Geometric model for gravitational and electroweak interactions

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A fiber bundle treatment for Kaluza–Klein-type geometric unification of gravitation with the bosonic sector of the standard electroweak theory is presented. The most general G-invariant quadratic Lagrangian is constructed explicitly, and it is shown that the Higgs field sector, including the symmetry-breaking potential, arise naturally from torsion in the fiber through an adequate choice of its transformation properties.

I. INTRODUCTION

This paper is part of a program intended to study spontaneously compactified solutions of gravitation—Yang–Mills–Higgs systems, where the Higgs scalar fields originate from the torsion and acquire dynamics through the introduction in the Lagrangian of quadratic terms in the curvature tensor.

A comprehensive geometrical treatment for Kaluza–Klein-type unification of gauge fields and gravitation has been developed by Cho.[1] The inclusion of torsion in this principal fiber bundle (PFB) formalism as a source of the Higgs fields was considered by Katanayev and Volovich.[2] Rosenbaum and Ryan[3] have applied the approach of Cremmer and Scherk[4] to study spontaneously compactified solutions to the field equations resulting from the most general quadratic Lagrangian that can be constructed from the curvature and torsion in the PFB. They showed that for SO(3) as a characteristic group, and a Gauss–Bonnet combination of the quadratic terms in the curvature, the compactified solutions that were obtained also led to direct predictions on the size of the dimensionless coupling constant of the Yang–Mills fields remarkably close to the value of the coupling constant for the SU(2) factor in the $SU(2)\times U(1)$ electroweak model. Since SU(2) is a covering group of SO(3), it is reasonable to expect that some of the salient features of the model of Rosenbaum and Ryan should be preserved when extending it to the structure group $SU(2)\times U(1)$ and, in particular, it is worthwhile to test if the agreement in the value of the coupling constant mentioned above is still preserved. This study is presently being completed and it will be the subject of a forthcoming paper.

Here we shall concentrate on the development of the appropriate fiber bundle formalism for $SU(2)\times U(1)$. This, we believe, is by itself an interesting result. First, because we arrive at a general G-invariant Kaluza–Klein-type Lagrangian that unifies geometrically the bosonic part of the electroweak model with gravitation, and second, because of the inherent mathematical elegance of the resulting theory.

As we will show, our construction cannot be based on a simple extension of the ideas contained in Refs. 2 and 3, where an $\mathscr{A}d$-invariant Lagrangian was obtained rather directly by allowing the torsion components (which generate the Higgs fields) to transform in the same way as the gauge field tensor, i.e., in accordance to the adjoint representation of the group. In the case of the direct product group $SU(2)\times U(1)$, since U(1) is Abelian, such an assumption, which is equivalent to the seemingly most natural requirement of right invariance of the torsion, would lead to the loss of important dynamical information on the real Higgs field which is associated with the generator of U(1).

In the phenomenological approach to the electroweak model this problem is resolved, of course, by having the group act on the gauge fields in accordance with the adjoint representation, while the scalar fields are required, in an *ad hoc* fashion, to transform as a complex spinor doublet under SU(2).

To arrive at this result from a geometrical point of view requires a generalization of the law of transformation of the torsion as well as a very careful choice of connections and representation of base vectors for our frame bundle. But once the proper choice is made all the terms in the bosonic part of the gravitation–electroweak Lagrangian follow unequivocally and in a geometrically unified manner from the curvature and torsion of the bundle.

It is important to stress that in its present stage our theory does not consider fermionic fields, and it must be regarded so far as an $SU(2)\times U(1)$ gauge theory coupled only to gravitation and a complex doublet of Higgs fields, where these fields, as well as the quartic scalar potential and the negative mass term required for spontaneous symmetry breaking, originate from torsion. Further remarks on the possibility of also including the fermion and Yukawa-type Lagrangians within the framework of our formalism, which is needed to complete the electroweak model, will be given in Sec. V.

This paper is organized as follows. Section II is dedicated to the construction of the several principal fiber bundles needed for our theory. In Sec. III we derive the different components of the curvature and torsion tensors on the bundle of frames and use the results to obtain a G-invariant Lagrangian which is, therefore, well defined on the base manifold. Section IV contains a procedure for relating the dynamical form of the Higgs Lagrangian in our formalism with the form which is commonly used by field theorists and particle physicists.

As far as notation is concerned, we will consistently use the following ranges for our indices: latin lower case letters

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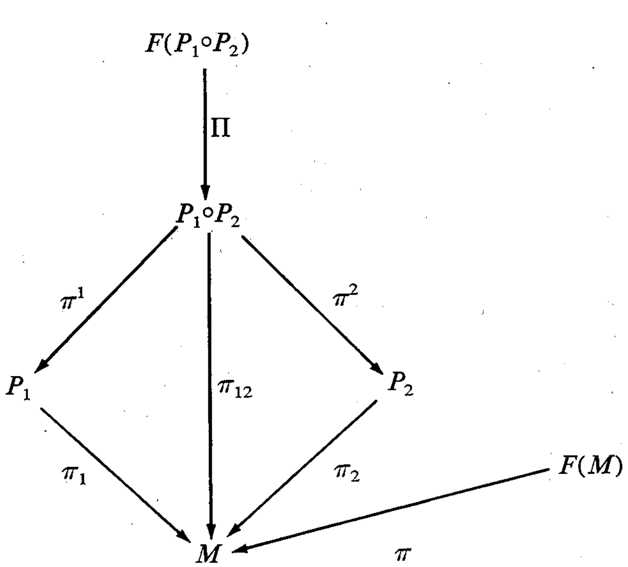
from the middle of the alphabet will have the range $1 \leqslant i, j, k, ..., \leqslant n$, greek lower case letters will have the range $1 \leqslant \alpha, \beta, \gamma, ..., \leqslant 3$, upper case latin letters from the beginning of the alphabet will have the range $1 \leqslant A, B, C, ..., \leqslant 4$, and lower case latin letters from the beginning of the alphabet will cover the full range $1 \leqslant a, b, c, d, ..., \leqslant n+4$. The spaces to which these indices refer will be self-evident from the text. With respect to sign conventions, we follow those of Landau–Lifshitz,[5] i.e., the signature of the base manifold metric is $\text{sgn}(g_{ij}) = (+, -, ..., -)$; the Riemann tensor is defined by $\xi^a{}_{;bc} - \xi^a{}_{;cb} = R^a{}_{dbc}\xi^d$, and the Ricci tensor by $R_{cd} = R^a{}_{cad}$. The Einstein equations then take the form $R_{ab} - \frac{1}{2}Rg_{ab} = 8\pi T_{ab}$ with $T_{00} \geqslant 0$.

II. THE BUNDLE FRAMEWORK FOR SU(2)×U(1)

As mentioned in the Introduction, the most adequate framework for a Kaluza–Klein theory that naturally unifies gravitation with the gauge and scalar fields is the principal fiber bundle formalism. In the specific case of $SU(2) \times U(1)$, the theory requires five different PFB's that are interrelated according to the diagram



For the description of these constructions we shall rely closely on the notation used by Bleecker.[6] Thus M denotes an n-dimensional oriented manifold, which we take to be space-time and which acts as the base space of the following PFB's:

(1) $\pi: F(M) \to M$, is the orthonormal frame bundle of M with group $O(r, s)$. For $u \in F(M)_x$ and the usual basis $\{e_i\}$, $i = 1, ..., n$ of $\mathfrak{R}^n$, we choose an orthonormal frame at $x \in U \subset M$ by means of the linear isomorphism $u: \mathfrak{R}^n \to T_x M$, i.e., $u(e_i) = \overline{E}_i$, $i = 1, ..., n$, are orthonormal vector fields with respect to the metric g on M, defined in a neighborhood U of $x = \pi(u)$ in such a way that the local section σ: $U \to F(M)$ determined by $\overline{E}_1, ..., \overline{E}_n$ is tangent to the horizontal subspace of $T_{\sigma(x)}F(M)$ relative to the connection $\theta(g)$. Consequently $\theta(g)(\sigma^*\overline{E}_i) = (\sigma^*\theta(g))(\overline{E}_i) = \overline{\theta}(g)(\overline{E}_i) = 0$ at x.

The curvature of the connection $\theta(g) \in \Lambda^1(F(M), \mathscr{O}(r, s))$ is given by

$$\Omega^{\theta(g)} \equiv D^{\theta(g)}\theta(g)$$
$$= d\theta(g) + \tfrac{1}{2}[\theta(g), \theta(g)] \in \overline{\Lambda}^2(F(M), \mathscr{O}(r,s)). \quad (2.1)$$

Note that, since $u^{-1}(\pi_*\sigma_*\overline{E}_i) = e_i$ and $\theta(g)(\sigma_*\overline{E}_i) = 0$, the vectors $\sigma_*\overline{E}_i \in T_u(F(M))$ are standard horizontal vectors relative to $\theta(g)$. Thus $\Omega^{\theta(g)}(\sigma_*\overline{E}_i, \sigma_*\overline{E}_j)(e_k)$ is the image of $e_k \in \mathfrak{R}^n$. We can therefore write

$$\Omega^{\theta(g)}(\sigma_*\overline{E}_i, \sigma_*\overline{E}_j)(e_k) = R^h{}_{kij}(\sigma(x))e_h. \quad (2.2)$$

Also note that for $X_u \in T_u F(M)$ we can define the canonical one-form $\varphi_M \in \overline{\Lambda}^1(F(M), \mathfrak{R}^n)$ by

$$\varphi_M(X_u) = u^{-1}(\pi_*(X_u)) \in \mathfrak{R}^n. \quad (2.3)$$

In terms of this canonical one-form the torsion two-form $\Theta^{\theta(g)}$ is given by

$$\Theta^{\theta(g)} = D^{\theta(g)}\varphi_M = d\varphi_M + \theta(g) \dot{\wedge} \varphi_M \in \overline{\Lambda}^2(F(M), \mathfrak{R}^n), \quad (2.4)$$

where the quantity $\theta(g) \dot{\wedge} \varphi_M$ is defined by

$$(\theta(g) \dot{\wedge} \varphi_M)(X_u, Y_u)$$
$$= \theta(g)(X_u) \cdot \varphi_M(Y_u) - \theta(g)(Y_u) \cdot \varphi_M(X_u), \quad (2.5)$$

and the "dot" operation denotes the left action of $O(r, s)$ on $\mathfrak{R}^n$.

Furthermore, since $\Theta^{\theta(g)}(\sigma_*\overline{E}_i, \sigma_*\overline{E}_j) \in \mathfrak{R}^n$, we can write

$$\Theta^{\theta(g)}(\sigma_*\overline{E}_i, \sigma_*\overline{E}_j) = S^k{}_{ij}(\sigma(x))e_k. \quad (2.6)$$

If we now let $\overline{\varphi}^i_M$ be the one-forms dual to $\overline{E}_i$, i.e., $\overline{\varphi}^i_M(\overline{E}_j) = \delta^i_j$, and we set $X_u = \sigma_*\overline{E}_i$ in (2.3), then

$$\varphi_M(\sigma_*\overline{E}_i) = (\sigma^*\varphi_M)(\overline{E}_i) = e_i = (\overline{\varphi}^1_M(\overline{E}_i), ..., \overline{\varphi}^n_M(\overline{E}_i)). \quad (2.7)$$

Thus

$$\sigma^*\varphi_M = \overline{\varphi}_M = \overline{\varphi}^i_M e_i. \quad (2.8)$$

The pullback with the local section σ of the canonical one-forms allows us to relate the curvature and torsion tensors in $F(M)$, as given by (2.2) and (2.6), to the corresponding tensors in $T_x(M)$. Indeed, acting with σ^* on (2.1) we get

$$(\overline{\Omega}^{\overline{\theta}(g)})^h{}_k \equiv (\sigma^*\Omega^{\theta(g)})^h{}_k$$
$$= D^{\overline{\theta}(g)}\overline{\theta}^h{}_k(g) = \tfrac{1}{2}\underline{R}^h{}_{kij}(x)\overline{\varphi}^i_M \wedge \overline{\varphi}^j_M \quad (2.9)$$

or

$$\overline{\Omega}^{\overline{\theta}(g)}(\overline{E}_i, \overline{E}_j) = \underline{R}^h{}_{kij}(x)e_h \otimes \hat{e}^k, \quad (2.10)$$

where $\hat{e}^k$ is the dual of e_k.

On the other hand, from (2.2) we get

$$\overline{\Omega}^{\overline{\theta}(g)}(\overline{E}_i, \overline{E}_j) = R^h{}_{kij}(\sigma(x))e_h \otimes \hat{e}^k. \quad (2.11)$$

Consequently, we have the following lemma.

Lemma 1: Let $R^h{}_{kij}$ be the components of an L tensor in $C(F(M), T^{1,3})$ (according to the definitions in Ref. 6), and let $\underline{R}^h{}_{kij} \in \mathscr{T}^{1,3}$ be the components of the curvature tensor of (M, g) relative to the orthonormal fields $\{\overline{E}_i\}$, then

$$R^h{}_{kij}(\sigma(x)) = \underline{R}^h{}_{kij}(x). \quad (2.12)$$

Proceeding in a similar fashion with the torsion, we get, from (2.4) and (2.8),

$$\overline{\Theta}^{\overline{\theta}(g)} = D^{\overline{\theta}(g)}\overline{\varphi}_M = d\overline{\varphi}_M + \overline{\theta}(g) \dot{\wedge} \overline{\varphi}_M$$
$$= \tfrac{1}{2}\underline{S}^i{}_{jk}(x)e_i \otimes (\overline{\varphi}^j_M \wedge \overline{\varphi}^k_M) \in \overline{\Lambda}^2(M, \mathfrak{R}^n), \quad (2.13)$$

while (2.6) yields

$$\overline{\Theta}^{\overline{\theta}(g)}(\overline{E}_j,\overline{E}_k) = S^i_{\ jk}(\sigma(x))e_i. \tag{2.14}$$

Therefore, we also have the following lemma.

Lemma 2: Let $S^i_{\ jk}$ be the components of an L tensor in $C(F(M), T^{1,2})$, and let $\underline{S}^i_{\ jk}\in\mathscr{T}^{1,2}$ be the components of the torsion tensor of (M, g) relative to the orthonormal fields $\{\overline{E}_i\}$, then

$$S^i_{\ jk}(\sigma(x)) = \underline{S}^i_{\ jk}(x). \tag{2.15}$$

In our calculations in the following sections, we will be frequently using the isomorphisms implied by Eqs. (2.12) and (2.15).

(2) π_1: $P_1\to M$, is a PFB with group $G_1 = \mathrm{SU}(2)$ and connection $\omega_1\in\Lambda^1(P_1, \mathscr{G}_1)$, where $\mathscr{G}_1$ is the Lie algebra of G_1.

The curvature of the connection ω_1 is

$$\Omega_1\equiv D^{\omega_1}\omega_1\equiv d\omega_1 + \tfrac{1}{2}[\omega_1,\omega_1]\in\overline{\Lambda}^2(P_1,\mathscr{G}_1). \tag{2.16}$$

Note that if l_α ($\alpha = 1,2,3$) is a basis for $\mathscr{G}_1$, we can write $\omega_1 = \omega_1^\alpha l_\alpha$, and

$$\Omega_1 = (d\omega_1^\alpha + \tfrac{1}{2}c^\alpha_{\ \beta\gamma}\omega_1^\beta\wedge\omega_1^\gamma)l_\alpha, \tag{2.17}$$

where $c^\alpha_{\ \beta\gamma}$ are the structure constants of G_1.

Moreover, if we let $E_1^{(1)},\ldots,E_n^{(1)}$ be an orthonormal basis of the horizontal subspace of $T_{p_1}P_1$ relative to ω_1, such that $\pi_{1*}E_i^{(1)} = \overline{E}_i$, and we also let $\overline{\varphi}^1_{(1)},\ldots,\overline{\varphi}^n_{(1)}$ be the one-forms dual to $E_1^{(1)},\ldots,E_n^{(1)}$, then we can also write

$$\Omega_1 = \tfrac{1}{2}(\Omega_1)^\alpha_{\ ij}l_\alpha\otimes(\overline{\varphi}^i_{(1)}\wedge\overline{\varphi}^j_{(1)}), \tag{2.18}$$

where $(\Omega_1)^\alpha_{\ ij}$ is a real function in $\pi_1^{-1}(U)$.

(3) π_2: $P_2\to M$ is a PFB with group $G_2 = \mathrm{U}(1)$ and connection $\omega_2\in\Lambda^1(P_2,\mathscr{G}_2)$, where $\mathscr{G}_2$ is the Lie algebra of G_2.

Since U(1) is Abelian, the curvature of the connection ω_2 is

$$\Omega_2\equiv D^{\omega_2}\omega_2 = d\omega_2\in\overline{\Lambda}^2(P_2,\mathscr{G}_2). \tag{2.19}$$

Taking $i=\sqrt{-1}$ as the basis for $\mathscr{G}_2$, we can write $\omega_2 = (-i\omega_2)i$, so that

$$(-i\Omega_2) = d(-i\omega_2)\in\overline{\Lambda}^2(P_2,\mathfrak{R}). \tag{2.20}$$

As an orthonormal basis of the horizontal subspace of $T_{p_2}P_2$ relative to ω_2, we take $\{E_i^{(2)}\}$, $i = 1,\ldots,n$, and the corresponding dual one-forms $\{\overline{\varphi}^i_{(2)}\}$ as above. We also require that $\pi_{2*}E_i^{(2)} = \overline{E}_i$. Thus

$$\Omega_2 = \tfrac{1}{2}(\Omega_2)_{ij}i\otimes\overline{\varphi}^i_{(2)}\wedge\overline{\varphi}^j_{(2)}, \tag{2.21}$$

where $(\Omega_2)_{ij}$ is a real function defined on $\pi_2^{-1}(U)$.

(4) π_{12}: $P_1\circ P_2\to M$ is the PFB with group SU(2) $\times$U(1), obtained by splicing the bundles π_i: $P_i\to M$. In this way we have that $P_1\circ P_2 = \{(p_1,p_2)\in P_1\times P_2|\pi_1(p_1) = \pi_2(p_2)\}$, and that $\pi_{12}(p_1,p_2) = \pi_1(p_1) = \pi_2(p_2)$. Also, for $(g_1,g_2)\in G_1\times G_2$ and $(p_1,p_2)\in P_1\circ P_2$ we define the right action of the product group by $(p_1,p_2)(g_1,g_2) = (p_1g_1,p_2g_2)$. The connections ω_1 and ω_2 may be used to construct a connection for $P_1\circ P_2$. To this end note first that corresponding to the projections π^i: $P_1\circ P_2\to P_i$ given by $\pi^i(p_1,p_2) = p_i$, $i = 1,2$, it can be shown that π^1: $P_1\circ P_2\to P_1$ and π^2: $P_1\circ P_2\to P_2$ are also PFB's with characteristic groups $\{1\}\times\mathrm{U}(1)\cong\mathrm{U}(1)$ and $\mathrm{SU}(2)\times\{1\}\cong\mathrm{SU}(2)$, respectively. Moreover, since the differential π^1_* maps tangent vectors in $T(P_1\circ P_2)$ onto $T(P_1)$, we have that for $X_{(p_1,p_2)}\in T(P_1\circ P_2)$, $\pi^{1*}\omega_1(X_{(p_1,p_2)}) = \omega_1(\pi^1_*X_{(p_1,p_2)}) = \omega_1(X_{p_1})$. However, ω_1 vanishes on horizontal vectors, so $\pi^1_*X_{(p_1,p_2)}$ has to be vertical on the fiber on which ω_1 acts, i.e., the pullback $\mathring{\omega}_1\equiv\pi^{1*}\omega_1$ is a connection for π^2: $P_1\circ P_2\to P_2$. Similarly, $\mathring{\omega}_2\equiv\pi^{2*}\omega_2$ is a connection for π^1: $P_1\circ P_2\to P_1$.

It is now a simple matter to show that $\pi^{1*}\omega_1\oplus\pi^{2*}\omega_2$ is a connection for the spliced bundle π_{12}: $P_1\circ P_2\to M$. One only needs to prove that, given $X = X_1\oplus X_2$ where $X_i\in\mathscr{G}_i$ (the Lie algebra of G_i, $i = 1,2$) and the fundamental vector field $(X_1\oplus X_2)^*$ defined by

$$\begin{aligned}(X_1\oplus X_2)^* &= \frac{d}{dt}((p_1,p_2)(\exp tX_1,\exp tX_2))|_{t=0}\\ &= X_1^*\oplus X_2^*,\end{aligned} \tag{2.22}$$

one gets

$$(\mathring{\omega}_1\oplus\mathring{\omega}_2)(X_1^*\oplus X_2^*) = X_1\oplus X_2, \tag{2.23}$$

and also

$$\begin{aligned}&(\mathring{\omega}_1\oplus\mathring{\omega}_2)(R_{(g_1,g_2)*}(X_1\oplus X_2)^*)\\ &\quad= \mathscr{A}\mathfrak{d}_{(g_1,g_2)^{-1}}(\mathring{\omega}_1\oplus\mathring{\omega}_2)((X_1\oplus X_2)^*).\end{aligned} \tag{2.24}$$

We introduce additional structure on $P_1\circ P_2$ by defining a nondegenerate bundle metric h as follows.

Let k_1 and k_2 be $\mathscr{A}\mathfrak{d}$-invariant metrics on $\mathscr{G}_1$ and $\mathscr{G}_2$, respectively, and set

$$h = \pi_{12}^*g + k_1\mathring{\omega}_1\oplus k_2\mathring{\omega}_2. \tag{2.25}$$

Note that for $X,Y\in T_{(p_1,p_2)}(P_1\circ P_2)$, we have

$$\begin{aligned}h(X,Y) = g(\pi_{12*}X,\pi_{12*}Y) &+ k_1(\mathring{\omega}_1(X),\mathring{\omega}_1(Y))\\ &+ k_2(\mathring{\omega}_2(X),\mathring{\omega}_2(Y)).\end{aligned} \tag{2.26}$$

It is easy to verify that for all $(g_1,g_2)\in G_1\times G_2$, the right action $R_{(g_1,g_2)}$: $P_1\circ P_2\to P_1\circ P_2$ on the fibers is an isometry of $(P_1\circ P_2, h)$.

Relative to this metric, an orthonormal frame at $(p_1,p_2)\in P_1\circ P_2$ is given by

$$\mathring{E}_1,\ldots,\mathring{E}_n,\mathring{E}_{n+1},\ldots,\mathring{E}_{n+4}.$$

Here, $\mathring{E}_1,\ldots,\mathring{E}_n$ [defined on $\pi_{12}^{-1}(U)\in P_1\circ P_2$] are horizontal lifts of the orthonormal basis $\overline{E}_1,\ldots,\overline{E}_n$ on (M,g) such that $\pi_{12*}\mathring{E}_i = \overline{E}_i$ and $(\mathring{\omega}_1\oplus\mathring{\omega}_2)(\mathring{E}_i) = 0$, while $\mathring{E}_{n+\alpha} = l^*_\alpha\oplus 0$ ($\alpha = 1,2,3$) and $\mathring{E}_{n+4} = 0\oplus l^*_4$ are fundamental vertical fields on $P_1\circ P_2$, i.e.,

$$\mathring{\omega}_1(l^*_\alpha\oplus 0) = l_\alpha\in\mathscr{G}_1,\quad \mathring{\omega}_2(0\oplus l^*_4) = l_4\in\mathscr{G}_2. \tag{2.27}$$

Furthermore, l_1,l_2,l_3,l_4 are chosen so that they constitute an orthonormal basis of $\mathscr{G}_1\oplus\mathscr{G}_2$ relative to $k_1\oplus k_2$.

Consequently,

$$\begin{aligned}h_{ij} &= h(\mathring{E}_i,\mathring{E}_j) = g(\overline{E}_i,\overline{E}_j) = g_{ij} = \pm\delta_{ij},\\ &\qquad i,j = 1,\ldots,n;\\ h_{\alpha\beta} &= h(\mathring{E}_{n+\alpha},\mathring{E}_{n+\beta}) = k_1(l_\alpha,l_\beta) = (k_1)_{\alpha\beta} = \delta_{\alpha\beta},\\ &\qquad \alpha,\beta = 1,2,3;\\ h_{44} &= h(\mathring{E}_{n+4},\mathring{E}_{n+4}) = k_2(l_4,l_4) = 1,\end{aligned} \tag{2.28}$$

and all other cross terms in the bundle metric components vanish.

If we set $l_\alpha = -(i/2)\sigma_\alpha$, where the σ_α's obey the Pauli algebra, then the SU(2)-manifold metric k_1 is given explicitly in terms of the corresponding structure constants by

$$(k_1)_{\alpha\beta} = -\tfrac{1}{2}c^{\lambda}{}_{\alpha\gamma}c^{\gamma}{}_{\beta\lambda} = -\tfrac{1}{2}\epsilon_{\lambda\alpha\gamma}\epsilon_{\gamma\beta\lambda} = \delta_{\alpha\beta}, \quad (2.29)$$

where $\epsilon_{\alpha\beta\gamma}$ is the usual Levi-Civita symbol.

Also, recalling that for the infinitesimal generator of U(1) we have $l_4 = i$, it follows that $k_2(l_4,l_4) = k_2(i,i) = 1$.

With this particular choice of an orthonormal basis on a neighborhood of (p_1,p_2), the calculations in the following sections will simplify considerably.

The curvature $\Omega^{\mathring{\omega}_1 \oplus \mathring{\omega}_2} \in \overline{\Lambda}^2(P_1 \circ P_2, \mathscr{G}_1 \oplus \mathscr{G}_2)$ of $\mathring{\omega}_1 \oplus \mathring{\omega}_2$ is given by

$$\begin{aligned}\Omega^{\mathring{\omega}_1 \oplus \mathring{\omega}_2} &\equiv D^{\mathring{\omega}_1 \oplus \mathring{\omega}_2}(\mathring{\omega}_1 \oplus \mathring{\omega}_2)\\ &= d(\mathring{\omega}_1 \oplus \mathring{\omega}_2) + \tfrac{1}{2}[\mathring{\omega}_1 \oplus \mathring{\omega}_2, \mathring{\omega}_1 \oplus \mathring{\omega}_2]\\ &= d\mathring{\omega}_1 + \tfrac{1}{2}[\mathring{\omega}_1,\mathring{\omega}_1] \oplus d\mathring{\omega}_2\\ &= \pi^{1*}(\Omega_1{}^{\omega_1}) \oplus \pi^{2*}(\Omega_2{}^{\omega_2}) \in \overline{\Lambda}^2(P_1 \circ P_2, \mathscr{G}_1)\\ &\quad \oplus \overline{\Lambda}^2(P_1 \circ P_2, \mathscr{G}_2). \end{aligned} \quad (2.30)$$

This result is a particular case of a theorem for spliced bundles which states that for any $\alpha \in \overline{\Lambda}^k(P_1 \circ P_2, \mathscr{G}_1 \oplus \mathscr{G}_2)$ and projections $\overline{\mathscr{G}}_i$: $\overline{\Lambda}^k(P_1 \circ P_2, \mathscr{G}_1 \oplus \mathscr{G}_2) \to \overline{\Lambda}^k(P_1 \circ P_2, \mathscr{G}_i)$ induced by the projections $\mathscr{G}_1 \oplus \mathscr{G}_2 \to \mathscr{G}_i$, there is a unique form $\alpha_i \in \overline{\Lambda}^k(P_i, \mathscr{G}_i)$ such that $\pi^{i*}\alpha_i = \overline{\mathscr{G}}_i(\alpha)$, and $\alpha = \overline{\mathscr{G}}_1(\alpha) \oplus \overline{\mathscr{G}}_2(\alpha) = \pi^{1*}(\alpha_1) \oplus \pi^{2*}(\alpha_2)$.

Moreover, if we let $\mathring{\overline{\varphi}}^1,...,\mathring{\overline{\varphi}}^{n+4}$ be one-forms dual to $\mathring{E}_1,...,\mathring{E}_{n+4}$, and recall that $\Omega^{\mathring{\omega}_1 \oplus \mathring{\omega}_2}$ vanishes on vertical vectors, we can write [making use of (2.18) and (2.21)]

$$\begin{aligned}\Omega^{\mathring{\omega}_1 \oplus \mathring{\omega}_2} &= (\mathring{\Omega}_1)^{\alpha}{}_{ij}(l_\alpha) \otimes (\pi^{1*}\overline{\varphi}^{i}{}_{(1)}) \wedge (\pi^{1*}\overline{\varphi}^{j}{}_{(1)})\\ &\quad \oplus (\mathring{\Omega}_2)_{ij}(i) \otimes (\pi^{2*}\overline{\varphi}^{i}{}_{(2)}) \wedge (\pi^{2*}\overline{\varphi}^{j}{}_{(2)}),\end{aligned} \quad (2.31)$$

where

$$(\mathring{\Omega}_1)^{\alpha}{}_{ij} = (\Omega_1)^{\alpha}{}_{ij} \circ \pi^1, \quad (\mathring{\Omega}_2)_{ij} = (\Omega_2)_{ij} \circ \pi^2.$$

Note that $\pi^1{}_*\mathring{E}_i = E_i^{(1)}$ and $\pi^2{}_*\mathring{E}_i = E_i^{(2)}$. In fact,

$$(\mathring{\omega}_1 \oplus \mathring{\omega}_2)(\mathring{E}_i) = \mathring{\omega}_1(\mathring{E}_i) \oplus \mathring{\omega}_2(\mathring{E}_i) = 0, \quad (2.32)$$

so $\mathring{\omega}_1(\mathring{E}_i) = 0$ and $\mathring{\omega}_2(\mathring{E}_i) = 0$. Moreover, $\pi_{1*}\pi^1_*\mathring{E}_i = (\pi_{12})_*\mathring{E}_i = \overline{E}_i$ and $\omega_1(\pi^1_*\mathring{E}_i) = (\pi^1)^*\omega_1(\mathring{E}_i) = \mathring{\omega}_1(\mathring{E}_i) = 0$. Therefore $\pi^1_*\mathring{E}_i$ is horizontal in $T(P_1)$ and projects onto $\overline{E}_i$; hence $\pi^1_*\mathring{E}_i = E_i^{(1)}$. In a similar way $\pi^2_*\mathring{E}_i = E_i^{(2)}$.

From the above it follows that

$$\begin{aligned}\delta^i_j = \overline{\varphi}^{i}{}_{(1)}(E_j^{(1)}) = \overline{\varphi}^{i}{}_{(1)}(\pi^1{}_*\mathring{E}_j) &= (\pi^{1*}\overline{\varphi}^{i}{}_{(1)})(\mathring{E}_j)\\ &= (\pi^{2*}\overline{\varphi}^{i}{}_{(2)})(\mathring{E}_j),\end{aligned}$$

i.e.,

$$\pi^{1*}\overline{\varphi}^{i}{}_{(1)} = \pi^{2*}\overline{\varphi}^{i}{}_{(2)} = \mathring{\overline{\varphi}}^i. \quad (2.33)$$

Consequently (2.31) becomes

$$\Omega^{\mathring{\omega}_1 \oplus \mathring{\omega}_2} = [(\mathring{\Omega}_1)^{\alpha}{}_{ij}l_\alpha \oplus (\mathring{\Omega}_2)_{ij}i] \otimes \mathring{\overline{\varphi}}^i \wedge \mathring{\overline{\varphi}}^j. \quad (2.34)$$

Note also that (2.27) implies

$$\mathring{\omega}_1^{\alpha} = \mathring{\overline{\varphi}}^{n+\alpha}, \quad (-i\mathring{\omega}_2) = \mathring{\overline{\varphi}}^{n+4}. \quad (2.35)$$

The one other construction that appears in our diagram is the orthonormal bundle of frames Π: $F(P_1 \circ P_2) \to P_1 \circ P_2$, for which the manifold $P_1 \circ P_2$, which has just been described, acts as a base manifold.

If we let $\theta(h) \in \Lambda^1(F(P_1 \circ P_2), \mathscr{O}(r+4,s))$ denote a general connection on $F(P_1 \circ P_2)$, we can now choose the vectors $\mathring{E}_1,...,\mathring{E}_{n+4}$ as an orthonormal frame for the horizontal subspace $T_{(p_1,p_2)}F(P_1 \circ P_2)$ relative to $\theta(h)$.

Furthermore, if $\Omega = D^{\theta(h)}\theta(h) \in \overline{\Lambda}^2(F(P_1 \circ P_2), \mathscr{O}(r+4,s))$ is the curvature of $\theta(h)$, $\tilde{\sigma}$: $\Pi^{-1}(U) \to F(P_1 \circ P_2)$ is a local section determined by the above orthonormal fields, and $\overline{e}_a$ are standard horizontal vectors on $F(P_1 \circ P_2)$ associated with $e_a \in \Re^{n+4}$, then

$$\Omega^{\theta(h)}(\overline{e}_c,\overline{e}_d)(e_b) = R^{a}{}_{bcd}(\tilde{\sigma}(p_1,p_2))e_a. \quad (2.36)$$

Note on the other hand that

$$\Pi_*\tilde{\sigma}_*\mathring{E}_c = \mathring{E}_c = \Pi_*\overline{e}_c. \quad (2.37)$$

Thus $\overline{e}_c$ and $\tilde{\sigma}_*\mathring{E}_c$ differ at most by a vertical vector on $T_{\tilde{\sigma}(p_1,p_2)}F(P_1 \circ P_2)$, but $\Omega^{\theta(g)}$ vanishes on vertical vectors, so

$$\begin{aligned}\Omega^{\theta(h)}(\overline{e}_c,\overline{e}_d) &= \Omega^{\theta(h)}(\tilde{\sigma}_*\mathring{E}_c,\tilde{\sigma}_*\mathring{E}_d)\\ &= (d\overline{\theta}(h) + \overline{\theta}(h) \dot{\wedge} \overline{\theta}(h))(\mathring{E}_c,\mathring{E}_d)\\ &= \Omega^{\overline{\theta}(h)}(\mathring{E}_c,\mathring{E}_d).\end{aligned} \quad (2.38)$$

If we now write

$$(\Omega^{\overline{\theta}(h)})^{a}{}_{b} = \tfrac{1}{2}\mathscr{R}^{a}{}_{bcd}(p_1,p_2)\mathring{\overline{\varphi}}^c \wedge \mathring{\overline{\varphi}}^d, \quad (2.39)$$

or

$$\Omega^{\overline{\theta}(h)} = \tfrac{1}{2}\mathscr{R}^{a}{}_{bcd}(p_1,p_2)e_a \otimes \hat{e}^b \otimes (\mathring{\overline{\varphi}}^c \wedge \mathring{\overline{\varphi}}^d), \quad (2.40)$$

and make use of (2.36), we get

$$R^{a}{}_{bcd}(\tilde{\sigma}(p_1,p_2)) = \mathscr{R}^{a}{}_{bcd}(p_1,p_2). \quad (2.41)$$

This last expression is the equivalent result in Π: $F(P_1 \circ P_2) \to (P_1 \circ P_2)$ to our previous Lemma 1.

In analogy to (2.7) and (2.8) we can use the local section $\tilde{\sigma}$ to define canonical one-forms $\mathring{\varphi} \in \overline{\Lambda}^1(F(P_1 \circ P_2), \Re^{n+4})$ such that

$$\tilde{\sigma}^*\mathring{\varphi} = \mathring{\overline{\varphi}}^a e_a. \quad (2.42)$$

Corresponding to $\mathring{\varphi}$, we have that the torsion two-form $\mathring{\Theta} \in \overline{\Lambda}^2(F(P_1 \circ P_2), \Re^{n+4})$ of $\theta(h)$ is given by

$$\mathring{\Theta}^{\theta(h)} \equiv D^{\theta(h)}\mathring{\varphi} = d\mathring{\varphi} + \theta(h) \dot{\wedge} \mathring{\varphi}. \quad (2.43)$$

Since $\mathring{\Theta}^{\theta(h)}$ is $\Re^{n+4}$ valued, we can write

$$\mathring{\Theta}^{\theta(h)}(\tilde{\sigma}_*\mathring{E}_c,\tilde{\sigma}_*\mathring{E}_d) = S^{a}{}_{cd}(\tilde{\sigma}(p_1,p_2))e_a. \quad (2.44)$$

However, we also have

$$\begin{aligned}\mathring{\Theta}^{\theta(h)}(\tilde{\sigma}_*\mathring{E}_c,\tilde{\sigma}_*\mathring{E}_d) &= (\tilde{\sigma}^*\mathring{\Theta}^{\theta(h)})(\mathring{E}_c,\mathring{E}_d)\\ &\equiv \mathring{\overline{\Theta}}^{\overline{\theta}(h)}(\mathring{E}_c,\mathring{E}_d)\\ &= (d\mathring{\overline{\varphi}} + \overline{\theta}(h) \dot{\wedge} \mathring{\overline{\varphi}})(\mathring{E}_c,\mathring{E}_d)\\ &= [\tfrac{1}{2}\mathscr{S}^{a}{}_{bf}(p_1,p_2)e_a\\ &\quad \otimes (\mathring{\overline{\varphi}}^b \wedge \mathring{\overline{\varphi}}^f)](\mathring{E}_c,\mathring{E}_d),\end{aligned} \quad (2.45)$$

where $\overline{\theta}(h) \in \Lambda^1(P_1 \circ P_2, \mathscr{O}(r+4,s))$ is the pullback with $\tilde{\sigma}$ of the connection $\theta(h)$. Comparing (2.44) and (2.45) we get

$$\mathscr{S}^{a}{}_{cd}(p_1,p_2) = S^{a}{}_{cd}(\tilde{\sigma}(p_1,p_2)), \quad (2.46)$$

which is the analog of Lemma (2.15) derived before.

As we mentioned in the Introduction, a judicious choice on the transformation properties of the torsion tensor is needed in order that the Higgs fields, which will originate from the torsion itself, couple correctly with the Yang–Mills fields.

For this purpose let $g\in \mathrm{SU}(2)\times \mathrm{U}(1)$ and define the linear transformation $t_0(g)\colon \mathscr{G}_1\oplus\mathscr{G}_2\to\mathscr{G}_1\otimes\mathscr{G}_2$, by

$$t_0(g)\,\hat{V}=\rho(g)\circ\mathscr{A}\mathfrak{d}_g V. \tag{2.47}$$

Here $V\in\mathscr{G}_1\oplus\mathscr{G}_2$ and $\rho(g)$ is a 4×4 real-matrix representation of g. The application of $\rho(g)$ on the basis element of $\mathscr{G}_1\oplus\mathscr{G}_2$ is given by

$$\rho(g)\circ\mathring{l}_A\equiv(\rho(g))_A{}^B\mathring{l}_B,\quad A,B=1,\dots,4, \tag{2.48}$$

where $\mathring{l}_\alpha=(l_\alpha\oplus 0)$, $\alpha=1,2,3$, and $\mathring{l}_4=(0\oplus i)$.

It is easy to verify that the following matrices are appropriate real linear representations of the infinitesimal generators of $\mathrm{SU}(2)\times\mathrm{U}(1)$:

$$\rho(\mathring{l}_1)=-\frac{1}{2}\begin{pmatrix}0&0&0&1\\0&0&-1&0\\0&1&0&0\\-1&0&0&0\end{pmatrix}=-\frac{1}{2}\sigma_1\otimes i\sigma_2, \tag{2.49a}$$

$$\rho(\mathring{l}_2)=\frac{1}{2}\begin{pmatrix}0&0&-1&0\\0&0&0&-1\\1&0&0&0\\0&1&0&0\end{pmatrix}=-\frac{1}{2}i\sigma_2\otimes I_2, \tag{2.49b}$$

$$\rho(\mathring{l}_3)=-\frac{1}{2}\begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&-1\\0&0&1&0\end{pmatrix}=-\frac{1}{2}\sigma_3\otimes i\sigma_2, \tag{2.49c}$$

$$\rho(\mathring{l}_4)=-\frac{1}{2}\begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix}=-\frac{1}{2}I_2\otimes i\sigma_2. \tag{2.49d}$$

Given $t_0(g)$ we can define a transformation $t(g)\colon T(P_1\circ P_2)\to T(P_1\circ P_2)$ in the following way: Let X be a vector field in $T(P_1\circ P_2)$, and at each $(p_1,p_2)\in P_1\circ P_2$ let $X_H(p_1,p_2)$ and $X_V(p_1,p_2)$ be the horizontal and vertical components of $X(p_1,p_2)$. Write $X_V(p_1,p_2)=V^*_{p_1,p_2}$ for some $V\in\mathscr{G}_1\oplus\mathscr{G}_2$. We then require that

$$t_{(p_1,p_2)}(g)(X_H(p_1,p_2))=X_H(p_1,p_2), \tag{2.50a}$$

and

$$t_{(p_1,p_2)}(g)(X_V(p_1,p_2))=(t_0(g)V)^*_{(p_1,p_2)}. \tag{2.50b}$$

Finally, we also have the isomorphism $\hat{t}(g)\colon TF(P_1\circ P_2)\to TF(P_1\circ P_2)$, which is in turn induced by $t(g)$ according to the commutative diagram

$$\begin{array}{ccc} TF(P_1\circ P_2) & \xrightarrow{\ \hat{t}(g)\ } & TF(P_1\circ P_2)\\ \big\downarrow \Pi_* & & \Pi_*\big\downarrow\\ T(P_1\circ P_2) & \xrightarrow{\ t(g)\ } & T(P_1\circ P_2) \end{array}.$$

Hence

$$\Pi_*(\hat{t}(g)\tilde{X})=t(g)(\Pi_*\tilde{X}),\quad \forall\tilde{X}\in TF(P_1\circ P_2). \tag{2.51}$$

Making use of the definitions (2.47)–(2.51), and letting $\hat{R}_g$ denote the right action diffeomorphism $\hat{R}_g\colon F(P_1\circ P_2)\to F(P_1\circ P_2)$, we now prescribe the following transformation for torsion:

$$\begin{aligned}\hat{R}_g{}^*\mathring{\Theta}(\tilde{X},\tilde{Y})&=\hat{t}(g^{-1})\circ\mathring{\Theta}(\tilde{X},\tilde{Y})\\&=\mathring{\Theta}(\hat{t}(g^{-1})\tilde{X},\hat{t}(g^{-1})\tilde{Y}).\end{aligned} \tag{2.52}$$

In order to calculate $\hat{R}^*_g\mathring{\Theta}(\tilde{\sigma}_*X,\tilde{\sigma}_*Y)$, with $X,Y\in T(P_1\circ P_2)$, note first that we can identify $\mathfrak{R}^{n+4}$ with $\mathfrak{R}^n\oplus\mathscr{G}_1\oplus\mathscr{G}_2$ by means of a vector space isomorphism

$$(e_i,<\ ,\ >)\to(e_i,<\ ,\ >),\quad i=1,\dots,n,$$

$$(e_{n+A},<\ ,\ >)\to(\mathring{l}_A,k_1\oplus k_2),$$

where $\{e_a,\ a=1,\dots,n+4\}$ is the canonical basis of $\mathfrak{R}^{n+4}$, $<\ ,\ >$ denotes the standard scalar product in $\mathfrak{R}^{n+4}$, and $\mathring{l}_A$, $A=1,\dots,4$ are the basis elements of the Lie algebra for $\mathscr{G}_1\oplus\mathscr{G}_2$ which we introduced earlier. This in turn allows us to define an application

$$a\colon \mathrm{SU}(2)\times\mathrm{U}(1)\to O(\mathfrak{R}^n\oplus\mathscr{G}_1\oplus\mathscr{G}_2)\cong O(\mathfrak{R}^{n+4}),$$

such that

$$a(g)=\begin{pmatrix}Id_{\mathfrak{R}^n}&0\\0&\mathscr{A}\mathfrak{d}_g\end{pmatrix}. \tag{2.53}$$

It then follows that

$$R_{a(g^{-1})}(\tilde{\sigma}\circ R_g)=\hat{R}_g(\tilde{\sigma}). \tag{2.54}$$

In fact, acting with the left side of the above expression on e_i gives

$$\begin{aligned}&R_{a(g^{-1})}(\tilde{\sigma}\circ R_g)(p_1,p_2)(e_i)\\&=\tilde{\sigma}_{(p_1,p_2)g}\circ a(g^{-1})(e_i)=\tilde{\sigma}_{(p_1,p_2)g}(e_i)\\&=\mathring{E}_i((p_1,p_2)g)=(R_{g*}\mathring{E}_i)_{(p_1,p_2)}\\&=R_{g*}\tilde{\sigma}_{(p_1,p_2)}(e_i)=\hat{R}_g(\tilde{\sigma})(p_1,p_2)(e_i),\end{aligned}$$

and acting with the left side of (2.54) on e_{n+A} yields

$$\begin{aligned}&R_{a(g^{-1})}(\tilde{\sigma}\circ R_g)(p_1,p_2)(e_{n+A})\\&=\tilde{\sigma}_{(p_1,p_2)g}\circ(\mathscr{A}\mathfrak{d}_{g^{-1}}l_A)=(\mathscr{A}\mathfrak{d}_{g^{-1}}l_A)^*_{(p_1,p_2)g}\\&=R_{g*}(l_A)^*_{(p_1,p_2)}=R_{g*}\tilde{\sigma}_{(p_1,p_2)}(e_{n+a})\\&=\hat{R}_g(\tilde{\sigma})(p_1,p_2)(e_{n+A}).\end{aligned}$$

Equation (2.54) then follows, and therefore

$$\begin{aligned}&\hat{R}^*_g\mathring{\Theta}(\tilde{\sigma}_*X,\tilde{\sigma}_*Y)\\&\equiv\mathring{\Theta}(\hat{R}_{g*}\tilde{\sigma}_*X,\hat{R}_{g*}\tilde{\sigma}_*Y)\\&=\mathring{\Theta}(R_{a(g^{-1})*}\tilde{\sigma}_*R_{g*}X,R_{a(g^{-1})*}\tilde{\sigma}_*R_{g*}Y)\\&=a(g)\cdot\underline{\underline{\mathring{\Theta}}}(R_{g*}X,R_{g*}Y).\end{aligned} \tag{2.55}$$

Consequently (2.52) is equivalent to

$$\mathring{\overline{\Theta}}(R_{g*}X,R_{g*}Y) = a(g^{-1})\cdot\mathring{\Theta}(\hat{t}(g^{-1})\tilde{\sigma}_*X,\hat{t}(g^{-1})\tilde{\sigma}_*Y). \tag{2.56}$$

Furthermore, $\hat{t}(g^{-1})\tilde{\sigma}_*X$ and $\tilde{\sigma}_*t(g^{-1})X$ are both in $T_{(p_1,p_2)}F(P_1\circ P_2)$, and

$$\Pi_*\hat{t}(g^{-1})\tilde{\sigma}_*X = t(g^{-1})\Pi_*\tilde{\sigma}_*X$$
$$= t(g^{-1})X = \Pi_*\sigma_*t(g^{-1})X.$$

Thus $\hat{t}(g^{-1})\tilde{\sigma}_*X$ and $\tilde{\sigma}_*t(g^{-1})X$ differ at most by a vertical vector. However, since $\mathring{\Theta}$ vanishes on vertical vectors, we may write (2.56) as

$$\mathring{\overline{\Theta}}_{(p_1,p_2)g}(R_{g*}X,R_{g*}Y) = a(g^{-1})\cdot\mathring{\overline{\Theta}}_{(p_1,p_2)}(t(g^{-1})X,t(g^{-1})Y). \tag{2.57}$$

We can carry our analysis of the transformation properties of the torsion further by recalling that the difference between any two connections vanishes on vertical vectors. Consequently, if we write $\theta(h) = \tau(h) + \theta(h)_{\rm LC}$, where $\theta(h)_{\rm LC}$ is the unique Levi-Civita connection and $\tau(h)\in\overline{\Lambda}^1(F(P_1\circ P_2),\mathscr{O}(r+4,s))$, we have

$$\begin{aligned}\mathring{\Theta}^{\theta(h)} &= d\mathring{\varphi} + (\theta(h)_{\rm LC} + \tau(h))\dot{\wedge}\mathring{\varphi}\\ &= \mathring{\Theta}^{\theta(h)_{\rm LC}} + \tau(h)\dot{\wedge}\mathring{\varphi}\\ &= \tau(h)\dot{\wedge}\mathring{\varphi}\end{aligned} \tag{2.58}$$

(since the torsion from the Levi-Civita connection vanishes).

If we now let $\tilde{\sigma}^*\tau(h) = \overline{\tau}(h)$, and observe that

$$(\overline{\tau}\dot{\wedge}\mathring{\overline{\varphi}})(X,Y) = \overline{\tau}(X)\cdot\mathring{\overline{\varphi}}(Y) - \overline{\tau}(Y)\cdot\mathring{\overline{\varphi}}(X),$$

we obtain from (2.56) the result

$$\begin{aligned}&\overline{\tau}_{(p_1,p_2)g}(R_{g*}X)\cdot\mathring{\overline{\varphi}}_{(p_1,p_2)g}(R_{g*}Y)\\ &\quad - \overline{\tau}_{(p_1,p_2)g}(R_{g*}Y)\cdot\mathring{\overline{\varphi}}_{(p_1,p_2)g}(R_{g*}X)\\ &= a(g^{-1})\cdot\overline{\tau}_{(p_1,p_2)}(t(g^{-1})X)\cdot\mathring{\overline{\varphi}}_{(p_1,p_2)}(t(g^{-1})Y)\\ &\quad - a(g^{-1})\cdot\overline{\tau}_{(p_1,p_2)}(t(g^{-1})Y)\cdot\mathring{\overline{\varphi}}_{(p_1,p_2)}(t(g^{-1})X).\end{aligned} \tag{2.59}$$

Furthermore, since $\overline{\tau}\in\Lambda^1(P_1\circ P_2,\mathscr{O}(r+4,s))$, we can write [comparing with (2.45)]

$$\overline{\tau}_{(p_1,p_2)} = \tfrac{1}{2}\mathscr{S}^a{}_{bc}(p_1,p_2)\mathring{\overline{\varphi}}{}^b{}_{(p_1,p_2)}e_a\otimes\hat{e}^c. \tag{2.60}$$

Substituting (2.60) in (2.59) and making use of (2.50), we get the following.

(i) For $X=\mathring{E}_i$, $Y=\mathring{E}_j$ (X,Y both horizontal)

$$\mathscr{S}^a{}_{ij}((p_1,p_2)g) = a(g^{-1})^a{}_b\mathscr{S}^b{}_{ij}(p_1,p_2), \tag{2.61}$$

with $a(g^{-1})^a{}_b = \hat{e}^a(a(g^{-1})(e_b))$.

(ii) For $X=\mathring{E}_i$, $Y=\mathring{E}_{n+A}$ (X horizontal and Y vertical),

$$\mathscr{S}^a{}_{i\,n+A}((p_1,p_2)g) = (\rho(g^{-1}))_A{}^Ba(g^{-1})^a{}_b\mathscr{S}^b{}_{i\,n+B}(p_1,p_2). \tag{2.62}$$

(iii) For $X=\mathring{E}_{n+A}$, $Y=\mathring{E}_{n+B}$ (X,Y both vertical),

$$\begin{aligned}&\mathscr{S}^a{}_{n+A\;n+B}((p_1,p_2)g)\\ &= (\rho(g^{-1}))_A{}^C(\rho(g^{-1}))_B{}^Da(g^{-1})^a{}_b\\ &\quad\times\mathscr{S}^b{}_{n+C\;n+D}(p_1,p_2).\end{aligned} \tag{2.63}$$

To be mathematically more precise, in (2.58) we should actually write $\theta(h) = \tau_1(h) + \tau_2(h) + (\theta(h))_{\rm LC}$. So that $\mathring{\Theta}^{\theta(h)} = \tau_1(h)\dot{\wedge}\mathring{\varphi} + \tau_2(h)\dot{\wedge}\mathring{\varphi} = \mathring{\Theta}^{\theta_1(h)} + \mathring{\Theta}^{\theta_2(h)}$, where $\theta_i(h) = \tau_i(h) + (\theta(h))_{\rm LC}$, $i=1,2$. We would then have that $\mathring{\Theta}^{\theta_1(h)}$ transforms according to (2.52), and for $\mathring{\Theta}_2 = \mathring{\Theta}^{\theta_2(h)}$ we have

$$\hat{R}{}^*_g\mathring{\Theta}_2(X,Y) = a(g)\mathring{\Theta}_2(\hat{t}_0(g^{-1})X,\hat{t}_0(g^{-1})Y)$$

with $t_0(g) = \mathscr{A}\mathfrak{d}_g$. Observe, however, that $\mathscr{S}^a{}_{bc} = \overline{\mathscr{S}}^a{}_{bc} + \overline{\overline{\mathscr{S}}}{}^a{}_{bc}$, where $\overline{\mathscr{S}}^a{}_{bc}$ and $\overline{\overline{\mathscr{S}}}{}^a{}_{bc}$ are the associated tensors of $\mathring{\Theta}^{\theta_1(h)}$ and $\mathring{\Theta}^{\theta_2(h)}$, respectively. The first one transforms according to (2.60)–(2.63), while the second one has to be constant on each fiber.

With these basic definitions and results we are now ready to compute the components relative to $\mathring{E}_1,\dots,\mathring{E}_n$, $\mathring{E}_{n+1},\dots,\mathring{E}_{n+4}$ of the curvature and torsion tensors for the metric h on $P_1\circ P_2$. This we shall do in the following section.

III. THE UNIFIED LAGRANGIAN

Recall that the components of the Riemann tensor on $P_1\circ P_2$ are related to the connection one-forms $\overline{\theta}(h) = \tilde{\sigma}^*\theta(h)$ by means of (2.38) and (2.39). In matrix notation these equations lead to

$$\tfrac{1}{2}\mathscr{R}^a{}_{bcd}(p_1,p_2)\mathring{\overline{\varphi}}{}^c\wedge\mathring{\overline{\varphi}}{}^d = d\overline{\theta}(h)^a{}_b + \overline{\theta}(h)^a{}_e\wedge\overline{\theta}(h)^e{}_b. \tag{3.1}$$

Therefore, in order to evaluate $\mathscr{R}^a{}_{bcd}$ we need first to calculate the various matrix terms $\overline{\theta}(h)^a{}_b$ for $1\leqslant a,\ b\leqslant n+4$ relative to the choice of orthonormal basis described in the preceding section. To do this we make use of (2.43) and (2.13) to write

$$d\mathring{\overline{\varphi}}{}^a = \mathring{\overline{\Theta}}{}^a - \overline{\theta}(h)^a{}_b\wedge\mathring{\overline{\varphi}}{}^b, \tag{3.2}$$

$$d\overline{\varphi}{}^i_M = \overline{\Theta}{}^i - \overline{\theta}(g)^i{}_j\wedge\overline{\varphi}{}^j_M. \tag{3.3}$$

Moreover, since $\overline{\varphi}{}^i_M(\overline{E}_j) = \delta^i_j = \overline{\varphi}{}^i_M(\pi_{12*}\mathring{E}_j)$, we have that $\pi_{12}{}^*\overline{\varphi}{}^i_M = \mathring{\overline{\varphi}}{}^i$. Consequently, pulling back (3.3) with π_{12} yields

$$d\mathring{\overline{\varphi}}{}^i = \mathring{\overline{\Theta}}{}^i(g) - \pi_{12}{}^*\overline{\theta}(g)^i{}_j\wedge\mathring{\overline{\varphi}}{}^j. \tag{3.4}$$

Note parenthetically that this last expression implies

$$-\mathring{\overline{\varphi}}{}^i([\mathring{E}_j,\mathring{E}_k]) = \underline{S}^i{}_{jk}(x). \tag{3.5}$$

Thus if we impose the restriction of vanishing torsion on the base space M, i.e.,

$$\underline{S}^i{}_{jk} = 0, \tag{3.6}$$

then it immediately follows that the commutator of the basis vectors $\mathring{E}_i$, $i=1,\dots,n$, has to be vertical. We shall use this result later on for deriving the form of the covariant derivative of the Higgs fields.

Now, from (2.35) and (3.2) we have

$$d(-i\mathring{\omega}_2) = \mathring{\overline{\Theta}}{}^{n+4} - \overline{\theta}(h)^{n+4}{}_b\wedge\mathring{\overline{\varphi}}{}^b,$$
$$d\mathring{\omega}^\alpha_1 = \mathring{\overline{\Theta}}{}^{n+\alpha} - \overline{\theta}(h)^{n+\alpha}{}_b\wedge\mathring{\overline{\varphi}}{}^b.$$

Substituting on the left side of these equations the pullback with π^2 and π^1, respectively, of (2.20) and (2.17), gives

$$\tfrac{1}{2}(\mathring{\Omega}_2)_{ij}\mathring{\overline{\varphi}}{}^i\wedge\mathring{\overline{\varphi}}{}^j = \mathring{\overline{\Theta}}{}^{n+4} - \overline{\theta}(h)^{n+4}{}_b\wedge\mathring{\overline{\varphi}}{}^b, \tag{3.7}$$

$$\begin{aligned}&\tfrac{1}{2}(\mathring{\Omega}_1)^\alpha{}_{ij}\mathring{\overline{\varphi}}{}^i\wedge\mathring{\overline{\varphi}}{}^j - \tfrac{1}{2}c^\alpha{}_{\beta\gamma}\mathring{\overline{\varphi}}{}^\beta\wedge\mathring{\overline{\varphi}}{}^\gamma\\ &= \mathring{\overline{\Theta}}{}^{n+\alpha} - \overline{\theta}(h)^{n+\alpha}{}_b\wedge\mathring{\overline{\varphi}}{}^b.\end{aligned} \tag{3.8}$$

Furthermore, since $\bar{\theta}(h)$ is $\mathscr{O}(r+4,s)$-valued, the matrix elements $\bar{\theta}(h)^a{}_b$ must satisfy the constraint

$$\bar{\theta}(h)_{ab}+\bar{\theta}(h)_{ba}=0. \qquad (3.9)$$

This condition is fulfilled if we require that

$$\bar{\theta}(h)^{n+A}{}_{n+B} = -(\rho(\mathring{l}_C))^A{}_B\bar{\mathring{\varphi}}^{n+C}, \qquad (3.10)$$

where the matrices $\rho(\mathring{l}_C)$ are the infinitesimal generators of SU(2)×U(1) which we explicitly displayed in (2.49a)–(2.49d).

It is important to note here the fact that (3.10) does not fully specify the connection. The remaining freedom is clearly manifest in the equations that relate some of the connection coefficients to undefined components of the torsion tensor.

Since in the end we want the Higgs fields to originate geometrically from torsion, we make the additional assumption that our connection is semisymmetric.[7] It has been shown (cf. Theorems 1 and 2 of Ref. 8) that this assumption is tantamount to essentially taking only the first two terms in a unique decomposition for the torsion tensor. Introducing further terms resulting from a spin-tensor H (described in the paper referred to above) provides a way to generalize our results in the sense that additional fields appear, whose physical meaning remains to be determined, and would make it also possible to investigate nonmetric theories within the framework of our formalism.

In the context of the semisymmetry assumption we have that

$$\mathscr{S}^{n+\alpha}{}_{n+4\,i} = \mathscr{S}^{n+4}{}_{n+\alpha\,i} = \mathscr{S}^{i}{}_{n+4\,n+\alpha} = \mathscr{S}^{i}{}_{n+\alpha\,n+\beta} = 0, \qquad (3.11)$$

and since by our argument following Eq. (2.63) it is reasonable to assume that each of the connections $\theta_1(h)$ and $\theta_2(h)$ in the decomposition $\mathring{\Theta}^{\theta(h)} = \mathring{\Theta}^{\theta_1(h)} + \mathring{\Theta}^{\theta_2(h)}$ are to be semisymmetric, it follows that the torsion components $\bar{\mathscr{S}}^{n+\alpha}{}_{n+\beta\,i}$, $\bar{\mathscr{S}}^{n+4}{}_{n+4\,i}$, $\bar{\bar{\mathscr{S}}}^{n+\alpha}{}_{n+\beta\,i}$, $\bar{\bar{\mathscr{S}}}^{n+4}{}_{n+4\,i}$, must be proportional to quantities of the form $\delta^{n+\alpha}_{n+\beta}\Phi_i$ and $\delta^{n+4}_{n+4}\Phi_i$, respectively, where Φ_i is an additional vector field. Using (2.62) we obtain that if $\bar{\mathscr{S}}^{n+\alpha}{}_{n+\beta\,i} = c\delta^\alpha_\beta\Phi_i \neq 0$, then $c\delta^\beta_\alpha\Phi_i = \rho(g^{-1})_\alpha{}^\lambda a(g^{-1})^\beta{}_\sigma\delta^\sigma_\lambda\Phi_i c$, from where it follows that $\rho(g^{-1})_\alpha{}^\lambda a(g^{-1})^\beta{}_\lambda = \delta^\beta_\alpha$, i.e., $\rho(g^{-1}) = a(g)$, which is impossible because G is a non-Abelian group. Thus we have $\bar{\mathscr{S}}^{n+\alpha}{}_{n+\beta\,i} = 0$ and, as a consequence, $\mathscr{S}^{n+\alpha}{}_{n+\beta\,i} = \bar{\bar{\mathscr{S}}}^{n+\alpha}{}_{n+\beta\,i}$ must be proportional to $\delta^\alpha_\beta\Phi_i$ with Φ_i constant on each fiber. This vector field could be considered if one wished to generalize our present results. However, for the purposes stated above, we choose to set Φ_i equal to zero in view of the fact that it is obviously not a Higgs field, and that we can use the remaining freedom that we still have in selecting our connection.

Using (3.10) and (3.11) in (3.7) and evaluating on $(\mathring{E}_{n+4},\mathring{E}_{n+4})$, $(\mathring{E}_{n+4},\mathring{E}_{n+\gamma})$, $(\mathring{E}_{n+4},\mathring{E}_i)$, $(\mathring{E}_{n+\alpha},\mathring{E}_i)$, $(\mathring{E}_{n+\alpha},\mathring{E}_{n+\beta})$, and $(\mathring{E}_i,\mathring{E}_j)$, yields, respectively,

$$\text{(a) } \mathscr{S}^{n+4}{}_{n+4\,n+4} = 0,$$
$$\text{(b) } \mathscr{S}^{n+4}{}_{n+4\,n+\gamma} = -(\rho(\mathring{l}_4))^4{}_\gamma, \qquad (3.12)$$
$$\text{(a) } \bar{\theta}(h)^{n+4}{}_i(\mathring{E}_{n+4}) = 0,$$
$$\text{(b) } \bar{\theta}(h)^{n+4}{}_i(\mathring{E}_{n+\alpha}) = 0, \qquad (3.13)$$

$$\mathscr{S}^{n+4}{}_{n+\alpha\,n+\beta} = (\rho(\mathring{\ell}_\beta))^4{}_\alpha - (\rho(\mathring{\ell}_\alpha))^4{}_\beta, \qquad (3.14)$$

$$(\mathring{\Omega}_2)_{ij} = \mathscr{S}^{n+4}{}_{ij} - \bar{\theta}(h)^{n+4}{}_j(\mathring{E}_i) + \bar{\theta}(h)^{n+4}_i(\mathring{E}_j). \qquad (3.15)$$

Making use of (3.13) and (3.15) it immediately follows that

$$\bar{\theta}(h)^{n+4}{}_i = \tfrac{1}{2}[(\mathring{\Omega}_2)_{ij} - \mathscr{S}^{n+4}{}_{ij}]\bar{\mathring{\varphi}}^j - \mathscr{S}_{ij}{}^{n+4}\bar{\mathring{\varphi}}^j, \qquad (3.16a)$$

with

$$\mathscr{S}_{ij}{}^{n+4} = \mathscr{S}_{ji}{}^{n+4}. \qquad (3.16b)$$

Also, because of (3.9),

$$\bar{\theta}(h)^i{}_{n+4} = -\tfrac{1}{2}[(\mathring{\Omega}_2)^i{}_j - \mathscr{S}_{n+4}{}^i{}_j]\bar{\mathring{\varphi}}^j + \mathscr{S}_j{}^i{}_{n+4}\bar{\mathring{\varphi}}^j. \qquad (3.17)$$

Similarly, evaluating (3.8) on $(\mathring{E}_{n+\gamma},\mathring{E}_{n+4})$, $(\mathring{E}_{n+4},\mathring{E}_{n+4})$, $(\mathring{E}_{n+4},\mathring{E}_i)$, $(\mathring{E}_{n+\beta},\mathring{E}_i)$, $(\mathring{E}_{n+\beta},\mathring{E}_{n+\gamma})$, and $(\mathring{E}_i,\mathring{E}_j)$, we get, respectively,

$$\text{(a) } \mathscr{S}^{n+\alpha}{}_{n+\gamma\,n+4} = -(\rho(\mathring{l}_\gamma))^\alpha{}_4 + (\rho(\mathring{l}_4))^\alpha{}_\gamma,$$
$$\text{(b) } \mathscr{S}^{n+\alpha}{}_{n+4\,n+4} = 0, \qquad (3.18)$$
$$\text{(a) } \bar{\theta}(h)^{n+\alpha}{}_i(\mathring{E}_{n+4}) = 0,$$
$$\text{(b) } \bar{\theta}(h)^{n+\alpha}{}_i(\mathring{E}_{n+\beta}) = 0, \qquad (3.19)$$

$$\mathscr{S}^{n+\alpha}{}_{n+\beta\,n+\gamma} = (\rho(\mathring{l}_\gamma))^\alpha{}_\beta - (\rho(\mathring{l}_\beta))^\alpha{}_\gamma - c^\alpha{}_{\beta\gamma}, \qquad (3.20)$$

$$(\mathring{\Omega}_1)^\alpha{}_{ij} = \mathscr{S}^{n+\alpha}{}_{ij} - \bar{\theta}(h)^{n+\alpha}{}_j(\mathring{E}_i) + \bar{\theta}(h)^{n+\alpha}{}_i(\mathring{E}_j). \qquad (3.21)$$

It is obvious from (3.19) and (3.21) that

$$\bar{\theta}(h)^{n+\alpha}{}_i = \tfrac{1}{2}[(\mathring{\Omega}_1)^\alpha{}_{ij} - \mathscr{S}^{n+\alpha}{}_{ij}]\bar{\mathring{\varphi}}^j - \mathscr{S}_{ij}{}^{n+\alpha}\bar{\mathring{\varphi}}^j, \qquad (3.22a)$$

with

$$\mathscr{S}_{ij}{}^{n+\alpha} = \mathscr{S}_{ji}{}^{n+\alpha}. \qquad (3.22b)$$

Moreover, from (3.9) we also have

$$\bar{\theta}(h)^i{}_{n+\alpha} = -\tfrac{1}{2}[(\mathring{\Omega}_1)_\alpha{}^i{}_j - \mathscr{S}_{n+\alpha}{}^i{}_j]\bar{\mathring{\varphi}}^j + \mathscr{S}^i{}_{j\,n+\alpha}\bar{\mathring{\varphi}}^j. \qquad (3.23)$$

The remaining expressions that we need for the connection coefficients are obtained by noting that (3.2) also implies that

$$d\bar{\mathring{\varphi}}^i = \mathscr{S}^i{}_{j\,n+A}\bar{\mathring{\varphi}}^j\wedge\bar{\mathring{\varphi}}^{n+A} - \bar{\theta}(h)^i{}_j\wedge\bar{\mathring{\varphi}}^j - \bar{\theta}(h)^i{}_{n+A}\wedge\bar{\mathring{\varphi}}^{n+A}. \qquad (3.24)$$

Thus evaluating on $(\mathring{E}_j,\mathring{E}_{n+4})$ results in

$$\bar{\theta}(h)^i{}_j(\mathring{E}_{n+4}) = -\tfrac{1}{2}[(\mathring{\Omega}_2)^i{}_j - \mathscr{S}_{n+4}{}^i{}_j]. \qquad (3.25)$$

Finally, note that substituting in (3.24) the values for $\bar{\theta}(h)^i{}_{n+\alpha}$ and $\bar{\theta}(h)^i{}_{n+4}$ given by (3.23) and (3.17), and equating the result to (3.4), yields

$$\bar{\theta}(h)^i{}_j = \pi^*_{12}\bar{\theta}(g)^i{}_j - \tfrac{1}{2}[(\mathring{\Omega}_1)_\alpha{}^i{}_j - \mathscr{S}_{n+\alpha}{}^i{}_j]\mathring{\varphi}^{n+\alpha} - \tfrac{1}{2}[(\mathring{\Omega}_2)^i{}_j - \mathscr{S}_{n+4}{}^i{}_j]\bar{\mathring{\varphi}}^{n+4}. \qquad (3.26)$$

As we mentioned previously, the Higgs fields originate directly from torsion by assuming that the connection $\bar{\theta}(h)$ is semisymmetric. With this in mind, we make the following additional ansatz on the torsion:

$$\mathscr{S}_{ij}{}^{n+4} = (1/n) g_{ij} \Phi^A, \tag{3.27}$$

and

$$\mathscr{S}^{n+\alpha}{}_{ij} = (\mathring{\Omega}_1)^\alpha{}_{ij}, \quad \mathscr{S}^{n+4}{}_{ij} = (\mathring{\Omega}_2)_{ij}. \tag{3.28}$$

Note that with these last assumptions the connection matrices $\bar{\theta}(h)^a{}_b$, as given by (3.9), (3.10), (3.16), (3.22), and (3.26), are uniquely specified. Also, as we will show in Sec. IV, the four scalar fields Φ^A introduced in (3.27) can be identified with the real Higgs fields.

We now have all the ingredients that are needed to evaluate from (3.1) the components $\mathscr{R}^a{}_{bcd}$ of the curvature tensor of the metric h on $P_1 \circ P_2$ relative to our orthonormal basis $\mathring{E}_1, \ldots, \mathring{E}_{n+4}$. Since the calculation, although lengthy, is fairly straightforward, we only state the final results here:

$$\begin{aligned}
\mathscr{R}^{n+\alpha}{}_{n+\beta\, n+\mu\, n+\nu} &= 2\epsilon_{\gamma\mu\nu} (\rho(\mathring{l}_\gamma))^\alpha{}_\beta, \\
\mathscr{R}^{n+\alpha}{}_{n+\beta\, ij} &= -(\rho(\mathring{l}_\gamma))^\alpha{}_\beta (\mathring{\Omega}_1)^\gamma{}_{ij} - (\rho(\mathring{l}_4))^\alpha{}_\beta (\mathring{\Omega}_2)_{ij}, \\
\mathscr{R}^{n+4}{}_{n+\alpha\, n+\mu\, n+\nu} &= 2\epsilon_{\beta\mu\nu} (\rho(\mathring{l}_\beta))^4{}_\alpha, \\
\mathscr{R}^{n+4}{}_{n+\alpha\, ij} &= -(\rho(\mathring{l}_\beta))^4{}_\alpha (\mathring{\Omega}_1)^\beta{}_{ij} - (\rho(\mathring{l}_4))^4{}_\alpha (\mathring{\Omega}_2)_{ij},
\end{aligned} \tag{3.29}$$

$$\begin{aligned}
\mathscr{R}^{n+4}{}_{ikj} &= (1/n) g_{ik} \mathring{E}_j[\Phi^4] - (1/n) g_{ij} \mathring{E}_k[\Phi^4], \\
\mathscr{R}^{n+\alpha}{}_{ijk} &= (1/n) g_{ik} \mathring{E}_j[\Phi^\alpha] - (1/n) g_{ij} \mathring{E}_k[\Phi^\alpha], \\
\mathscr{R}^i{}_{jkm} &= \underline{R}^i{}_{jkm} + (1/n)[\delta^i_m g_{jk} - \delta^i_k g_{jm}] \Phi_A \Phi^A,
\end{aligned}$$

and all other components vanish. In the above expressions terms of the form $\mathring{E}_i[\Phi^A] = d\Phi^A(\mathring{E}_i)$ denote directional derivatives. In Sec. IV we will show that this directional derivative is a covariant derivative, i.e.,

$$\mathring{E}_i[\Phi^A] = d\Phi^A(\mathring{E}_i) \equiv D_i \Phi^A. \tag{3.30}$$

Moreover, we will also establish the relation between the covariant derivatives of our four real-scalar fields Φ_A and the covariant derivative of the Higgs complex spin doublet as it commonly appears in the electroweak model.

For the construction of the Lagrangian density we also need the nonvanishing components of the Ricci tensor on $P_1 \circ P_2$ as well as the Ricci scalar. These follow directly from (3.29) and are given by

$$\begin{aligned}
\mathscr{R}_{jm} &= \underline{R}_{jm} + [(1-n)/n^2] g_{jm} \Phi_A \Phi^A, \\
\mathscr{R}_{n+4\, n+\alpha} &= 2\epsilon_{\alpha\gamma\beta} (\rho(\mathring{l}_\gamma))^\beta{}_4, \\
\mathscr{R}_{n+4\, i} &= [(1-n)/n] \mathring{E}_i[\Phi_4], \\
\mathscr{R}_{n+\alpha\, n+\beta} &= 2\epsilon_{\lambda\gamma\beta} (\rho(\mathring{l}_\lambda))^\gamma{}_\alpha, \\
\mathscr{R}_{n+\alpha\, i} &= [(1-n)/n] \mathring{E}_i[\Phi_\alpha], \\
\mathscr{R} &= \underline{R} + [(1-n)/n] \Phi^A \Phi_A + 2.
\end{aligned} \tag{3.31}$$

General Lagrangian density: We construct the most general G-invariant Lagrangian density on $P_1 \circ P_2$ up to quadratic terms in the Riemann, Ricci, and torsion tensors as well as in the Ricci scalar by adding up all the G-invariant terms that can be obtained from Eqs. (3.14), (3.18), (3.20), (3.27)–(3.31). The result is

$$\begin{aligned}
\mathscr{L} = \frac{\sqrt{|g|}}{V_I} \Bigg\{ &\alpha_0 \Big(\underline{R} - \frac{n-1}{n} \Phi^A \Phi_A + 2\Big) + \alpha_1 \Big[\frac{(n-1)^2}{n^2} (\Phi^A \Phi_A)^2 - \frac{2(n-1)}{n} \underline{R} \Phi_A \Phi^A - 4 \frac{(n-1)}{n} \Phi^A \Phi_A + (\underline{R} + 2)^2 \Big] \\
&+ \alpha_2 \Big[\underline{R}_{ijkm} \underline{R}^{ijkm} - \frac{4}{n^2} \underline{R} (\Phi_A \Phi^A) + \frac{2}{n^3} (n-1) (\Phi_A \Phi^A)^2 \Big] - \alpha_3 (\mathring{\Omega}_1)^\gamma{}_{ij} (\mathring{\Omega}_1)_\gamma{}^{ij} - \alpha_4 (\mathring{\Omega}_2)_{ij} (\mathring{\Omega}_2)^{ij} + \alpha_5 (D^i \Phi^A)(D_i \Phi_A) \\
&+ \alpha_6 \Big[\underline{R}_{ij} \underline{R}^{ij} - \frac{2}{n^2} (n-1) \underline{R} (\Phi_A \Phi^A) + \frac{(n-1)^2}{n^3} (\Phi_A \Phi^A)^2 \Big] + \alpha_7 \frac{n-1}{n} \Phi_A \Phi^A + K \Bigg\},
\end{aligned} \tag{3.32}$$

where V_I is the volume of the n–4 compact "internal" coordinates of the base manifold, and K is a constant that contributes to the cosmological constant.

Before proceeding with the proper dimensioning and physical interpretation of the different terms and parameters in the above Lagrangian density, we show explicitly that all the entries in (3.32) are indeed G invariant. Clearly the terms containing the several contractions of the Riemann tensor are G invariant since, according to (2.9) or (2.10), the components $\underline{R}^h{}_{ijk}$ are defined on M and are therefore independent of the choice of point on the fiber. The quantities $(\mathring{\Omega}_1)_{\alpha ij} (\mathring{\Omega}_1)^{\alpha ij}$ and $(\mathring{\Omega}_2)_{ij} (\mathring{\Omega}_2)^{ij}$ are also G invariant because

$$\begin{aligned}
(\mathring{\Omega}_1)_{\alpha ij} (\mathring{\Omega}_1)^{\alpha ij} &= g^{ik} g^{jl} k_1((\Omega_1)(p_1)(E_i^{(1)}, E_j^{(1)}), (\Omega_1)(p_1)(E_k^{(1)}, E_l^{(1)})) \\
&= g^{ik} g^{jl} k_1(\mathscr{A}\mathfrak{d}_{g_1^{-1}}(\Omega_1)(p_1)(E_i^{(1)}, E_j^{(1)}), \mathscr{A}\mathfrak{d}_{g_1^{-1}}(\Omega_1)(p_1)(E_k^{(1)}, E_l^{(1)})) \\
&= g^{ik} g^{jl} k_1((\Omega_1)(p_1 g_1)(R_{g_1 *} E_i^{(1)}, R_{g_1 *} E_j^{(1)}), (\Omega_1)(p_1 g_1)(R_{g_1 *} E_k^{(1)}, R_{g_1 *} E_l^{(1)})) \\
&= g^{ik} g^{jl} k_1((\Omega_1)(p_1 g_1)((E_i^{(1)}) p_1 g_1, (E_j^{(1)}) p_1 g_1), (\Omega_1)(p_1 g_1)((E_k^{(1)}) p_1 g_1, (E_l^{(1)}) p_1 g_1)),
\end{aligned} \tag{3.33}$$

i.e., $k_1(\Omega_1, \Omega_1)$ is well defined on M since it is independent of the choice of p_1. An even simpler argument applies to $k_2(\Omega_2, \Omega_2)$ since in this case the group is Abelian.

To prove that $\Phi_A \Phi^A$ is G-invariant we need the transformation properties of the fields Φ^A. These follow readily from (2.62) and (3.27). We thus have

$$\Phi_A((p_1, p_2) g) = \rho(g^{-1})_A{}^B \Phi_B(p_1, p_2), \tag{3.34}$$

and since the matrices $\rho(g)$ are orthogonal [cf. Eqs. (2.49) for the infinitesimal generators], it immediately follows that

$$(\Phi_A \Phi^A)_{(p_1, p_2) g} = (\Phi_A \Phi^A)_{(p_1, p_2)}.$$

Finally, since $D_i \Phi_A \equiv \mathring{E}_i[\Phi_A]$, it is obvious that the term $(D_i \Phi_A)(D^i \Phi^A)$ is also independent of the point in the fiber where it is evaluated.

In summary, the Lagrangian density (3.32) is a well-defined function on the base manifold M, and we can write an action by integrating it over a volume element μ_g on M determined by g and the orientation of M, i.e.,

$$I=\int_U \mathscr{L}\mu_g, \tag{3.35}$$

where U is an open subset of M with compact closure.

We now turn to the physical interpretation of the curvatures Ω_1 and Ω_2. Recall that by (2.17) and (2.20) we have

$$(\Omega_1)^\alpha{}_{ij}=(d\omega_1^\alpha)(E_i^{(1)},E_j^{(1)})+\epsilon_{\alpha\beta\gamma}\omega_1^\beta(E_i^{(1)})\omega_1^\gamma(E_j^{(1)}), \tag{2.17'}$$

$$(\Omega_2)_{ij}=d(-i\omega_2)(E_i^{(2)},E_j^{(2)}). \tag{2.20'}$$

If we let $(\sigma_1)_u$ and $(\sigma_2)_u$ be local sections $(\sigma_1)_u$: $M\to P_1$, $(\sigma_2)_u$: $M\to P_2$, such that $(\sigma_1)_{u*}\overline{E}_i\in T_{p_1}P_1$ and $(\sigma_2)_{u*}\overline{E}_i\in T_{p_2}P_2$, and if we further choose the orthonormal basis at each $x\in U\subset M$ to be a coordinate basis $\overline{E}_i=\partial_i$, we then have

$$\begin{aligned}(\Omega_1)^\alpha{}_{ij}&=\partial_i((\sigma_1)_u^*\omega_1^\alpha(\partial_j))-\partial_j((\sigma_1)_u^*\omega_1^\alpha(\partial_i))\\&\quad+\epsilon_{\alpha\beta\gamma}((\sigma_1)_u^*\omega_1^\beta(\partial_i))((\sigma_1)_u^*\omega_1^\gamma(\partial_j))\\&=g(\partial_iW^\alpha{}_j-\partial_jW^\alpha{}_i+g\epsilon_{\alpha\beta\gamma}W^B{}_iW^\gamma{}_j),\end{aligned} \tag{3.36}$$

$$\begin{aligned}(\Omega_2)_{ij}&=\partial_i((\sigma_2)_u^*(-i\omega_2)(\partial_j))-\partial_j((\sigma_2)_u^*(-i\omega_2)(\partial_i))\\&=g'(\partial_iB_j-\partial_jB_i).\end{aligned} \tag{3.37}$$

Here we have used the definitions

$$gW^\alpha{}_j\equiv((\sigma_1)_u^*\omega_1^\alpha)(\partial_j),\quad g'B_j\equiv((\sigma_2)_u^*(-i\omega_2))(\partial_j), \tag{3.38}$$

and g, g' denote the dimensionless coupling constants for the SU(2) and U(1) factors, respectively.

Hence

$$F^\alpha{}_{ij}\equiv(1/g)(\Omega_1)^\alpha{}_{ij}=\partial_iW^\alpha{}_j-\partial_jW^\alpha{}_i+g\epsilon_{\alpha\beta\gamma}W^\beta{}_iW^\gamma{}_j, \tag{3.39}$$

$$F_{ij}\equiv(1/g')(\Omega_2)_{ij}=\partial_iB_j-\partial_jB_i, \tag{3.40}$$

are the field tensors for the SU(2) and U(1) vector bosons, respectively.

To conclude this section we have only to properly dimension and interpret the parameters that occur in (3.32) in order to bring it into the usual form of Einstein–Cartan gravity coupled to the Yang–Mills and Higgs fields for the electroweak model.

For this purpose, assume that $\hbar=c=1$ so that the action integral (3.35) is dimensionless. This in turn implies that the Lagrangian density has to have units of (length)$^{-4}$. Since all our quantities in (3.32) are so far dimensionless, we need to introduce appropriate powers of a mass scale factor τ [in units of (length)$^{-1}$] into each of the terms. Thus the Riemann and gauge field tensors have to be multiplied by τ^2, while D_i and Φ_A require a factor of τ. We will use, however, the same notation for the newly dimensioned quantities as there is no risk of confusion.

Therefore, after combining terms our action becomes

$$\begin{aligned}I=\frac{1}{V_I}\int\sqrt{|g|}\Big\{&-\kappa\underline{R}+\alpha_1\underline{R}^2+\alpha_2\underline{R}_{ijkm}\underline{R}^{ijkm}+\alpha_6\underline{R}_{ij}\underline{R}^{ij}\\&-\frac14F^\alpha{}_{ij}F_\alpha{}^{ij}-\frac14F_{ij}F^{ij}+\frac12(D_i\Phi_A)(D^i\Phi^A)\\&+\frac{m^2}{2}\Phi_A\Phi^A-\frac{\lambda}{4}(\Phi_A\Phi^A)^2+\frac{n}{2(n-1)}\lambda\underline{R}\Phi_A\Phi^A\\&-\kappa\Lambda\Big\}d^nx,\end{aligned} \tag{3.41}$$

where we have made the following obvious identifications in order to fix the physical parameters:

$(\alpha_0+4\alpha_1)\tau^2=-\kappa$

(the proportionality factor in the Einstein–Hilbert Lagrangian), (3.42)

$2[(n-1)/n](\alpha_7\tau^2+\kappa)=m^2>0$

(square of the mass parameter associated with the Higgs field), (3.43)

$4[(1-n)/n^3][\alpha_1n(n-1)+2\alpha_2+\alpha_6(n-1)]=\lambda>0$

(coupling constant of the self-interaction term of the scalar field). (3.44)

$\Lambda=$ the cosmological constant. (3.45)

Also, in order to normalize the free Lagrangians of the Yang–Mills and the Higgs fields to their customary values, we have set

$$\alpha_3g^2=\alpha_4g'^2=\tfrac14, \tag{3.46}$$

$$\alpha_5=\tfrac12. \tag{3.47}$$

Note that (3.46) provides a relation between the parameters α_3 and α_4 and the Weinberg angle. Indeed,

$$\tan\theta_W=g'/g=\sqrt{\alpha_3/\alpha_4}, \tag{3.48}$$

so we see that the deviation of the Weinberg angle from $\pi/4$ measures the relative extent by which the SU(2) and U(1) sectors of the theory deviate from an Einstein–Cartan model in which torsion only occurs implicitly in the curvature terms. For this latter case, α_3 would equal α_4.

IV. COVARIANT DERIVATIVE OF THE HIGGS SPIN COMPLEX DOUBLET

In the preceding section we defined the operator D_i acting on the scalar fields Φ_A as their directional derivative [Eq. (3.30)]. Here we want to obtain an explicit expression for this differential operator that will allow us to relate the Lagrangian of the scalar fields, as given in (3.41), to the form in which it usually appears in the electroweak model.

In order to accomplish this, we first locally trivialize the fiber bundle π_{12}: $P_1\circ P_2\to M$ (i.e., we choose a gauge) by taking the local section $\sigma_{g_1\times g_2}(x)$, $x\in U\subset M$, defined as the set of points $(p_1,p_2)=(x,g_1\times g_2)$ with fixed g_1 and g_2.

Since $\sigma_{g_1\times g_2}(x)$ is a submanifold of $P_1\circ P_2$ which is diffeomorphic to U, the basis vectors $\mathring{\partial}_i\equiv\sigma_{g_1\times g_2*}\overline{E}_i=\sigma_{g_1\times g_2*}\partial_i$ of the tangent space of $\sigma_{g_1\times g_2}(x)$ form a closed Lie algebra. We can therefore take as a new basis in $\pi_{12}^{-1}(U)$ the local external direct sum of $\{\mathring{\partial}_i\}$ and $\{\mathring{E}_{n+A}\equiv l_A^*, A=1,...,4\}$.

In terms of this basis we can write

$$\mathring{E}_i = \mathring{\partial}_i - gW^{\alpha}{}_i\mathring{E}_{n+\alpha} - g'B_i\mathring{E}_{n+4}, \tag{4.1}$$

where $W^{\alpha}{}_i$, B_i are the gauge potentials defined in (3.38) with $(\sigma_i)_u = \pi^{i}\circ\sigma_{g_1\times g_2}$.

Note the (4.1), together with (2.34) and (3.38), implies

$$\begin{aligned}\bar{\mathring{\varphi}}^{n+A}(\mathring{E}_i) &= \delta^{n+A}_{n+\beta}\mathring{\omega}^{\beta}_1(\mathring{\partial}_i) + \delta^{n+A}_{n+4}(-i\mathring{\omega}_2)(\mathring{\partial}_i) \\ &\quad - gW^{\beta}{}_i\bar{\mathring{\varphi}}^{n+A}(\mathring{E}_{n+\beta}) - g'B_i\bar{\mathring{\varphi}}^{n+A}(\mathring{E}_{n+4}) \\ &= 0,\end{aligned} \tag{4.2}$$

as required by the definition of $\bar{\mathring{\varphi}}^{n+A}$. Also note that, since

$$g\mathring{E}_{n+\alpha}[W^{\beta}{}_j] = \pounds_{(\ell^*_\alpha\oplus 0)}\mathring{\omega}^{\beta}_1(\mathring{\partial}_j)$$

and

$$\begin{aligned}\pounds_{(l^*_\alpha\oplus 0)}[\mathring{\omega}^{\beta}_1(\mathring{E}_{n+\gamma})] = 0 &= (\pounds_{(l^*_\alpha\oplus 0)}\mathring{\omega}^{\beta}_1)(\mathring{E}_{n+\gamma}) \\ &\quad + \mathring{\omega}^{\beta}_1([\mathring{E}_{n+\alpha},\mathring{E}_{n+\gamma}]),\end{aligned}$$

it follows that

$$\mathring{E}_{n+\alpha}[W^{\beta}{}_j] = -\epsilon_{\beta\alpha\gamma}W^{\gamma}{}_j. \tag{4.3}$$

By the same line of reasoning we find [since U(1) is Abelian]

$$\mathring{E}_{n+4}[B_i] = 0. \tag{4.4}$$

It is now a simple matter to verify that (4.1) leads to the correct expression for the commutator $[\mathring{E}_i, \mathring{E}_j]$. Indeed, making use of (4.3) and (4.4), we obtain

$$[\mathring{E}_i,\mathring{E}_j] = -((\sigma_1)^*_u\Omega_1)^{\alpha}{}_{ij}\mathring{E}_{n+\alpha} - ((\sigma_2)^*_u\Omega_2)_{ij}\mathring{E}_{n+4}. \tag{4.5}$$

Thus the commutator is vertical, as required by (3.5) and (3.6). Furthermore, from (2.30) we have

$$d\bar{\mathring{\varphi}}^{n+\alpha} = \pi^{1*}(\Omega_1) - \tfrac{1}{2}\epsilon_{\alpha\beta\gamma}\bar{\mathring{\varphi}}^{n+\beta}\wedge\bar{\mathring{\varphi}}^{n+\gamma}, \tag{4.6}$$

$$d\bar{\mathring{\varphi}}^{n+4} = \pi^{2*}(-i\Omega_2), \tag{4.7}$$

and evaluating these two expressions on $(\mathring{E}_i, \mathring{E}_j)$, we get

$$-\bar{\mathring{\varphi}}^{n+\alpha}([\mathring{E}_i,\mathring{E}_j]) = ((\sigma_1)^*_u\Omega_1)^{\alpha}{}_{ij}, \tag{4.8a}$$

$$-\bar{\mathring{\varphi}}^{n+4}([\mathring{E}_i,\mathring{E}_j]) = ((\sigma_2)^*_u\Omega_2)_{ij}, \tag{4.8b}$$

respectively. But (4.8a) and (4.8b) are the same as what we derive from applying $\bar{\mathring{\varphi}}^{n+A}$ to (4.5). Consequently, the expression (4.1) for $\mathring{E}_i$ in terms of the external direct sum basis is consistent with our previous results.

By virtue of (4.1) the directional derivatives of our scalar fields become

$$\begin{aligned}D_i\Phi_A \equiv \mathring{E}_i[\Phi_A] &= \mathring{\partial}_i\Phi_A - gW^{\alpha}{}_i\mathring{E}_{n+\alpha}[\Phi_A] \\ &\quad - g'B_i\mathring{E}_{n+4}[\Phi_A].\end{aligned} \tag{4.9}$$

Thus we now need to evaluate the quantities $\mathring{E}_{n+\alpha}[\Phi_A]$ and $E_{n+4}[\Phi_A]$. These follow directly by noting that

$$\begin{aligned}(\mathring{E}_{n+B}[\Phi_A])_{(p_1,p_2)} &= (\pounds_{\mathring{E}_{n+B}}\Phi_A)_{(p_1,p_2)} \\ &= \lim_{t\to 0}(1/t)[\Phi_A((p_1,p_2)g(t)) \\ &\quad - \Phi_A(p_1,p_2)],\end{aligned} \tag{4.10}$$

and making use of (3.34). We get

$$\mathring{E}_{n+B}[\Phi_A] = -(\rho(\mathring{l}_B))_A{}^C\Phi_C, \tag{4.11}$$

where $(\rho(\mathring{l}_B))_A{}^C$ are the matrices given in (2.49).

Substituting (4.11) into (4.9) and operating explicitly with the representation given in (2.49), we arrive at the following matrix expressions for the directional derivatives:

$$D_i\mathbf{\Phi} = \mathring{\partial}_i\mathbf{\Phi} - g\mathbf{W}_i\mathbf{\Phi} + g'B_i\rho(\mathring{l}_4)\mathbf{\Phi}, \tag{4.12}$$

where

$$\mathbf{\Phi} = \begin{pmatrix}\Phi_1\\ \Phi_2\\ \Phi_3\\ \Phi_4\end{pmatrix}, \quad \mathbf{W}_i = \frac{1}{2}\begin{pmatrix}0 & W^3_i & W^2_i & W^1_i\\ -W^3_i & 0 & -W^1_i & W^2_i\\ -W^2_i & W^1_i & 0 & -W^3_i\\ -W^1_i & -W^2_i & W^3_i & 0\end{pmatrix}, \tag{4.13}$$

and $\rho(\mathring{l}_4)$ is defined in (2.49d).

Since the torsion is a real tensor, the model calls naturally for the real representation of the Higgs fields that we have been using, but in order to cast the Lagrangian in the *usual* form (i.e., the way it most commonly appears in the literature of the standard model), we make the following transformation:

$$\begin{pmatrix}\phi\\ \phi^*\end{pmatrix} = U\mathbf{\Phi}, \tag{4.14}$$

where

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i & 0 & 0\\ 0 & 0 & 1 & -i\\ 1 & i & 0 & 0\\ 0 & 0 & 1 & i\end{pmatrix} \tag{4.15}$$

is a unitary matrix. The quantity ϕ in (4.14) is the complex doublet scalar field of the standard electroweak model and is related to our real scalar fields by means of

$$\phi \equiv \begin{pmatrix}\varphi^+\\ \varphi^0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}\Phi_1 - i\Phi_2\\ \Phi_3 - i\Phi_4\end{pmatrix}, \tag{4.16}$$

while ϕ^* stands for its ordinary complex conjugate.

From (4.12) it is a simple matter to verify that

$$U(D_i\Phi) = UD_iU^\dagger U\Phi = \begin{pmatrix} \mathscr{D}_i\phi \\ (\mathscr{D}_i\phi)^* \end{pmatrix}, \tag{4.17}$$

where

$$\mathscr{D}_i\phi = \mathring{\partial}_i\phi - (i/2)gW^\alpha{}_i\sigma_\alpha\phi - (i/2)g'B_i\phi \tag{4.18}$$

corresponds to the covariant derivative of the standard model, and the 2×2 matrices σ_α are the usual Pauli matrices. Using (4.14), and remembering that the components of Φ are real fields, we obtain

$$\Phi_A\Phi^A = \Phi^\dagger\Phi = (U\Phi)^\dagger(U\Phi) = \phi^\dagger\phi + (\phi^\dagger\phi)^* = 2\phi^\dagger\phi. \tag{4.19}$$

Similarly, (4.17) gives

$$D^i\Phi^A D_i\Phi_A = (D^i\Phi)^\dagger(D_i\Phi) = 2(\mathscr{D}^i\phi)^\dagger(\mathscr{D}_i\phi). \tag{4.20}$$

Finally, substituting (4.19) and (4.20) into (3.41) yields the following form for our action integral:

$$I = \frac{1}{V_I}\int\sqrt{|g|}\Big\{-\kappa\underline{R} + \alpha_1\underline{R}^2 + \alpha_2\underline{R}_{ijkm}\underline{R}^{ijkm} + \alpha_6\underline{R}_{ij}\underline{R}^{ij} - \frac{1}{4}F^\alpha{}_{ij}F_\alpha{}^{ij} - \frac{1}{4}F_{ij}F^{ij} + (\mathscr{D}_i\phi)^\dagger(\mathscr{D}^i\phi) + m^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2 + \frac{n}{n-1}\lambda\underline{R}\phi^\dagger\phi - \kappa\Lambda\Big\}d^nx. \tag{4.21}$$

Note that in the action (4.21) the complex Higgs doublet still has four degrees of freedom, which in turn implies the existence of spurious Goldstone bosons. To eliminate these unphysical states one may still resort to a unitary gauge choice (although it is not even certain that local unitary gauges exist about every $x\in M$) such that

$$\phi = \begin{pmatrix} 0 \\ [\rho(x) + \varphi_0]/\sqrt{2} \end{pmatrix}, \tag{4.22}$$

where $\rho(x)$ denotes the remaining massive Higgs boson and φ_0 is the vacuum value of the scalar field.

An interesting point to note in (4.21) is the appearance of an extra curvature dependent "mass" term

$$[4n/(n-1)]\lambda\underline{R}\phi^\dagger\phi,$$

with its coefficient determined by the dimension of the base manifold M. This term will play an important role in the compactification analysis to be implemented in a forthcoming paper.

V. CONCLUSIONS

We have developed a formalism based on fiber bundle structures, which makes possible a geometric unification of the Yang–Mills and Higgs field sector of the standard electroweak model with gravitation. The theory requires a non-Levi-Civita connection on the bundle of frames and the ensuing torsion on the frame acts as a source for the scalar field Lagrangian, including the symmetry breaking potential.

In order to give torsion a dynamical character, the theory has to include terms quadratic in the fiber-bundle curvature. Quadratic Lagrangians in the curvature are however of interest, both because they appear naturally in the low energy limit of superstring theory, and also because through compactification of the extra dimensions of the base manifold the solutions of the modified field equations suggest a possible means of predicting values for the coupling constants of the theory.

The mathematical structures are necessarily more complicated than those used in the literature. First, because of the need to include spliced bundles in order to accommodate the direct product of two groups; and second, because the characteristic group of the spliced bundle is not semisimple, which requires in turn a careful choice of the transformation properties for the different components of the torsion, as opposed to a mere action of the adjoint representation of the group (as done in previous works). These new requirements on torsion seem to be essential for more realistic models such as the one considered here, as well as others which would include the SU(3) color gauge fields.

As pointed out in the Introduction, our theory does not yet encompass the fermionic fields needed to obtain the fermion and Yukawa Lagrangians which would complete the description of the electroweak interactions coupled to gravitation. One could, of course, resort to the phenomenological approach found in some of the literature[9] on modern Kaluza–Klein theories, where fermions are included by means of an additional Lagrangian term of the generic form

$$(\det e_i{}^A)\bar{\psi}e^i{}_A\Gamma^A D_i\psi, \tag{5.1}$$

where $e_i{}^A$ is a vielbein, $i = 1,\ldots,n$ are general coordinate indices of the base manifold, Γ^A are the generators of the Clifford algebra relative to the standard inner product in $\Re^n$ with signature $(+ , - ,..., -)$, and $D_i\psi$ are spinor connections.

Note that by allowing the spinor connections in (5.1) to contain torsion again, a Yukawa-type scalar–spinor interaction may be obtained without having to insert it in an *ad hoc* manner. This approach for introducing fermions is, however, rather unsatisfactory from a unification goal point of view, first, because the fermionic terms in the Lagrangian do not derive from a "pure" Einstein–Hilbert action principle, and second, because in addition to having to put in the terms of the form (5.1) by construction, the assignments of the left and right-handed fermions to multiplets of SU(2) in the present state of the electroweak model must rely heavily on experimental data. Furthermore, the standard model would have to be extended in order to determine the values of the Yukawa coupling constants. Attempts to resolve these drawbacks have led to a variety of alternative theories of supergravity, including a combination of these with Kaluza–Klein theories, as well as to the ongoing massive effort in superstring theory.

One should not rule out the possibility of an altogether different conception on the structure of the space-time manifold in order to achieve a theory of grand unification, such as the one implied in the twistor program. Work along this line of research is presently being pursued by our group, which might lead to an adequate incorporation of fermions within the framework of our theory, consisting essentially on the use of supertwistors for the frame bundle of the base manifold in the construction of fiber bundle spaces.

Regardless of such aspects of a more speculative nature, the results presented here suggest that torsion, in addition to its already acquired importance in supergravity theories, may also play a determinant role as a geometric source of the Higgs fields required for the symmetry breaking process in gauge theories, independently of which theory will ultimately prove to be the right one.

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