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Bifurcations of meromorphic vector fields on the Riemann sphere

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Abstract. Let $\{X_\theta\}$ be a family of rotated singular real foliations in the Riemann sphere which is the result of the rotation of a meromorphic vector field X with zeros and poles of multiplicity one. We prove that the set of bifurcation values, in the circle $\{\theta\}$, is for each family a set with at most a finite number of accumulation points. A condition which implies a finite number of bifurcation values is given. We also show that the property of having an infinite set of bifurcation values defines open but not dense sets in the space of meromorphic vector fields with fixed degree.

1. *Introduction*

Let X be a meromorphic vector field over the Riemann sphere $\mathbb{P}^1$, with poles and zeros of multiplicity one which are called generic. Using the complex structure in $\mathbb{P}^1$, the rotation by an angle θ in the unit circle S^1, is well defined in $\mathbb{P}^1$. So we have a family of rotated meromorphic vector fields $\{X_\theta \mid \theta \in S^1\}$. In this work we study the bifurcations of the real singular foliations in $\mathbb{P}^1$ given by the real trajectories (or solutions) of the families $\{X_\theta\}$.

We prove the following result:

THEOREM 3.2. *Let $\{X_\theta\}$ be a family of rotated generic meromorphic vector fields in the Riemann sphere $\mathbb{P}^1$. The bifurcation values of the family $\{X_\theta\}$ form a set in S^1 that has at most a finite number of accumulation points.*

For every family $\{X_\theta\}$ we have a singular flat Riemannian metric g in $\mathbb{P}^1$ (where the singularities of the metric come from the zeros and poles of X). The trajectories of the meromorphic vector fields $\{X_\theta\}$ are geodesics in this Riemannian metric. The construction of g is related to the theory of quadratic differentials, [**5, 6, 10**]. However, we look at the objects from the point of view of flows as dynamical systems. Now we describe a condition which implies that the bifurcation values is a finite set. In these singular flat Riemannian metrics we have flat cylinders that are isometric to the product of an open interval and $S^1(r)$, the circle of radius r. We say that a cylinder of g is trivial

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iff it is of the form $(0, \infty) \times S^1(r)$, where the end $\{\infty\} \times S^1(r)$ of this cylinder corresponds to a zero of the vector field and the extremum $\{0\} \times S^1$ contains poles of the field. The set of bifurcation values is finite iff all the cylinders of the Riemannian metric are trivial (see Corollary 3.4). As a consequence, there are only a finite number of bifurcation values for families of rotated polynomial vector fields (see Corollary 3.5). If we fix the number of zeros (say $n \in \mathbb{N}$) of the vector fields, then the space of all such meromorphic vector fields forms, in a natural way, an analytic space $GQ(n)$ so that the union of these spaces parametrize the set of all non-identically zero generic meromorphic vector fields. In §4 we prove that meromorphic vector fields with at least a non-trivial cylinder form an open set in these analytic spaces. As a consequence, the property of having an infinite set of bifurcations is a stable property under small perturbations of the family $\{X_\theta\}$ in $GQ(n)$. On the other hand, there exist non-empty open sets of meromorphic vector fields in $GQ(n)$, where $n \geq 4$, without non-trivial cylinders. The above is in contrast with other types of quadratic differentials, see [**1**]. Hence the property of having an infinite number of bifurcations is not a dense property in $GQ(n)$ (see Proposition 5.6). In §5 we give the explicit description of some families of flat singular Riemannian metrics which correspond to meromorphic vector fields.

Our work is motivated by the work of Duff [**2**] (see also [**7**]), for bifurcations in families of rotated real smooth vector fields. The results show that for the solutions and the bifurcations of meromorphic vector fields in the sphere it is possible to have simple descriptions. The flow of a meromorphic vector field has several nice properties, for example, it has a transverse measure (see [**8**]), and gives examples of, singular geodesible flows (see [**3**]).

2. *Flat metrics and meromorphic fields*

We recall some preliminary results on meromorphic vector fields, mainly from [**4**] and [**7**]. Through this paper we will work on a compact connected Riemann surface M, let TM be its holomorphic tangent bundle. A meromorphic vector field X over M is a meromorphic section of TM. Given a meromorphic vector field X on M we have its associated divisor of zeros and poles: $(X) = (X)_0 - (X)_\infty$, which is the formal sum of the zeros and poles of X, counted with their multiplicities. A meromorphic vector field X over a Riemann surface is *generic* iff its zeros and poles are of multiplicity one. The degree of a divisor is the sum of its multiplicities, recall that;

$$\text{degree}(X)_0 - \text{degree}(X)_\infty = 2 - 2\,\text{genus}(M).$$

Therefore, the only surfaces which admit holomorphic vector fields are: the Riemann sphere $\mathbb{P}^1$, where holomorphic vector fields are given by polynomial vector fields of degree two, on some affine chart; and the torus, with any complex structure, where constant vector fields are well defined.

Using the complex structure of any M, we define rotation operators:

$$e^{i\theta} : TM \to TM,$$

given in local coordinates $\{z\}$ for M as

$$\left(z, \frac{\partial}{\partial z}\right) \mapsto \left(z, e^{i\theta}\frac{\partial}{\partial z}\right),$$

where $\theta \in \mathbb{R}/2\pi\mathbb{Z} := S^1$. These rotation operators $\{e^{i\theta}\}$ are complex automorphisms of TM. Given a meromorphic vector field X, the operators $\{e^{i\theta}\}$ transform X in a natural way as a section of TM. Hence there exist a family of rotated vector fields

$$\{X_\theta\} = \{X_\theta := e^{i\theta}(X) \mid \theta \in S^1\}.$$

Note that for any θ, X_θ has the same zeros and poles as X.

For each meromorphic vector field X in M we have an associated singular, real, one-dimensional, oriented foliation $\mathcal{F}(X)$ given by the real trajectories (or solutions) of X as a real vector field in $M - \mathrm{Sing}(X)$, where the *singular set of* X, $\mathrm{Sing}(X)$, is by definition the set of the zeros and poles of X. Note that the flow of X is not well defined at its poles. Our main tool for studying the dynamics of $\{\mathcal{F}(X_\theta)\}$ is the following result.

LEMMA 2.1. *Let X be a meromorphic vector field over a compact Riemann surface M. Then there exists a flat Riemannian metric g, on $M - \mathrm{Sing}(X)$ such that the real solutions of X are geodesics for g.*

Proof. The vector fields X and $X_{\pi/2}$ are real, smooth and non-singular in $M - \mathrm{Sing}(X)$. If we fix the complex structure of M, given X in local charts, by $X = (u(x, y), v(x, y))$, it is easy to see that:

$$[X, X_{\pi/2}] = \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)(-u, v) + \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)(v, u) = 0,$$

where $[\ ,\ ]$ is the usual bracket of vector fields. The existence of such a pair of vector fields is equivalent to the existence of such a g, where X and $X_{\pi/2}$ give an orthonormal frame for g, see [**9**] p. 261. □

Remark 2.2. For all the fields in $\{X_\theta\}$ the construction gives the same Riemannian metric g. Hence g is associated with the whole family $\{X_\theta\}$. We denote the corresponding Riemannian manifold by $(M - \mathrm{Sing}(X), g)$.

Now we suppose that X is generic and study the Riemannian metric g near its singularities.

Let $p \in M$ be a simple zero of X in M. Since we are interested in the local problem, without loss of generality we can suppose that the meromorphic vector field X is defined in a neighborhood of $0 \in \mathbb{C}$ and that $X(0) = 0$. A straightforward computation shows that under a holomorphic change of coordinates X is given in a neighborhood of 0 by the function $z \mapsto az$, where $a \in \mathbb{C} - \{0\}$. Hence, in a neighborhood of p the Riemannian manifold $(M - \mathrm{Sing}(X), g)$ is isometric to a flat cylinder $(0, \infty) \times S^1(r)$, where $r = 1/|X'(0)|$, and the end $\{\infty\} \times S^1(r)$ of this cylinder corresponds to p.

In a similar way, in a neighborhood of a pole of order one, the vector field X looks like $z \to 1/z$ and $(M - \mathrm{Sing}(X), g)$ is locally isometric to a 4-fold branch isometric covering of $\mathbb{R}^2/\pm Id$, where the branch point corresponds to the pole of X. The cone angle of g at the pole is 4π, note that g is not geodesically complete at the poles of X.

There is a map of meromorphic sections $i : \Gamma(M, TM) \to \Gamma(M, \Omega(M)^{\otimes 2})$, to the meromorphic quadratic differentials, locally defined by

$$f(z)\frac{\partial}{\partial z} \mapsto \frac{1}{f(z)^2}dz^2,$$

hence $i(X)$ is a meromorphic quadratic differential. The metric g just constructed corresponds to the usual metric associated with the quadratic differential $i(X)$, see [**10**]. The quadratic differentials $i(X)$ are of infinite norm (or area) iff X has zeros and are orientable in the sense of [**6**].

In order to apply some of the ideas of smooth dynamical systems we redefine the flow of X.

Remark 2.3. Let M be a compact Riemann surface and X a generic meromorphic vector field over M. Then there exists a smooth vector field $\overline{X}$ with the same trajectories as X (as subsets of M).

From the existence of g we obtain:

LEMMA 2.4. *Let X be a meromorphic vector field on the Riemann sphere $\mathbb{P}^1$. The ω-limit of any trajectory γ is:*

(i) *A singular point of* Sing(X), *or*

(ii) *γ itself, when γ is a periodic trajectory.*

Proof. According to the Poincaré–Bendixson theory for smooth vector fields in the sphere, we only need to prove that limit cycles and graphics, with non-trivial Poincaré maps, are impossible. This follows from the fact that X has an invariant transverse measure given by the path integral $\int [X_{\pi/2}]^* \, ds$ along transverse curves of X, where $[X_{\pi/2}]^*$ is the 1-form dual to $X_{\pi/2}$, using g. □

The following example will be useful.

2.5. A meromorphic vector field in $\mathbb{P}^1$ with only two zeros p_1, p_2 of multiplicity one, induces in $\mathbb{P}^1 - \{p_1, p_2\}$ a flat metric which makes it isometric to $\mathbb{R} \times S^1(r)$, a complete flat cylinder.

3. *Bifurcation values*

Let $\{X_\theta\}$ be a family of rotated generic meromorphic vector fields in M. Given $\lambda, \mu \in S^1$, we say that X_λ, $X_\mu \in \{X_\theta\}$ are *topologically equivalent* iff there exists an orientation-preserving homeomorphism $h: M \to M$ that sends the foliation $\mathcal{F}(X_\lambda)$ to $\mathcal{F}(X_\mu)$. We say that $X_\lambda \in \{X_\theta\}$ is *structurally stable* iff there exists a neighborhood $V(\lambda) \subset S^1$ of λ such that for any $\mu \in V(\lambda)$, X_λ and X_μ are topologically equivalent.

Definition 3.1. We say that $\lambda \in S^1$ is a *bifurcation value* of the family $\{X_\theta\}$ iff X_λ is not structurally stable.

Note that by the description of Sing(X) in a neighborhood of a zero of $\{X_\theta\}$ we have only two bifurcation values: when the zero changes from source to sink or vice versa. On the other hand, a pole of $\{X_\theta\}$ does not have bifurcations for any $\theta \in S^1$ in a neighborhood of it.

THEOREM 3.2. *Let $\{X_\theta\}$ be a family of rotated generic meromorphic vector fields in the Riemann sphere $\mathbb{P}^1$. The bifurcation values of the family $\{X_\theta\}$ form a set in S^1 that has at most a finite number of accumulation points.*

Proof. Let λ be a bifurcation value for $\{X_\theta\}$. Since the elements of the family are generic, by use of Remark 2.3 and Peixoto's classical theorem for the structural stability of real smooth vector fields in $\mathbb{P}^1$, see [7], we have that X_λ has at least one of the following local dynamics:

(i) A saddle connection.
(ii) A graphic.
(iii) A closed non-singular trajectory.
(iv) A center.

We consider the following cases separately:

Case 1. X_λ has only a finite number of saddle connections, but no graphics, closed non-singular trajectories or centers. The zeros of X_λ (or equivalently of $\overline{X_\lambda}$ in 2.3) are hyperbolic and by Lemma 2.4 the α or ω-limits of the trajectories of X_λ near a separatrix of a saddle connection are zeros of X_λ (which are sources or sinks). Using the flat Riemannian metric g associated with the family $\{X_\theta\}$ we have a picture like that in Figure 1.

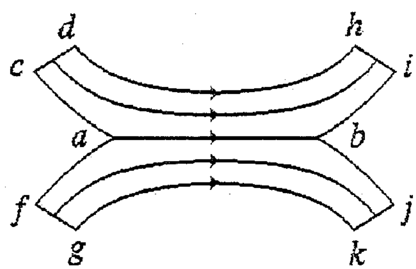

FIGURE 1.

$\overline{ab}$ is the separatrix for the saddle connection and $\overline{cd}$, $\overline{fg}$, $\overline{hi}$ and $\overline{jk}$ are the α or ω-limits of the other trajectories and correspond to pieces of cylinders around the zeros of X_λ. For each separatrix we have an $\varepsilon \in \mathbb{R}^+$ such that for $\mu \in (\lambda - \varepsilon, \lambda + \varepsilon) \subset S^1 = \mathbb{R}/2\pi\mathbb{Z}$ the separatrix breaks, and the foliations in $(\lambda - \varepsilon_0, \lambda) \cup (\lambda, \lambda + \varepsilon_0) \subset S^1$ are structurally stable, where ε_0 is the minimum of $\{\epsilon\}$. Hence the number of saddle connections for X_λ is finite, we have that λ is an isolated bifurcation value.

For the next cases, note that if X_λ has closed non-singular trajectories hence, by Lemma 2.4 these closed trajectories are not isolated. If γ is a periodic orbit then $\varphi_t(\gamma)$ is also a periodic orbit, where φ_t is the local flow of $X_{\lambda+\pi/2}$ and $t \in (-\epsilon, \epsilon)$, for small ϵ. For a closed trajectory γ, define $A(\gamma)$ as the maximal, open, connected, saturated set (by closed trajectories of X_λ) which contains γ. Each $A(\gamma)$ is isometric to an open flat cylinder $(a, b) \times S^1(r)$, where $a, b \in \mathbb{R} \cup \{\pm\infty\}$, $r \in \mathbb{R}^+$, and is maximal with respect to the flat cylinders in $(\mathbb{P}^1 - \mathrm{Sing}(X_\lambda), g)$ which contains it. We have three posibilities for $A(\gamma)$ as a subset of $\mathbb{P}^1$:

(1) $\mathbb{P}^1$ minus two points, isometric to $\mathbb{R} \times S^1(r)$, $r \in \mathbb{R}^+$.
(2) A punctured disk, isometric to $(0, \infty) \times S^1(r)$.
(3) An annulus, isometric to $(0, h) \times S^1(r)$, $h \in \mathbb{R}$.

Definition 3.3. We say that $(M - \text{Sing}(X), g)$ has a *trivial cylinder* when $A(\gamma)$ is a punctured disk. If $A(\gamma)$ is an annulus then $(M - \text{Sing}(X), g)$ is said to have a *non-trivial cylinder*.

If $A(\gamma)$ is $\mathbb{P}^1$ minus two points, then X_λ has zeros at these points and the whole family $\{X_\theta\}$ has only two bifurcation values, see 2.5. Note that the last two options for $A(\gamma)$ always imply the existence of graphics at the boundary of the closure $\overline{A(\gamma)} \subset \mathbb{P}^1$ (in the case of the punctured disk, the point removed corresponds to a zero of the field).

Case 2. Suppose that X_λ has only closed non-singular trajectories which are always in trivial cylinders $A(\gamma)$ which are punctured disks in $\mathbb{P}^1$. Inside of each disk we have a zero of X_λ which is a center and on its boundary we have a graphic. For small $\varepsilon \in \mathbb{R}$ we can break the centers and the graphics. The centers become sources or sinks and there are no saddle conections, because under small rotations the singular trajectories of the graphics change to trajectories inside trivial cylinders. Hence λ is an isolated bifurcation value.

Case 3. Suppose now that X_λ has closed non-singular trajectories which are always in non-trivial cylinders $A(\gamma)$ which form an annulus in $\mathbb{P}^1$ isometric to cylinders of the form $(0, h) \times S^1(r)$. The boundary of each closure $\overline{A(\gamma)} \subset \mathbb{P}^1$ consists of two graphics Γ_1, Γ_2, and all the singular points in Γ_1, Γ_2 are poles of X_λ. Inside $A(\gamma)$ for small $\{\varepsilon\} \in \mathbb{R}$ note that $X_{\lambda+\varepsilon}$ has saddle connections between the singular points in Γ_1, Γ_2. The set of $\{\varepsilon\} \subset \mathbb{R}/2\pi\mathbb{Z}$ with this property is infinite and has only $0 \in \mathbb{R}/2\pi\mathbb{Z}$ as an accumulation point. If γ_1 and γ_2 are different closed orbits corresponding to two vector fields X_{θ_1}, X_{θ_2} in $\{X_\theta\}$ then, by using the Jordan curve theorem, γ_1 intersection γ_2 is empty or is a finite set with an even number of points. However the rotated vector field of a closed orbit goes inside or outside the closed orbit and the intersection of the orbits is empty (this is also proved in Theorem 4 of [**2**]), hence the cylinders $A(\gamma_1)$ and $A(\gamma_2)$ are disjoint. Since the sets $A(\gamma)$ must contain poles at their boundaries and the vector field X has only a finite number of them, we conclude (by the above observation) that $(M - \text{Sing}(X), g)$ contains at most a finite number of non-trivial cylinders and hence, the values $\{\lambda\} \subset S^1$ at which the bifurcation values accumulate are finite.

Case 4. By simple inspection (using Peixoto's theorem), we can see that any other posibility of local dynamics reduces to Cases 1, 2 or 3 and bifurcations at the zeros, in a local way. □

COROLLARY 3.4. *A family $\{X_\theta\}$ of rotated generic meromorphic vector fields in the Riemann sphere (with at least three zeros), has a finite number of bifurcation values iff all the cylinders in the space $(M - \text{Sing}(X), g)$ are trivial.*

Proof. In the proof of Theorem 3.2 the non-trivial cylinders are the origin of accumulation points in the bifurcation set. □

Similar results to 3.2 are true when we remove the hypothesis of generic singularities. For example, a meromorphic vector field X in $\mathbb{P}^1$ is a *polynomial vector field* iff it has only one pole (of any multiplicity), see also 5.7.

COROLLARY 3.5. *Let $\{X_\theta\}$ be a family of rotated polynomial vector fields in the Riemann sphere $\mathbb{P}^1$. The bifucation values of $\{X_\theta\}$ form a finite set in S^1.*

Proof. If the multiplicity of the pole p of X is $n \in \mathbb{N}$, then the singular flat metric g is locally isometric with a $(2+2n)$-fold branch isometric covering of $\mathbb{R}^2/\pm I\,d$. Note that there are no bifurcations in a neighborhood of p. We apply the same ideas as in Theorem 3.2 and use the fact that $(\mathbb{P}^1 - \mathrm{Sing}(X), g)$ does not contain non-trivial cylinders when X is a polynomial vector field. □

4. *Cylinders are open*

In this section we shall show that the non-identically zero generic meromorphic vector fields on $\mathbb{P}^1$ with n zeros, form an analytic space $GQ(n)$, and we prove that the meromorphic vector fields such that their associated singular flat Riemannian metrics have non-trivial cylinders, form an open set in $GQ(n)$.

We define $Q(n)$ as the space of non-identically zero sections of the tangent bundle $T\mathbb{P}^1$ with at most n zeros and $n-2$ poles, counted with multiplicty. A meromorphic vector field is determined by its divisor of zeros and poles, and by one scalar factor in $\mathbb{C}^*$. Hence $Q(n)$ is a finite-dimensional analytic space isomorphic to:

$$\mathrm{Sym}^n(\mathbb{P}^1) \times \mathrm{Sym}^{n-2}(\mathbb{P}^1) \times \mathbb{C}^*,$$

where $\mathrm{Sym}^k(\mathbb{P}^1)$ is the k-fold symetric product of $\mathbb{P}^1$ (which are parameter spaces for the divisors of meromorphic vector fields), and $\mathbb{C}^* = \mathbb{C} - \{0\}$. We are only interested in giving a parametrization of the generic meromorphic vector fields $GQ(n)$. Since $GQ(n)$ is an open set in $Q(n)$, then it is a smooth complex analytic space of complex dimension $2n-1$.

PROPOSITION 4.1. *The set of all meromorphic vector fields in $GQ(n)$ such that $(M - \mathrm{Sing}(X), g)$ has at least one non-trivial cylinder, form an open set in this space.*

Proof. Let X be a generic meromorphic vector field with n zeros such that $(M - \mathrm{Sing}(X), g)$ has a non-trivial cylinder $C = (-a, a) \times S^1(r)$. We will prove the existence of a number $\delta \in \mathbb{R}^+$ such that for each $Y \in B_\delta(X) \subset GQ(n)$ (the ball of radius δ centered in X) there exists at least one angle $\theta(Y)$, which depends on Y, with the property that $e^{i\theta(Y)}Y$ has a closed non-singular trajectory contained in $C_{1/2} = (-a/2, a/2) \times S^1(r)$. Choosing δ sufficiently small we can ensure that the cylinder generated by this trajectory in the corresponding flat space, will be non-trivial (since no new poles can appear and the ones that are present only move slightly).

We will compute the first return map of X_θ in the non-trivial cylinder C. Suppose that the trajectories of X correspond to the horizontal lines $\{y\} \times S^1(r)$ in C. Then $T = (-a, a) \times \{p\}$ (where $p \in S^1(r)$) is transversal to X, for y and θ small enough.

The first return map of the foliations associated with X_θ is

$$\begin{array}{rcl} j : T \times (-\pi, \pi) & \to & T \\ (y, \theta) & \mapsto & y + 2\pi r \tan(\theta). \end{array}$$

The equation which gives the fixed points for j is $f(y, \theta) = 2\pi r \tan(\theta) = 0$ and the graph of this function in transversal to $T \times (-\pi, \pi) \times \{0\}$ in $T \times (-\pi, \pi) \times T$.

We can construct a real one-dimensional foliation $\mathcal{F}$ in the manifold $\mathbb{P}^1 \times B_\delta(X) - \{\mathrm{Sing}(Y) \times Y | Y \in B_\delta(X)\}$ by using the orbits of all the fields in $B_\delta(X)$. This foliation is transversal to the product $T \times B_\delta(X)$ for δ small enough. Using $\mathcal{F}_\theta$ (which is constructed in the same way as $\mathcal{F}$ but using Y_θ instead of Y) we can define the first return map:

$$J : T \times B_\delta(X) \times (-\pi, \pi) \to T \times B_\delta(X),$$

which depends smoothly on $Y \in B_\delta(X)$.

So we obtain a smooth family of maps $F(y, Y, \theta)$ which give the fixed points of J, and such that the graph of $F(y, X, \theta) = f(y, \theta)$ in $T \times (-\pi, \pi) \times T$ is transversal to $T \times (-\pi, \pi) \times \{0\}$ with non-empty intersection. Transversality with non-empty intersection is an open condition, hence $F(y, Y, \theta)$ remains transversal to $T \times (-\pi, \pi) \times \{0\}$ for δ small enough, so there exists a $\theta(Y) \in (-\pi, \pi)$ such that $e^{i\theta(Y)}Y$ has a non singular trajectory. Finally, making another reduction to δ and by continuity, we can assure that this trajectory will be contained in $C_{1/2}$. □

A dynamical interpretation of the above result is:

COROLLARY 4.2. *The property of $\{X_\theta\}$ of having an infinite set of bifurcation values, is a stable property under small perturbations of the family $\{X_\theta\}$ in $GQ(n)$.*

5. *Explicit examples of flat metrics*

In this section we describe some open sets of flat metrics which come from $GQ(n)$. Note that $GQ(n)$ has two natural actions: one given by the multiplication of a non-zero constant $\lambda \in \mathbb{C}^*$, and the other given by the group of complex automorphisms of $\mathbb{P}^1$, which acts by changing the positions of three of the points in the divisor of a meromorphic vector field.

For simplicity we suppose that $\mathrm{Aut}(\mathbb{P}^1)$ changes three zeros.

If our main interest is the associated flat metrics, then note that the action of $\mathrm{Aut}(\mathbb{P}^1)$, for some $X \in GQ(n)$, does not change the Riemannian metric $(\mathbb{P}^1 - \mathrm{Sing}(X), g)$. The action of $\mathbb{C}^*$ changes the metric by a real scalar factor $|\lambda|$. Hence, if $GQ(n)^*$ is the set of classes of elements in $GQ(n)$ under the actions of $S^1 \subset \mathbb{C}^*$ and $\mathrm{Aut}(\mathbb{P}^1)$, then $GQ(n)^*$ can be considered as the *set of singular flat Riemannian metrics which correspond to meromorphic vector fields in $GQ(n)$.*

Thus, for $n \geq 3$ the action of S^1 is free and $\mathrm{Aut}(\mathbb{P}^1)$ has at most finite isotropy groups for some meromorphic vector fields. So $GQ(n)^*$ is a real smooth open orbifold with real dimension $4n - 9$, when $n \geq 3$. If $n = 2$, then $GQ(n)^*$ is diffeomporphic to $\mathbb{R}^+$, since in this case we have no poles. Note that if $n = 0, 1$, then $GQ(n)^*$ is empty.

Basic example 5.1. Let T be a triangle with vertices $a, b, c \in \mathbb{C}$ and sides of euclidean length $\{r_i | i = 1, 2, 3\}$. We identify a, b, c as a single point p, thus obtaining a topological sphere minus three open disks Δ_i, $i = 1, 2, 3$. At the boundary of each disk we glue isometrically a cylinder of the form $[0, h_i) \times S^1(r_i)$ where $h_i \in \mathbb{R}^+ \cup \{\infty\}$. We name this type of object a *singular flat pant* and denote it by $\{(r_i, h_i) | i = 1, 2, 3\}$, see Figure 2.

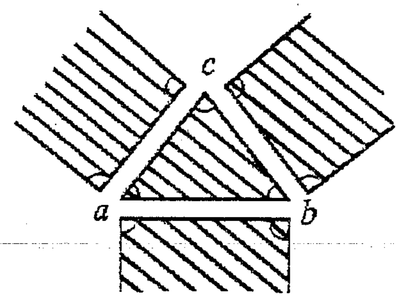

FIGURE 2.

The cylinders are represented in Figure 2 by rectangles with sides of length $2\pi r_i$ and h_i. Note that the cone angle around the pole corresponding to a, b, c is always 4π and the family of rotated foliations is induced by the family of foliations in $\mathbb{C}$ by parallel lines with fixed slope.

5.2. Consider the sets of singular flat pants as in 5.1, with $h_i = \infty$ for $i = 1, 2, 3$. These metrics in $\mathbb{P}^1 - \{4 \text{ points }\}$, have three real parameters $\{r_i\}$ and come from meromorphic vector fields in $GQ(3)$. The real dimension of $GQ(3)^*$ is also three. By simple inspection it is possible to prove that these types of pants produce an open dense set in $GQ(3)^*$. The degenerate case when the three points a, b, c are in a line is also possible, however this family has two parameters. Moreover, these pants do not contain non-trivial cylinders.

Note that if one trajectory falls inside a trivial cylinder, then it goes to the corresponding zero. Consider the following construction. Take a singular flat metric $(\mathbb{P}^1 - \text{Sing}(X), g)$ which comes from a meromorphic vector field X and remove the interior of all the trivial cylinders so we obtain a connected set (but not always with interior). This is called the *dynamical locus* of the singular flat metric g. The dynamical complexity in the behavior of the solutions is given by the geometrical complexity of the dynamical locus. If there exists a non-trivial cylinder then it is always inside the dynamical locus. In 5.2 the dynamical locus is a flat triangle. In what follows we construct several types of dynamical locus and obtain dynamical information from them.

5.3. Consider two singular flat pants $P = \{(r_1, \infty), (r_2, \infty), (r_3, h_3)\}$ and $Q = \{(p_1, \infty), (p_2, \infty), (r_3, h_3)\}$. Glue isometrically the ends of the cylinders in P and Q that have finite height h_3 and radius r_3. This process is well-defined up to a rotation by an angle $\psi \in S^1$. This construction gives a familiy of flat metrics in $\mathbb{P}^1 - \{6 \text{ points}\}$, with seven real parameters $\{r_1, r_2, r_3, p_1, p_2, h_3, \psi\}$, so this family must correspond to an open set in $GQ(4)^*$. The dynamical locus for a metric in this family is the union of a non-trivial cylinder and two triangles.

The generalization of the above example for $n \geq 5$ is obvious (glueing several singular flat pants to obtain $\mathbb{P}^1 - \{2n-2 \text{ points}\}$, and provides us with open sets of flat metrics in $GQ(n)^*$ that have non-trivial cylinders.

For $n \geq 4$ several degenerations may occur, for example: the height of some non-trivial cylinders becomes zero or some of the triangles may degenerate into segments. However, these cases are not reproduced here since they do not seem to provide qualitative information which is different from that already described.

5.4. One example in coordinates $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$, $z = x + iy \in \mathbb{C}$, is:

$$X(z) = \frac{(z-a_1)(z-\overline{a_1})(z-a_2)(z-\overline{a_2})}{(z-a_3)(z-\overline{a_3})} \frac{\partial}{\partial z},$$

where $a_1, a_2, a_3 \in \mathbb{C} - \mathbb{R}$ are all different. Here X is a meromorphic vector field in $GQ(4)$. Note that $\gamma = \mathbb{R} \cup \{\infty\}$ is a non-singular closed orbit in $\mathbb{P}^1$ which generates a non-trivial cylinder $A(\gamma)$, with poles $a_3, \overline{a_3}$ at its boundary. In fact, the flat metric associated with X is of the type described in 5.3.

5.5. Consider a hexagon in $\mathbb{C}$ like the one in Figure 3(a), such that a, b, d, e define a rectangle R and c is on the left-hand side of the segment $\overline{ae}$.

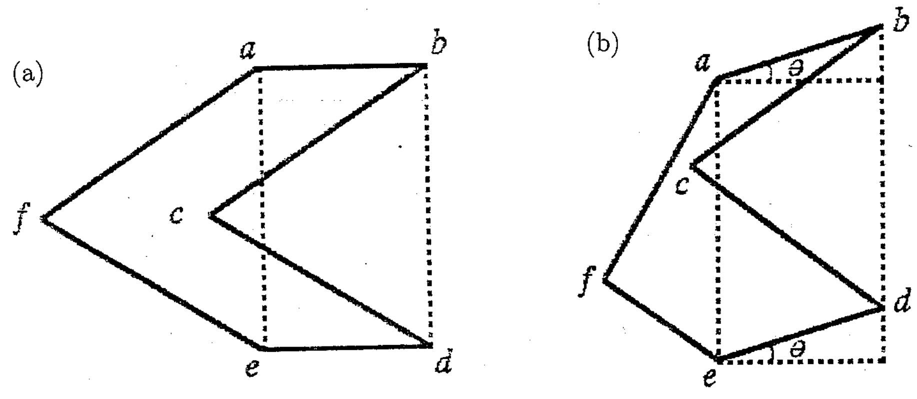

FIGURE 3.

We identify $\overline{ab}$ and $\overline{ed}$ and glue infinite cylinders of the form $[0, \infty) \times S^1(r_i)$ at the segments $\overline{bc}$, $\overline{cd}$, $\overline{ef}$ and $\overline{fa}$ with isometries. The result is a singular flat metric in $GQ(4)^*$, with the hexagon $\{a, b, c, d, e, f\}$ as dynamical locus. This contruction has seven real parameters: the length of $\{\overline{bc}, \overline{cd}, \overline{bd}, \overline{ef}, \overline{fa}\}$, the distance h between $\overline{ae}$ and $\overline{bf}$ and the rotation angle θ, see Figure 3(b). Hence, this example gives an open set $\mathcal{D}$ in $GQ(4)^*$. Note that the family contains only trivial cylinders. Using $\pi : GQ(n) \to GQ(n)^*$, the natural projection, we can see that $\pi^{-1}(\mathcal{D})$ defines an open set in $GQ(4)$.

PROPOSITION 5.6. *The set of metrics $g \in GQ(n)^*$ such that g has non-trivial cylinders is not dense in $GQ(n)^*$.*

Proof. For $n = 2, 3, 4$, the result follows from 2.5, 5.2 and 5.5. We give an extension of 5.5 for degree $n \geq 5$. Consider the family of metrics in 5.5. We remove one of the trivial cylinders around a zero and introduce a new piece as follows, see Figure 4.

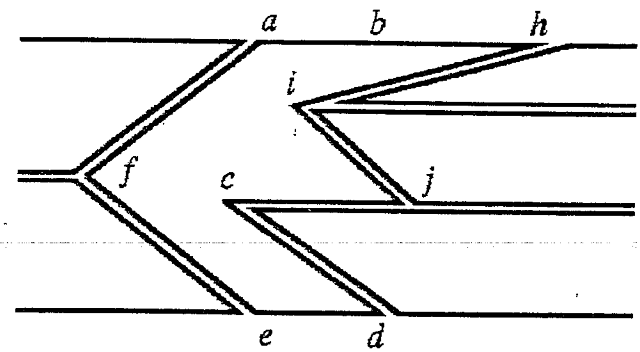

FIGURE 4.

Insert three new points h, i, j such that i is in the left-hand side of $\overline{bc}$ (which is the boundary of the removed trivial cylinder). If we glue $\overline{ab}$ with $\overline{ed}$ and $\overline{bh}$ with $\overline{cj}$, using isometries, and glue five trivial cylinders (represented in the figure by the open bands), then we have a family of flat metrics in $GQ(5)^*$. Note that there are three poles given by the identification of $\{a, e, f\}$, $\{b, c, d, \}$ and $\{h, i, j\}$, the cone angles are always 4π. The family of rotated foliations in these flat metrics comes from foliations in the plane formed by parallel lines with fixed slope. The non-existence of non-trivial cylinders is equivalent to the non-existence of straight segments inside the dynamical locus (which is the polygon $\{a, b, h, i, j, c, d, e, f\}$), having end points in $\overline{ab}$ and $\overline{ed}$, or $\overline{bh}$ and $\overline{cj}$. The number of parameters in the new family of flat metrics has been incremented by four: the distance between $\overline{bc}$ and $\overline{hj}$, the lengths of $\overline{hi}$, $\overline{ij}$, and one additional rotation parameter θ that comes from the choice of h and j over the same line defined by the segment $\overline{hj}$. Hence we have constructed a family $\mathcal{D}(5)$ of flat metrics without non-trivial cylinders and open in $GQ(5)^*$. Note that the real dimensions of $GQ(n)$ and $GQ(n)^*$ also change by four when n changes by $n + 1$. Using the same ideas it is possible to obtain similar families $\mathcal{D}(n) \subset GQ(n)^*$ for all $n \geq 6$. We have left the details for the reader. □

5.7. *The case of polynomials.* If we fix an affine chart in $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$, a polynomial vector field of degree n is given in this chart by $n+1$ complex parameters. Let us denote by $PQ(n)$ the space of these polynomial vector fields. Using the manifold $GPQ(n)$ of polynomials with zeros of multiplicity one we construct the space $GPQ(n)^*$, given by $GPQ(n)$ modulus the actions of $S^1 \subset \mathbb{C}^*$ and $\mathrm{Aut}(\mathbb{C})$. The set $GPQ(n)^*$ of singular flat metrics in $\mathbb{C} - \{n \text{ points}\}$ which comes from polynomial vector fields has real dimension $2n - 3$.

Consider an n-polygon in $\mathbb{C}$ with internal angles $\psi_1, \ldots, \psi_n$ and sides of euclidean length $r_1, \ldots, r_n$. If we identify the vertices of this polygon as a single point we obtain a topological sphere minus n open disks $\{\Delta_i\}$. We glue isometrically a cylinder of the form

$[0, \infty) \times S(r_i)$ at the boundary of each disk, where $2\pi r_i$ is the length of the corresponding side. The parameters in this type of flat metrics are $\{r_1, \ldots, r_{n-1}, \psi_1, \ldots, \psi_{n-2}\}$ and the dimension of this family is $2n - 3$. Hence this family gives an open set in $GPQ(n)^*$.

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