \to \mathbb{C} \ ,$$

having as transition functions euclidean translations $F_i \circ F_j^{-1} : z \mapsto \ z + c_{ij}$. □

As usual the zeros and poles of global meromorphic objects obey some rules:

2.9 Corollary. *Let $\mathcal{L}$ be a compact connected Riemann surface of genus* g*. For a meromorphic vector field X on $\mathcal{L}$:*

$$\text{zeros}(X) + \text{poles}(X) = 2 - 2\text{g} = c_1(T\mathcal{L}) \ .$$

For ω a meromorphic form on $\mathcal{L}$:

$$\text{zeros}(\omega) + \text{poles}(\omega) = 2\text{g} - 2 = c_1(\mathcal{K}_{\mathcal{L}}) \ .$$

Here zeros and poles are counted with multiplicities, $T\mathcal{L}$, $\mathcal{K}_{\mathcal{L}}$ are the tangent and cotangent holomorphic line bundles respectively, and c_1 represents the Chern class, see [16] *p. 139.*

Proof. There are several ways, depending on the reader's background. For example, compute the Poincaré–Hopf index for the real singular foliation given by the real trajectories of $\Re e(X)$. □

In consequence, for compact Riemann surfaces only the Riemann sphere and the tori admit holomorphic vector fields.

3. Complete holomorphic vector fields on Riemann surfaces.

Now we are ready to classify complete holomorphic vector fields on arbitrary Riemann surfaces.

3.1 Lemma. *Let $\mathcal{L}$ be a connected Riemann surface, and X a non–identically zero complete holomorphic vector field in $\mathcal{L}$. Then, up to biholomorphism, X and $\mathcal{L}$ are as follow:*

1.- $\lambda z\frac{\partial}{\partial z}$ *in* $\mathbb{C}P^1$.
2.- $z^2\frac{\partial}{\partial z}$ *in* $\mathbb{C}P^1$.
3.- $\frac{\partial}{\partial z}$ *in* $\mathbb{C}$.
4.- $\lambda z\frac{\partial}{\partial z}$ *in* $\mathbb{C}$.
5.- $\lambda z\frac{\partial}{\partial z}$ *in* $\mathbb{C}^*$.
6.- $\lambda\frac{\partial}{\partial z}$ *in* $\mathbb{C}/\Lambda$.
Here $\lambda \in \mathbb{C}^*$, *and* Λ *is a rank two lattice.*

Proof. Start by noting that if $\mathcal{L}$ is covered by the Poincaré disk Δ, then it does not have a non–identically zero complete holomorphic vector field. By contradiction, assume that a such X exists. Let $\Phi(t, p_0) : \mathbb{C} \to \mathcal{L}$ be the complex flow, for $p_0 \in \mathcal{L}$ such that $X(p_0) \neq 0$. By analytic continuation we can lift Φ to a non constant holomorphic function $\widetilde{\Phi}(t, p_0) : \mathbb{C} \to \Delta$, which is a contradiction.

It follows that $\mathcal{L}$ supporting a nontrivial complete holomorphic vector field has as universal cover $\mathbb{C}$ or $\mathbb{C}P^1$.

If $\mathcal{L}$ is compact we have two possibilities. First $\mathcal{L} = \mathbb{C}/\Lambda$ is a torus, where the holomorphic vector fields are the constant (the existence of a zero will imply the existence of poles, by Corollary 2.9), we have the case (6). Second $\mathcal{L} = \mathbb{C}P^1$, by Corollary 2.9 X has two simple zeros giving case (1), or one double zero giving case (2).

In fact, if two simple zeros appear, we can move them by a biholomorphism of $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$ to the points 0 and ∞. In the affine chart $\mathbb{C}$ containing 0, the resulting vector field assumes the explict form in (1). If one double zero appears, moving it to the point $0 \in \mathbb{C}P^1$, the expression (2) follows.

If $\mathcal{L}$ is noncompact then it is a copy of $\mathbb{C}$ or the cylinder $\mathbb{C}^*$. Applying suitable biholomorphisms as above, we get normal forms (3), (4) and (5). □

To be more familiar with the above complete vector fields, we recognize them as the Lie algebra of complex Lie groups of automorphisms on Riemann surfaces:

Riemann surface:	Lie algebra of complete holomorphic vector fields:	Dimension of the Lie algebra:	Lie group of holomorphic automorphisms:
$\mathbb{C}$	$(\lambda z + \mu)\frac{\partial}{\partial z}$	2	$\text{Affine}(\mathbb{C}) = \{z \mapsto az + b\}$
$\mathbb{C}^*$	$\lambda z\frac{\partial}{\partial z}$	1	$\mathbb{C}^* = \{z \mapsto az\}$
$\mathbb{C}P^1$	$\lambda(z-\mu)(z-\eta)\frac{\partial}{\partial z}$	3	$PSL(2,\mathbb{C}) = \{z \mapsto \frac{az+b}{cz+d}\}$
$\mathbb{C}/\Lambda$	$\lambda\frac{\partial}{\partial z}$	1	$\mathbb{C}/\Lambda = \{z \mapsto z + a\}$

where $\lambda, \mu, \eta, a, b, c, d \in \mathbb{C}$, $a \neq 0$ in the first and second line, and $ad - bc = 1$ in the third line.

In the negative sense we have:

3.2 Corollary. *Let X be a holomorphic vector field on a Riemann surface $\mathcal{L}$. If X has a zero of order greater than or equal to three, then the vector field is incomplete.* □

The usual theory of differential equations can not be applied to vector fields having poles or essential singularities at its singular points. However the next result will be very useful in the last section.

3.3 Corollary. *Let $\mathcal{L}$ be a Riemann surface, and $p \in \mathcal{L}$. If X is a holomorphic vector field on $\mathcal{L} - \{p\}$, which extends to p having a pole or an essential singularity, then the vector field on $\mathcal{L} - \{p\}$ is incomplete.*

Proof. By contradiction assume that X is complete over $\mathcal{L} - \{p\}$. Since p is a conformal puncture, X near p looks like the conformal punctures of the Riemann surfaces in Lemma 3.1, cases (3), (4) or (5). But in any case the local model in the punctured neighborhood of p implies that X has a zero of order 1 or 2 at p, which is a contradiction.

For the case of poles at p, a direct proof follows from the local model of a pole having order $-k$ (the glue of $2k+2$ flat hyperbolic sectors). This singular flat model is not geodesically complete. See also examples 2.3 and 2.4. □

For more examples of meromorphic vector fields and the dynamical study of their $(\mathbb{R}, +)$ and $(\mathbb{C}, +)$–actions, see [24], [25].

On the other hand we have:

3.4 Example. *Two families of complex manifolds without complete holomorphic vector fields.* It is well known that a compact complex manifold M is Kobayashi hyperbolic if and only if every holomorphic map from $\mathbb{C}$ to M is constant, see [21] p. 166. Hence in Kobayashi hyperbolic or moreover in Brody hyperbolic manifolds, the unique complete holomorphic vector field is the identically zero vector field, see [21] for explicit examples. Another large family of complex manifolds without non–identically zero complete holomorphic vector fields are the manifolds covered by bounded domains in $\mathbb{C}^n$. Use the idea in the proof of Lemma 3.1.

4. Separatrices of complete holomorphic vector fields.

Following an idea of J. C. Rebelo [28], in order to get the existence of zeros of order ≥ 3 as an obstruction to the completeness for vector fields in higher dimensional complex manifolds, it is necessary compute the flow at the separatrices, i.e., complex trajectories across the zeros of the vector field. The main problems are: the existence of separatrices, and that separatrices can be singular.

Let M be a complex manifold, provided with a holomorphic vector field X. A complex analytic curve $\mathcal{L} \subset M$ is a *separatrix* of the vector field iff:

i) $\mathcal{L}$ has only one irreducible component.

ii) $\mathcal{L}$ is tangent to the vector field X.
iii) $\mathcal{L} \cap \text{zeros}(X)$ is a discrete and nonempty set.

Since any complex analytic curve admits a finite number of irreducible components, see [20] II.5, we assume (i) only for simplicity. Condition (ii) says that $\mathcal{L}$ is union of complex orbits or equivalently invariant under the local complex flow of X.

The problem of deciding whether a holomorphic vector field has a separatrix is famous and very difficult. It was first proposed by Briot and Bouquet in 1854. In 1982, C. Camacho and P. Sad proved that a germ of a two dimensional holomorphic vector field with an isolated zero always admits a separatrix [7]. X. Gómez–Mont and I. Luengo showed in [14] examples of germs of holomorphic vector fields with isolated zeros in $\mathbb{C}^3$ without separatrix. Moreover, J. Olivares–Vásquez has constructed additional examples without separatrix in $\mathbb{C}^{3n}$ [26], [27].

In what follows of this section we assume that X has at least one separatrix $\mathcal{L}$. See the work of J. C. Rebelo [28] for further results in the absence of separatrices. As first step we want to define the order of the zero of a holomorphic vector field at a separatrix.

Consider two cases:

Let $p \in M$ be a zero of X. If $\mathcal{L}$ is smooth by p, then X restricts to $\mathcal{L}$ given a holomorphic vector field $X|_{\mathcal{L}}$. In particular the order of the zero for $X|_{\mathcal{L}}$ is well defined. Denote it by $\text{order}(X|_{\mathcal{L}}, p)$.

Assume $\mathcal{L}$ is non-smooth for some $p \in \mathcal{L} \cap \text{zeros}(X)$. It is well known that given a germ $(\mathcal{L}, p)$ of a singular complex analytic curve in M at p as above, we can always resolve p, see [33]. That is, there exists a non singular complex analytic curve $(\widetilde{\mathcal{L}}, 0)$ called the strict transform and a holomorphic map $\alpha : (\widetilde{\mathcal{L}}, 0) \to (\mathcal{L}, p)$ which is a local biholomorphic map from some punctured neighborhood of $\widetilde{\mathcal{L}} - \{0\}$ to $\mathcal{L} - \{p\}$, for each point $0 \in \alpha^{-1}(p)$. Hence, the vector field X on $\mathcal{L} \subset M$ can be lifted to a unique holomorphic vector field $\alpha^*(X) = \widetilde{X}$ on $\widetilde{\mathcal{L}}$. However resolutions are not at all unique.

4.1 Remark. *The positive number* $\text{order}(X|_{\mathcal{L}}, p)$ *is independent of the resolution.* Note that $\alpha^{-1}(p)$ is a finite set, we fix some point $0 \in \alpha^{-1}(p)$, and define the order for this point. By abuse of notation we do not make explicit this in the notation. Now the order of the zero of $\widetilde{X}$ at some $0 \in \{\alpha^{-1}(p)\}$ is equal to the Poincaré–Hopf index of the associated real vector field, by Lemma 2.6.4. Every resolution α is a local biholomorphism from some punctured neighborhood of $\widetilde{\mathcal{L}} - \{0\}$ to $\mathcal{L} - \{p\}$. The Poincaré–Hopf index is independent of the resolution, and hence the order is also independent. Also note that $\alpha^*(X)$ is holomorphic and has a zero at 0 (using for example the removable singularity Theorem). In particular the order is positive.

4.2 Definition. Let $\mathcal{L}$ be a separatrix for the vector field X at $p \in M$. The *order of X restricted to $\mathcal{L}$ at p* is the order of the vector field $\alpha^*(X)$ at 0, for some resolution $\alpha : (\widetilde{\mathcal{L}}, 0) \to (\mathcal{L}, p)$. By simplicity, in the singular or smooth case we denote the order by:

$$\text{order}(X|_{\mathcal{L}}, p) \in \mathbb{N} .$$

Recall that for X any non–identically zero holomorphic vector field in M, given $p \in \text{zeros}(X)$, the *order of* X *at* p is the lowest degree of the monomials that appear in the power series expansion of X around p. We denote this number as $\text{order}(X,p) \in \mathbb{N}$. It is a very simple invariant of vector fields under holomorphic change of coordinates.

4.3 Proposition. *Let M be a complex manifold, X a non–identically zero holomorphic vector field, and $p \in M$ a zero of X. Assume that $\mathcal{L} \subset M$ is a separatrix of X by p, and $\alpha : (\widetilde{\mathcal{L}}, 0) \to (\mathcal{L}, p)$ is a local resolution of $\mathcal{L}$ at p.*
1.- Then

$$\text{order}(X|_{\mathcal{L}}, p) \geq \text{order}(\alpha, 0) \cdot \text{order}(X, p) - \text{order}(\alpha, 0) + 1,$$

where $\text{order}(\alpha, 0)$ *is the order of the resolution at* p.
2.- In particular, if $\mathcal{L}$ is smooth at p, then

$$\text{order}(X|_{\mathcal{L}}, p) \geq \text{order}(X, p) .$$

Proof. For convenience of the freshman reader we give an elementary proof for (2). Assume without loss of generality that there exists a local holomorphic coordinate in M such that p corresponds to $0 \in \mathbb{C}^n$ and the separatrix trajectory $\mathcal{L}$ is the z_1–axis in $(\mathbb{C}^n, 0)$. Suppose that X looks like

$$X(z_1, ..., z_n) = X_1(z_1, ..., z_n)\frac{\partial}{\partial z_1} + ... + X_n(z_1, ..., z_n)\frac{\partial}{\partial z_n} ,$$

in $(\mathbb{C}^n, 0)$, for holomorphic functions $X_j : (\mathbb{C}^n, 0) \to \mathbb{C}$. The restriction of the vector field to the separatrix $\mathcal{L}$ is:

$$X|_{\mathcal{L}}(z_1) = X_1(z_1, 0, ..., 0)\frac{\partial}{\partial z_1} .$$

Since $\mathcal{L}$ intersects the zero set of X in a discrete set, the above vector field has an isolated zero at $z_1 = 0$. Note that the power series of $X_1(z_1, 0, ..., 0)$ contains at least one monomial of type $a_i z_1^i$, for $i \in \mathbb{N}$ and $a_i \in \mathbb{C}^*$. Hence $\text{order}(X|_{\mathcal{L}}, 0)$, that is by definition the lowest power of the series $\Sigma a_j z_1^j$ of $X_1(z_1, 0, ..., 0)$, is greater or equal than the original $\text{order}(X(z_1, ..., z_n), 0)$. This finishes part (2).

For the singular case (1), introduce local coordinates $(\mathbb{C}^n, 0)$ for M at p. Let α be a local representative of the resolution of $\mathcal{L}$ at the singular point, this means a germ of a holomorphic parametrization

$$\begin{array}{rcl} \alpha : (\mathbb{C}, 0) & \to & \mathcal{L} \subset (\mathbb{C}^n, 0) \\ t & \mapsto & (z_1(t), z_2(t), ..., z_n(t)) \end{array}$$

such that
i) $\alpha(0) = 0$ and $\frac{d\alpha}{dt}(0) = 0$,
ii) $\alpha(t)$ is a local biholomorphic mapping from a punctured neighborhood in $(\mathbb{C}, 0)$ to $\mathcal{L} - \{p\}$.

Define the order$(\alpha, 0)$ as the lowest number in $\{\text{order}(z_i(t), 0)\}$. Since $\mathcal{L}$ is singular at 0, it follows that order$(\alpha, 0) \geq 2$.

Consider an auxiliary holomorphic vector field $f(t)\frac{\partial}{\partial t}$ in $(\mathbb{C}, 0)$ which satisfies

$$\alpha_* \left(f(t)\frac{\partial}{\partial t} \right) = X(\alpha(t)) \ .$$

Hence by definition order$(X|_{\mathcal{L}}, p) = $ order$(f(t), 0)$. The above vectorial equation is equivalent with the system of equations

$$\frac{dz_i(t)}{dt} f(t) = X_i(z_1(t), z_2(t), ..., z_n(t)) \quad , \text{ for } \ i = 1, 2, ..., n \ .$$

Since X and $\frac{d\alpha}{dt}$ are $\mathbb{C}$–linearly dependent along $\mathcal{L}$, a holomorphic function $f(t)$ as above exists.

Assume that for some index i, the function $z_i(t)$ realizes the order of α at 0, and moreover X_i is non–identically zero. In fact, if $X_i(z_1(t), ..., z_n(t)) \equiv 0$, where $t \in (\mathbb{C}, 0)$, then $dz_i(t)/dt \equiv 0$. This is impossible, since the order of $z_i(t)$ is finite and positive.

We have that

$$\text{order}(\frac{dz_i(t)}{dt}, 0) + \text{order}(f(t), 0) = \text{order}(X_i(z_1(t), z_2(t), ..., z_n(t)), 0) \ ,$$

which implies

$$\begin{aligned} \text{order}(f(t), 0) &= \text{order}(X_i(z_1(t), z_2(t), ..., z_n(t)), 0) - \text{order}(\tfrac{dz_i(t)}{dt}, 0) \\ &\geq \text{order}(\alpha(t), 0) \cdot \text{order}(X, 0) - \text{order}(\tfrac{d\alpha(t)}{dt}, 0) \ . \end{aligned}$$

This implies

$$\text{order}(X|_{\mathcal{L}}, p) \geq \text{order}(\alpha, 0) \cdot \text{order}(X, p) - \text{order}(\frac{d\alpha(t)}{dt}, 0) \ ,$$

and the result follows. □

4.4 Example. Consider in $\mathbb{C}^2$, the polynomial vector field

$$X(z_1, z_2) = z_1^n \frac{\partial}{\partial z_1} + z_2^m \frac{\partial}{\partial z_2} \ ,$$

where n, m are natural numbers. At $(0,0) \in \mathbb{C}^2$, the axes $\{z_1 = 0\}$ and $\{z_2 = 0\}$ are smooth separatrices for the field X. If $m > n$, then exist separatrices $\mathcal{L}$, $\mathcal{J}$ such that order$(X|_{\mathcal{L}}, 0) = $ order$(X, 0) = n$ and $m = $ order$(X|_{\mathcal{J}}, 0) > $ order$(X, 0) = n$, hence part (2) in the Proposition is the best possible.

4.5 Example. Consider the holomorphic function $H : \mathbb{C}^2 \to \mathbb{C}$ given by $H(z_1, z_2) = z_1^n - z_2^m$, where $(n, m) = 1$. Its Hamiltonian vector field is

$$X(z_1, z_2) = mz_2^{m-1} \frac{\partial}{\partial z_1} + nz_1^{n-1} \frac{\partial}{\partial z_2} \ .$$

The origin is a zero of X having as singular separatrix $\mathcal{L} = \{z_1^n - z_2^m = 0\}$. The usual resolution is given by the Puiseux parametrization

$$\begin{array}{rcl} \alpha : (\mathbb{C}, 0) & \to & \mathcal{L} \subset (\mathbb{C}^2, 0), \\ t & \mapsto & (t^m, t^n). \end{array}$$

Since

$$\begin{aligned} \alpha_* f(t)\frac{\partial}{\partial t} &= mt^{m-1} f(t)\frac{\partial}{\partial z_1} + nt^{n-1} f(t)\frac{\partial}{\partial z_2} \\ &= mt^{(m-1)n}\frac{\partial}{\partial z_1} + nt^{(n-1)m}\frac{\partial}{\partial z_2} = X(\alpha(t)) \ , \end{aligned}$$

it follows that

$$\alpha^* X = t^{(n-1)(m-1)}\frac{\partial}{\partial t} \qquad \text{in } (\mathbb{C}, 0) \ .$$

4.6 Theorem. *Let M be a complex manifold, X a complete non–identically zero holomorphic vector field, and $p \in M$ a zero of X. Assume that $\mathcal{L} \subset M$ is a separatrix of X by p. Then*
1.- $\mathrm{order}(X|_{\mathcal{L}}, p)$ *is 1 or 2.*
2.- $\mathrm{order}(X, p)$ *is 1 or 2.*
3.- If $\mathrm{order}(X|_{\mathcal{L}}, p) = 1$*, then the separatrix trajectory contains an embedded copy of $\mathbb{C}^* \subset M$.*
4.- If $\mathrm{order}(X|_{\mathcal{L}}, p) = 2$*, then the separatrix trajectory contains an embedded copy of $\mathbb{C} \subset M$.*

Proof. Since X is complete, it follows that its flow must be complete restricted to the resolution of the separatrix $\widetilde{\mathcal{L}}$. This Riemann surface belongs to the list in Section 3, so (1) follows.

For (2), using the same notation as in 4.3, completeness hypothesis gives

$$2 \geq \mathrm{order}(X|_{\mathcal{L}}, p) \geq \mathrm{order}(\alpha, 0) \cdot \mathrm{order}(X, p) - \mathrm{order}(\alpha, 0) + 1 \geq 1,$$

which implies

$$1 \geq \mathrm{order}(\alpha, 0)(\mathrm{order}(X, p) - 1) \geq 0,$$

so the assertion (2) follows. Also by Lemma 3.1, (3) and (4) follow by simple inspection. Note that the copies of $\mathbb{C}$, $\mathbb{C}^*$ are embedded, since X is nonzero there. □

Note that Examples 4.4 and 4.5 are complete if and only if $n = 1 = m$.

5. Some examples.

5.1 Example. *Linear vector fields.* Let $A = (a_{ij})$ be a linear function in $\mathbb{C}^{n+1}$, having $n+1$ different eigenvectors. Consider the associated complete holomorphic vector field in $\mathbb{C}^{n+1}$, given by

$$X(z_0, ..., z_n) = (\sum_i a_{i1} z_i)\frac{\partial}{\partial z_0} + ... + (\sum_i a_{in+1} z_i)\frac{\partial}{\partial z_n} \ .$$

From each eigenvector we have an smooth separatrix across the zero at $0 \in \mathbb{C}^{n+1}$. Under the usual projection $\pi : \mathbb{C}^{n+1} - \{0\} \to \mathbb{C}P^n$ the vector field X defines a holomorphic vector field in complex projective space. Each eigenvector produces a zero, assume that they are $\{[0, ..., 1, ..., 0]\}$. The projective lines $\mathbb{C}P^1 \subset \mathbb{C}P^n$ given by the 2–planes "$z_i z_j$" in $\mathbb{C}^{n+1}$ are smooth separatrices, as in case (1) of the Lemma 3.1. See [6] for a dynamical description in $\mathbb{C}^{n+1}$ and $\mathbb{C}P^n$.

5.2 Example. *Vector fields from overshears.* In $\mathbb{C}^n$ for $n \geq 2$, let

$$X(z_1, ..., z_n) = (f(z_2, ..., z_n)z_1 + g(z_2, ..., z_n))\frac{\partial}{\partial z_1}$$

be a holomorphic vector field, where $f,\ g : \mathbb{C}^{n-1} \to \mathbb{C}$ are entire functions. The vector field X has zeros at the hypersurface $\{f(z_2, ..., z_n)z_1 + g(z_2, ..., z_n) = 0\}$. The lines $\{z_2 = c_2,\ ...,\ z_n = c_n\}$ intersecting the hypersurface are smooth separatrices for the vector field. By simple integration, the flow is given by overshears, see [2], in $\mathbb{C}^n$ of the form

$$\Phi(t, (z_1, \ldots, z_n))$$
$$= \left(\frac{[f(z_2, \ldots, z_n)z_1 + g(z_2, \ldots, z_n)]e^{f(z_2,\ldots,z_n)t} - g(z_2, \ldots, z_n)}{f(z_2, \ldots, z_n)}, z_2, \ldots, z_n \right),$$

when $f(z_2, ..., z_n) \neq 0$. Also by Lemma 3.1, it follows that X is complete.

5.3 Example. *Product manifolds.* Let M be any complex manifold and $\mathcal{L}$ a Riemann surface as in Lemma 3.1. The product $M \times \mathcal{L}$ has a complete holomorphic vector field. If $\mathcal{L} = \mathbb{C}$ this family of examples are known as cylinder–like manifolds, see [8].

5.4 Corollary. *Let M be a complex manifold, X a complete holomorphic vector field (non–identically zero). Assume that $\mathcal{L} \subset M$ is a separatrix of X. The intersection $\mathcal{L} \cap \{zeros\ of\ X\}$ has one or two points.*

Proof. Apply Lemma 3.1 to the resolution of the separatrix. □

In manifolds without rational curves we can say a little bit more:

5.5 Corollary. *Let M be a complex manifold, assume that every holomorphic map from $\mathbb{C}P^1$ to M is constant, for example for $M = \mathbb{C}^n$ or M a Stein manifold. Let X be a complete non–identically zero holomorphic vector field and $\mathcal{L} \subset M$ a separatrix of X. Then*

1.- $\text{order}(X, p) = 1$, *where p is a zero of X.*

2.- The intersection $\mathcal{L} \cap \{$zeros of $X\}$ is one point. □

For example, holomorphic vector fields in $\mathbb{C}^n$ having a complex saddle conection (i.e., a complex analytic curve having at least two zeros in its closure) are incomplete.

As simple applications we have:

5.6 Example. *The sum of two complete vector fields does not need to be complete.* Consider two complete holomorphic vector fields in $\mathbb{C}^2$:

$$z_1 z_2 \frac{\partial}{\partial z_1} \quad , \quad z_2 z_1 \frac{\partial}{\partial z_2} \, .$$

The sum

$$X(z_1, z_2) = z_1 z_2 (\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2}) \, ,$$

has as zero set $\{z_1 z_2 = 0\}$, and the lines $\{z_1 - z_2 = c\}$ are separatrices. By 5.5 part (2), it follows that X is incomplete.

There are very interesting examples of complete holomorphic vector fields without zeros in $\mathbb{C}^n$ but non holomorphically equivalent to the trivial $\frac{\partial}{\partial z_1}$, see [34].

Obviously, holomorphic vector fields having trajectories or separatrices with fundamental group different from $\{e\}$, $\mathbb{Z}$, or $\mathbb{Z} \oplus \mathbb{Z}$ are incomplete. For example:

5.7 Corollary. *Let (M, ω) be a complex holomorphic symplectic surface, $H : M \to \mathbb{C}$ a nonconstant holomorphic function. If some level set curve $\{H^{-1}(c)\} \subset M$ has fundamental group nonisomorphic with $\{e\}$, $\mathbb{Z}$, or $\mathbb{Z} \oplus \mathbb{Z}$, then the complex Hamiltonian vector field X_H is incomplete.* □

A consequence of the classical genus formula for algebraic curves in $\mathbb{C}P^2$, see [16] p. 220, is that generically polynomial functions H in $\mathbb{C}^2$, of degree at least four, produce incomplete Hamiltonian vector fields. See also [10] Section 7, and our Proposition 7.9.

6. Periodic points of time–1 maps.

The inspiration for this Section is the work of G. T. Buzzard and J. E. Fornæss [4]. In fact, from the explicit knowledge of the complex trajectories of complete vector fields in Section 3 we can study periodic points for time-1 maps.

Let us recall that $p \in M$ is of *minimal period* $n \geq 2$ for some holomorphic automorphisms $\Phi : M \to M$, if $\Phi^{(n)}(p) = p$, but $\Phi^{(m)}(p) \neq p$ for $m = 1, 2, \ldots, n-1$.

6.1 Proposition. *Let M be a complex manifold, X a non–identically zero complete holomorphic vector field, Φ its flow and $\Phi_1 : M \to M$ its time–1 map. Let $p \in M$ be a periodic point of Φ_1 having minimal period $n \geq 2$.*
1.- If the Φ–orbit of p is non compact, then it is an embedded cylinder $\mathbb{C}^* \subset M$ of periodic points.
2.- If the Φ–orbit of p is compact, then it is an embedded torus $\mathbb{C}/\Lambda \subset M$ of periodic points.

Proof. Consider the real vector field $\Re e(X)$, as in Section 2. The trajectory of p under $\Re e(X)$ is a circle $S^1 \subset M$. Every point on this circle has the same period as p under Φ_1. Recall that the above circle is a closed geodesic in $(\mathcal{L}, g_X)$ for the flat metric g_X as in Section 2. Use the classification in Lemma 3.1 to show that S^1 is in $\mathcal{L}$ biholomorphic to $\mathbb{C}^*$ or $\mathbb{C}/\Lambda$. □

As one consequence of the existence of cylinders of periodic points we have the following:

6.2 Corollary. *Let $\Phi_1 : M \to M$ be the time–1 map of X a complete non–identically zero holomorphic vector field, with a periodic point $p \in M$ of minimal period $n \geq 2$. Then the differential $D\Phi_n(p)$ has 1 as eigenvalue.*

Proof. Since p is not a fixed point, $X(p) \neq 0$ and the complex trajectory of p under Φ is an embedded copy of $\mathbb{C}^*$ or $\mathbb{C}/\Lambda \subset M$. The time–$n$ map has p as fixed point and is the identity map restricted to the complex Φ trajectory. Hence the eigenvalue corresponding to eigenvector tangent to the Φ complex trajectory is 1. □

Separatrix trajectories for X also give origin to periodic points in the flow:

6.3 Corollary. *Assume that the complete vector field X has a separatrix trajectory $\mathcal{L}$ by p, such that order$(X|_{\mathcal{L}}, p) = 1$. Given a number $n \geq 2$, there exists a complex time $T(n) \in \mathbb{C}^*$ such that $\Phi_{T(n)} : M \to M$, the time–$T(n)$ map, has a cylinder $\mathbb{C}^*$ of periodic points of minimal period n.*

Proof. Since order$(X|_{\mathcal{L}}, p) = 1$, by Lemma 3.1 the normalization of $\mathcal{L}$ has an embedded copy of $\mathbb{C}^*$. Assume that the vector field X looks like $\lambda z \frac{\partial}{\partial z}$ in this copy of $\mathbb{C}^*$, for λ a non zero complex number (recall that by definition of separatrix p is a isolated zero in $X|_{\mathcal{L}}$). Making the choice $T(n) = (2\pi\sqrt{-1}/n\lambda) \in \mathbb{C}^*$, the time–$T(n)$ map of X is the same as the time–1 map of $T(n)X$. Note that

$$T(n)X = \frac{2\pi\sqrt{-1}}{n} z \frac{\partial}{\partial z} \quad \text{on} \quad \mathbb{C}^* \subset \mathcal{L} \, ,$$

and hence its flow has only periodic points of primitive period n. See also [25] for applications of the idea of rotating the vector field X by some complex number. □

If in the above Corollary M is a complex Stein surface and X has an isolated zero, then by the Camacho–Sad theorem [7], a separatrix of X by p always exists. It follows that many time–$T(n)$ maps give origin to cylinders of periodic points.

7. Indetermination points for rational vector fields.

In this part we consider polynomial vector fields in $\mathbb{C}^2$, giving some simple ideas on the flow behavior at infinity (for simplicity we restrict our attention to dimension two).

It is elementary that every polynomial vector field in $\mathbb{C}^2$ extends in a unique way to a rational vector field on $\mathbb{C}P^2$.

Recall some notation. Let $\mathbb{C}^3 = \{(z_0, z_1, z_2)\}$ be the complex space, the complex projective plane is $\mathbb{C}P^2 = \mathbb{C}^3 - \{0\}/\mathbb{C}^*$. We consider on the affine chart

$\mathbb{C}^2 = \{z_0 = 1\} \subset \mathbb{C}P^2$ the polynomial vector field P. The coordinate charts are as usual $\{\phi_i : U_i \subset \mathbb{C}P^2 \to \mathbb{C}^2 \mid i = 0,1,2\}$, where $U_i = \{z_i \neq 0\}$, and

$$\begin{array}{ccc} \phi_0 : U_0 & \to & \mathbb{C}^2 \\ (z_0, z_1, z_2) & \mapsto & \left(\frac{z_1}{z_0}, \frac{z_2}{z_0}\right) , \end{array}$$

etc.

The changes of coordinates applied to P give rational vector fields on the other charts

$$\{(\phi_j \circ \phi_0^{-1})_* P \ \ in \ \ \mathbb{C}^2 = \{z_j = 1\} \mid for \ j = 1,2\} .$$

The *associated singular holomorphic foliation* $\mathcal{F}(P)$ is defined by the above vector fields. To do this, we remove the poles at $\mathbb{C}P^1_\infty = \{z_0 = 0\}$ from each vector field $(\phi_j \circ \phi_0^{-1})_* P$ (multiplying by a suitable z_0^k), obtaining holomorphic vector fields on each coordinate chart U_i, the associated foliations glue to give $\mathcal{F}(P)$ on $\mathbb{C}P^2$. See [13] for the explicit computation.

The behavior of polynomial vector fields at infinity is well understood, let us recall also from [13].

7.1 Proposition. *Let*

$$P(z_1, z_2) = P_1(z_1, z_2)\frac{\partial}{\partial z_1} + P_2(z_1, z_2)\frac{\partial}{\partial z_2}$$

be a polynomial vector field in $\mathbb{C}^2 = \{z_0 = 1\} \subset \mathbb{C}P^2$ *of degree* m *(i.e.* m *is the highest degree of the polynomials* P_j*).*
1.- Then P *has a pole of order* $1-m$ *at the line at infinity* $\mathbb{C}P^1_\infty = \{z_0 = 0\} \subset \mathbb{C}P^2$*, unless the terms of degree* m *of* P *have the form*

$$g(z_1, z_2)(z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2}) ,$$

for g *a polynomial of degree* $m-1$*, in which case it has a pole of order* $2-m$ *at the line at infinity.*
2.- Then the singular holomorphic foliation $\mathcal{F}(P)$ *has* $\mathbb{C}P^1_\infty$ *as a leaf if and only if the terms of degree* m *of* P *can not be expressed as*

$$g(z_1, z_2)(z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2}) ,$$

for g *a polynomial of degree* $m-1$*.*
3.- If $\mathbb{C}P^1_\infty$ *is a leaf of* $\mathcal{F}(P)$*, then* $\mathcal{F}(P)$ *has a finite number of singularities in* $\mathbb{C}P^1_\infty$*.* □

Strictly speaking, $\mathbb{C}P^1_\infty$ *is a leaf of* $\mathcal{F}(P)$, means that $\mathbb{C}P^1_\infty - \{$ *singular points of* $\mathcal{F}(P)\}$ is a leaf in the usual sense.

The application to our problem is as follows. Given a polynomial vector field P on $\mathbb{C}^2$ of degree $m \geq 2$ (the case $m = 1$ always defines complete vector fields). A *separatrix of the singular foliation* $\mathcal{F}(P)$ is a complex analytic curve that is a finite union of non-singular leaves of $\mathcal{F}(P)$ and having singularities of $\mathcal{F}(P)$ in its closure. Roughly speaking, we consider each separatrix $\{\mathcal{L}\}$ of the singular foliation

$\mathcal{F}(P)$ intersecting the line of poles $\mathbb{C}P^1_\infty$ in a discrete set, and we understand how the restricted vector field is on $\mathcal{L}$.

As a first problem, note that some $\mathcal{L}$ can be singular at points on the line at infinity. This is solved as in Section 4 by considering its resolution.

Following 7.1 we have two possibilities for the behavior at infinity of singular holomorphic foliations in $\mathbb{C}P^2$ coming from polynomial vector fields. The first is:

7.2 Corollary. *Let P be a polynomial vector field in $\mathbb{C}^2$ of degree $m \geq 2$. If $\mathbb{C}P^1_\infty$ is not a leaf of the singular holomorphic foliation defined by P in $\mathbb{C}P^2$, then P is incomplete on the trajectories across $\mathbb{C}P^1_\infty - \{$singular points of $\mathcal{F}(P)\}$.*

Proof. Assume the degree of P is at least three. Let $\mathcal{L}$ be a leaf of the foliation $\mathcal{F}(P)$ in $\mathbb{C}P^2$. Since the singularities of $\mathcal{F}(P)$ are isolated, it is easy to choose a leaf such that $p = \mathcal{L} \cap \mathbb{C}P^1_\infty$ is a nonsingular point of $\mathcal{F}(P)$. The vector field X defines on $\mathcal{L}$ near p a meromorphic vector field with a pole at p (since $\mathbb{C}P^1_\infty$ is not a leaf). From Corollary 3.3 it follows that the flow of X along $\mathcal{L}$ is incomplete in $\mathbb{C}^2$.

For P of degree two, an explicit computation in the other affine charts shows that the vector field is transverse and holomorphic at the line at infinity. Hence it is incomplete in $\mathbb{C}^2$. □

7.3 Example. Consider the vector field

$$P(z_1, z_2) = g(z_1, z_2)(z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2}) \quad \text{at } z_0 = 1,$$

where g is a polynomial of degree $m - 1 \geq 1$, and $\{g = 0\}$ as an algebraic curve does not contain complex lines through $(z_1, z_2) = (0, 0)$. The foliation $\mathcal{F}(P)$ in $\mathbb{C}P^2$ is a pencil of lines through $[1, 0, 0]$. The associated rational vector field has a pole of order $2 - m$ in the line at infinity. Every complex line $\mathcal{L}$ through $[1, 0, 0]$ is such that

$$\text{zeros}\,(P|_{\mathcal{L}}) + \text{poles}\,(P|_{\mathcal{L}}) = (1 + m - 1) + (2 - m) = 2 \ ,$$

according to Corollary 2.9. Following 7.2 the vector field is incomplete in $\mathbb{C}^2$. For the particular subcase

$$P(z_1, z_2) = (az_1 + bz_2)(z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2}) \quad \text{at } z_0 = 1.$$

The affine degree is two; however the associated rational vector field in $\mathbb{C}^3$ is the linear:

$$-(az_1 + bz_2)\frac{\partial}{\partial z_0} \ ,$$

giving a holomorphic vector field in $\mathbb{C}P^2$. The line $\mathbb{C}P^1_\infty$ is not invariant under its flow, hence this vector field is incomplete in the affine plane $\mathbb{C}^2 = \{z_0 = 1\}$.

As second possibility; the foliation defined by P has $\mathbb{C}P^1_\infty$ as a leaf, we know that $\mathbb{C}P^1_\infty$ is always a pole for P.

Roughly speaking the flow of P will be incomplete along the leaves of $\mathcal{F}(P)$ that are separatrices for the singular points of $\mathcal{F}(P)$ in $\mathbb{C}P^1_\infty$. The next example shows that the above can be false.

7.4 Example. Let

$$P(z_1, z_2) = g(z_2)\frac{\partial}{\partial z_1}$$

be a polynomial vector field, where g is a polynomial of degree $m \geq 2$. The associated singular holomorphic foliation in $\mathbb{C}P^2 = \{[z_0, z_1, z_2]\}$ has a unique singularity in $[0, 1, 0]$, and the leaves are given by the pencil of lines by this point. Every leaf of $\mathcal{F}(P)$ is a separatrix across $[0, 1, 0]$. However the vector field is holomorphic and complete in $\mathbb{C}^2$.

What kind of point is $[0, 1, 0]$ for the associated rational vector field on $\mathbb{C}P^2$?

On Riemann surfaces the types of singularities of a complex analytic vector field are zeros, poles or essential singularities. However on complex manifolds of dimension two or more, there can exist points where the zeros of a rational vector field intersect its poles. This is the phenomenon of indetermination points, well known in several complex variables for functions and maps, see [16] p. 490–491. For simplicity the next definition is formulated locally in $(\mathbb{C}^2, 0)$, hence can be applied to meromorphic vector fields in complex surfaces:

7.5 Definition. Let $A, B, C, D : (\mathbb{C}^2, 0) \to \mathbb{C}$ be holomorphic functions. The point $0 \in \mathbb{C}^2$ is an *indetermination point* of the meromorphic vector field

$$X(z_1, z_2) = \frac{A(z_1, z_2)}{B(z_1, z_2)}\frac{\partial}{\partial z_1} + \frac{C(z_1, z_2)}{D(z_1, z_2)}\frac{\partial}{\partial z_2},$$

iff $\{A = 0\} \cap \{B = 0\} = 0$ and/or $\{C = 0\} \cap \{D = 0\} = 0$.

Moreover, the singular holomorphic foliation $\mathcal{F}(X)$ associated to X can be described by the holomorphic one form

$$\mathcal{F}(X) = \{\ C(z_1, z_2)B(z_1, z_2)dz_1 - A(z_1, z_2)D(z_1, z_2)dz_2 = 0\ \}.$$

However for our problem it is necessary to understand the original vector field at the separatrices at the origin for this foliation $\mathcal{F}(X)$.

It is possible to show that for a polynomial vector field P in $\mathbb{C}^2$, the indetermination points of its associated rational vector field in $\mathbb{C}P^2$ always define (probably removable) singularities of the singular holomorphic foliation $\mathcal{F}(P)$ in $\mathbb{C}P^2$.

7.6 Theorem. *Let P be a polynomial vector field on $\mathbb{C}^2$ of degree $m \geq 2$, having $\mathbb{C}P^1_\infty$ as a leaf of its associated foliation $\mathcal{F}(P)$. Let $\mathcal{L}$ be a separatrix of $\mathcal{F}(P)$ through a singularity $p \in \mathbb{C}P^1_\infty$ of $\mathcal{F}(P)$. Assume that $P|_{\mathcal{L}}$ at p has:*
1.- A regular point.
2.- A zero of order greater or equal to three.
3.- A pole.
4.- An essential singularity.
Then P is incomplete (in $\mathbb{C}^2$).

An easy exercise shows that possibility (4) never appears, our proof is independent of this.

Proof. Consider $\alpha : (\mathbb{C}, 0) \to (\mathcal{L}, p)$ a local representative of the resolution of $\mathcal{L}$ at p. It follows that $\alpha^* P$ is a holomorphic vector field in a punctured neighborhood of 0 in $(\mathbb{C}, 0)$. Following the classical theory of complex analytic functions in one variable, $\alpha^* P$ has at 0 a regular point, a zero, a pole or an essential singularity.
Case (1) follows from the classical theory of ordinary differential equations, since $p \in \mathbb{C}P^1_\infty$ the flow leaves the affine $\mathbb{C}^2 = \{z_0 = 1\}$ in finite time. Moreover by Corollaries 3.2, 3.3, results (2)–(4) follow. $\square$

7.7 Example. Consider in $\mathbb{C}^2$ the polynomial vector field

$$P(z_1, z_2) = z_1^2 \frac{\partial}{\partial z_1} + z_2^2 \frac{\partial}{\partial z_2} \quad \text{at } z_0 = 1 .$$

This defines an incomplete vector field in $\mathbb{C}^2$. The associated rational vector fields in the other affine charts of $\mathbb{C}P^2$ are:

$$-\frac{\partial}{\partial z_0} + \frac{z_2(z_2 - 1)}{z_0} \frac{\partial}{\partial z_2} \quad \text{at } z_1 = 1,$$

$$-\frac{\partial}{\partial z_0} + \frac{z_1(z_1 - 1)}{z_0} \frac{\partial}{\partial z_1} \quad \text{at } z_2 = 1.$$

It follows that $[1,0,0]$ is a zero, and $\{[0,1,0],\ [0,1,1],\ [0,0,1]\}$ are indetermination points, for P as a rational vector field. Moreover, the vector field has a rational first integral

$$\frac{z_0 z_1 - z_0 z_2}{z_1 z_2} : \mathbb{C}P^2 \to \mathbb{C}P^1 .$$

Hence, the topology of the associated foliation is given by a pencil of quadrics in $\mathbb{C}P^2$, having as base locus $\{[1,0,0], [0,1,0], [0,0,1]\}$. Note that $[0,1,1]$ is a critical point for the rational first integral.

It is easy to compute that each separatrix of $\mathcal{F}(P)$ is isomorphic to $\mathbb{C}P^1$ having a holomorphic vector field $\alpha^*(P)$ with a double zero coming from $[1,0,0]$, and that $\mathcal{L} \cap \mathbb{C}P^1_\infty$ is a regular point for $\alpha^*(P)$.

The affirmative result is as follows:

7.8 Theorem. *Let P be a polynomial vector field in $\mathbb{C}^2$ of degree $m \geq 2$, having $\mathbb{C}P^1_\infty$ as a leaf of its associated foliation $\mathcal{F}(P)$, and a polynomial first integral. The following assertions are equivalent:*
1.- P is complete (in $\mathbb{C}^2$).
2.- For every separatrix $\mathcal{L}$ of $\mathcal{F}(P)$ through the singularities of $\mathcal{F}(P)$ in $\mathbb{C}P^1_\infty$, the restrictions $P|_\mathcal{L}$ have zeros.

Proof. Assume (1), it follows by Theorem 7.6 that any other possibility for $P|_\mathcal{L}$ in a separatrix through a point at infinity, must imply that X is incomplete.

For the converse, let $\mathcal{L} \subset \mathbb{C}P^2$ be a leaf of $\mathcal{F}(P)$ irreducible as a complex analytic curve. Since P has a polynomial first integral, the closure $\overline{\mathcal{L}}$ is an algebraic curve in $\mathbb{C}P^2$.

Note that $\overline{\mathcal{L}} \cap \mathbb{C}P^1_\infty$ is non empty. P induces a complete holomorphic vector field on the resolution of $\overline{\mathcal{L}}$ if and only if assertion (2) is true. The result follows. □

The existence of a polynomial first integral is a strong hypothesis. Recall in this direction the nice result of G. Darboux; a polynomial vector field having enough algebraic curves as solutions has in fact a polynomial first integral, [30] p. 440.

7.4 Example (revised). For the vector field

$$P(z_1, z_2) = g(z_2)\frac{\partial}{\partial z_1}$$

the expression at the indetermination point $[0,1,0]$ is given as:

$$-z_0 g(\frac{z_2}{z_0})(z_0\frac{\partial}{\partial z_0} + z_2\frac{\partial}{\partial z_2}) \quad \text{at } z_1 = 1.$$

It follows that the vector field restricted to each separatrix by $(z_0, z_2) = (0,0)$, given by a parametrization

$$\alpha : t \mapsto (t, \lambda t) \ , \text{ for } \lambda \in \mathbb{C} \ , \ t \in (\mathbb{C}, 0) \ ,$$

is such that

$$\alpha^* X = -t^2 g(\lambda)\frac{\partial}{\partial t},$$

which is holomorphic at $\alpha(0) = [0,1,0]$. Hence the indetermination point is a double zero for each complex line through it, except the line at infinity $\mathbb{C}P^1_\infty = \{z_0 = 1\}$, or the lines that are zeros of g.

The simplest examples of polynomial vector fields with polynomial first integral are Hamiltonian vector fields. We give a very simple result for the incompleteness of Hamiltonian vector fields in $\mathbb{C}^2$, compare with [10], 7.1.

7.9 Proposition. *Let H be a polynomial in $\mathbb{C}^2$. If a projectivized curve $\{H = \lambda_0/\mu_0\} \subset \mathbb{C}P^2$, for $\lambda_0/\mu_0 \in \mathbb{C}$, intersects $\mathbb{C}P^1_\infty$ in at least three different points (set theoretically), then its Hamiltonian vector field X_H is incomplete in $\mathbb{C}^2$, for all the complex leaves $\{H = \lambda/\mu\}$.*

Proof. Consider H as a rational function, this means

$$H = \frac{\widetilde{H}(z_0, z_1, z_2)}{z_0^m} : \mathbb{C}P^2 \to \mathbb{C}P^1 \ ,$$

where $\widetilde{H}$ is the homogenized associated polynomial, and m is the degree of H. The indetermination points of this function are by definition

$$\{\widetilde{H} = 0\} \cap \{z_0^m = 0\} \subset \mathbb{C}P^2 \ .$$

Every curve $\{H=\lambda/\mu\}$, for $\lambda/\mu \in \mathbb{C}$, can be written in $\mathbb{C}P^2$ as $\{\mu\widetilde{H}-\lambda z_0^m = 0\}$. Hence all the curves intersect the indetermination points of the function. By hypothesis, the indetermination set of $\widetilde{H}/z_0^m$ has at least three different points. They give origin to the same number of indetermination points for the Hamiltonian vector field X_H, as a rational vector field in $\mathbb{C}P^2$.

Case 1. Assume $\{H=\lambda/\mu\}$ is an irreducible curve in $\mathbb{C}^2$. The resolution for every $\{H=\lambda/\mu\}$ has at least three points coming from $\mathbb{C}P^1_\infty$. If the three points are zeros for the Hamiltonian vector field, we have a Riemann surface with three zeros, and by Lemma 3.1, the vector field is not complete. In any other case (regular points, poles, essential singularities), the result follows from Theorem 7.6.

Case 2. Assume $\{H=\lambda/\mu\}$ is a reducible curve of degree three. Suppose it has two irreducible components, say $\mathcal{L}$, $\mathcal{J} \subset \mathbb{C}P^2$. By the hypothesis the intersection $\{H=\lambda/\mu\} \cap \mathbb{C}P^1_\infty$ is given by three different points, hence one of the components is a quadric, for example say $\mathcal{L}$. Then $\mathcal{L}$ has three special points:

Two points from $\mathcal{L} \cap \mathbb{C}P^1_\infty$.

One additional point from $\mathcal{L}\cap\mathcal{J}$ that is a point in the affine $\mathbb{C}^2 = \{z_0 = 1\}$, giving origin to a zero in the restricted vector field.

Hence P restricted to $\mathcal{L}$ is an incomplete vector field in $\mathbb{C}^2$.

All other cases: $\{H=\lambda/\mu\}$ with three components of degree one, or $\{H=\lambda/\mu\}$ of higher degree, follow from the same type of argument. □

From the classical Bezout's Theorem in $\mathbb{C}P^2$, it follows that for H of degree $m \geq 3$, the hypothesis in the Proposition is generically true. We leave to the interested reader the problem of finding necessary and sufficient conditions.

7.10 Corollary. *Let P be a polynomial vector field on $\mathbb{C}^2$ having degree $m \geq 2$. If the associated foliation $\mathcal{F}(P)$ in $\mathbb{C}P^2$ has a leaf intersecting $\mathbb{C}P^1_\infty \cup \{zeros(P)\ in\ \mathbb{C}^2\}$ in at least three points (set theoretically), then P is incomplete in $\mathbb{C}^2$.*

Proof. Follows the same idea as for the leaves of the Hamiltonian in 7.9. □

Note that the hypothesis of P having a polynomial first integral or P Hamiltonian, avoid complex recurrences on the leaves of $\mathcal{F}(P)$ near the leaf at infinity. The presence of these recurrences also give origin to incompleteness. It is probably the most difficult step in the characterization of complete polynomial vector fields.

Acknowledgments. We express our gratitude to the editors and to professor R. Michael Porter for their assistance in the preparation of this paper.

References

[1] D. Akhiezer, *Lie Group Actions in Complex Analysis,* Vieweg (1995).

[2] E. Andersen, L. Lempert, *On the group of holomorphic automorphisms of* $\mathbb{C}^n$, Invent. Math. 110 (1992) 371–388.

[3] L. Brickman, E. S. Thomas, *Conformal equivalence of analytic flows,* Journal of Differential Equations 25 (1977) 310–324.

[4] G. T. Buzzard, J. E. Fornæss, *Complete holomorphic vector fields and time–1 maps*, Indiana University Mathematics Journal 44 (1995) 1175–1182.

[5] J. Carrell, A. Howard, C. Kosniowski, *Holomorphic vector fields on complex surfaces*, Math. Ann. 204 (1973) 303–309.

[6] C. Camacho, N. Kuiper, J. Palis, *The topology of holomorphic flows with singularity*, Pub. Math. IHES 48 (1978) 5–38.

[7] C. Camacho, P. Sad, *Invariant varieties through singularities of holomorphic vector fields*, Annals of Math. 115 (1982) 579–595.

[8] K. Fieseler, *On complex affine surfaces with $\mathbb{C}^+$–action*, Comment. Math. Helvetici 69 (1994) 5–27.

[9] J. E. Fornæss, *Dynamics in several complex variables.* Regional Conference Series in Mathematics A. M. S., 87 (1996).

[10] F. Forstneric, *Actions of* $(\mathbb{R},+)$ *and* $(\mathbb{C},+)$ *on complex manifolds*, Math. Z. 223 (1996) 123–153.

[11] E. Ghys, J.-C. Rebelo, *Singularités des flots holomorphes II.* Ann. Inst. Fourier, Grenoble 47, 4 (1997) 1117–1174.

[12] A. A. Glutsyuk, *Hyperbolicity of leaves of a generic one–dimensional holomorphic foliation on a nonsingular projective algebraic manifold*, Proceedings of the Steklov Institute of Mathematics, 213 (1996) 90–111.

[13] X. Gómez–Mont, *On families of rational vector fields*, Coloquio de Sistemas Dinámicos, J. Seade and G. Sienra (eds.) Aportaciones Matemáticas 1 (1985) 36–65.

[14] X. Gómez–Mont, I. Luengo, *Germs of holomorphic vector fields in $\mathbb{C}^3$ without a separatrix*, Inv. Math 109 (1990).

[15] A. Goriely, C. Hyde, *Necessary and sufficient conditions for finite time blow–up in ordinary differential equations*, Preprint (1997).

[16] Ph. Griffiths, J. Harris, *Principles of Algebraic Geometry*, Wiley Interscience (1978).

[17] N. Hitchin, *Vector fields in the circle*, 200 *Years after Lagrange*, M. Francaviglia (ed.), Elsevier Science Publishers (1991) 359–378.

[18] S. Kobayashi, *Transformation Groups in Differential Geometry*, Springer–Verlag (1972).

[19] H. Kraft, *Challenging problems on affine n–space*, Séminaire Bourbaki, Astérisque 237 (1996) 295-317.

[20] S. Lojasiewicz, *Introduction to Complex Analytic Geometry,* Birkhaüser (1991).

[21] S. Lang, *Hyperbolic and Diophantine analysis.* Bulletin (New Series) of the A. M. S. 14, 2 (1986) 159–205.

[22] D. Lieberman, *Holomorphic vector fields on projective varieties*, Proceedings of Symposia in Pure Mathematics 30 (1997) 273–276.

[23] Y. Matsushima, *Holomorphic Vector Fields on Compact Kähler Manifolds*, Regional Conference Series on Mathematics 7, A. M. S. (1971).

[24] J. Muciño–Raymundo, *Complex structures adapted to smooth vector fields*, Preprint (1997).

[25] J. Muciño–Raymundo, C. Valero–Valdés, *Bifurcations of meromorphic vector fields on the Riemann sphere.* Ergodic Theory and Dynamical Systems 15 (1995) 1211–1222.

[26] J. Olivares–Vázquez, *On vector fields in* $\mathbb{C}^3$ *without separatarix.* Revista Matemática de la Universidad Complutense de Madrid. vol. 5 núm. 1 (1992) 13–34.

[27] J. Olivares–Vázquez, *On the problem of existence of germs of holomorphic vector fields in* $\mathbb{C}^m$*, without separatrix,* $(m \geq 3)$, Ecuaciones Diferenciales Singularidades, J. Mozo (ed.), Universidad de Valladolid (1997) 317–351.

[28] J. C. Rebelo, *Singularités des flots holomorphes*, Ann. Inst. Fourier, Grenoble 46, 2 (1996) 411-428..

[29] D. G. Saari, Z. Xia, *Off to infinity in finite time*, Notices of the AMS, 42, 5 (1995) 538-546.

[30] D. Schlomiuk, *Algebraic and geometric aspects of the theory of polynomial vector fields*, Bifurcations and periodic orbits of vector fields, D. Schlomiuk (ed.), Kluwer (1993) 429–467.

[31] M. Spivak, *A Comprehensive Introduction to Differential Geometry II*, Publish or Perish (1979).

[32] K. Strebel, *Quadratic Differentials*, Springer–Verlag (1984).

[33] B. Teissier, *Introduction to curve singularities*, Singularity Theory, D. T. Le, K. Saito, B. Teissier (eds.) World Scientific (1995) 866–893.

[34] J. Winkelmann, *On free holomorphic* $\mathbb{C}$*–actions on* $\mathbb{C}^n$ *and homogeneous Stein manifolds*, Math. Ann. 286, (1990) 593–612.

Jorge L. López
Instituto de Física y Matemáticas,
Universidad Michoacana,
Morelia, 58060,
Michoacán, México
E-mail address: jorge@itzel.ifm.umich.mx

Jesús Miciño Raymundo
Instituto de Matemáticas UNAM,
Unidad Morelia,
Nicolás Romero 150,
Col. Centro, Morelia 58000,
Michoacán, Mexico
E-mail address: jmucino@zeus.ccu.umich.mx