GROUP INVARIANT CONNECTIONS ON PRINCIPAL FIBER BUNDLES

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Abstract

We consider group-invariant connection forms on bundles with arbitrary characteristic groups, and give an algebraic procedure for constructing gauge fields from them possessing the symmetry of the underlying manifold. A general definition for the local invariance of a connection is proposed, and the conditions for two group-invariant connections to be related by a gauge transformation are examined. This, together with a characterization of gauge-equivalence of connections in terms of their holonomy groups, provides us with a means of classifying our construction of symmetric gauge fields into classes modulo gauge transformations. The theory has been developed both in the local and global domains.

1. Introduction

Symmetry, as a group of motions in a base manifold M, plays an important role in gauge theories and leads naturally to the study of connections on a principal fiber bundle $P \longrightarrow M$ which are invariant under the action of the symmetry group S (see e.g. Forgacs and Manton [1980], Jackiw [1980]). Physically, one is therefore interested in studying the internal symmetry group of *gauge-equivalent* classes of connections.

A gauge-equivalent characterization of connections, in terms of their associated holonomy groups, is here presented, as well as a study of the conditions which result from imposing the additional requirement that the two gauge-related connections should both be S–invariant, for S and arbitrary group with a given action on M. Due to limitations of space, only the results will be presented here while the details will be given elsewhere.

2. Holonomy, and S-Invariance

Let $P(M,G)$ denote a principal fiber bundle with structure group G and projection operator $\pi : P \to M$. Denote by $C(P,G)$ the space of all maps $\tau : P \to G$ which satisfy $\tau(pg) = g^{-1}\tau(p)g$ for all $g \in G,\ p \in P$. This space is isomorphic to the space of sections of the associated bundle $P \times_G G \to M$ with standard fiber G. A diffeomorphism $f : P \to P$ which satisfies $f(pg) = f(p)g$ for all $p \in P, g \in G$, is called a *fiber bundle automorphism.* Note that such an automorphism induces a diffeomorphism $\bar{f} : M \to M$ given by $\bar{f}(\pi(p)) = \pi(f(p))$. We define a *gauge transformation* to be an automorphism $f : P \to P$ such that $\bar{f} = 1_M$, and shall denote the group of gauge transformations on P by $GA(P)$.

Now let ω be a connection 1–form on P, and $C(x,y)$ denote the collection of paths in M from x to y. Thus, $\alpha \in C(x,x)$ is a loop based at $x \in M$, i.e. $\alpha(0) = \alpha(1) = x$, and if $\hat{\alpha}(t)$ denotes the ω–horizontal lift of $\alpha(t)$ which passes through $p \in \pi^{-1}(x)$ then there exists an $h_p^\omega(\alpha) \in G$ such that $\hat{\alpha}(1) = \hat{\alpha}(0)h_p^\omega(\alpha)$. The *holonomy group* $Hol_p(\omega)$ of ω at p consists of all such elements for all possible loops based at $x = \pi(p)$, i.e. $Hol_p(\omega) = \{h_p^\omega(\alpha) | \alpha \in C(x,x), x = \pi(p)\}$. The *restricted holonomy group* $Hol_p^0(\omega)$ is the subgroup of $Hol_p(\omega)$ generated by loops at x which are homotopic to the identity.

With the notation introduced above we can then prove the following

Proposition 2.1: Let ω_1, ω_2 be two connections on a principal fiber bundle $P(M,G)$. Then a gauge transformation f with the property $f^*\omega_2 = \omega_1$ exists if and only if at some point $p \in P$ we have

$$h_p^{\omega_2} = u h_p^{\omega_1} u^{-1} \tag{2.1}$$

with $u \in C(P,G)$ such that $f(p) = pu(p)$. For a fixed p, and f such that $f^*\omega_2 = \omega_1$ and $u(p) = u$, f is unique.

This general result has as immediate corollaries two interesting results due to Fischer (1987):

Corollary 2.2: Let $p \in P$ be fixed, $f \in GA(P)$, and suppose that $f^*\omega = \omega$. There then exists $u = u(p) \in C_G(Hol_p(\omega))$ with $f(p) = pu$. Conversely, for every $u \in C_G(Hol_p(\omega))$ there exists a unique gauge transformation $f : P \to P$ such that $f^*\omega = \omega$ and $f(p) = pu$. (Here, $C_G(Hol_p(\omega))$ denotes the centralizer in G of the holonomy group of ω with reference point p.)

Corollary 2.3: For $f \in GA(P)$ with associated function $\tau \in C(P,G)$, the following conditions are equivalent:

i) $f^*\omega = \omega$.
ii) τ is constant on each ω–horizontal curve in P.
iii) τ is constant on the holonomy subbundle $P(p_0)$ of P.

We now look at the following problem: given two connections, both required to be invariant under certain group S, what are the conditions for them to be related by a gauge transformation? The answer to this question will provide us with a means of classifying a construction of symmetric gauge fields into classes modulo gauge-equivalence. We have the following two definitions:

Definition 2.4: Let $U \subset M$ be an open subset of the base manifold and ω_1, ω_2 two connection 1–forms in P. We then say that ω_2 *is gauge-equivalent to* ω_1 *on* U iff there exists a gauge transformation $f \in GA(\pi^{-1}(U))$ such that $f^*\omega_1|_{\pi^{-1}(U)} = \omega_2|_{\pi^{-1}(U)}$.

Definition 2.5: Let $W \subset M$ be an open set, $x_o \in W$, and ω a connection defined on $\pi^{-1}(W)$. We say that ω is *locally S–invariant at* x_0 iff for all $s \in S$ with $sx_0 \in W$ there exists a connected neighborhood V_s of x_0 contained in $W \cap s^{-1}W$ and such that

$$s^*\omega|_{V_s} = \omega|_{V_s}.$$

Let $\mathcal{W} = \{s \in S / sx_0 \in W\}$. Clearly, we have $\mathcal{W}x_0 = W$. Note also that given any $x \in M$, $x = \pi(p)$, there exists a neighborhood $U_0 \subset M$ of x such that $Hol_p^0(\omega) = Hol_p(\omega)(\pi^{-1}(U_0)) = Hol_p(\omega)(\pi^{-1}(V))$ for any simply connected neighborhood V of x contained in U_0. In what follows we shall take neighborhoods V of x_0 such that $Hol_{p_o}^0(\omega) = Hol_{p_o}(\omega)(\pi^{-1}(V))$, for $x_0 = \pi(p_0)$.

Following Wang [1958] (see also Kobayashi and Nomizu [1963]) we may associate, to any given S–invariant connection ω, a linear transformation Λ defined as follows: if $X \in L(S)$ (the Lie algebra of S) then $\Lambda(X) = [\omega(\hat{X})]_{p_o}$ where $\hat{X}_p = \frac{d}{dt}\,(exp\; tX \cdot p)|_{t=0}$. It turns out that the answer to the question posed at the beginning is more easily dealt with in terms of these associated linear transformations. Indeed, if $J \subset S$ denotes the isotropy subgroup which fixes x_0 (given the action of S on M), and the action of $j \in J$ on any $p \in \pi^{-1}(x_0)$ is expressed as $jp = p\mu(j)$ with $\mu(j) \in G$, then it is easy to show that $\mu(j_1, j_2) = \mu(j_1)\mu(j_2)$ (so that $\mu : J \to G$ is a morphism of groups) and one can prove the following

Proposition 2.6: Let ω_1 and ω_2 be two S-invariant connections, and let Λ_1 and Λ_2 be their associated linear transformations respectively. Then an open set $V_s \subset M$ containing x_0 exists, such that ω_1 and ω_2 are gauge-equivalent over $\pi^{-1}(V_s)$ if and only if there exists $u \in G$ with the following properties:

i) $\mu(j)^{-1}u\mu(j)u^{-1} \in C_G(Hol^0_{p_0}(\omega))$ for all $j \in J$.

ii) There exists a local section $\sigma : V_s \to Q_s(p_0) = \pi^{-1}(V_s)$.

iii) There exists a function $\nu : \mathcal{W} \to C_G(Hol^0_{P_0}(\omega))$ satisfying the following conditions: Given $x \in V_s$ and $s \in S$, with $sx \in V_s$, and writing $s\sigma(x) = \sigma(sx)\varphi_x(s)$ for some $\varphi_x(s) \in G$, then

$$\nu(rt) = \varphi_{x_0}(t)^{-1}\nu(r)\varphi_{x_0}(t)\nu(t), \text{ for } r \in \mathcal{W}, t \in V_s \tag{2.2}$$

$$\nu(j) = \mu(j)^{-1}u\mu(j)u^{-1}, \text{ for } j \in J \tag{2.3}$$

iv) $\Lambda_2 = u^{-1}(\Lambda_1 + \nu_*|_e)u$ (2.4)

A global version of the local result above may also be formulated, and for generic connections the conditions simplify considerably.

The finding of $u \in G$ and $\nu : \mathcal{W} \to C_G(Hol^0_{p_0}(\omega))$ with the properties required in Proposition 2.6 may, however, prove untractable in certain circunstances. Nevertheless, and alternative approach in terms of integro–differential conditions, which may prove more amenable to actual calculations in such cases, can be obtained in the local domain.

Let $U \subset M$ be an open neighborhood of x_o, and let X_i, X_j be any two elements of a coordinate basis for $\Xi(U)$, the space of vector fields on U. (If U is a coordinate neighborhood, with coordinates $(x^1, \ldots, x^n)$, then we may take $X_i = \partial/\partial x^i$, etc.) Starting at a point $y_1 \in U$, move a distance $\varepsilon > 0$ along the integral curve of X_i passing through y_1, reaching a point y_2. From there move a distance ε along the integral curve of X_j passing through y_2; and then back along X_i and X_j to form a "rectangle" which we call $\gamma : [0,1] \to U$. Then, making use of Ado's and Frobenius' theorems we obtain, after a very lengthy proof, the desired result:

Proposition 2.7: Let ω_1 and ω_2 be two S-invariant connection 1–forms in $\pi^{-1}(U)$. Then ω_1 and ω_2 are locally gauge-equivalent iff there exists $u \in G$ such that, for $\tau(\sigma_U(x)g) =$

$g^{-1}\varphi_{x_0}(s)u\varphi_{x_0}(s)^{-1}g$, with $\sigma_U(x)$ a local section in the ω_1-holonomy subbundle and $\sigma_U(x)g \in \pi^{-1}(U)$, one has

$$\text{i)} \quad -\lim_{\varepsilon\to 0}\Big(\frac{1}{\varepsilon^2}\oint_\gamma (\sigma_U^*\omega_2)(\gamma'(t))dt\Big) + [(\sigma_U^*\omega_2)X_j, (\sigma_U^*\omega_2)X_i] =$$
$$= \tau(\sigma_U(x))^{-1}\{\lim_{\varepsilon\to 0}\Big(\frac{1}{\varepsilon^2}\oint_\gamma (\sigma_U^*\omega_1)(\gamma'(t))dt\Big) + [(\sigma_U^*\omega_1)X_j, (\sigma_U^*\omega_1)X_i]\}\tau(\sigma_U(x)), \tag{2.5}$$

$$\text{ii)} \quad \tau(\sigma_U(x))^{-1}\tau(\sigma_U(x))_*\sigma_{U*}X_k + a\delta_{\tau(\sigma_U(x))^{-1}\omega_1}(\sigma_{U*}X_k) - \omega_2(\sigma_{U*}X_k) \in C_G(Hol_{\sigma_U(x)}(\omega_2)), \tag{2.6}$$

for $k = i, j$.

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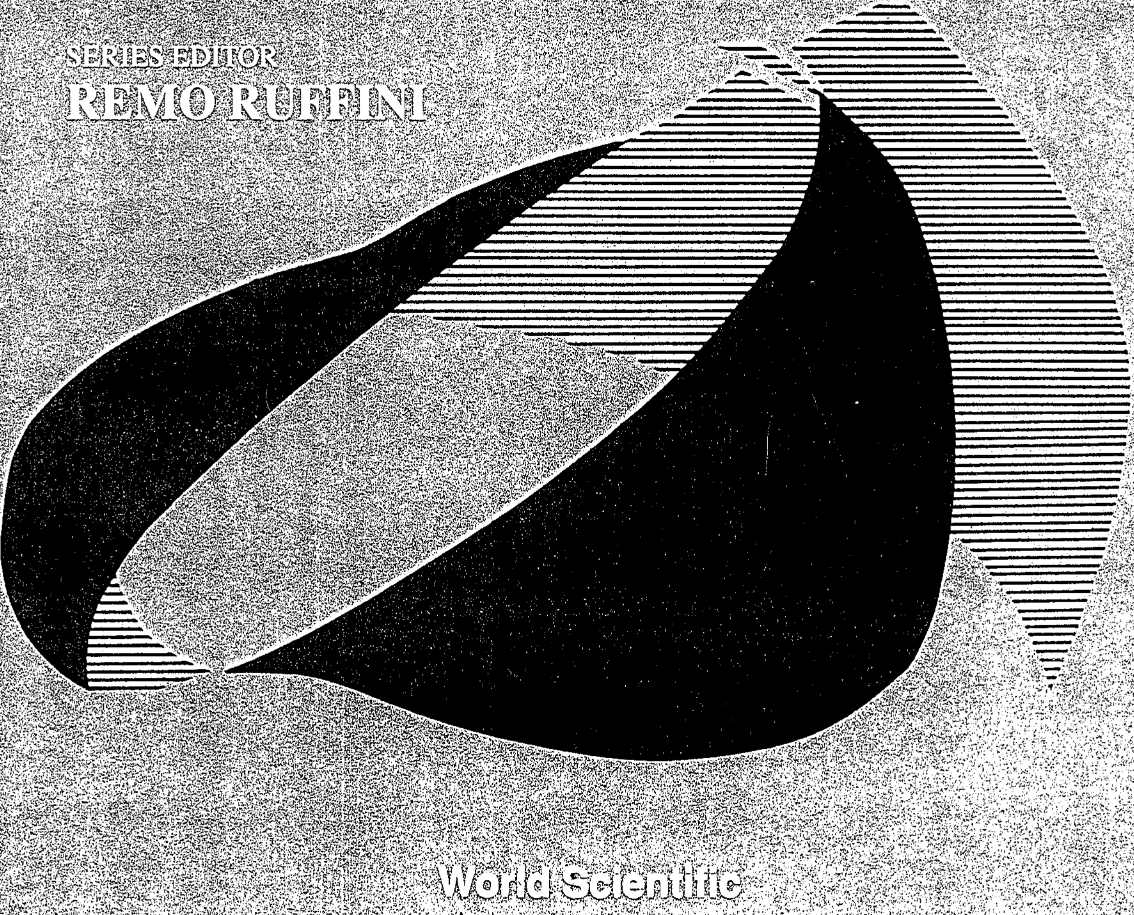

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