J. reine angew. Math. **461** (1995), 189—219

Journal für die reine und angewandte Mathematik

Deformations of holomorphic foliations having a meromorphic first integral

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§ 0. Introduction

In this paper we study singular holomorphic foliations $\mathscr{F}$ of codimension one defined on complex projective connected manifolds M with $H^1(M, \mathbb{C}) = 0$ and having generic rational functions f/g as first integrals, where generic means that f/g is a Lefschetz pencil, see [A-F]. We consider holomorphic deformations of the foliation $\mathscr{F}$ where the complex structure on M is fixed. The set of singular holomorphic foliations of codimension one on M (where the singular sets of the foliations are at least of codimension two), form in a natural way complex projective varieties $\mathrm{Fol}(M, \varepsilon)$, where ε is the Chern class of a complex line bundle L over M, and the foliations are defined by holomorphic bundle maps $L \to T^* M$ satisfying the integrability condition (see [GM-M]). If the singular holomorphic foliation $\mathscr{F}$ has a first integral f/g, then it is given by a rational differential one-form ω on M with a pole of order two along $\{g = 0\}$. We can represent an infinitesimal deformation by a rational differential one-form η on M with a pole of order two along $\{g = 0\}$ and satisfying an infinitesimal integrability condition. Some of the families $\mathrm{Fol}(M, \varepsilon)$ have analytic subspaces of foliations which are Lefschetz pencils. We denote the sets of these foliations by $LP(M, \varepsilon)$. The main result is:

2.1. Theorem. *Let M be a complex connected projective manifold such that*

$$H^1(M, \mathbb{C}) = 0$$

and let $\mathscr{F}$ be a singular holomorphic foliation of codimension one on M with a first integral which is a Lefschetz pencil. An infinitesimal deformation $[\eta]$ of $\mathscr{F}$ is tangent to the space formed by those foliations which have a first integral iff $\int_\gamma \eta = 0$ for all the closed loops γ contained in the leaves of the foliation $\mathscr{F}$.

The holonomy of a foliation having a Lefschetz pencil as first integral is trivial (all the holonomy maps are the identity). In the result the hypothesis of the vanishing for the integrals is a necessary condition, since it means that the holonomy map of the infinitesimal deformation associated to η is also trivial. The result asserts that it is also sufficient.

In [I1] Yu. S. Ilyashenko studied polynomial deformations of foliations on $\mathbb{C}^2$ with polynomial first integral, see [A-I], p. 110 for a review on Ilyashenko's results. This work was motived by Ilyashenko's work, and is a natural generalization of it. Our contribution is the use of the universal families of foliations $\mathrm{Fol}(M, \varepsilon)$ in order to consider infinitesimal deformations of foliations, i.e. tangent vectors in $T_{\mathscr{F}} \mathrm{Fol}(M, \varepsilon)$. Instead of restricting to foliations of $\mathbb{C}P^2$ which have the line at infinity as a leaf, as is the case in Ilyashenko's work.

We give some applications of the main theorem:

In a foliation having a Lefschetz pencil as first integral the closure of the generic leaves are compact submanifolds, under deformations of the foliation (not tangent to the space of foliations with first integral), the closure of the leaves change to give non compact leaves, however some homotopy loops in the leaves can persist under deformation. When the dimension of M is two, we extend to our case, Ilyashenko's study on the persistence of non trivial homotopy loops in the leaves of foliations, and give some particular computations for Lefschetz pencils in $\mathbb{C}P^2$. As another application we characterize infinitesimal deformations of Lefschetz pencils in $\mathbb{C}P^2$ such that an algebraic leaf persists, up to first order, (see Theorem 3.6). In Section 4 we compute an explicit bound for the multiplicity of the zeros in the abelian integrals $\int_\gamma \eta$ which arise on the study of infinitesimal deformations of Lefschetz pencils (see Theorem 4.1), essentially we follow the idea of Ilyashenko in [I3]. This bound gives an extension of the work of Varchenko [V], [A-I], p. 110, to the case of rational abelian integrals. We also bound locally the number of limit cycles that arise from deformations of Lefschetz pencils (see Corollary 4.6).

For higher dimension of M, many foliations with meromorphic first integrals have leaves with small fundamental groups, this type of foliations have simple behavior under deformation, in particular the existence of first integrals persist under deformation, as the following results say.

If M has dimension at least three, we show that foliations having Lefschetz pencils as first integrals determine irreducible components of the space of foliations. Moreover we proof the structural stability of Lefschetz pencils as foliations (see Theorem 5.1). A similar result was obtained by X. Gómez-Mont and A. Lins-Neto in [GM-LN] by a different way. In Section 6 we show that polynomial foliations in $\mathbb{C}^m$ (where m is at least three), having k generic polynomial first integrals and leaves of dimension $m-k$ at least two, give irreducible components in the space of polynomial foliations of codimension k (see Theorem 6.4).

The Theorems 5.1 and 6.4 can be considered as generalizations for the case of singular holomorphic foliations of the theory of G. Reeb, A. Haefliger and W. Thurston on stability of smooth real non singular foliations, see [T].

Part of this work comes from the author's doctoral thesis at The National University of México (UNAM). The author wants to thank X. Gómez-Mont for his guidance and encouragement along the elaboration of this work. To Ch. Bonatti for help on the proof of the main theorem, and to: E. Paul, P. Mardešić, A. Marin, O. Calvo-Andrade for useful discussions. Also to the referee for comments. The main result was announced in [Mu2].

§ 1. Foliations with first integral

A codimension one singular holomorphic foliation in a complex connected manifold M may be given by a family of integrable holomorphic one-forms $\{\omega_\alpha\}$ defined on an open cover $\{U_\alpha\}$ of M such that $\omega_\alpha \wedge d\omega_\alpha = 0$ and satisfying $\omega_\alpha = \xi_{\alpha\beta}\omega_\beta$ in $U_\alpha \cap U_\beta$ where $\{\xi_{\alpha\beta}\}$ are never vanishing holomorphic functions. If L denotes the holomorphic line bundle on M obtained with the cocycle $\{\xi_{\alpha\beta}^{-1}\}$, then the one-forms $\{\omega_\alpha\}$ glue to give a holomorphic line bundle map $\omega: L \to T^*M$, from L to the cotangent bundle T^*M of M. We say that ω is equivalent to $\omega': L' \to T^*M$ if there is a holomorphic bundle isomorphism $\varrho: L \to L'$ such that $\omega = \omega' \circ \varrho$.

1.1. Definition. A *codimension one singular holomorphic foliation* $\mathscr{F}$ in the complex manifold M is an equivalence class of holomorphic bundle maps $\omega: L \to T^*M$ from the line bundle L to the cotangent bundle of M such that ω does not vanish identically on M and in local coordinates we have $\omega_\alpha \wedge d\omega_\alpha = 0$, which is the integrability condition. The *singular set of the foliation* $\mathscr{F}$ is the set of points on M where a defining map $\omega: L \to T^*M$ vanishes, we denote it by $\mathrm{Sing}(\mathscr{F}) = \{p \in M \mid \omega(p) = 0\}$. The *leaves of the foliation* $\mathscr{F}$ are the leaves of the associated non singular foliation in $M - \mathrm{Sing}(\mathscr{F})$.

Let M be a complex projective manifold with $H^1(M, \mathbb{C}) = 0$. Using the Hodge decomposition of cohomology (see [G-H], p. 116), these hypoteses imply that $H^1(M, \mathcal{O}_M) = 0$, where $\mathcal{O}_M$ is the structure sheaf on M. This implies that if L is a holomorphic line bundle on M, then the holomorphic structure on the topological bundle is unique (see [G-H], p. 140). The set of holomorphic maps from L to T^*M modulus multiplication by non-vanishing constants is a finite dimensional projective space

$$\mathrm{Proj}\, H^0(M, \mathrm{Hom}(L, T^*M)).$$

The set of those maps which satisfy the integrability condition form a closed algebraic subset that we will denote by $\mathrm{Fol}(M, \varepsilon)$, where $\varepsilon \in H^2(M, \mathbb{Z})$ is the Chern class of L (see [GM-M]). They are parameter spaces for the universal families of codimension one singular holomorphic foliations in M with Chern class ε. In fact, it may be shown that $\bigcup_\varepsilon \mathrm{Fol}(M, \varepsilon)$ parametrizes the universal family of codimension one singular holomorphic foliations in M (see [GM]).

Now we recall some facts about first integrals and Lefschetz pencils.

1.2. Definition. The fibers $\{h^{-1}(\lambda/\mu)\}$ of a meromorphic map $h: M \to \mathbb{C}P^1$, where $\lambda/\mu \in \mathbb{C}P^1$, defined on a complex connected projective manifold M form a *Lefschetz pencil* if there is an embedding $i: M \to \mathbb{C}P^k$ and linear homogeneous polynomials f and g on $\mathbb{C}P^k$ with $h = (f/g) \circ i: M \to \mathbb{C}P^1$ satisfying:

1. The linear subspace of codimension two defined by $\{f = g = 0\}$ intersects $i(M)$ transversally in a smooth manifold K, called the center of the pencil, K is the indeterminancy locus of h as a rational map on M.

2. The hyperplanes $\lambda f + \mu g = 0$ intersect $i(M)$ transversally, except for a finite number of values $\lambda/\mu \in \mathbb{C}P^1$, where it has just one non degenerate tangency (i.e. h has only a Morse-type singular point on $M - K$ for each critical value).

The main results about Lefschetz pencils that we will use are The Lefschetz's Theorems on hyperplane sections, see [A-F] and [Mi]:

1.3. Theorem. 1. *Let $h: M \to \mathbb{C}P^1$ be a Lefschetz pencil on M, and $V = h^{-1}(\lambda/\mu)$ a smooth member of the pencil, then the inclusion $V \to M$ induces isomorphisms on the homotopy groups $\pi_j(V) \to \pi_j(M)$ for $j < \dim(M) - 1$ and a surjection for $j = \dim(M) - 1$.*

2. *Let $i: M \to \mathbb{C}P^k$ be an embedding and let $G(1,k)$ be the Grassmann manifold of lines in the dual space $(\mathbb{C}P^k)^*$ of hyperplanes in $\mathbb{C}P^k$. The set of points in $G(1,k)$ that define Lefschetz pencils on M form a Zariski open dense subset in $G(1,k)$.* □

Recall that line bundles on a projective manifold M obtained by pulling back the line bundle of hyperplane sections of $\mathbb{C}P^k$ under an embedding $i: M \to \mathbb{C}P^k$ are called very ample. A line bundle L is ample if $L^{\otimes n}$ is very ample for some $n > 0$. An ample class in $H^2(M, \mathbb{Z})$ is the first Chern class of an ample line bundle on M, and it is always non zero (see [G-H], p.148).

Let M be a complex projective manifold and let L_1 and L be holomorphic line bundles on M with Chern classes $c, a \in H^2(M, \mathbb{Z})$, where c is an ample class. If

$$H^0(M, \mathcal{O}_M(L_1^{\otimes n} \otimes L))$$

denotes the holomorphic sections of $L_1^{\otimes n} \otimes L$ and $nc + a$ is the Chern class of $L_1^{\otimes n} \otimes L$, then for n sufficiently large the space of these sections is non-empty. Let $G(2, \varepsilon)$ be the Grassmann manifold of two planes in $H^0(M, \mathcal{O}_M(L_1^{\otimes n} \otimes L))$. If (f, g) denotes the point in $G(2, \varepsilon)$ representing the plane generated by f, g and $\mathscr{F}$ is the foliation whose leaves are the irreducible components of the fibers of the meromorphic map $f/g: M \to \mathbb{C}P^1$, then the differential $d(f/g)$ is a rational one-form on M describing the foliation $\mathscr{F}$, with a pole of order two along $\{g = 0\}$ (for example in the classical example of $M = \mathbb{C}P^n$ we can take f and g as homogeneous polynomials of the same degree). Hence $g^2 d(f/g)$ determines a holomorphic bundle map tangent to $\mathscr{F}$. This construction gives rise to a holomorphic injective map

$$\Phi: G(2, \varepsilon) \to \mathrm{Fol}(M, -2(nc + a)),$$

whose image we denote as $\mathrm{FI}(M, -2(nc + a))$, the foliations with a first integral.

1.4. Definition. A foliation $\mathscr{F} \in \mathrm{Fol}(M, -2(nc + a))$ *is a Lefschetz pencil* if it is of the form $\Phi(f, g)$ where (f, g) is a Lefschetz pencil. We denote all these foliations by $\mathrm{LP}(M, -2(nc + a))$.

By Theorem 1.3.2, $\mathrm{LP}(M, -2(nc + a)) \subset \mathrm{FI}(M, -2(nc + a))$ is a Zariski open dense set.

1.5. Definition. Let $\mathcal{F} \in \mathrm{Fol}(M, \varepsilon)$ be a singular holomorphic foliation of codimension one on a complex connected projective manifold M. An *infinitesimal deformation of the foliation* $\mathcal{F}$ is a tangent vector in $T_{\mathcal{F}} \mathrm{Fol}(M, \varepsilon)$.

Now we construct in an explicit way infinitesimal deformations of foliations. Let A be the analytic space associated to the dual numbers $\mathbb{C}[t]/(t^2)$ (see [H], p. 265). Consider the analytic space $A \times M$. This is topologically the same manifold M, but as ringed space it has $\mathcal{O}_M \oplus t\mathcal{O}_M$ (where $t^2 = 0$) as structure sheaf. If L denotes a holomorphic line bundle on M, then using the projection $\Pi : A \times M \to M$ to the second factor, we obtain $\Pi^* L = L \oplus tL$ a holomorphic line bundle over $A \times M$ and the relative cotangent bundle

$$\Pi^* T^* M = T^* M \oplus t T^* M .$$

A bundle map $\phi : L \oplus tL \to T^* M \oplus t T^* M$ over $A \times M$ is described in a unique way by $\phi = \omega + t\eta$, where $\omega, \eta \in H^0(M, \mathrm{Hom}(L, T^* M))$.

1.6. Lemma. *Let M be a complex connected projective manifold, L a holomorphic line bundle on M with Chern class ε, and $\omega \in H^0(M, \mathrm{Hom}(L, T^* M))$ a bundle map which represents a foliation $\mathcal{F} \in \mathrm{Fol}(M, \varepsilon)$.*

1. *A bundle map $\phi = \omega + t\eta : L \oplus tL \to T^* M \oplus t T^* M$ over $A \times M$, with*

$$\eta \in H^0(M, \mathrm{Hom}(L, T^* M)) ,$$

represents a tangent vector in $T_{\mathcal{F}} \mathrm{Fol}(M, \varepsilon)$ if and only if for local holomorphic one-forms $\{\omega_\alpha\}, \{\eta_\alpha\}$ describing ω and η with respect to a trivializing covering $\{U_\alpha\}$ of L, we have

$$(*) \qquad \omega_\alpha \wedge d\eta_\alpha + \eta_\alpha \wedge d\omega_\alpha = 0 .$$

2. *We have*

$$T_{\mathcal{F}} \mathrm{Fol}(M, \varepsilon) = \frac{\{\eta \in H^0(M, \mathrm{Hom}(L, T^* M)) \mid \eta \text{ satisfies } (*)\}}{\mathbb{C} \cdot \omega} . \qquad \square$$

For the proof, see [GM-M], sec. 2.

1.7. Remark. In particular, if we work with a line bundle of the form $L_1^{\otimes n} \otimes L$ and $\mathcal{F}$ is a foliation given by the Lefschetz pencil f/g, where $f, g \in H^0(M, \mathcal{O}_M(L_1^{\otimes n} \otimes L))$, each vector $[\eta] \in T_{\mathcal{F}} \mathrm{Fol}(F, -2(nc + a))$ can be represented by a rational one-form on M which has a pole of order two along $\{g = 0\}$. To see this it is sufficient to consider the trivialization of $L_1^{\otimes n} \otimes L$ in $U_1 = M - \{g = 0\}$ given by the section g. All

$$\eta \in H^0(M, \mathrm{Hom}(L_1^{\otimes n} \otimes L, T^* M))$$

can be given by a holomorphic one-form η_1 in U_1 and such that this holomorphic one-form has a unique rational extension to all M. An explicit way to make these constructions can be given using the explicit expressions for rational one-forms over $\mathbb{C}P^k$ given in [G], we leave the details to the reader.

1.8. Remark. Let $\omega, \eta \in H^0(M, \mathrm{Hom}(L_1^{\otimes n} \otimes L, T^*M))$ be given by local holomorphic one-forms $\{\omega_1\}$, $\{\eta_1\}$ with respect to $U_1 = M - \{g = 0\}$, as in Remark 1.7, such that $[\omega_1] \in \mathrm{Fol}(M, -2(nc+a))$ and $[\eta_1] \in T_{\mathscr{F}}\,\mathrm{Fol}(M, -2(nc+a))$. If $h_1 + th_2 \in \mathcal{O}_{U_1} \oplus t\mathcal{O}_{U_1}$ is a holomorphic function on $U_1 \times A$, then

$$(h_1 + th_2)(\omega_1 + t\eta_1) = h_1\omega_1 + t(h_2\omega_1 + h_1\eta_1) \quad (\mathrm{mod}\, t^2 = 0)\,.$$

In particular, $[\omega_1 + t\eta_1]$ and $[h_1\omega_1 + t(h_2\omega_1 + h_1\eta_1)]$ represent the same infinitesimal deformation of the foliation associated to $[\omega]$ in M.

We work in order to prove that $\mathrm{LP}(M, -2(nc+a))$ is a smooth analytic subspace in $\mathrm{Fol}(M, -2(nc+a))$.

1.9. Lemma. *Let $\mathscr{F}$ be a Lefschetz pencil in $\mathrm{LP}(M, -2(nc+a))$ given by*

$$f/g : M \to \mathbb{C}P^1$$

where $f, g \in H^0(M, \mathcal{O}_M(L_1^{\otimes n} \otimes L))$. Then each tangent vector in $T_{\mathscr{F}}\,\mathrm{LP}(M, -2(nc+a))$ can be represented by a rational one-form on M of type

$$d\left(\frac{r}{g}\right) + \left(\frac{s}{g}\right)^2 d\left(\frac{f}{s}\right),$$

where $r, s \in H^0(M, \mathcal{O}_M(L_1^{\otimes n} \otimes L))$.

Proof. Given (f, g) and (r, s) two points in $G(2, \varepsilon)$, consider the one-parameter family of rational functions on M given by $(f + tr)/(g + ts)$, where $t \in \mathbb{C}$. We compute the tangent vector associated to this curve in $T_{\mathscr{F}}\,\mathrm{Fol}(M, -2(nc+a))$ using the trivialization of $L_1^{\otimes n} \otimes L$ given by the section g on $U_1 = M - \{g = 0\}$. We define $\{f, g\}^d = gdf - fdg$. Modulus $t^2 = 0$ we have

$$d\left(\frac{f + tr}{g + ts}\right) = \frac{\{f, g\}^d}{g^2} + \frac{t}{g^2}\left[\{r, g\}^d + \{f, s\}^d - \frac{2s}{g}\{f, g\}^d\right]$$

and taking the evaluation of the differential one-form in $\partial/\partial t$ at $t = 0$, we obtain

$$d\left(\frac{f + tr}{g + ts}\right)\Big|_{t=0} = d\left(\frac{r}{g}\right) + \left(\frac{s}{g}\right)^2 d\left(\frac{f}{s}\right) - \left(\frac{2s}{g}\right) d\left(\frac{f}{g}\right).$$

By Remark 1.8 the last term vanishes and we arrive to the result. □

1.10. Remark. Let $\mathscr{F} \in \mathrm{LP}(M, -2(nc+a))$ be a Lefschetz pencil, given by f/g. A rational one-form η on M which has a pole of order two along $\{g = 0\}$ represents the $0 \in T_{\mathscr{F}}\,\mathrm{Fol}(M, -2(nc+a))$, iff

$$\eta = \frac{\{r, g\}^d + \{f, s\}^d}{g^2} = \lambda d\left(\frac{f}{g}\right)$$

for $\lambda \in \mathbb{C}$ and some $r, s \in H^0(M, \mathcal{O}_M(L_1^{\otimes n} \otimes L))$. This follows from Lemma 1.6.2 and Remark 1.8.

1.11. Proposition. *The space of Lefschetz pencils* $\mathrm{LP}(M, -2(nc+a))$ *is a smooth analytic subspace of* $\mathrm{Fol}(M, -2(nc+a))$.

Proof. Let f/g be a Lefschetz pencil and let $\{f, g, r_1, \ldots, r_\beta\}$ be a basis for

$$H^0(M, \mathcal{O}_M(L_1^{\otimes n} \otimes L)),$$

such that the rational functions $f/(g+r_j)$ and $(f+r_j)/g$ are Lefschetz pencils for all $j = 1, \ldots, \beta$ (this can be done by Theorem 1.3.2). We have local coordinate charts for the Grassmannian $G(2, n)$ of two planes over $H^0(M, \mathcal{O}_M(L_1^{\otimes n} \otimes L))$, given by

$$\Psi : \mathbb{C}^{2\beta} \to G(2, \varepsilon),$$

$$\Psi(x_1, \ldots, x_\beta, y_1, \ldots, y_\beta) = (f + x_1 r_1 + \cdots + x_\beta r_\beta, g + y_1 r_1 + \cdots + y_\beta r_\beta),$$

such that $\Psi(0, \ldots, 0) = (f, g) \in G(2, \varepsilon)$. We take the composition $\Phi \circ \Psi$ and obtain two families of curves on $\mathrm{Fol}(M, -2(nc+a))$ defined by:

$$x_j \mapsto d\left(\frac{f + x_j r_j}{g}\right) \quad \text{and} \quad y_i \mapsto d\left(\frac{f}{g + y_j r_j}\right).$$

Note that two rational functions determine the same foliation in $\mathrm{Fol}(M, -2(nc+a))$ if and only if they differ by a Möbius transformation (see [J], p. 109). From this and Remark 1.8 we can, without loss of generality, change the second family by

$$y_i \mapsto \left(\frac{f}{g}\right)^2 d\left(\frac{g + y_j r_j}{f}\right).$$

We compute the associated tangent vectors:

$$\frac{d}{dx_j} d\left(\frac{f + x_j r_j}{g}\right)\Big|_{x_j = 0} = \frac{\{r_j, g\}^d}{g^2},$$

$$\frac{d}{dy_j}\left(\frac{f}{g}\right)^2 d\left(\frac{g + y_j r_j}{f}\right)\Big|_{x_j = 0} = \frac{\{f, r_j\}^d}{g^2}.$$

These vectors are different from $0 \in T_{\mathscr{F}} \mathrm{Fol}(M, -2(nc+a))$ and they are linearly independent, by Remark 1.10 and the fact that $f, g, r_1, \ldots, r_\beta$ is a basis for $H^0(M, \mathcal{O}_M(L_1^{\otimes n} \otimes L))$. Thus, Φ is an embedding in the Lefschetz pencils. □

§ 2. Proof of the main theorem

Now we can write the main theorem in a more technical form.

2.1. Theorem. *Let M be a complex connected projective manifold such that*

$$H^1(M, \mathbb{C}) = 0$$

and let $\mathscr{F}$ be a singular holomorphic foliation of codimension one on M which is a Lefschetz pencil in $\mathrm{LP}(M, -2(nc+a))$. *An infinitesimal deformation* $[\eta] \in T_{\mathscr{F}}\,\mathrm{Fol}(M, -2(nc+a))$ *is in* $T_{\mathscr{F}}\,\mathrm{LP}(M, -2(nc+a))$ *iff* $\int_\gamma \eta = 0$ *for all the closed loops γ contained in the leaves of the non singular foliation associated to $\mathscr{F}$ in* $M - \mathrm{Sing}(\mathscr{F})$.

Proof of the part "if". Given $[\eta] \in T_{\mathscr{F}}\,\mathrm{LP}(M, -2(nc+a))$, we consider a holomorphic curve $t \mapsto \mathscr{F}_t$ in $\mathrm{LP}(M, -2(nc+a))$, where $t \in \mathbb{C}$, such that $[\eta]$ is the tangent vector at $t=0$. We have a holomorphic family of foliations $\{\mathscr{F}_t\}$ with first integral. According to [GM-M], Theorem 2.9, if a loop γ is contained in a leaf of the Lefschetz pencil f/g, then the linear part of the holonomy of the family of foliations $\{\mathscr{F}_t\}$ at $t=0$ has the form

$$(t, z) \mapsto \left(t, z + t \int_\gamma \eta\right).$$

Hence the holonomy map of the family of foliations is the identity map and the integral vanishes.

For the proof of the part "only if" we use essentially Ilyashenko's method in [I1]. We consider two vector spaces associated to the Lefschetz pencil f/g as follows: Let $\Omega_g^1(M)$ be the rational one-forms on M which have a pole of order two along $\{g=0\}$ and give vectors in $T_{\mathscr{F}}\,\mathrm{Fol}(M, -2(nc+a))$. We define

$$W_{\mathscr{F}} = \left\{\eta \in \Omega_g^1(M) \,\middle|\, \int_\gamma \eta = 0 \text{ for all loops } \gamma \text{ contained on the leaves of } f/g\right\}$$

and

$$V_{\mathscr{F}} = \{\eta \in \Omega_g^1(M) \mid [\eta] \in T_{\mathscr{F}}\,\mathrm{LP}(M, -2(nc+a))\}.$$

We have that $V_{\mathscr{F}}$ and $W_{\mathscr{F}}$ are vector subspaces of $T_{\mathscr{F}}\,\mathrm{Fol}(M, -2(nc+a))$ and $V_{\mathscr{F}} \subset W_{\mathscr{F}}$. For the proof of the theorem we construct a linear map $\vartheta : W_{\mathscr{F}} \to N$, where N is a certain finite-dimensional complex vector space. We will show that $\vartheta(W_{\mathscr{F}}) = \vartheta(V_{\mathscr{F}})$ and that $\mathrm{kernel}(\vartheta) = 0$, from this we conclude that $V_{\mathscr{F}} = W_{\mathscr{F}}$.

Now we construct ϑ. If $\mathscr{F}$ is a Lefschetz pencil, then there exists an embedding $i : M \to \mathbb{C}P^k$ and an associated pencil of hyperplanes in $\mathbb{C}P^k$ such that it induces the foliation $\mathscr{F}$ in M. We can take a linear subspace $T' \cong \mathbb{C}P^{k-b} \subset \mathbb{C}P^k$, where b depends on i, which is not contained in any hyperplane of the pencil and such that $T = i(M) \cap T'$ is a smooth connected complex curve in $M - \mathrm{Sing}(\mathscr{F})$. Note that T intersects all the leaves of the non-singular foliation associated to $\mathscr{F}$ in $M - \mathrm{Sing}(\mathscr{F})$, and the intersection with each leaf is a finite number of points (to see this, consider f/g restricted to T as a rational map to $\mathbb{C}P^1$).

If $p \in M - \{g=0\} \cup \mathrm{Sing}(\mathscr{F})$ and $\mathscr{L}(p)$ is the leaf of the foliation $\mathscr{F}$ in $M - \mathrm{Sing}(\mathscr{F})$ which contains p, then we denote $T \cap \mathscr{L}(p)$ by $\{p_1, \ldots, p_\kappa\}$. Note that we always consider the multiplicity of the points in the intersection $T \cap \mathscr{L}(p)$, in particular for every leaf $\mathscr{L}(p)$ its intersection with T has exactly κ points (not necessarily different), where κ depends on $\mathscr{F}$. Given $\eta \in W_{\mathscr{F}}$, we define the function

$$I_\eta : M - \{g=0\} \cup \mathrm{Sing}(\mathscr{F}) \to \mathbb{C},$$

$$I_\eta(p) = \sum_{\nu=1}^{\kappa} \int_{p_\nu}^{p} \eta,$$

where the integrals are computed using paths contained in the leaves $\mathscr{L}$ of $\mathscr{F}$, and the restriction of η to $\mathscr{L}$. Using the fact $\int_\gamma \eta = 0$ for every closed loop γ contained in a leaf of $\mathscr{F}$, it follows that the function I_η is well defined and holomorphic, namely the path integrals depend only on the extreme points. By Hartog's Theorem this function extends to $\mathrm{Sing}(\mathscr{F}) - \{g = 0\}$.

Thus we can define $\vartheta(\eta) = I_\eta$. Now we study each I_η as meromorphic function on all M. We recall that $K = \{f = g = 0\} \subset M$.

2.2. Lemma. *If $\eta \in W_{\mathscr{F}}$ is non-zero then each function I_η has a natural extension to a meromorphic function on $M - K$:*

1. *with a pole of order two at $\{g = 0\}$ if η does not vanish on $\{g = 0\}$.*

2. *with a pole of order one at $\{g = 0\}$ if η does vanish on $\{g = 0\}$.*

Proof. We make a local computation, namely we consider only one summand in I_η and we let p go to $\{g = 0\}$. Let $q \in \{g = 0\} \cap T$, note that $T \cap K = \emptyset$, and let (U, ϕ) be a local foliated chart $\phi : U \subset M \to \mathbb{C}^m$ for the foliation with $\phi(U)$ a polydisk in $\mathbb{C}^m$. We assume that ϕ is such that $\phi(q) = (0, \ldots, 0)$, $\phi(T \cap U) \subset \{(0, \ldots, 0, z_m)\}$ and

$$\phi(\{g = 0\} \cap U) \subset \{(z_1, \ldots, z_{m-1}, 0)\},$$

where $(z_1, \ldots, z_m)$ are local coordinates for $\mathbb{C}^m$. We consider $p \in U - (\{g = 0\} \cup T)$, such that $\phi(p) = (z_{1,0}, \ldots, z_{m,0})$. Hence we can write

$$\phi_* \eta = \sum_{i=1}^{m} \frac{A_i(z_1, \ldots, z_m)\, dz_i}{z_m^2},$$

with $A_i(z_1, \ldots, z_m)$ holomorphic functions in $\phi(U)$. We make $p_1 = \phi^{-1}(0, \ldots, 0, z_{m,0})$ and consider the first sumand in the definition of I_η as:

$$\int_{\phi(p_1)}^{\phi(p)} \phi_* \eta \, .$$

We make also $p \to \{g = 0\}$ using the family of points $(z_{1,0}, \ldots, z_{m-1,0}, \varepsilon z_{m,0})$, where $\varepsilon \to 0$. Let $\delta_\varepsilon : [0,1] \to \mathbb{C}^m$ be a family of paths defined by

$$\delta_\varepsilon(t) = (tz_{1,0}, \ldots, tz_{m-1,0}, \varepsilon z_{m,0}) \, .$$

Thus

$$\lim_{p \to \{g=0\}} \int_{\phi(p_1)}^{\phi(p)} \phi_* \eta = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \int_0^1 \left(\sum_{i=1}^{m} \frac{A_i(tz_{1,0}, \ldots, tz_{m-1,0}, \varepsilon z_{m,0})}{z_{m,0}^2} z_{i,0} \right) dt \, .$$

If η does not vanish at $\{g = 0\}$, then the function

$$\int_{\phi(p_1)}^{\phi(p)} \phi_* \eta$$

has a pole of order two at $\{g = 0\}$. If η does vanish then the pole is of order one or zero. However as $\eta \in W_{\mathscr{F}}$ is non-zero, the associated function I_η is a non-constant meromorphic function and the pole of order zero is impossible. □

2.3. Lemma. *If $\eta \in W_{\mathscr{F}}$ is non-zero, then the meromorphic function I_η has locus of indetermination at K.*

Proof. We show that a finite-level set of the function I_η cross K. Let $q \in K$ be a point in the center of the pencil and consider (U, ϕ) a local chart for the foliation, $q \in U$. The Lefschetz pencil f/g is described by z_1/z_m in $\phi(U)$, where $(z_1, \ldots, z_m)$ are coordinates for $\phi(U)$. We consider the local expression

$$\phi_* \eta = \sum_{i=1}^{m} \sum_{j=0}^{\infty} \frac{A_{i,j}(z_1, \ldots, z_m)\, dz_i}{z_m^2},$$

where each $A_{i,j}(z_1, \ldots, z_m)$ is a homogeneous polynomial of degree j on $z_1, \ldots, z_m$. We choose $\lambda \in \mathbb{C}$ such that it satisfies $A_{1,0} + \lambda A_{m,0} = 0$. Consider the leaf of the foliation associated to $z_1/z_m = \lambda$, and take a path inside of it, $\delta : [0,1] \to \mathbb{C}^m$ given by

$$\delta(t) = (\lambda t, 0, \ldots, 0, t).$$

Without loss of generality we can assume that $\delta[0,1] \subset \phi(U)$ and denote $p = \phi^{-1}(\delta(1))$. Note that the $\lim_{p \to q} I_\eta(p)$ is in $\mathbb{C}$ if and only if $\lim_{\varepsilon \to 0} \int_\varepsilon^1 \phi_* \eta$ is in $\mathbb{C}$. Finaly we compute this limit by considering the series expansion in powers of t:

$$\begin{aligned}\lim_{\varepsilon \to 0} \int_\varepsilon^1 \phi_* \eta = \lim_{\varepsilon \to 0} \Bigg[& \int_\varepsilon^1 \frac{(A_{1,0} + \lambda A_{m,0})\, dt}{t^2} \\ & + \int_\varepsilon^1 \frac{A_{1,1}(\lambda, 0, \ldots, 0, 1)\, \lambda dt + A_{m,1}(\lambda, 0, \ldots, 0, 1)\, dt}{t} \\ & + \int_\varepsilon^1 \sum_{j=2}^{\infty} \frac{A_{1,j}(\lambda t, 0, \ldots, 0, t)\, \lambda dt + A_{m,j}(\lambda t, 0, \ldots, 0, t)\, dt}{t^2} \Bigg].\end{aligned}$$

Now we observe that by the selection of λ the first term of the right side vanishes and the second one is zero (by hypothesis the residues of η in the leaves of the foliation are zero i.e. $\int_\gamma \eta = 0$ for all loops γ in a leaf of $\mathscr{F}$). Since the limit is in $\mathbb{C}$ the result follows. □

Now we can write an explicit form for the maps I_η.

2.4. Lemma. *Each rational map I_η has an expression as*

$$\frac{rg + sf}{g^2},$$

where $f, g, r, s \in H^0(M, \mathcal{O}_M(L_1^{\otimes n} \otimes L))$.

Proof. By Lemma 2.2 each rational map $I_\eta : M \to \mathbb{C}P^1$ has denominator g^2 and its numerator vanishes at K by Lemma 2.3. Now we apply a generalization of Noether's "AF + BG" Theorem to compute the numerator (see [G-H], p. 703). Let $I \subset \mathcal{O}_M$ be the sheaf of ideals of K. The Koszul complex gives the short exact sequence

$$0 \to \mathcal{O}_M \to \mathcal{O}_M(L_1^{\otimes n} \otimes L) \oplus \mathcal{O}_M(L_1^{\otimes n} \otimes L) \to I \otimes \mathcal{O}_M(4nc + 4a) \to 0,$$

where the maps in this sequence are:

$$\alpha \to \alpha g \oplus -\alpha f, \quad \alpha \in \mathcal{O}_M,$$

$$\psi \oplus \phi \to \psi g + \phi f, \quad \psi, \phi \in \mathcal{O}_M(L_1^{\otimes n} \otimes L).$$

Using $H^1(M, \mathcal{O}_M) = 0$, we have the exact cohomology sequence

$$0 \to H^0(M, \mathcal{O}_M) \to H^0(M, \mathcal{O}_M(L_1^{\otimes n} \otimes L)) \oplus H^0(M, \mathcal{O}_M(L_1^{\otimes n} \otimes L))$$
$$\to H^0(M, I \otimes \mathcal{O}_M(4nc + 4a)) \to 0.$$

The denominator of I_η vanishes at K and $H^0(M, I \otimes \mathcal{O}_M(4nc + 4a))$ is the space of global sections of the ideal determined by K. Hence the numerator of I_η is of the form $rg + sf$. □

2.5. Lemma. *The dimension of $\vartheta(W_{\mathcal{F}})$ is such that*

$$\dim \vartheta(W_{\mathcal{F}}) \leqq 2 \dim H^0(M, \mathcal{O}_M(L_1^{\otimes n} \otimes L)) - 4.$$

Proof. The dimension of $\{I_\eta\}$ by Lemma 2.4 is bounded by the dimension of

$$H^0(M, I \otimes \mathcal{O}_M(4nc + 4a)).$$

Using the exact cohomology sequence in Lemma 2.4 we have:

$$\dim \vartheta(W_{\mathcal{F}}) \leqq 2 \dim H^0(M, \mathcal{O}_M(L_1^{\otimes n} \otimes L)) - \dim H^0(M, \mathcal{O}_M).$$

If $\eta \neq 0$ then the associated I_η is non-constant along the leaves of the foliation $\mathcal{F}$. On the other hand all the rational maps which are constant along the leaves of the foliation and have a pole of order two along $\{g = 0\}$ may be written as $a\left(\frac{f}{g}\right)^2 + b\left(\frac{f}{g}\right) + c$ for $a, b, c \in \mathbb{C}$, which is a three parameter family. From this observation and the previous inequality we arrive to the result. □

Proof of the "only if" part of Theorem 2.1. Given the foliation $\mathcal{F}$, we construct the linear function $\vartheta : W_{\mathcal{F}} \to N$. It is not difficult to see (using the definition of I_η by integrals and Stokes' Theorem), that $I_\eta = 0$ if and only if $\eta = 0 \in W_{\mathcal{F}}$. Hence, ϑ is injective. On the other hand by Proposition 1.11

$$\dim(V_{\mathcal{F}}) = \dim \mathrm{LP}(M, -2(nc + a)) = 2[\dim H^0(M, \mathcal{O}_M(L_1^{\otimes n} \otimes L)) - 2].$$

Recall that $V_{\mathcal{F}} \subset W_{\mathcal{F}}$, and by Lemma 2.5, $\vartheta(V_{\mathcal{F}}) = \vartheta(W_{\mathcal{F}})$. We conclude that $V_{\mathcal{F}} = W_{\mathcal{F}}$ this proves Theorem 2.1. □

Another proof of the main theorem for the classical case when $M = \mathbb{C}P^2$ can be obtained following the ideas of [I2] and the Riemann Roch Theorem.

We give two applications of Lemma 2.2. The next result is related to the problem of determining where a holomorphic foliation with many algebraic leaves has a first integral, see [J] and [Ce] for related results.

2.6. Definition. Let $\mathscr{F} \in \mathrm{Fol}(M, -2(nc+a))$ be a foliation and a holomorphic section $g \in H^0(M, \mathcal{O}_M(L_1^{\otimes n} \otimes L))$, as in §1, where $Q = \{g = 0\} \subset M$ is an algebraic hypersurface in M. The *foliation* $\mathscr{F}$ *has an algebraic leaf* Q iff one of its leaves on $M - \mathrm{Sing}(\mathscr{F})$ has Q as closure.

2.7. Corollary. *Let* $\mathscr{F} \in \mathrm{LP}(M, -2(nc+a))$ *be a foliation with a first integral given by* f/g *and a foliation* $\mathscr{F}' = [\eta] \in \mathrm{Fol}(M, -2(nc+a))$. *Suppose that* $\int_\gamma \eta = 0$ *for every closed loop contained on the leaves of* $\mathscr{F}$. *If* $\{g = 0\}$ *is a common algebraic leaf for* $\mathscr{F}$ *and* $\mathscr{F}'$, *then* $\mathscr{F}'$ *has a first integral.*

Proof. We represent $[\eta]$ by a rational one-form η on M with pole along $\{g = 0\}$. By 2.2.2 we have that $I_\eta = s/g$, where $s, g, f \in H^0(M, \mathcal{O}_M(L_1^{\otimes n} \otimes L))$. Thus along the leaves of $\mathscr{F}$ we have that $dI_\eta - \eta = 0$, since over all $M: d(s/g) - \eta = \lambda d(f/g)$, where λ is a meromorphic function on M. By a degree consideration λ is a constant function and

$$\eta = d(s/g - \lambda f/g) . \quad \square$$

Now we have a rigidity result.

2.8. Corollary. *Let* $\mathscr{F}, \mathscr{F}'$ *be two foliations as in Corollary* 2.7. *If they have two common algebraic leaves then* $\mathscr{F} = \mathscr{F}'$.

Proof. Use Corollary 2.7 and the fact that Lefschetz pencils are determined by two of their leaves. $\square$

§ 3. Some applications

We shall extend some results of Ilyashenko on [I1]. Suppose that the manifold M has dimension two. The case when the dimension of M is greater than two will be considered in Theorem 5.1.

Let $\mathscr{F} \in \mathrm{Fol}(M, -2(nc+a))$ be a Lefschetz pencil given by the holomorphic map $h = f/g: M \to \mathbb{C}P^1$ and let Λ be the set of $\{$*critical values of* $h\} \subset \mathbb{C}P^1$. The restricted map

$$h: M - (K \cup h^{-1}(\Lambda)) \to \mathbb{C}P^1 - \Lambda ,$$

with $K = \{f = g = 0\}$, is the center of the pencil, has the structure of a smooth fibre bundle, associated to $\mathscr{F}$, with fibre the generic leaf of $\mathscr{F}$. In particular we have a monodromy representation

$$\mathscr{M}: \pi_1(\mathbb{C}P^1 - \Lambda, \lambda_0) \to \mathrm{Aut}(H_1(\mathscr{L}(\lambda_0), \mathbb{Z})) ,$$

where $\lambda_0 \in \mathbb{C}P^1 - \Lambda$ is a fixed basis point for the classes in $\pi_1(\mathbb{C}P^1 - \Lambda)$ and

$$\mathscr{L}(\lambda_0) = h^{-1}(\lambda_0) \cap (M - K),$$

is the leaf associated to λ_0. We call $\mathscr{L}(\lambda_0)$ a *generic leaf of* $\mathscr{F}$. Note that a generic leaf is a non-singular Riemann surface.

Let $\tau : \Delta \to \mathbb{C}P^1 - \Lambda$ be the universal cover of $\mathbb{C}P^1 - \Lambda$ (where Δ is the Poincaré disc). We can pull back the fibre bundle h to Δ and obtain a new fiber bundle $H: M' \to \Delta$, each fibre is homeomorphic to $\mathscr{L}(\lambda_0)$, the generic leaf of $\mathscr{F}$.

In the homology groups $H_1(\mathscr{L}(\lambda_0), \mathbb{Z})$ we call *indeterminacy homology classes* the classes which arise from the center of the pencil K, i.e. are in the kernel of the homology map associated to natural embedding $\mathscr{L}(\lambda_0) \to \overline{\mathscr{L}(\lambda_0)} \subset M$, where the closure is taken on all M. We denote these associated classes by $I(\mathscr{L}(\lambda_0))$. The indeterminacy homology classes are invariant under monodromy, see [GM-M], p. 158.

Suppose that $0 \in \mathbb{C}P^1$ is a critical value of h in M and let p be its associated critical point. In a neighborhood of p, the Lefschetz pencil has a local description of the type $\bar{h}(z_1, z_2) = z_1 z_2$. Let $\lambda_0 \in \mathbb{C}P^1$ be a point near 0. A loop on the leaf $\bar{h}^{-1}(\lambda_0)$ given by $t \mapsto (e^{it}, \lambda_0/e^{it})$ for $t \in [0, 2\pi]$ is called a *vanishing cycle* of the Lefschetz pencil (see [I1] or [L] for a pure homological description). Denote the set of *vanishing homology classes* by $V(\mathscr{L}(\lambda_0))$. As in [L], p. 29, 30, we have that:

$$H_1(\mathscr{L}(\lambda_0), \mathbb{Z}) = I(\mathscr{L}(\lambda_0)) \oplus V(\mathscr{L}(\lambda_0))$$

and this decomposition is invariant by the monodromy representation $\mathscr{M}$ (see [L], p. 45). Note that the origin of our invariant cycles is not the same of [L], however our invariant cycles have intersection number zero with all the vanishing cycles which is the condition (3.9.5) in [L], it follows that our vanishing cycles behave as the vanishing cycles in [L]. In order to work only with the homology classes which are vanishing cycles we make the following construction. Fix a Lefschetz pencil $\mathscr{F}$ as above and consider the decomposition:

$$T_{\mathscr{F}}\mathrm{Fol}(M, -2(nc+a)) = T_{\mathscr{F}}\mathrm{LP}(M, -2(nc+a)) \oplus N_{\mathscr{F}}\mathrm{LP}(M, -2(nc+a)),$$

where $N_{\mathscr{F}}\mathrm{LP}(M, -2(nc+a))$ is the normal space to the Lefschetz pencils.

Define the vector space

$$N^0_{\mathscr{F}}\mathrm{LP}(M, -2(nc+a)) = \{[\eta] \in N_{\mathscr{F}}\mathrm{LP}(M, -2(nc+a)) \mid \int_\gamma \eta = 0 \text{ for every } [\gamma] \in I(\mathscr{L}) \text{ and all generic leaves } \mathscr{L} \text{ of } \mathscr{F}\}.$$

Represent $[\eta] \in N^0_{\mathscr{F}}\mathrm{LP}(M, -2(nc+a))$ by rational one-forms which have a pole along $\{g = 0\}$. Hence we have induced meromorphic one-forms on the leaves of $\mathscr{F}$, with residues equal to zero.

Now let $\lambda_0 \in \Delta$ be a fixed basis point. Without loss of generality we use the same notation for $\lambda_0 \in \Delta$ and its image under $\tau : \Delta \to \mathbb{C}P^1 - \Lambda$. Given a fixed $\gamma \in H_1(\mathscr{L}(\lambda_0), \mathbb{Z})$, for each $\lambda \in \Delta$ there is a well defined class $\gamma(\lambda) \in H_1(\mathscr{L}(\lambda), \mathbb{Z})$ given by the monodromy $\mathscr{M}$ on the bundle $H : M' \to \Delta$. Let $\{\eta_1, \ldots, \eta_\mu\}$ be rational one-forms, with pole along $\{g = 0\}$, such that $[\{\eta_1], \ldots, [\eta_\mu]\}$ are a basis for $N^0_{\mathscr{F}}\mathrm{LP}(M, -2(nc+a))$. If we take $[\gamma] \in V(\mathscr{L})$ then define the associated Ilyashenko's curve as follows:

$$\mathscr{I}_\gamma : \Delta \to \mathbb{C}^\mu \subset \mathbb{C}^b,$$
$$\lambda \mapsto \left(\int_{\gamma(\lambda)} \eta_1, \ldots, \int_{\gamma(\lambda)} \eta_\mu, 0, \ldots, 0 \right),$$

where $\mu = \dim N^0_{\mathscr{F}}\mathrm{LP}(M, -2(nc+a))$ and $b = \dim N_{\mathscr{F}}\mathrm{LP}(M, -2(nc+a))$, see [I1] and [GM-M]. The Ilyashenko's curves associated to the indeterminacy homology class were studied on [GM–M].

3.1. Theorem. *Let M be a two dimensional compact complex connected projective manifold such that $H^1(M, \mathbb{C}) = 0$. Let $\mathscr{F} \in \mathrm{Fol}(M, -2(nc+a))$ be a Lefschetz pencil and $\{[\eta_1], \ldots, [\eta_\mu]\} \in T_{\mathscr{F}}\mathrm{Fol}(M, -2(nc+a))$ a set of linearly independent vectors generating $N^0_{\mathscr{F}}\mathrm{LP}(M, -2(nc+a))$, as above. If $\gamma \in H_1(\mathscr{L}(\lambda_0), \mathbb{Z})$, where $\lambda_0 \in \mathbb{C}P^1 - \Lambda$ is a vanishing cycle, then the associated Ilyashenko's curve $\mathscr{I}_\gamma$ is not contained in any hyperplane of $\mathbb{C}^\mu \subset \mathbb{C}^b$.*

Proof. Following Ilyashenko's idea of Theorem 2 in [I1], we have that the closure of $\mathscr{I}_\gamma$ contains $0 \in \mathbb{C}^\mu$. Consider the hyperplane given by the equation $\sum a_j z_j = 0$, where $(z_1, \ldots, z_\mu)$ are coordinates of $\mathbb{C}^\mu$ and $j = 1, \ldots, \mu$. If the Ilyashenko's curve is contained in this hyperplane, then we can take an infinitesimal deformation $\eta = \sum a_j \eta_j$ which does not belong to $T_{\mathscr{F}}\mathrm{LP}(M, -2(nc+a))$. Since $H^1(M, \mathbb{C}) = 0$ we have that, for each $\lambda \in \mathbb{C}P^1 - \Lambda$, the space $H_1(\overline{\mathscr{L}(\lambda)}, \mathbb{Z}) = V(\mathscr{L}(\lambda))$ has a basis of vanishing cycles (see (4.1.3) and (4.1.4) of [L]). By (7.3.2) of [L] this basis can be obtained by the monodromy $\{\gamma(\lambda)\}$ of $\gamma \in H_1(\mathscr{L}(\lambda_0), \mathbb{Z})$. We obtain that $\int_{\gamma(\lambda)} \eta = 0$, which is the hypothesis of Theorem 2.1. This is a contradiction, because $[\eta]$ is not tangent to Lefschetz pencils. Thus we have shown that $\mathscr{I}_\gamma$ can not be contained in any hyperplane of $\mathbb{C}^\mu$. □

To obtain a weaker hypotesis for Theorem 2.1 we need the following:

3.2. Remark. Let $\mathscr{F}$ be a Lefschetz pencil given by h in M as above. Let γ be a vanishing cycle on a leaf $\mathscr{L}(\lambda_0)$, with $\lambda_0 \in \mathbb{C}P^1 - \Lambda$ a regular value for h. Given $\varepsilon > 0$ an *ε-family of the vanishing cycle γ in $\mathscr{F}$* is the set of associated vanishing cycles of $\mathscr{F}$ on the leaves $\mathscr{L}(\tilde{\lambda})$ which are obtained by monodromy of γ for regular values $\tilde{\lambda} \in V(\lambda_0)$ of h, where $V(\lambda_0)$ is an ε-neighborhood of λ_0 in $\mathbb{C}P^1 - \Lambda$.

Let $\mathscr{F} \in \mathrm{Fol}(M, -2(nc+a))$ be a Lefschetz pencil and let

$$[\eta] \in N^0_{\mathscr{F}}\mathrm{LP}(M, -2(nc+a))$$

be an infinitesimal deformation of the foliation.

If $\int_\gamma \eta = 0$ for all γ in an ε-family of the vanishing cycle $\gamma \in \mathscr{L}(\lambda_0)$ then the integral vanishes for all closed loops contained on the leaves of the foliation $\mathscr{F}$ on $M - \mathrm{Sing}(\mathscr{F})$. For the proof of this last assertion we can use the same argument as in the proof of Theorem 3.1.

For the case $M = \mathbb{C}P^2$ we have a more careful computation of the dimensions.

If $\mathscr{F}$ is a Lefschetz pencil $h = f/g : \mathbb{C}P^2 \to \mathbb{C}P^1$, where f and g are homogeneous polynomials of degree c, then by [GM-M], p. 155:

$$\dim T_{\mathscr{F}} \mathrm{Fol}(\mathbb{C}P^2, -2c) = 4c^2 - 2\,,$$

$$\dim T_{\mathscr{F}} \mathrm{LP}(\mathbb{C}P^2, -2c) = c^2 + 3c - 2\,,$$

$$\dim N_{\mathscr{F}} \mathrm{LP}(\mathbb{C}P^2, -2c) = 3c^2 - 3c\,.$$

There are c^2 points of indetermination on each generic leaf $\mathscr{L}$ of $\mathscr{F}$. The genus of $\mathscr{L}$ is $\frac{1}{2}(d-1)(d-2)$ by the classical genus formula for plane curves in terms of their degrees. Using the decomposition $H_1(\mathscr{L}, \mathbb{C}) = I(\mathscr{L}) \oplus V(\mathscr{L})$, we have that the dimension of the vanishing homology classes $V(\mathscr{L})$ is $(c-1)(c-2)$.

If we take $[\gamma] \subset I(\mathscr{L})$, then the associated Ilyashenko's curve

$$\mathscr{I}_\gamma : \Delta \to \mathbb{C}^{3c^2 - 3c}$$

is a curve where each coordinate is a polynomial function of degree 2 (see [GM-M], p.159). Hence it is possible to show that its image is contained in at most a two dimensional space $\mathbb{C}^2 \subset \mathbb{C}^{3c^2-3c}$. This curve is given by the residues of $[\eta] \in N_{\mathscr{F}} \mathrm{LP}(\mathbb{C}P^2, -2c)$. The only relation on these residues is that their sum over each leaf $\mathscr{L}(\lambda)$ is equal to zero (the residue theorem). Thus, we have that the dimension of $N^0_{\mathscr{F}} \mathrm{LP}(M, -2c)$ is the dimension of $N_{\mathscr{F}} \mathrm{LP}(\mathbb{C}P^2, -2c)$ minus the dimension of the space generated by the Ilyashenko's curves associated to $I(\mathscr{L})$, which is $2(c^2-1)$. That is $\dim N^0_{\mathscr{F}} \mathrm{LP}(M, -2c) = c^2 - 3c + 2$.

For example for, $c = 1$ we have that $\mathscr{F}$ is a pencil of lines in $\mathbb{C}P^2$. Hence,

$$T_{\mathscr{F}} \mathrm{Fol}(\mathbb{C}P^2, -2) = T_{\mathscr{F}} \mathrm{LP}(\mathbb{C}P^2, -2)$$

are spaces of dimension two (the pencils of lines are parametrized by their basis points). For $c = 2$, $\mathscr{F}$ is a pencil of quadrics in $\mathbb{C}P^2$. In this case $\dim T_{\mathscr{F}} \mathrm{Fol}(\mathbb{C}P^2, -4) = 14$ and $\dim T_{\mathscr{F}} \mathrm{LP}(\mathbb{C}P^2, -4) = 8$, so $\dim N^0_{\mathscr{F}} \mathrm{LP}(\mathbb{C}P^2, -4) = 0$ since all the homology in the leaves of $\mathscr{F}$ is generated by indeterminacy homology classes.

To study the persistence of algebraic leaves we need some generalities.

Let $\mathscr{F} \in \mathrm{Fol}(M, -2(nc+a))$ be a Lefschetz pencil and γ a vanishing cycle on a generic leaf $\mathscr{L}$ of $\mathscr{F}$. By a *one parameter holomorphic deformation* $\{\mathscr{F}_t\}$ of $\mathscr{F}$, we mean a holomorphic curve $t \mapsto \mathscr{F}_t$ in $\mathrm{Fol}(M, -2(nc+a))$ where t is a parameter in $\Delta \subset \mathbb{C}$. We say that *$\gamma$ is a persistent cycle under the deformation* if there exists a continuous family (with respect to t) of loops $\{\gamma_t\}$ on the leaves of $\{\mathscr{F}_t\}$ such that $\gamma_0 = \gamma$. The definition of persistent cycles is a generalization, to the case of foliations, of the concept of generating cycles in differential equations (see [A-I], p. 109).

3.3. Proposition. *Let γ be a vanishing cycle on a leaf of a Lefschetz pencil $\mathscr{F}$ and $\{\mathscr{F}_t\}$ a one-parameter holomorphic deformation of $\mathscr{F}$ with $[\eta] \in T_{\mathscr{F}}\mathrm{Fol}(M, -2(nc+a))$ its associated infinitesimal deformation. If γ is a persistent cycle then $\int_\gamma \eta = 0$. Conversely if we have that $\int_{\gamma(\lambda)} \eta$ vanishes at $\lambda = \lambda_0$, where $\gamma(\lambda_0) = \gamma$, but is not identically zero as function of λ, then γ is a persistent cycle.*

A similar result to Proposition 3.3 was firstly proved by Ilyashenko in the case of polynomial deformations of a polynomial first integral on $\mathbb{C}^2$ in [I1] and extended to Lefschetz pencils in [GM-M], Proposition 2.12.

3.4. Corollary. *Given a Lefschetz pencil $\mathscr{F} \in \mathrm{LP}(\mathbb{C}P^2, -2c)$ and $c^2 - 3c + 1$ vanishing cycles γ_i in different leaves of $\mathscr{F}$. There exists a one parameter holomorphic deformation $\{\mathscr{F}_t\}$ of $\mathscr{F}$ such that the vanishing cycles γ_i are persistent cycles under $\{\mathscr{F}_t\}$, and give origin to limit cycles in the foliations $\mathscr{F}_t$, for t different of zero.*

Proof. The result follows from the equality

$$\dim N^0_{\mathscr{F}}\mathrm{LP}(\mathbb{C}P^2, -2c) - 1 = c^2 - 3c + 2$$

and the idea of [I1], Corollary 2′. □

Now we use Proposition 3.3 to study infinitesimal deformations of Lefschetz pencils in $\mathbb{C}P^2$ where one of the leaves persists up to first order as a complex curve.

Let Q be an algebraic non singular curve of degree c in $\mathbb{C}P^2$ given by f, an homogeneous polynomial of degree c in $\mathbb{C}^3$. A foliation $\mathscr{F} \in \mathrm{Fol}(\mathbb{C}P^2, -2c)$ has an algebraic leaf Q if one of its leaves on $M - \mathrm{Sing}(\mathscr{F})$ has Q as closure (see Definition 2.6). We consider foliations $\mathscr{F} \in \mathrm{Fol}(\mathbb{C}P^2, -2c)$ that have Q as an algebraic leaf. We denote the set of such foliations as:

$$\mathrm{Fol}(\mathbb{C}P^2, -2c, Q) = \{[\omega] \in \mathrm{Fol}(\mathbb{C}P^2, -2c) \mid (\omega \wedge df)|_{f=0} = 0\}.$$

The set of all singular holomorphic foliations in $\mathbb{C}P^2$ with at least one algebraic non singular leaf of degree c is:

$$\mathrm{FAL}(\mathbb{C}P^2, c) = \bigcup_Q \mathrm{Fol}(\mathbb{C}P^2, -2c, Q) \subset \mathrm{Fol}(\mathbb{C}P^2, -2c),$$

where the union is taken over all the algebraic, non singular curves Q of degree c in $\mathbb{C}P^2$. Note that $\mathrm{LP}(\mathbb{C}P^2, -2c) \subset \mathrm{FAL}(\mathbb{C}P^2, c)$. Very little is known about this type of foliations and their global structure in $\mathrm{Fol}(\mathbb{C}P^2, -2c)$ (see [J], p. 158 and [LN] for related results).

3.5. Lemma. *The closure of the space of foliations with at least one algebraic non singular leaf $\overline{\mathrm{FAL}(\mathbb{C}P^2, c)}$ is an analytic subspace of* $\mathrm{Fol}(\mathbb{C}P^2, -2c)$.

Proof. Denote by $\mathbb{C}P^s$ the projective space of all the algebraic curves in $\mathbb{C}P^2$ of degree c. Consider the subspace $\mathscr{C} \subset \mathbb{C}P^s$ of all the algebraic, non singular curves. Take in $\mathbb{C}P^s \times \mathrm{Fol}(\mathbb{C}P^2, -2c)$ the subset given by all the pairs $(Q, [\omega])$ for $Q \in \mathscr{C}$ such that the foliation associated to $[\omega]$ has as algebraic non singular leaf Q. Each set

$$\mathrm{Fol}(\mathbb{C}P^2, -2c, Q)$$

is a linear analytic subspace of $\mathrm{Fol}(\mathbb{C}P^2, -2c)$ (i.e. if $[\omega]$, $[\omega']$ are in $\mathrm{Fol}(\mathbb{C}P^2, -2c, Q)$ then $[\omega + \omega']$ is also in $\mathrm{Fol}(\mathbb{C}P^2, -2c, Q)$). Using this local description of the set $\{(Q, [\omega])\}$ as a ruled set in $\mathbb{C}P^s \times \mathrm{Fol}(\mathbb{C}P^2, -2c)$ it is possible to show that this is a constructible and hence analytic set (see [F], p. 43). If we take the image of $\{(Q, [\omega])\}$ under the natural projection

$$\Pi : \mathbb{C}P^s \times \mathrm{Fol}(\mathbb{C}P^2, -2c) \to \mathrm{Fol}(\mathbb{C}P^2, -2c),$$

then by a theorem of Chevalley (see [H], p. 94), we obtain a constructible set on $\mathrm{Fol}(\mathbb{C}P^2, -2c)$. Finaly take the closure, which is a closed constructible set, hence an analytic subspace (see [F], p. 43). □

In order to give some description of $\mathrm{FAL}(\mathbb{C}P^2, c)$, as a first step we have:

3.6. Theorem. *Let $\mathscr{F} \in \mathrm{Fol}(\mathbb{C}P^2, -2c)$ be a Lefschetz pencil and an infinitesimal deformation $[\eta] \in T_{\mathscr{F}} \mathrm{Fol}(\mathbb{C}P^2, -2c)$. The vector $[\eta]$ is tangent to $\mathrm{Fol}(\mathbb{C}P^2, -2c, Q)$ iff for the algebraic non singular leaf Q of $\mathscr{F}$ we have $\int_\gamma \eta = 0$, for each $\gamma \in H_1(\mathscr{L}, \mathbb{Z})$, where $\mathscr{L}$ is the leaf of $\mathscr{F}$ such that its closure is Q.*

Proof. Recall that $\mathrm{Fol}(\mathbb{C}P^2, -2c, Q)$ define a linear subspace in the projective space of foliations $\mathrm{Fol}(\mathbb{C}P^2, -2c)$, hence it is a smooth analytic subspace.

Given $[\eta]$ a tangent vector to $\mathrm{Fol}(\mathbb{C}P^2, -2c, Q)$ we consider a holomorphic curve $t \mapsto \mathscr{F}_t$ in $\mathrm{FAL}(\mathbb{C}P^2, c)$, where $t \in \mathbb{C}$ and such that $[\eta]$ is its tangent vector in $t = 0$. Hence we have a holomorphic deformation $\{\mathscr{F}_t\}$ and each one of these foliations $\mathscr{F}_t$ in $\mathbb{C}P^2$ has the algebraic leaf Q (in a continuous way with respect to the parameter t). All the cycles $\gamma \in H_1(\mathscr{L}, \mathbb{Z})$ are persistent cycles and hence by Proposition 3.3, the integrals vanish.

For the other part, let $[\eta] \in T_{\mathscr{F}} \mathrm{Fol}(\mathbb{C}P^2, -2c)$ be such that the integrals vanish. We represent $[\eta]$ by a rational one-form η on $\mathbb{C}P^2$ with a pole of order two along $\{g = 0\}$. Recall that f/g is the Lefschetz pencil associated to $\mathscr{F}$ and that Q is given by $\{f = 0\}$. Similarly as in the proof of Theorem 2.1 for $q \in Q - \mathrm{Sing}(\mathscr{F})$ a fixed basis point, we define a map

$$J_\eta : Q \to \mathbb{C}P^1,$$

$$J_\eta(p) = \int_q^p \eta,$$

where the integral is computed using paths in Q and the restriction of η to Q. Since J_η is a well defined meromorphic function and has divisor of poles along $Q \cap \{g = 0\}$ of order one, see Lemma 2.2.2. We can consider its zero divisor $Z \subset Q$. Note that J_η has extension to

$\mathbb{C}P^2$ as rational map to $\mathbb{C}P^1$. Hence there exists a curve S in $\mathbb{C}P^2$ such that $S \cap Q = Z$, if S is given by the homogeneous equation $\{s = 0\}$ (where s is a homogeneous polynomial of the same degree of g). Consider the rational map $s/g : \mathbb{C}P^2 \to \mathbb{C}P^1$ and the associated vector in $T_{\mathscr{F}}\mathrm{LP}(\mathbb{C}P^2, -2c)$ given by the rational one-form $\eta_1 = d(s/g)$. Since J_η and J_{η_1} have the same zeros and poles in Q, there exists $\mu \in \mathbb{C}$ such that $J_\eta = \mu J_{\eta_1}$ in Q. This implies that

$$(\eta - \mu\eta_1)|_{f=0} = 0,$$

and if we define $\eta_2 = \eta - \mu\eta_1$ then,

$$\eta_2 \wedge df|_{f=0} = 0$$

which implies that $[\eta_2]$ is a vector in $T_{\mathscr{F}}\mathrm{Fol}(\mathbb{C}P^2, -2c, Q)$. Hence the theorem follows from the expression $[\eta] = [\eta_1] + [\eta_2]$. □

§ 4. An explicit bound for the multiplicity of the zeros of abelian integrals

Here we give an explicit bound for the multiplicity of the zeros of the rational abelian integrals which arise from the study of infinitesimal deformations of foliations. The bound that we obtain is related to the work of Varchenko (see [V], [A-I], p. 111), which shows the existence of a bound for the multiplicity of the zeros of polynomial abelian integrals. We adapt to our case the computations of [I3] and [M].

We suppose that the manifold M is of dimension two, for dimension greather than two the situation is given by Theorem 5.1. Let $\mathscr{F} \in \mathrm{LP}(M, -2(nc + a))$ be a Lefschetz pencil, given by $h = f/g : M \to \mathbb{C}P^1$, where $f, g \in H^0(M, \mathcal{O}_M(L))$ and L is a line bundle over M with Chern class $(nc + a)$.

4.1. Theorem. *Let M be a two dimensional compact complex connected projective manifold such that $H^1(M, \mathbb{C}) = 0$ and let $\mathscr{F} \in \mathrm{LP}(M, -2(nc + a))$ be a Lefschetz pencil. Given $\mathscr{L} = \{h^{-1}(\lambda_0)\}$ with $\mathscr{L}$ a generic leaf of $\mathscr{F}$, $[\gamma] \in H_1(\mathscr{L}, \mathbb{Z})$ a vanishing cycle. Let $\{\gamma(\lambda)\}$ be its family of cycles associated by monodromy. For every infinitesimal deformation $[\eta] \in N_{\mathscr{F}}\mathrm{LP}(M, -2(nc + a))$, if the integral $\int_{\gamma(\lambda)} \eta$ has a zero at $\lambda = \lambda_0$ then the multiplicity of the zero is at most*

$$\frac{1}{2}[(r+1)(r+2) - 2] + s(r-2) - (r-1)$$

where $r = \dim H_1(\mathscr{L}, \mathbb{Z})$ and s is the number of singular Morse points of h (the function associated to the Lefschetz pencil).

Proof. We follow Ilyashenko ideas of [I3].

Using a Möbius transformation as change of coordinates in $\mathbb{C}P^1$, we can suppose that the generic leaf $\mathscr{L}$ is given by the equation $\{f = 0\} = \{h^{-1}(0)\}$ in M and that $\{g = 0\} = \{h^{-1}(\infty)\}$ determines a generic leaf of $\mathscr{F}$. Take $\{\gamma = \gamma_1, \ldots, \gamma_r\}$ a basis of vanishing cycles for $H_1(\mathscr{L}, \mathbb{Z})$. Such basis exists because $H^1(M, \mathbb{C}) = 0$ (see [L], 4.13 and 4.14). For $\lambda \in \mathbb{C}P^1 - \Lambda$ we define the functions

$$I_{ij}(\lambda) = \frac{d^j}{d\lambda^j} \int_{\gamma_i(\lambda)} \eta$$

where $i = 1, \ldots, r$ and $j = 0, \ldots, r-1$, for η a rational one-form in M with a pole of order two along $\{g = 0\}$ such that $[\eta] \in N_{\mathscr{F}} \mathrm{LP}(M, -2(nc + a))$.

In general the functions $I_{ij}(\lambda)$ are holomorphic and multivalued. We shall study the function

$$W : \mathbb{C}P^1 - \Lambda \to \mathbb{C}P^1,$$
$$W(\lambda) = \det(I_{ij}(\lambda)).$$

Note that this function is holomorphic and univalued. If we take analytical extension of $W(\lambda)$ around a critical value $p \in \Lambda$ of h, the function $W(\lambda)$ changes by the determinant of the monodromy transformation associated to p (i.e. the monodromy representation of the loop in $\pi_1(\mathbb{C}P^1 - \Lambda)$ around p). By the Picard-Lefschetz formula this determinant is equal to one (see [A-GZ-V], p. 320). We need to show that $W(\lambda)$ is a rational function over $\mathbb{C}P^1$.

4.2. Lemma. *$W(\lambda)$ has a pole of order at most $\frac{1}{2}(r+1)(r+2) - 1$ at $\infty \in \mathbb{C}P^1$.*

Proof. Since $\{h^{-1}(\infty)\} = \overline{\mathscr{L}(\infty)}$, it gives a generic leaf of $\mathscr{F}$. Take a neighborhood $V(\infty) \subset \mathbb{C}P^1 - \Lambda$ of ∞, such that the restriction of the bundle

$$h : M - (K \cup h^{-1}(\Lambda)) \to \mathbb{C}P^1 - \Lambda$$

to $V(\infty)$ has the structure of a trivial product:

$$h^0 : \mathscr{L}(\infty) \times V(\infty) \to V(\infty).$$

Hence, for $\lambda \in V(\infty)$ each class $\gamma(\lambda) \in H_1(\mathscr{L}(\lambda), \mathbb{Z})$ can be pushed by monodromy on $H_1(\mathscr{L}(\infty), \mathbb{Z})$ in a trivial way. Let (U, ϕ) be a local foliated chart

$$\phi : U \subset \mathscr{L}(\infty) \times V(\infty) \to \mathbb{C}^2,$$

such that $\phi(U)$ is a polydisk $\Delta^2 \subset \mathbb{C}^2$ with coordinates (z_1, z_2) and the first integral h is given by z_2 in Δ^2, where $\phi(\mathscr{L}(\infty) \cap U)$ is $\{z_2 = 0\}$ in Δ^2. The integrals of $I_{i0}(\lambda)$ are finite sums of local integrals of η over sets of the type $\gamma(\lambda) \cap U = \gamma(\lambda, U)$. We get expressions of the type

$$\int_{\gamma(\lambda, U)} \phi_* \eta = \int_{\gamma(\lambda, U)} \frac{A_1(z_1, z_2)\, dz_1}{z_2^2}$$

where $A_1(z_1, z_2)$ is a holomorphic function in Δ^2 (as in Lemma 2.2). If λ goes to ∞ in $\mathbb{C}P^1 - \Lambda$ then z_2 goes to 0 in Δ^2. Since the loops $\gamma(\lambda)$ remain compacts when λ goes to ∞, each function $I_{i0}(\lambda)$ has a pole of order at most 2 as the integrals $\int \phi_* \eta$ do (see Lemma

2.2). By a direct computation with the matrix $(I_{ij}(\lambda))$ we get the corresponding bound for the order of the pole at ∞ of $W(\lambda)$. □

4.3. Remark. For foliations in $\mathbb{C}^2 \subset \mathbb{C}P^2$ defined by a generic polynomial first integral and polynomial non-conservative infinitesimal deformations, P. Mardešić in [M], obtains that the bound for the order of the pole at ∞, for the similar functions $I_{i0}(\lambda)$, is at most one. In our context this follows because a generic polynomial first integral has the line at infinite $\mathbb{C}P^1 = \mathbb{C}P^2 - \mathbb{C}^2$ as a leaf and this leaf is persistent up to first order (see Theorem 3.6) under polynomial deformations of the foliation. Hence the integral $\int \phi_* \eta$ has a pole of order at most one (see Lemma 2.2).

4.4. Lemma. *$W(\lambda)$ has poles of order at most $r-2$ in the set of singular values Λ, of h.*

Proof. Given $\{\gamma = \gamma_1, \ldots, \gamma_r\}$ the basis of $H_1(\mathscr{L}, \mathbb{Z})$ and $\mu \in \Lambda$ we consider a neighborhood $V(\mu) \subset \mathbb{C}P^1 - \{\infty\}$ of μ such that $V(\mu) \cap \Lambda = \mu$. Using monodromy we push $\{\gamma = \gamma_1, \ldots, \gamma_r\}$ to a basis of $H_1(\overline{\mathscr{L}(\lambda_0)}, \mathbb{Z})$, for $\lambda_0 \in V(p)$. Change the basis of $H_1(\overline{\mathscr{L}(\lambda_0)}, \mathbb{Z})$ such that γ_1 is the vanishing cycle in $H_1(\overline{\mathscr{L}(\lambda_0)}, \mathbb{Z})$ canonically associated to μ as in the definition of vanishing cycles in Section 3. Using $H^1(M, \mathbb{C}) = 0$ we have that $H_1(\overline{\mathscr{L}(\lambda_0)}, \mathbb{Z})$ has a basis of vanishing cycles (by [L] 4.13 and 4.14) and the change of basis does not change the order of the pole of $W(\lambda)$. Without loss of generality we can denote by $\{\gamma_1, \ldots, \gamma_r\}$ the new basis. Thus, by the Picard Lefschetz Theorem on local monodromy (see [A-GZ-V], Ch. I or [L], 6.3.3), we have local the expressions:

$$I_{10}(\lambda) = (\lambda - \mu) h_1(\lambda),$$

$$I_{20}(\lambda) = h_2(\lambda) - (1/2\pi i)(\lambda - \mu) \log(\lambda - \mu) h_1(\lambda),$$

$$I_{\ell 0}(\lambda) = h_\ell(\lambda) \qquad \text{for } \ell = 3, \ldots, r$$

(see [M], p. 850, for details), where $\{h_1(\lambda), \ldots, h_\ell(\lambda)\}$ are holomorphic univalued functions in $V(\mu)$. From [M], p. 850, and a direct computation with the matrix $(I_{ij}(\lambda))$, we may obtain that the order of the poles of $W(\lambda)$ is at most $r-2$. □

Conclusion of the proof of Theorem 4.1. From Lemmas 4.2, 4.4 we conclude that $W(\lambda)$ has poles of bounded order and hence it is a rational function in $\mathbb{C}P^1$. We get a bound for the degree of the denominator of $W(\lambda) = \dfrac{W_1(\lambda)}{W_2(\lambda)}$, where W_1 and W_2 are polynomial functions, from the inequality:

$$\text{degree } W_2(\lambda) \leqq \frac{1}{2}[(r+1)(r+2) - 2] + s(r-2),$$

where $r = \dim H_1(\mathscr{L}, \mathbb{Z})$ and $s =$ number of singular Morse points of h (the function associated to the Lefschetz pencil). Since the degree of the numerator $W_1(\lambda)$ has the same bound and the multiplicity of a zero of $I_{1(r-1)}(\lambda)$ is equal or less than the degree of $W(\lambda)$, we have that the multiplicity of a zero of $I_{10}(\lambda)$ is bounded. □

In order to make some application of this result we need:

4.5. Proposition. *Let M be a two dimensional compact complex connected projective manifold, such that $H^1(M,\mathbb{C}) = 0$ and let $\mathscr{F} = [\omega] \in \mathrm{Fol}(M, -2(nc+a))$ be a Lefschetz pencil. Given $\mathscr{L} = \{h^{-1}(\lambda_0)\}$, with $\mathscr{L}$ a generic leaf of $\mathscr{F}$, $[\gamma] \in H_1(\mathscr{L},\mathbb{Z})$ a vanishing cycle such that $\int_\gamma \eta = 0$ and their family $\{\gamma(\lambda)\}$ of cycles associated by monodromy. For an infinitesimal deformation $[\eta] \in N_{\mathscr{F}}\mathrm{LP}(M, -2(nc+a))$, the multiplicity of a zero of the integral $\int_{\gamma(\lambda)} \eta$ at $\lambda = \lambda_0$ is equal to the number of cycles (counted with their multiplicities) that are persistent and arise from $\gamma(\lambda_0)$ under the infinitesimal deformation $\{\omega + t\eta\}$ of the Lefschetz pencil.*

Proof. Since the dimension of M is equal to two, the integrability conditions of Lemma 1.6 for $\{\omega + t\eta\}$ vanish and $\{\omega + t\eta\}$ can be interpreted as a family of foliations $\{\mathscr{F}_t\}$ in M, where $t \in \Delta$, with associated infinitesimal deformation

$$[\eta] \in T_{\mathscr{F}}\mathrm{LP}(M, -2(nc+a)).$$

The proposition follows from Proposition 3.3, [GM-M], Proposition 2.12, and the Theorem in Limit Cycles of [V], p. 100. □

4.6. Corollary. *Let M be a two dimensional compact complex connected projective manifold, such that $H^1(M,\mathbb{C}) = 0$, and let $\mathscr{F} \in \mathrm{LP}(M, -2(nc+a))$ be a Lefschetz pencil. For every infinitesimal deformation $[\eta] \in N_{\mathscr{F}}\mathrm{LP}(M, -2(nc+a))$ and a set of linearly independent vanishing cycles $\{\gamma_1 \cdots \gamma_\ell\} \subset H_1(\mathscr{L},\mathbb{Z})$, where $\mathscr{L}$ is a generic leaf of $\mathscr{F}$, the number of cycles which are persistent and arise from $\{\gamma_1, \ldots, \gamma_\ell\}$ under the infinitesimal deformation $\{\omega + t\eta\}$ is at most*

$$\frac{1}{2}[(r+1)(r+2) - 2] + s(r-2) - (r-1),$$

with r and s as in Theorem 4.1.

Proof. By the construction in Theorem 4.1, the sum of the multiplicities of the zeros of the integrals

$$\int_{\gamma_1(\lambda)} \eta, \ldots, \int_{\gamma_\ell(\lambda)} \eta$$

for $\lambda = \lambda_0$ is bounded by the multiplicity of the zero of the associated function $W(\lambda)$. □

4.7. Corollary. *For $c \geqq 3$, let $\mathscr{F} \in \mathrm{LP}(\mathbb{C}P^2, -2c)$ be a Lefschetz pencil in $\mathbb{C}P^2$, given by $h = f/g : \mathbb{C}P^2 \to \mathbb{C}P^1$, where f and g are homogeneous polynomials of degree c. Let $[\gamma] \in H_1(\mathscr{L},\mathbb{Z})$ be a vanishing cycle in a generic leaf $\mathscr{L}$ of $\mathscr{F}$ and $\{\gamma(\lambda)\}$ their associated family of cycles under monodromy. Then for every infinitesimal deformation*

$$[\eta] \in N_{\mathscr{F}}\mathrm{LP}(\mathbb{C}P^2, -2c),$$

the abelian integral $\int_{\gamma(\lambda)} \eta$ has zeros at $\lambda = \lambda_0$ of multiplicity at most

$$\frac{1}{2}(7c^4 - 36c^3 + 56c^2 - 33c + 12).$$

Proof. From [GM-M], p. 155, we have that:

$$r = \dim H_1(\bar{\mathscr{L}}, \mathbb{Z}) = \dim V(\mathscr{L}) = (c-1)(c-2),$$
$$s = 3c^2 - 6c + 3. \quad \square$$

We do not know if for $\mathbb{C}P^2$ this bound is the best possible. If $c = 1, 2$ the Lefschetz pencils do not contain vanishing cycles (see Section 3), and the Corollary 4.7 not works.

§ 5. Lefschetz pencils in higher dimension

We shall now study the case where the dimension of M is at least three. In this case $\mathrm{Fol}(M, -2(nc+a))$ might be quite complicated as an analytic subspace of the projective space over $H^0(M, \mathrm{Hom}(L, T^*M))$. The following result asserts that Lefschetz pencils form open dense sets in some irreducible components of $\mathrm{Fol}(M, -2(nc+a))$. This result was obtained independently using different technics by X. Gómez-Mont and A. Lins-Neto in [GM-LN] and is related with results of O. Calvo-Andrade in [C].

5.1. Theorem. *Let M be a complex connected projective manifold such that $H^1(M, \mathbb{C}) = 0$, of dimension m at least three and let $\mathscr{F} \in \mathrm{LP}(M, -2(nc+a))$ be a Lefschetz pencil. Then $\mathrm{LP}(M, -2(nc+a))$ is an open dense set of irreducible components of $\mathrm{Fol}(M, -2(nc+a))$. In particular $\mathscr{F}$ is C^0-structurally stable as foliation.*

Proof. We shall prove first that for $\mathscr{L}$, a generic leaf of $\mathscr{F}$, all the elements of $\pi_1(\mathscr{L})$, are of finite torsion order.

Suppose that $m = 3$. The closure $\bar{\mathscr{L}}$ of $\mathscr{L}$ in M is a two dimensional complex algebraic manifold (an hyperplane section of M) and $K \cap \bar{\mathscr{L}}$ is a one dimensional complex algebraic irreducible curve. We have an exact sequence of groups

$$0 \to N \to \pi_1(\bar{\mathscr{L}} - K) \to \pi_1(\bar{\mathscr{L}}) \to 0,$$

where N is the normal subgroup generated by loops around K. By the assumption on the dimension of M and the Lefschetz's hyperplane theorem, $\pi_1(\bar{\mathscr{L}})$ is trivial. Note that K is an ample and non singular curve in $\bar{\mathscr{L}}$. Hence, by Nori's work on fundamental groups, $N = \pi_1(\bar{\mathscr{L}} - K)$ is abelian (see [N] or [Fu], p. 40), and we have that

$$\pi_1(\bar{\mathscr{L}} - K) = H_1(\bar{\mathscr{L}} - K).$$

By Lefschetz's duality, similar as in [F-L], p. 67, we obtain the equality:

$$H_1(\bar{\mathscr{L}} - K) = H^3(\bar{\mathscr{L}}, K) = \mathrm{coker}\,(i : H^2(\bar{\mathscr{L}}) \to H^2(K)),$$

where $i : H^2(\bar{\mathscr{L}}) \to H^2(K)$ is the natural inclusion. Since, $H^2(K) = \mathbb{Z}$ and $\bar{\mathscr{L}}$ is a submanifold of a projective space where $K \subset \bar{\mathscr{L}}$ is a complex submanifold, we have that the fundamental class of K is nonzero in the homology $H^2(\bar{\mathscr{L}})$ (see [G-H], p. 64). Hence, $\mathrm{coker}\,(i) = \mathbb{Z}_d$, for some $d \in \mathbb{N}$, and $\pi_1(\bar{\mathscr{L}} - K) = \pi_1(\mathscr{L})$ is finite.

If the dimension $m > 3$, in order to use the above argument, we need to compute $\pi_1(\mathscr{L} - K)$, where K is a complex analytic hypersurface of $\mathscr{L}$. Like in the previous case, by Lefschetz's hyperplane theorem, we get that $N = \pi_1(\mathscr{L} - K)$. Let γ be a loop that represents a class in N around K and let $p \in \mathscr{L}$ be their basis point. Take additional hyperplane sections $H_1, \ldots, H_s$ in M that contain the basis point p, and are transversal two by two of them and with $\mathscr{L}$ and K. If we take a sufficient large number of hyperplane sections, then $H_1 \cap \ldots \cap H_s \cap (\mathscr{L} - K)$ is a complex surface and γ can be pushed in it. Hence we may use the above case, this is, $[\gamma]$ has finite torsion order in $\pi_1(H_1 \cap \ldots \cap H_s \cap (\mathscr{L} - K))$ this implies that $[\gamma]$ also has finite torsion in $\pi_1(\mathscr{L} - K)$.

Let $\eta \in T_{\mathscr{F}}\mathrm{Fol}(M, -2(nc + a))$ be an infinitesimal deformation of $\mathscr{F}$. Given a loop γ contained in a leaf $\mathscr{L}$, we represent η by a rational one-form in M such that its restriction to $\mathscr{L}$ has a pole localized on K. Since $\omega = d(f/g)$ is closed, the infinitesimal integrability condition $(*)$ in Lemma 1.6 reduces to $\omega \wedge d\eta = 0$ in $M - \{g = 0\}$, which means that η is closed along the leaves of $\mathscr{F}$. We have that $\int_\gamma \eta = \frac{1}{p} \int_{\gamma^p} \eta$. Hence,

$$\int_\gamma \eta = \frac{1}{p} \int_{\gamma^p} \eta = 0\,,$$

where p is the torsion order of $[\gamma] \in \pi_1(\mathscr{L})$. These integrals are zero because γ^p is trivial in $\pi_1(\mathscr{L})$. Thus, we can apply Theorem 2.1 to show that every infinitesimal deformation of $\mathscr{F}$ is in $T_{\mathscr{F}}\mathrm{LP}(M, -2(nc + a))$, i.e.:

$$T_{\mathscr{F}}\mathrm{LP}(M, -2(nc + a)) = T_{\mathscr{F}}\mathrm{Fol}(M, -2(nc + a))\,.$$

From the above equality the Lefschetz pencils form open dense sets in irreducible components in the space of foliations (Lefschetz pencils of any degree are dense in the spaces of rational functions in $\mathbb{C}P^k$).

Finally in order to proof the structural stability we use the fact that all the Lefschetz pencils in $\mathrm{LP}(M, -2(nc + a))$ near $\mathscr{F}$ have the same topology as foliations. By the first assertion of the theorem every foliation $\mathscr{F}'$ near $\mathscr{F}$ is also a Lefschetz pencil. We can take a continuous path in the space of Lefschetz pencils, from f/g to f'/g'. Using the path as a family of functions we show that $\mathscr{F}$ and $\mathscr{F}'$ has the same topology as foliations, hence $\mathscr{F}$ is C^0-structural stable, as foliation. □

Let us remark that X. Gómez-Mont and A. Lins-Neto prove in [GM-LN] that if $H^1(M, \mathbb{C}) \neq 0$ then every Lefschetz pencil on M has holomorphic deformations such that they are not contained in the space of Lefschetz pencils.

§ 6. Irreducible components for several polynomial first integrals

As another application of all these technics we study the most classical case of algebraic foliations on $\mathbb{C}^m$ of higher codimension.

Given $k, c \in \mathbb{N}$, let $\Omega_c^k(\mathbb{C}^m)$ be the complex vector space of polynomial k-forms of degree c in $\mathbb{C}^m$ i.e. $\omega \in \Omega_c^k(\mathbb{C}^m)$ has an expression of the form:

$$\omega = \sum_{i_1, \dots, i_k} A_{i_1, \dots, i_k}(z_1, \dots, z_m)\, dz_{i_1} \wedge \cdots \wedge dz_{i_k},$$

where $A_{i_1, \dots, i_k}(z_1, \dots, z_k)$ are polynomials of degree at most c.

6.1. Definition. A *k-codimensional singular polynomial foliation in $\mathbb{C}^m$ of degree c* is an equivalence class $[\omega] \in \mathrm{Proj}(\Omega_c^k(\mathbb{C}^m))$ of polynomial k-forms such that it satisfies the integrability conditions:

1. There exist $\psi_1, \dots, \psi_k \in \Omega^1(\mathbb{C}^m)$ polynomial one-forms with $\omega = \psi_1 \wedge \cdots \wedge \psi_k$ in $\mathbb{C}^m$.

2. $\omega \wedge d\psi_i = 0$, for all $i = 1, \dots, k$.

See [J], p. 137 for a similar definition of algebraic foliations of higher codimension, more in the spirit of bundle maps, as in Section 1.

Let $\mathrm{Fol}(\mathbb{C}^m, c, k) \subset \mathrm{Proj}(\Omega_c^k(\mathbb{C}^m))$ be the analytic subspace of all the singular algebraic foliations in $\mathbb{C}^m$ of degree c and codimension k. Our problem is (as before): give a description of the irreducible components of $\mathrm{Fol}(\mathbb{C}^m, c, k)$.

We say that a polynomial $f\colon \mathbb{C}^m \to \mathbb{C}$ is *generic* iff it is a Morse function. Let

$$\mathscr{F} = [\omega] \in \mathrm{Fol}(\mathbb{C}^m, c, k)$$

be a singular polynomial foliation, as usual we say that $\mathscr{F}$ *has k polynomial first integrals* iff there exist k polynomials

$$f_1, \dots, f_k : \mathbb{C}^m \to \mathbb{C}$$

of degrees $d_1, \dots, d_k$ respectively, such that $[\omega] = [df_1 \wedge \cdots \wedge df_k]$ represents $\mathscr{F}$ and $d_1 + \cdots + d_k - k = c$, here $D = \{d_1, \dots, d_k\}$ is called the multidegree. Moreover we are interested in the case where each polynomial f_i is generic, irreducible and there are no common factors between two of them.

The closure of the non singular leaves of some of the above foliations $\mathscr{F} = [\omega]$ are algebraic submanifolds of $\mathbb{C}^m$ which come from the complete intersections given by equations:

$$\{f_1 = \mu_1, \dots, f_k = \mu_k \mid \mu_1, \dots, \mu_k \in \mathbb{C}\}.$$

The singular set $\mathrm{Sing}(\mathscr{F}) = \{p \in \mathbb{C}^m \mid \omega(p) \equiv 0\}$ is an analytic subspace of $\mathbb{C}^m$, defined by equations:

$$\{df_{i(1)} \wedge \cdots \wedge df_{i(s)} \equiv 0 \mid 1 \leqq s \leqq k,\ 1 \leqq i(1) < \cdots < i(s) \leqq k\}.$$

Define the analytic spaces

$$\mathrm{Sing}(\mathscr{F})^s = \{df_{i(1)} \wedge \cdots \wedge df_{i(s)} \equiv 0 \mid 1 \leqq i(1) < \cdots < i(s) \leqq k\}.$$

Note that $\mathrm{Sing}(\mathscr{F})^s \subset \mathrm{Sing}(\mathscr{F})^{s+1}$, and $\mathrm{Sing}(\mathscr{F})^1$ is the union of the Morse points of $\{f_i\}$. The dimension of $\mathrm{Sing}(\mathscr{F})^s$ at the points p in $\mathrm{Sing}(\mathscr{F})^s - \mathrm{Sing}(\mathscr{F})^{s-1}$ is higher when $s = k$. Using a local foliated chart (U, ϕ) at p (a point as above), we suppose that

$$df_1 = dz_1, \ldots, df_{k-1} = dz_{k-1}, \quad df_k = g_1 dz_1 + \cdots + g_m dz_m,$$

where $\{g_i\}$ are holomorphic functions and $df_1 \wedge \cdots \wedge df_k \equiv 0$ reduces to $m - k + 1$ conditions.

6.2. Definition. Let $\mathscr{F} = [\omega] \in \mathrm{Fol}(\mathbb{C}^m, c, k)$ a singular polynomial foliation. We say that $\mathscr{F}$ *has k generic polynomial first integrals of multidegree D* iff it has k polynomial first integrals, which are Morse functions, irreducible, without non common factors between two of them, and:

1. the codimension of $\mathrm{Sing}(\mathscr{F})$ is $m - k + 1$,

2. the closures of the generic non singular leaves of $\mathscr{F}$ are affine irreducible complete intersections of multidegree $D = \{d_1, \ldots, d_k\}$.

In fact, generic means that the k first integrals are itself generic and in general position between them. Given D a multidegree, where we always assume the order $d_1 \leqq d_2 \leqq \cdots \leqq d_k$. Now we consider the problem to give a parametrization of all the foliations having k generic polynomial first integrals of multidegree D.

6.3. Proposition. *For every multidegree D the space of singular polynomial foliations having k generic polynomial first integrals of multidegree D, which we denote by* $\mathrm{FI}(\mathbb{C}^m, D, k)$, *is a smooth analytic subspace of the space of foliations* $\mathrm{Fol}(\mathbb{C}^m, c, k)$.

Proof. Let $P(m, d_i)$ be the vector space of polynomials of degree d_i in $\mathbb{C}^m$ without constant terms. Consider the map

$$P(m, d_1) \times \cdots \times P(m, d_k) \to \mathrm{Fol}(\mathbb{C}^m, c, k) \cup \{0\}$$

defined by

$$f_1, \ldots, f_k \mapsto df_1 \wedge \cdots \wedge df_k.$$

The image of the map is the set of the foliations having k polynomial first integrals union the zero, and the map is onto over it. However this map is not one to one over its image, we want to construct a restricted domain for it.

In fact suppose that we have two collections of polynomials $(f_1, \ldots, f_k)$, $(g_1, \ldots, g_k)$ of multidegree D given the same foliation, hence

$$\mathscr{F} = [df_1 \wedge \cdots \wedge df_k] = [\lambda dg_1 \wedge \cdots \wedge dg_k]$$

where $\lambda \in \mathbb{C}^*$. Using that they define the same holomorphic foliation $\mathscr{F}$ in $\mathbb{C}^m$ it is possible to show that there exists a holomorphic automorphism $A : \mathbb{C}^k \to \mathbb{C}^k$ such that the first integrals satisfy

$$(f_1, \ldots, f_k) = A \circ (g_1, \ldots, g_k) : \mathbb{C}^m \to \mathbb{C}^k.$$

Since we represent the foliations by polynomial k-forms of multidegree D it follows that A is a polynomial automorphism of degree one, i.e. a linear map (all the first integrals $\{f_i, g_i\}$ are considered without constant terms). Moreover note that all $A = (a_{ij})$ as above satisfy that

$$\operatorname{degree}(f_i) = \operatorname{degree}(\sum_j a_{ij} g_j), \quad 1 \leqq i, j \leqq k,$$

(recall that we fix the order $d_1 \leqq \cdots \leqq d_k$ in D).

In conclusion, $(f_1, \ldots, f_k)$ and $(g_1, \ldots, g_k)$ give origin to the same foliation if and only if there exists one matrix $A = (a_{ij}) \in \mathrm{GL}(k, \mathbb{C})$ such that

$$(f_1, \ldots, f_k) = (a_{11} g_1 + \cdots + a_{1k} g_k, \ldots, a_{k1} g_1 + \cdots + a_{kk} g_k),$$

and where if $d_i < d_j$, then $a_{ii+r} = 0$ for all $r \geqq 1$.

The set of all the linear holomorphic automorphisms which send integrals of multidegree D to integrals of the same multidegree preserving order as say the above equality, define a subgroup $G(D)$ of $\mathrm{GL}(k, \mathbb{C})$. Note that the diagonal matrices are always in $G(D)$. For example, if D is such that $d_1 = \cdots = d_k$ then $G(D) = \mathrm{GL}(k, \mathbb{C})$, and if D is such that $d_1 < \cdots < d_k$ then $G(D)$ is the subgroup of lower triangular matrices in $\mathrm{GL}(k, \mathbb{C})$.

The action of $G(D)$ produces a quotient space

$$P(m, d_1) \times \cdots \times P(m, d_k) / G(D),$$

here $P(m, d_1) \times \cdots \times P(m, d_k)$ is a complex manifold and $G(D)$ acts by holomorphic transformations. In order to make the quotient space a complex manifold we show that the isotropy subgroup of all $(f_1, \ldots, f_k)$, given origin to a foliation, is the identity. Fix $(f_1, \ldots, f_k)$ such that $df_1 \wedge \cdots \wedge df_k$ is non identically zero. Assume by contradiction that there exists some $A = (a_{ij}) \in G(D)$ different of Id and such that it is in the isotropy i.e.

$$(f_1, \ldots, f_k) = A \circ (f_1, \ldots, f_k).$$

Since $A \neq \mathrm{Id}$, then we have that for some $1 \leqq r \leqq k$ there exists one relation of type

$$f_r = a_{r1} f_1 + \cdots + a_{rr} f_r + \cdots + a_{rk} f_k,$$

where at least some $a_{rl} \neq 0$ for $l \neq r$. Note that $a_{rr} = 0, 1$ implies that there is some linear dependence between the polynomials $\{f_i\}$, hence $df_1 \wedge \cdots \wedge df_k$ is identically zero, which is a contradiction. In any case if $a_{rr} \neq 0, 1$, then it also implies the contradiction

$$(1 - a_{rr})(df_1 \wedge \cdots \wedge df_k) = df_1 \wedge \cdots \wedge (1 - a_{rr}) df_r \wedge \cdots \wedge df_k \equiv 0.$$

It follows that $(P(m, d_1) \times \cdots \times P(m, d_k))/G(D)$ has a natural structure of complex manifolds restricted to the polynomials which define foliations. Now define a map Ψ using the action of $G(D)$ as

$$\Psi : (P(m, d_1) \times \cdots \times P(m, d_k))^\circ / G(D) \to \mathrm{Fol}(\mathbb{C}^m, c, k),$$

$$[(f_1, \ldots, f_k)] \mapsto [df_1 \wedge \cdots \wedge df_k],$$

here $(P(m, d_1) \times \cdots \times P(m, d_k))^\circ$ means the open dense set of all the $\{(f_1, \ldots, f_k)\}$ in the space $P(m, d_1) \times \cdots \times P(m, d_k)$ and such that it defines a foliation ($df_1 \wedge \cdots \wedge df_k$ is not identically zero) with k generic polynomial first integrals. This new function is one to one over its image. For example if D is such that $d_1 = \cdots = d_k$ then the domain of Ψ is the Grassmannian of k planes in the vector space $P(m, d_1)$.

It is easy to see that Ψ is holomorphic, a long and straightforward computation shows that Ψ is an embedding (or use that a one to one holomorphic map always has holomorphic inverse, see [G-H], p. 19), we leave the details to the reader. □

Now we have an analogous of Theorem 5.1 that gives some irreducible components of the spaces of singular holomorphic foliations of higher codimension in $\mathbb{C}^m$ (for related results in the case of codimension one, see for example the results of D. Cerveau and F. Maghous in [C-M], [Ce], and Calvo-Andrade in [C]).

6.4. Theorem. *Given a multidegree $D = \{d_1, \ldots, d_k\}$, let $\mathrm{FI}(\mathbb{C}^m, D, k)$ be the space of all singular polynomial foliations in $\mathbb{C}^m$ having k generic polynomial first integrals of multidegree D. If m is at least three and k is such that the leaves of the foliations have dimension $m-k$ at least two, then $\mathrm{FI}(\mathbb{C}^m, D, k)$ is an open dense set in the irreducible components of $\mathrm{Fol}(\mathbb{C}^m, c, k)$.*

Note that in the above result the existence of several polynomial first integrals imply that the foliations have as leaves algebraic varieties of dimension at least two, moreover the fundamental groups of the non singular leaves are finite. Theorems 5.1 and 6.4 show that holomorphic foliations having first integrals and where the leaves have small fundamental group have simple behavior under deformation, first integrals persist under deformation. On the other hand in the case of holomorphic foliations having leaves of codimension $k = m - 1$ and k first integrals, the above type of result not holds, for this case the leaves are algebraic curves (with large fundamental group), and in general under deformation the existence of k first integrals disappears. For example in $\mathbb{C}^2$ deformations of hamiltonian systems are not in general hamiltonian.

Proof. We use the idea of Theorem 5.1, by an infinitesimal computation we show that:

$$T_{\mathscr{F}} \mathrm{FI}(\mathbb{C}^m, D, k) = T_{\mathscr{F}} \mathrm{Fol}(\mathbb{C}^m, c, k) .$$

Similarly as in Definition 1.5, an infinitesimal deformation of the foliation $\mathscr{F}$ is a tangent vector in $T_{\mathscr{F}} \mathrm{Fol}(\mathbb{C}^m, c, k)$. We construct now in an explicit way, the infinitesimal deformations for this case as in Lemma 1.9.

Let $\mathscr{F} = [\omega] = [df_1 \wedge \cdots \wedge df_k] \in \mathrm{FI}(\mathbb{C}^m, D, k)$ as above and take a polynomial one form $\phi = \omega + t\eta$ over the analytic space $(\mathbb{C}[t]/t^2) \times (\mathbb{C}^m)$, where $\eta \in \Omega_c^k(\mathbb{C}^m)$.

We impose the infinitesimal integrability conditions, in according with Definition 6.1:

(1) $\omega + t\eta = (df_1 + t\phi_1) \wedge \cdots \wedge (df_k + t\phi_k) \quad (\mathrm{mod}\, t^2 = 0)$.

(2) $(\omega + t\eta) \wedge d(df_i \wedge t\phi_i) = 0 \quad$ for all $i = 1, \ldots, k \quad (\mathrm{mod}\, t^2 = 0)$.

Since we are only interested in infinitesimal deformations for spaces of polynomial foliations, we can suppose that $\phi_1, \ldots, \phi_k$ are also polynomial one-forms in $\mathbb{C}^m$. From (1) we obtain:

$$(1') \quad \eta = \phi_1 \wedge df_2 \wedge \cdots \wedge df_k + df_1 \wedge \phi_2 \wedge \cdots \wedge df_k + \cdots + df_1 \wedge \cdots \wedge df_{k-1} \wedge \phi_k .$$

Now using that $d(df_i) = 0$, from (2) we get $\omega \wedge d(df_i \wedge \phi_i) = 0$ hence:

$$(2') \qquad df_1 \wedge \cdots \wedge df_k \wedge d\phi_i = 0 \quad \text{for all } i = 1, \ldots, k .$$

6.5. Remark. For a generic leaf $\mathcal{L}$ of $\mathcal{F} = [\omega]$, the restriction of each one-form ϕ_i to $\mathcal{L}$ is locally closed.

Proof. Let $p \in \mathcal{L} \subset \mathbb{C}^m$ a non singular point of $\mathcal{F}$. By Frobenius integrability theorem we can take a local foliated chart $\psi: U \subset \mathbb{C}^m \to \Delta^m$, where U is a neighborhood of p and Δ^m is a polydisk, such that the first integrals $f_1, \ldots, f_k$ are given by the local coordinates $z_1, \ldots, z_k$. Let $\psi_* \phi_i = g_1 d_{z_1} + \cdots + g_m dz_m$ the holomorphic one-form associated to some ϕ_i on the new coordinates, where $g_1, \ldots, g_m$ are holomorphic functions. Condition (2′) is now equivalent to:

$$dz_1 \wedge \cdots \wedge dz_k \wedge d(\psi_* \phi_i) = 0 ,$$

which means that $d(\psi_* \phi_i) = 0$ over sets of the type $\{z_1 = \mu_1, \ldots, z_k = \mu_k\}$. □

Let T be a complex $(m-k)$-dimensional plane in $\mathbb{C}^m$, which is a generic transversal of $\mathcal{F}$ in $\mathbb{C}^m$. This means that on $\mathbb{C}^m$, for a generic leaf $\mathcal{L}(p)$ of $\mathcal{F}$ we have that $T \cap \mathcal{L}(p) = \{p_1, \ldots, p_s\}$ is a finite number of points, always counted with multiplicity, in fact $s = d_1 + \cdots + d_k$ is the degree of the non singular leaves as complete intersections. Given a fixed $\phi \in \{\phi_1, \ldots, \phi_k\}$, we define the function

$$\mathcal{I}_\phi : \mathbb{C}^m - \operatorname{Sing}(\mathcal{F}) \to \mathbb{C}$$

by

$$\mathcal{I}_\phi(p) = \sum_{v=1}^{s} \int_{p_v}^{p} \phi ,$$

where the integrals are computed over paths which belong to each $\mathcal{L}(p)$ and with end points p_v and p. By 6.4 the integrals are locally well defined.

6.6. Lemma. *For every $\phi \in \{\phi_1, \ldots, \phi_k\}$, the functions $\mathcal{I}_\phi$ are polynomial functions in $\mathbb{C}^m$.*

Proof. First we show that $\mathcal{I}_\phi$ is independent of the choice of the paths for all $p \in \mathbb{C}^m - \operatorname{Sing}(\mathcal{F})$. A generic leaf $\mathcal{L}$ is a complex submanifold of $\mathbb{C}^m$. If we take the natural compactification $\bar{\mathcal{L}}$ of $\mathcal{L}$ in $\mathbb{C}P^m$ (where $\mathbb{C}^m \hookrightarrow \mathbb{C}P^m$ is the usual affine chart and consider $\mathbb{C}P^m = \mathbb{C}^m \cup \mathbb{C}P^{m-1}$, with $\mathbb{C}P^{m-1}$ the hyperplane to infinite), then we have:

$$\pi_1(\mathcal{L}) = \pi_1(\bar{\mathcal{L}} - \mathcal{L} \cap (\mathbb{C}P^{m-1} \cup \operatorname{Sing}(\mathcal{F}))) .$$

Now, $\bar{\mathscr{L}} \cap \mathbb{C}P^{m-1}$ is a submanifold of $\bar{\mathscr{L}}$ of complex codimension one in $\bar{\mathscr{L}}$. $\bar{\mathscr{L}} \cap \operatorname{Sing}(\mathscr{F})$ is an analytic subspace of $\bar{\mathscr{L}}$ of complex codimension greather than one; from this observation

$$\pi_1(\mathscr{L}) = \pi_1(\bar{\mathscr{L}} - \bar{\mathscr{L}} \cap \mathbb{C}P^{m-1}) .$$

Using that $\bar{\mathscr{L}}$ is a complete intersection, by the same argument in the proof of Theorem 5.1, $\pi_1(\mathscr{L})$ is finite since $H^1(\mathscr{L}, \mathbb{C}) = 0$, the functions $\mathscr{I}_\phi$ are well defined in $\mathbb{C}^m - \operatorname{Sing}(\mathscr{F})$.

We note that $\mathscr{I}_\phi$ is holomorphic and locally bounded, near $\operatorname{Sing}(\mathscr{F})$, which is an analytic set of codimension greather than two. Hence by Riemann's removable singularity theorem in several complex variables, see [K-K], p. 22, $\mathscr{I}_\phi$ has a unique holomorphic extension to all $\mathbb{C}^n$.

We take $p \in \mathbb{C}^m$, if p goes to $\mathbb{C}P^{m-1}$ (the hyperplane at infinite), then $\mathscr{I}_\phi(p)$ also goes to infinite. By a coordinate change we can compute the growth of $\mathscr{I}_\phi$ when p goes to infinite, use the same argument of Lemma 2.2 (the leaves of $\mathscr{F}$ are given by the intersection of the rational functions $\hat{f}_1, \ldots, \hat{f}_k : \mathbb{C}P^m \to \mathbb{C}P^1$ which extend the polynomials $f_1, \ldots, f_k$). Since the growth of $\mathscr{I}_\phi$ is bounded, $\mathscr{I}_\phi$ is a rational function from $\mathbb{C}P^m$ to $\mathbb{C}P^1$ which only has a pole of finite order at $\mathbb{C}P^{m-1}$. □

Conclusion of the proof of Theorem 6.3. Along the leaves of $\mathscr{F}$, $d(\mathscr{I}_{\phi_i}) = a_i \phi_i$, for some $a_i \in \mathbb{C}$, hence $a_i \phi_i = d(\mathscr{I}_{\phi_i}) + \psi_i$ over all $\mathbb{C}^m$, where each ψ_i is a polynomial one-form which vanishes along the tangent bundle $T\mathscr{F}$ of the leaves of $\mathscr{F}$. This means that $df_1 \wedge \cdots \wedge df_k \wedge \psi_i \equiv 0$ on all $\mathbb{C}^n$. Using a local foliated chart $\psi : U \subset \mathbb{C}^m \to \Delta^m$ for p in $\mathbb{C}^m - \operatorname{Sing}(\mathscr{F})$ we may write:

$$\psi_i = g_{i1}\, df_1 + \cdots + g_{ik}\, df_k$$

in U, with $g_{i1}, \ldots, g_{ik}$ holomorphic functions in U. By analytic continuation and unicity of the coefficients $\{g_{ij}\}$ it is possible to show that the above expression is well defined in all $\mathbb{C}^m - \operatorname{Sing}(\mathscr{F})$. Note that ϕ_i and $d\mathscr{I}_{\phi_i}$ are polynomial forms, using the equation that defines ψ_i we see that ψ_i is also a polynomial form. On the other hand $\{df_1, \ldots, df_k\}$ are linearly independent, since if we have that $a_i \phi_i = d(\mathscr{I}_{\phi_i}) + \psi_i$ is satisfied then each g_{ij} must be a polynomial. Hence $(g_{i1}, \ldots, g_{ik})$ are defined over all $\mathbb{C}$ and are polynomials.

Since we are interested in working only with tangent vectors in $T_{\mathscr{F}} \operatorname{Fol}(\mathbb{C}^m, c, k)$, using (1′) we may change each ϕ_i by:

$$\bar{\phi}_i = \frac{1}{a_i}\left(d(\mathscr{I}_{\phi_i}) + g_{ii}\, df_i\right)$$

and:

$$\omega + t\eta = (df_1 + t\phi_1) \wedge \cdots \wedge (df_k + t\phi_k) = (df_1 + t\bar{\phi}_1) \wedge \cdots \wedge (df_k + t\bar{\phi}_k)$$
$$= df_1 \wedge \cdots \wedge df_k + t(\bar{\phi}_1 \wedge df_2 \wedge \cdots \wedge df_k + \cdots + df_1 \wedge \cdots \wedge df_{k-1} \wedge \bar{\phi}_k) \quad (\operatorname{mod} t^2 = 0) .$$

Now η is a k-form of degree at most c. We may use the fact that the f_i are generic polynomials with no common factors between them and each $d\mathscr{I}_{\phi_i}$ is of degree at most d_i, in order to show that each form

$$df_1 \wedge \cdots \wedge \frac{1}{a_i}(d\mathscr{I}_{\phi_i} + g_{ii}\, df_1) \wedge \cdots \wedge df_k$$

for all i, is at most of degree c. This implies that the degree of g_{ii} is zero hence g_{ii} is a constant function. We conclude that each $\bar{\phi}_i$ is exact. We denote by $\tilde{\phi}_i$ the polynomial function such that $d\tilde{\phi}_i = \bar{\phi}_i$. Now the family of infinitesimal deformations may be written as:

$$\omega + t\eta = (df_1 + td\tilde{\phi}_1) \wedge \cdots \wedge (df_k + td\tilde{\phi}_k) \quad (\operatorname{mod} t^2 = 0)$$

which is a deformation in the space of foliations with k generic polynomial first integrals, and $[\eta] \in T_{\mathscr{F}}\mathrm{FI}(\mathbb{C}^m, D, k)$. The above arguments prove that

$$T_{\mathscr{F}}\mathrm{Fol}(\mathbb{C}^m, c, k) \subset T_{\mathscr{F}}\mathrm{FI}(\mathbb{C}^m, D, k).$$

The other inclusion is obvious and hence $\mathrm{FI}(\mathbb{C}^m, D, k)$ are open dense sets in the irreducible components of $\mathrm{Fol}(\mathbb{C}^m, c, k)$. □

For the case of only one generic polynomial first integral f we have one additional conclusion:

6.7. Corollary. *Consider the singular polynomial foliations* $\mathrm{FI}(\mathbb{C}^m, d, 1)$ *of codimension one with one generic polynomial first integral of degree d in $\mathbb{C}^m$, where $m \geqq 3$. Each singular holomorphic foliation* $\mathscr{F} \in \mathrm{FI}(\mathbb{C}^m, d, 1)$ *is C^0-structural stable as foliation.*

Proof. Structural stability follows because Morse functions give structural stable foliations, see for example [GM-LN]. □

If we consider the polynomials as a branched Lefschetz pencils, then we have that Corollary 6.7 is a similar result to Theorem 4.3 of [GM-LN].

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Eingegangen 18. Mai 1993, in revidierter Fassung 15. November 1994