SYMMETRY AND REDUCED HAMILTONIANS METHODS IN GAUGE THEORIES

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ABSTRACT

On a principal fiber bundle $P \to M$ we consider the action of a Lie group S. Yang-Mills and Yang-Mills-Higgs equations on P can be reduced to algebraic equations or ordinary differential equations by the use of S-invariant connections. An example of Yang-Mills-Higgs equations on $M = S^3 \times \mathbb{R}$ is given.

The study of the Yang-Mills and Yang-Mills-Higgs system of equations has been relevant both to physics and mathematics. In physics it led to the introduction of gauge theories as a basis for the grand unification schemes of the fundamental interactions, while in mathematics the analysis of the Yang-Mills equations has unfoled valuable information relating to the topology of 4 simply connected manifolds as well as on stable bundles in the field of algebraic geometry.

The essential objects of study in Yang-Mills theories are the connections of principal fiber bundles (PFB) and Higgs fields which are particle fields or, equivalently, cross sections of the associated vector bundle. The space $C(P)$ of all connections is an affine space and hence has a simple structure. However, since in the language of fiber bundles the principle of minimal coupling in physical theories translates into the requirement of invariance of the action under the gauge group $GA(P)$ of base-preserving automorphisms of a PFB P, the space of interest in physics is the moduli space $C(P)/GA(P)$ which is infinite dimensional and of an extremely complicated structure. One way to circumvent this difficulty is presented to us by the presence of symmetry in many problems of interest, both in physics as well as in mathematics. Hence, if we restrict consideration to the subspace of connections which are invariant under a finite dimensional Lie group S of symmetry transformations, which project to a given action on the base manifold, the moduli space $\mathcal{M} = \{\omega \in C(P)/GA(P) \mid s^*\omega_1 = \omega_2,\ s \in S\}$ is finite dimensional and the classification of orbits in this space can

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be reduced to an algebraic procedure, provided the symmetry group S acts transitively on fibers.

Consequently, impossing symmetry on the connections leads to the following possibilities:

Let M/S be the space of orbits of the action of S in M.

1) If M/S = point, M is homogeneous under the action of S and the Yang-Mills field equations reduce to an algebraic system.

2) If M/S is a 1-manifold, we say that the action is of cohomogeneity one. Here the field equations yield a system of ordinary differential equations in M/S which may be solved by means of Hamiltonian reduction methods after applying the Symmetric Criticality Principle of Palais.[2]

3) If M/S is a manifold of $dim\ (M-1) < dim\ (M/S) < dim\ M$ then the Yang-Mills field equations reduce to a system of partial differential equations of simpler structure than the original ones.

The use of these techniques allows us to investigate some basic problems such as the existence, interpretation and classification of critical points to the Yang-Mills system on Riemannian 4-manifolds that are neither self-dual nor antiself-dual; and the construction of non-trivial Yang-Mills-Higgs fields on compact 4-Riemannian manifolds, for which there are only topological theorems of existence but no explicit non-trivial solutions have been found so far.

Because of the non-trivial relationship between local trivializations and symmetry transformations in gauge theories, the details of the analysis require us to consider several related technical questions such as the classification of all the inequivalent lifts to the PFB of a given action of S on the base manifold, the identification of connections which are gauge equivalent in the more general sense: $f^*\omega_1 = \omega_2$, and the necessary and sufficient conditions for the local gauge-equivalence of two S-invariant connections. Since these issues have been addressed in detail elsewhere,[1] in what follows we shall only cite and use the main results as needed.

The algebraic procedure for constructing gauge fields from S-invariant connection 1-forms on PFB's such that they posses the symmetry of the underlying base manifold for the case when S acts transitively on M, makes use essentially of the following theorem due to Wang:[3]

Theorem 1: There is a bijective correspondence between S-invariant connections and the linear transformations $\Lambda : L(S) \to L(G)$ of Lie algebras which satisfy the conditions:

(A) $\Lambda(Y) = \mu_{p_0 *}(Y)$, for $Y \in L(J_{x_0})$

(B) $\Lambda(ad_j(X)) = ad_{\mu_{p_0}}(j)(\Lambda(X))$, for $X \in L(S)$ $j \in J$, where $J \subset S$ is the isotropy subgroup and $\mu : J \to G$ is an homomorphism of Lie groups.

If ω is the S-invariant 1-form connection corresponding to Λ, we have

$$\Lambda(X) = \omega_{p_0}(\hat{X}_{p_0}), \tag{1}$$

with $\hat{X}_{p_0} = \frac{d}{dt}(\exp\ tX \cdot p_0)\ |_{t=0}, X \in L(S)$. If we define $\tilde{X}_x = \frac{d}{dt}(\exp\ tX \cdot x)\ |_{t=o}$, the relation between $\tilde{X}_x$ and $\hat{X}_p$ $(p = \sigma_\alpha(x))$ is

$$\hat{X}_{\sigma_\alpha(x)} = (\sigma_\alpha)_*(\tilde{X}_x) + [W_x^\alpha(X)]^*_{\sigma_\alpha(x)}\ , \tag{2}$$

where $\varphi_x(s) \in G$ describes how the action of S has been lifted to the fibers and $W_x^\alpha \equiv (\varphi_x^\alpha)_*$. Using the S-invariance of the connection, we can show that

$$A_\alpha(\tilde{X}_x) \equiv (\sigma_\alpha^*\omega)_x(\tilde{X}_x) = a\delta_{\varphi^\alpha_{x_o}}(s)\ \Lambda(a\delta_s^{-1}X) - W_x^\alpha(X). \tag{3}$$

This is the general algebraic expression for the S-invariant gauge-fields. It clearly follows from the R. H. S. that it is uniquely determined by the matrices Λ and by the lifting to the bundle of the action of S on M. Λ is derived by solving Wang's equations, while $\varphi_x^\alpha(s)$ is given by:

Proposition 2: Corresponding to the section $\tau_\alpha : U_\alpha \subset M \to S$ we have

$$\varphi_x^\alpha(s) = \mu_{p_0}(\tau_\alpha^{-1}(sx)\ s\tau_\alpha(x)),$$

with

$$\tau_\alpha^{-1}(sx)\ s\tau_\alpha(x) \in J_{x_0}\ , \tag{4}$$

which provides all the inequivalent actions of S on P. That is for S acting transitively on M the problem of the lifting of the action of S and P reduces to one of classification of group homomorphisms.

In order to have a means of classifying symmetric gauge fields into classes modulo gauge transformations, and to describe orbits in this moduli space, we need to know when, given two connections both required to be S-invariant, they are related by a gauge transformation. There are local and global theorems[1] which provide the answer. Here we only state the theorem in the global domain for the case of generic connections. Since it is known[4] that when P is a connected and compact manifold which admits at least one generic connection, the space of generic connections forms an open and dense subset of $\mathcal{C}(\mathcal{P})$, we then have:

Theorem 3: Two generic S-invariant connections ω_1 and ω_2 are gauge–related iff $\exists\ u\ \in\ G$ with

1. $\mu(j)^{-1}u\mu(j)u^{-1}\ \in Z(G)$, the center of $G\ \forall\ j\ \in\ J$
2. there is a group homomorphism $\nu : S \to Z(G)$ such that, for

$$j \in J,\ \nu(j) = \mu(j)^{-1}u\mu(j)u^{-1} \tag{5}$$

3. $\Lambda_2 = u^{-1}(\Lambda_1 + \nu_* |_e)u$, where Λ_1 and Λ_2 are the linear transformations associated to ω_1 and ω_2, respectively.

Note that here ν_* is a Lie algebra homomorphism from $L(S)$ onto an abelian subalgebra of $L(G)$, so if S or G are simple, then $\nu = e$ and the necessary and sufficient conditions for gauge equivalence of ω_1 and ω_2 reduce to

$$\Lambda_2 = u^{-1}\Lambda_1 u, \ \mathit{with}\ u \in C_G(\mu(J)). \tag{6}$$

As an application of the above described procedure, consider the following examples which, taken in sequence, constitute an illustration of how first the Yang-Mills part of the Yang-Mills-Higgs system can be reduced to an homogeneous base space problem and thus to a strictly algebraic formulation and subsequently, by means of the Principle of Symmetric Criticality, the full Yang-Mills-Higgs action is reduced to a cohomogeneity one problem, i.e., to a system of ordinary differential equations.

Example 1. Let S^3 be the homogeneous base space of a trivial SU(2)-PFB. We can write

$$S^3 = \frac{Spin(4)}{Spin(3)} \cong \frac{\mathrm{SU}(2)\times\mathrm{SU}(2)}{(\mathrm{SU}(2)\times\mathrm{SU}(2))_D} =]rmSU(2), \tag{7}$$

where $S = \mathrm{SU}(2)\times\mathrm{SU}(2)$ is the symmetry group acting transitively on M, and $J = (\mathrm{SU})(2)\times\mathrm{SU}(2))_D \cong \mathrm{SU}(2)$ in the group of isotropy. Using Theorem 1 we have

$$\Lambda = \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \\ 1+k & 0 & 0 \\ 0 & 1+k & 0 \\ 0 & 0 & 1+k \end{pmatrix}, \qquad k \in \mathbb{R}. \tag{8}$$

Now from Theorem 11.8 in Kobayashi and Nomizu,[5] it is easy to verify that $C_G(Hol_p\omega) = e$ so, the connection is generic. Furthermore since in this case $\mu(J) = \mathrm{SU}(2)$ and $C_G(\mu(J)) = e$, it follows that $u = e$ and thus, by virtue of Theorem 3, there are no other gauge-related (non-trivial) bi-invariant connections appart from (8). In addition, since the bundle is trivial, μ extends to a smooth morphism, and we can choose for the lifting the natural map $\varphi_x(s) = e$, so $s\sigma(x) = \sigma(sx)$. This in turn implies that $(\varphi_x)_* = 0$, and $W_x = 0$. Consequently we have that the gauge fields are given by

$$A = k\theta^i \otimes Y_i, \quad i = 1,2,3, \tag{9}$$

where θ^i is a basis of left-invariant 1-forms and $Y_i := \frac{-i\sigma_i}{2}(\sigma_i = \text{Pauli matrices})$.

Example 2. Yang-Mills-Higgs Fields of Cohomogeneity One.

Consider now $M = S^3 \times \mathbb{R}$ and a trivial PFB P with gauge group SU(2). Given an associated vector bundle E over the Riemannian manifold M, then for every pair (A, Φ), where A is the gauge field associated to the connection on P and Φ is a section of E, the Euclidean Yang-Mills-Higgs functional is:

$$YMH(A,\Phi) = \frac{1}{2}\int_M [\frac{1}{2} \mid F \mid^2 + \mid D\Phi \mid^2 + \frac{\lambda}{z} \mid \Phi \mid^4 - m^2 \mid \Phi \mid^2] d\mu. \qquad (10)$$

Here F and D are the curvature and the covariant derivative, respectively, associated with A, m is the mass parameter, and λ is a real positive constant. We take $S = \mathrm{SU}(2) \times \mathrm{SU}(2)$ as the symmetry group in M acting on the factor S^3 as above, and we are interested on pairs (A, Φ) of S-invariant objects. Now since, $\Lambda^1(S^3 \times \mathbb{R}, L(\mathrm{SU}(2))) = \Lambda^1(S^3, L(\mathrm{SU}(2))) \oplus \Lambda^1(\mathbb{R}, L(\mathrm{SU}(2)))$ where Λ^1 is the space of 1-forms valued in the Lie algebra of SU(2), we can write

$$\hat{A} = A(k(t)) + \varphi(t)dt, \qquad (11)$$

where $A(k(t))$ is the S-invariant connection on $S^3 \times \{t\}$ given by Eq. 9 and $\varphi(t) \in L(\mathrm{SU}(2))$. Applying next the following Lemma due to Parker:[6]

Lemma 4: There exists a path $g = g(t) \in \mathrm{SU}(2)$, unique up to a constant $g \in \mathrm{SU}(2)$, that defines a gauge transformation taking $\hat{A}$ to $\widetilde{A}(k(t))$, with $\varphi = 0$ (temporal gauge); it is easy to show that in this parametrized gauge

$$\widetilde{F} = F^{\widetilde{A}} + dt \wedge \dot{\widetilde{A}}\,, \qquad (12)$$

where

$$\dot{\widetilde{A}} = k(t)\, \theta^i \otimes Y_i,$$
$$F^{\widetilde{A}}_{ij} = (k^2 - k)[Y_i, Y_j]. \qquad (13)$$

On the other hand the action of S on S^3 can be lifted to the vector bundle E, by choosing $\rho : L(S) \to L(\mathrm{SU}(2))$, as the natural representation given by the Pauli matrices, so E is the trivial $\mathbb{C}^2$ or $\mathbb{R}^4$-bundle over S^3. If we further assume that Φ is S-invariant, then by definition it satisfies the infinitesimal relation

$$\mathcal{L}_{\xi_\alpha} \Phi = -\rho(X_\alpha) \cdot \Phi, \qquad (14)$$

where ξ_α are the Killing vector fields on S^3. The set of differential equations (14) can be readily solved under the ansatz that Φ is linear in the coordinates of $S^3 \subset \mathbb{R}^4 = \{(x_1, \ldots, x_4)\}$. We thus obtain

$$\Phi = \mathbf{W}(t) \cdot \mathrm{x}(t), \qquad (15)$$

where

$$\mathbf{W}(t) = ae_0 - be_1 - ce_2 - de_3, \; \mathrm{x}(t) = x_1 e_0 + x_2 e_1 + x_3 e_2 + x_4 e_3,$$

with $\{e_0, e_1, e_2, e_3\}$ a 4-dimensional representation of the basis of quaternions.

We can take the metric for $M = S^3 \times \mathbb{R}$ to be given by

$$g = \sum_{i=1}^{3} (\theta^i \otimes \theta^i) + dt \otimes dt. \tag{16}$$

Hence the action of S on S^3 is in fact an action by isometries, and the derivation of the field equations for our problem simplifies extensively if we first insert Eq. 9, Eq. 15 and Eq. 16 into the action Eq. 10 and, making use of the Principle of Symmetric Criticality of Palais,[2] we consider the equivalent Hamiltonian mechanics system which results from such a reduced mechanical Lagrangian. We thus obtain

$$\begin{aligned} YMH(A(t), \Phi(t)) = \pi^2 \textstyle\int_{-\infty}^{\infty} \{&3[k(t)^2 - k(t)]^2 + (\dot{k}(t))^2 + \tfrac{3}{4}(1 - k(t))^2\, \overline{\mathbf{W}}(t) \cdot \mathbf{W}(t) \\ &+ (\dot{\overline{\mathbf{W}}}(t) \cdot \dot{\mathbf{W}}(t)) + \tfrac{\lambda}{2}(\overline{\mathbf{W}}(t) \cdot \mathbf{W}(t))^2 - m^2(\overline{\mathbf{W}}(t) \cdot \mathbf{W}(t))\}\, dt. \end{aligned} \tag{17}$$

Varying Eq. 17 with respect to k gives

$$k(k-1)(2k-1) + \frac{(k-1)}{4}(a^2(t) + b^2(t) + c^2(t) + d^2(t)) - \ddot{k} = 0, \tag{18}$$

while variation with respect to $\mathbf{W}$ yields

$$\frac{3}{4}(k-1)^2 \overline{\mathbf{W}}(t) - \ddot{\overline{\mathbf{W}}}(t) + \lambda(\overline{\mathbf{W}}(t) \cdot \mathbf{W}(t))\mathbf{W}(t) - m^2 \overline{\mathbf{W}}(t) = 0. \tag{19}$$

Observe that if we now substitute the conjugate of $\mathbf{W}$ into Eq. 19, we get the same ordinary differential equation for each of the components (a, b, c, d). This implies in turn that for simplicity we can set

$$a(t) = a_0 h(t), \quad b(t) = b_0 h(t), \quad c(t) = c_0 h(t), \quad d(t) = d_0 h(t), \tag{20}$$

where $h(t)$ is a solution of

$$\frac{3}{4}(1-k)^2\, h(t) - \ddot{h}(t) + \lambda(a^2(t) + b^2(t) + c^2(t) + d^2(t))\, h(t) - m^2 h(t) = 0. \tag{21}$$

Thus the problem has been reduced to solving the simultaneous pair of non-linear ordinary differential equations Eq. 18 and Eq. 21, which certainly can be approached by numerical techniques.

Periodic solutions (on t) will give solutions on $S^3 \times S^1$, while solutions on $S^3 \times \mathbb{R}$ with finite energy will extend to solutions in S^4 via stereographic projection and Uhlenbeck-Parker[7] theorem on removable singularities for Yang-Mills-Higgs fields.

References

1. R. Bautista, J. Muciño, E. Nahmad-Achar and M. Rosenbaum, *Reviews in Mathematical Physics* **5** (1993) 69–103.

2. R. S. Palais, *Commun. Math. Phys.* **69** (1979) 19–30.
3. H. Wang, *Nagoya Math. J.* **13** (1958) 1.
4. I. M. Singer, *Commun. Math. Phys.* **60** (1978) 7.
5. S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry* (Interscience, New York, 1963).
6. T. H. Parker, *Invent. Math.* **107** (1992) 397–420.
7. T. H. Parker, *Commun. Math. Phys.* **85** (1982) 563–602.