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Dinatural numbers

Robert Paré [a,*], Leopoldo Román [b]

[a] *Department of Mathematics, Statistics and Computing Science, Dalhousie University, Halifax, Nova Scotia, Canada B3H 3J5*

[b] *Instituto de Matemáticas, UNAM, México D.F., Mexico*

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Abstract

The notion of strong Barr dinatural transformation is introduced which, when taken between Hom functors, gives a notion of natural number specifically adapted to the category under consideration. We call these dinatural numbers and we study their arithmetic which depends in a nice way on the structure of the category. We also consider families of dinatural numbers, which leads to a new universal property for natural numbers object as classifying object for dinatural numbers. When there is a natural numbers object, its arithmetic defined by recursion corresponds to the arithmetic of dinatural numbers. Examples are given with a particular emphasis on the category of finite sets. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

In [1], Bainbridge et al. introduced a semantics for the polymorphic lambda calculus in terms of functors and generalized natural transformations. This theme is picked up again in [7]. Types are interpreted as bivariant functors on a suitable cartesian closed category and terms as dinatural transformations. It was already known to Dubuc and Street when they introduced them in [6] that dinatural transformations do not compose in general. This embarrassment is circumvented in [1] by working only with the category **PER** of partial equivalence relations on **N**. There, the dinatural transformations do compose, as there are relatively few of them and they have a very special form.

Still, the general idea is attractive enough to make one wonder how far it might go using such nice categories as the category of sets or the category of finite sets. In particular, inspired by the notion of Church numeral, one might wonder, as Bainbridge et al.

* Corresponding author. Tel.: +1 902 494 2354; e-mail: pare@cs.dal.ca.

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did, whether every dinatural transformation on the Hom set is given by iteration a fixed number of times. The answer is "no" and counterexamples are given in Section 2.2.

We introduce a stronger notion of dinatural transformation suggested by Barr, which is much better behaved. They are closed under composition, in particular. For the Hom functor on the category of sets, they correspond exactly to iteration a fixed number of times. For other categories, such as finite sets, there are more. We view them as a notion of natural number specifically adapted to the category in question. We call them dinatural numbers.

There is a certain amount of arithmetic which can be done with dinatural numbers which depends quite nicely on the structure of our category. If the category is merely monoidal we can define addition and prove certain nice properties; for cartesian categories we also have a well-behaved multiplication; and for cartesian closed categories we can define exponentiation of dinatural numbers.

We also study families of dinatural numbers and their arithmetic. This gives us a new universal property for natural numbers objects as classifying objects for families of dinatural numbers. That is to say, taking families of dinatural numbers gives a contravariant functor into the category of sets and there is a natural numbers object if and only if this functor is representable.

In the last section we make a detailed study of dinatural numbers for finite sets and relate this to the counterexamples of Section 2.2. The work of Bénabou and Loiseau [2] suggests how these results might be extended to an arbitrary elementary topos.

2. Barr dinatural transformations

2.1. Dinatural transformations

Dinatural transformations were introduced by Dubuc and Street in [6]. Given two functors of mixed variance, $F, G : \mathbf{A}^{\mathrm{op}} \times \mathbf{A} \to \mathbf{B}$, a *dinatural transformation*, $t : F \stackrel{..}{\longrightarrow} G$, consists of a morphism $t(A) : F(A,A) \to G(A,A)$ for every A in $\mathbf{A}$, such that for every morphism $f : A \to A'$, the following hexagon commutes:

$$\begin{tikzcd}[column sep=small]
 & F(A,A) \arrow[rr, "t(A)"] & & G(A,A) \arrow[dr, "{G(A,f)}"] & \\
F(A',A) \arrow[ur, "{F(f,A)}"] \arrow[dr, "{F(A',f)}"'] & & & & G(A,A') \\
 & F(A',A') \arrow[rr, "t(A')"'] & & G(A',A') \arrow[ur, "{G(f,A')}"'] &
\end{tikzcd}$$

Note: We used Mac Lane's convention of placing two dots over the arrow when we want to emphasise that what we have is a dinatural transformation. The prefix "di" was chosen to reflect the fact that it is defined on the diagonal, i.e. on pairs (A,A) in $\mathbf{A}^{\mathrm{op}} \times \mathbf{A}$.

Examples. For any category $\mathbf{A}$, the function $\mathrm{id}_A : 1 \to \mathrm{Hom}(A,A)$ which picks out the identity morphism on A gives a dinatural transformation from the constant functor $\triangle(1) : \mathbf{A}^{\mathrm{op}} \times \mathbf{A} \to \mathbf{Set}$ with value 1 to the hom functor $\mathrm{Hom}_{\mathbf{A}}(-,-) : \mathbf{A}^{\mathrm{op}} \times \mathbf{A} \to \mathbf{Set}$.

For finite-dimensional vector spaces, the trace $\mathrm{tr}_V : \mathrm{Hom}(V,V) \to K$ is dinatural. In any cartesian closed category, the evaluation $\mathrm{ev}_{A,B} : A^B \times B \to A$ is natural in A and dinatural in B. Any natural transformation $F \to G$ restricts to a dinatural transformation.

A dinatural transformation $t : \mathrm{Hom}_{\mathbf{A}} \xrightarrow{\cdot\cdot} \mathrm{Hom}_{\mathbf{A}}$ assigns to each endomorphism $A \xrightarrow{f} A$ another endomorphism $t(f) : A \to A$, in such a way that for all $g : A \to B$ and $h : B \to A$, $t(hg)h = ht(gh)$ (the hexagon condition for g). E.g.

$$t(f) = f^{(n)} = f \circ f \circ f \circ \cdots \circ f \quad (n \text{ times}).$$

The question posed by Bainbridge et al. was whether every dinatural $t : \mathrm{Hom}_{\mathbf{A}} \xrightarrow{\cdot\cdot} \mathrm{Hom}_{\mathbf{A}}$ was of this form for $\mathbf{A} = \mathbf{Set}_0$.

2.2. The counterexamples

Our counterexample is given by iteration but the number of times depends on the cardinality of the set. Thus, for $f : A \to A$, we define $t(f) = f^{(n!!)}$ where $n!! = 1!2!3! \cdots n!$ and $n = \#A$.

The point is that if $m \geq n$, then $f^{(m!!)} = f^{(n!!)}$ so that as far as the hexagon, which involves only two sets, is concerned, one can think of t as iteration $m!!$ times where m is the maximum of the cardinalities of the two sets. Thus the hexagon does commute and t is dinatural. On the other hand, if f is defined on $\{1,2,3,\ldots,n\}$ by

$$f(i) = \begin{cases} n & \text{if } i = n, \\ i+1 & \text{otherwise,} \end{cases}$$

then $t(f)$ is the constant function with value n. So t is not iteration any fixed number of times.

Peter Johnstone has given a much more conceptual version of this example. For any finite set A, $\mathrm{Hom}(A,A)$ is a finite monoid so any $f \in \mathrm{Hom}(A,A)$ has a unique power which is idempotent. (Indeed, there must exist $k < l$ such that $f^{(k)} = f^{(l)}$ from which it follows that $f^{(m)} = f^{(m+n(l-k))}$ for all $m \geq k$ and $n \geq 0$. If we let $m = k(l-k)$ and $n = k$, we see that $f^{(k(l-k))}$ is idempotent. If we had two powers of f which were idempotent, $f^{(r)}$ and $f^{(s)}$ say, then $f^{(r)} = f^{(rs)} = f^{(s)}$.) Define $t(f)$ to be that power of f which is idempotent, $f^{(r)}$. Then given f and g such that $t(fg) = (fg)^{(r)}$ and $t(gf) = (gf)^{(s)}$, then also $t(fg) = (fg)^{(rs)}$ and $t(gf) = (gf)^{(rs)}$ so for the same reason as above t is dinatural but not iteration a fixed number of times.

Freyd also gave a large class of examples. Let K be any set of cardinal numbers, and define

$$t(f) = \begin{cases} f & \text{if } \#\operatorname{Fix}(f) \in K, \\ 1_A & \text{otherwise.} \end{cases}$$

Then as $\operatorname{Fix}(fg)$ and $\operatorname{Fix}(gf)$ have the same cardinality (f gives a bijection between the two sets), t is dinatural.

Our example, which is the same as Peter Johnstone's, works for finite sets, whereas Freyd's works as well for all sets. Furthermore, Freyd's are all different so on **Set** he has a proper class of them (even more!).

2.3. BDNs

It is well-known that dinatural transformations do not compose. The problem is this: given $t : F \xrightarrow{..} G$ and $u : G \xrightarrow{..} H$ and $f : A \to A'$, we get two commutative hexagons but if we try to paste them together, we get

$$\begin{tikzcd}
 & F(A,A) \arrow[rr, "t(A)"] & & G(A,A) \arrow[rr, "u(A)"] \arrow[dr] & & H(A,A) \arrow[dr] & \\
F(A',A) \arrow[ur] \arrow[dr] & & G(A',A) \arrow[ur] \arrow[dr] & & G(A,A') & & H(A,A') \\
 & F(A',A') \arrow[rr, "t(A')"'] & & G(A',A') \arrow[rr, "u(A')"'] \arrow[ur] & & H(A',A') \arrow[ur] &
\end{tikzcd}$$

and there is no way in general to conclude that the outside hexagon commutes.

If t is natural (not merely dinatural) then we can fit $t(A',A) : F(A',A) \to G(A',A)$ into the diagram above, and a simple diagram chase shows that the outside hexagon does commute now. Thus, if one of t or u is natural then $u \cdot t$ is again dinatural. In a similar vein, if G has the property that all the squares

$$\begin{tikzcd}
G(A',A) \arrow[r, "G(f,A)"] \arrow[d, "G(A',f)"'] & G(A,A) \arrow[d, "G(A,f)"] \\
G(A',A') \arrow[r, "G(f,A')"'] & G(A,A')
\end{tikzcd}$$

are pullbacks (or pushouts), then $u \cdot t$ is again dinatural (because, once again, we get a fill-in for one of the chevron-shaped regions). This leads us to Barr's strengthening of the notion of dinatural transformation (oral communication) which we call Barr dinaturals.

Definition. Let $F, G : \mathbf{A}^{\mathrm{op}} \times \mathbf{A} \to \mathbf{B}$ be functors where $\mathbf{B}$ has pullbacks. A *Barr dinatural transformation* (BDN) consists of a family of morphisms $t(A) : F(A,A) \to G(A,A)$, one for each object of $\mathbf{A}$, such that for each morphism $f : A \to A'$,

$$\begin{tikzcd}
& F(A,A) \arrow[r, "t(A)"] & G(A,A) \arrow[dr, "G(A,f)"] & \\
F_f \arrow[ur] \arrow[dr] & & & G(A,A') \\
& F(A',A') \arrow[r, "t(A')"'] & G(A',A') \arrow[ur, "G(f,A')"'] &
\end{tikzcd}$$

commutes, where F_f is given by the pullback

$$\begin{tikzcd}
& F(A,A) \arrow[dr, "F(A,f)"] & \\
F_f \arrow[ur] \arrow[dr] & & F(A,A') \\
& F(A',A') \arrow[ur, "F(f,A')"'] &
\end{tikzcd}$$

It is not hard to see that a Barr dinatural is dinatural and that a natural transformation restricts to a Barr dinatural. Furthermore, Barr dinaturals are closed under composition because the pullback property gives a fill-in τ

$$\begin{tikzcd}
& F(A,A) \arrow[rr, "t(A)"] & & G(A,A) \arrow[rr, "u(A)"] \arrow[dr] & & H(A,A) \arrow[dr] & \\
F_f \arrow[ur] \arrow[dr] \arrow[rr, dotted, "\tau"] & & G_f \arrow[ur] \arrow[dr] & & G(A,A') & & H(A,A') \\
& F(A',A') \arrow[rr, "t(A')"'] & & G(A',A') \arrow[rr, "u(A')"'] \arrow[ur] & & H(A',A') \arrow[ur] &
\end{tikzcd}$$

from which commutativity of the outside hexagon is easily seen.

If F has the property that all the squares

$$\begin{tikzcd}
& F(A,A) \arrow[dr, "F(A,f)"] & \\
F(A',A) \arrow[ur, "F(f,A)"] \arrow[dr, "F(A',f)"'] & & F(A,A') \\
& F(A',A') \arrow[ur, "F(f,A')"'] &
\end{tikzcd}$$

are pullbacks, then Barr dinaturals are the same as dinaturals, obviously. This holds, for example, if F is constant in one of the variables. It is also the case that in any cartesian closed category, evaluation $e_{A,B}: A^B \times B \to A$ is Barr dinatural in B; the reason is the same as above, i.e. the squares obtained by varying B in $A^B \times B$ are pullbacks (which comes from the more basic fact that a product of two pullback diagrams is a pullback, a special case of limits commuting with limits).

This argument breaks down in the monoidal closed case and the evaluation is not in general Barr dinatural (e.g. it is not for finite-dimensional vector spaces – take f to be the unique $k \to 0$).

Note: The condition that $\mathbf{B}$ have pullbacks can be removed by the usual trick of embedding $\mathbf{B}$ in $\mathbf{Set}^{\mathbf{B}^{\mathrm{op}}}$ via the Yoneda functor, and then reformulating the conditions solely in terms of morphisms of $\mathbf{B}$. Thus, t would be Barr dinatural if for every b, b' making $(*)$ commute in

$$\begin{tikzcd}
 & F(A,A) \arrow[rr, "t(A)"] \arrow[dr] & & G(A,A) \arrow[dr] & \\
B \arrow[ur, "b"] \arrow[dr, "b'"'] & (*) & F(A,A') & (**) & G(A,A') \\
 & F(A',A') \arrow[ur] \arrow[rr, "t(A')"'] & & G(A',A') \arrow[ur] &
\end{tikzcd}$$

the exterior hexagon also commutes. We might take this as a definition of what it would mean for a diagram such as $(**)$ to commute. However, as will become clear below, we will be concerned only with the case $\mathbf{B} = \mathbf{Set}$.

2.4. *Iterators*

Our main concern will be with Barr dinatural transformations

$$t: \mathrm{Hom}_{\mathbf{A}} \xrightarrow{-} \mathrm{Hom}_{\mathbf{A}},$$

where $\mathrm{Hom}_{\mathbf{A}}$ is the hom functor $\mathbf{A}^{\mathrm{op}} \times \mathbf{A} \to \mathbf{Set}$. (We have some fun with notation and replace Mac Lane's two dots with a bar when we wish to emphasise that we are in the presence of a Barr dinatural transformation.) Such a BDN associates to each endomorphism of $\mathbf{A}$, $f: A \to A$, another endomorphism $t(f): A \to A$ such that if $\varphi f = g\varphi$ then $\varphi t(f) = t(g)\varphi$, i.e.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\varphi"'] & A \arrow[d, "\varphi"] \\
B \arrow[r, "g"'] & B
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
A \arrow[r, "t(f)"] \arrow[d, "\varphi"'] & A \arrow[d, "\varphi"] \\
B \arrow[r, "t(g)"'] & B
\end{tikzcd}.$$

For example, $t(f) = f^{(n)}$ gives a BDN; if $\varphi f = g\varphi$ then $\varphi f^{(n)} = g^{(n)}\varphi$.

Proposition 1 (Barr). *Every BDN* $t : \mathrm{Hom}_{\mathbf{Set}} \to \mathrm{Hom}_{\mathbf{Set}}$ *is of the form* $t(f) = f^{(n)}$ *for a unique* $n \in \mathbf{N}$.

Proof. Let $n = t(s)(0)$ where $s : \mathbf{N} \to \mathbf{N}$ is the successor function. Then for any $f : A \to A$ and $a \in A$, there exists a unique h such that

$$\begin{tikzcd} & N \arrow[r, "s"] \arrow[d, "h"] & N \arrow[d, "h"] \\ 1 \arrow[ur, "0"] \arrow[dr, "a"'] & & \\ & A \arrow[r, "f"'] & A . \end{tikzcd}$$

This h is given by $h(k) = f^{(k)}(a)$. As t is a BDN we have

$$\begin{tikzcd} & N \arrow[r, "t(s)"] \arrow[d, "h"] & N \arrow[d, "h"] \\ 1 \arrow[ur, "0"] \arrow[dr, "a"'] & & \\ & A \arrow[r, "t(f)"'] & A \end{tikzcd}$$

so $t(f)(a) = ht(s)(0) = h(n) = f^{(n)}(a)$, i.e. $t(f) = f^{(n)}$. The uniqueness of n is obvious. □

Remark. As all of Freyd's examples extend to **Set**, they cannot be BDNs. However, our examples based on iteration as they are, are BDNs.

We phrased Barr's proof in terms of natural numbers objects because this way it is clear that this is a much more general result.

2.5. NNOs

Let **A** be a cartesian category, i.e. a category with finite products. Recall that in this setting Lawvere's definition of natural numbers object must be strengthened to include parameters. Thus, a *natural numbers object* in **A** is a diagram $1 \xrightarrow{0} N \xrightarrow{s} N$ such that for every diagram $A \xrightarrow{g} B \xrightarrow{f} B$ there exists a unique h such that

$$\begin{tikzcd} A \times 1 \arrow[r, "A \times 0"] \arrow[d, "\cong"'] & A \times N \arrow[r, "A \times s"] \arrow[d, "h"] & A \times N \arrow[d, "h"] \\ A \arrow[r, "g"'] & B \arrow[r, "f"'] & B. \end{tikzcd}$$

It is well-known that when $\mathbf{A}$ is cartesian closed, it is sufficient to state the definition with $A = 1$, as Lawvere did.

The definition can be reformulated in terms of adjoints. Let $E(\mathbf{A})$ be the category of endomorphisms in $\mathbf{A}$. Its objects are endomorphisms $f : A \to A$ and its morphisms $\varphi : (A, f) \to (B, g)$ are morphisms $\varphi : A \to B$ such that

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ A \arrow[r, "\varphi"'] & B. \end{tikzcd}$$

$E(\mathbf{A})$ is the functor category $\mathbf{A}^{\mathbf{N}}$ where $\mathbf{N}$ is the monoid of (ordinary) natural numbers. If $\mathbf{A}$ has a natural numbers object, then the forgetful functor $U : E(\mathbf{A}) \to \mathbf{A}, U(A, f) = A$ has a left adjoint F given by $F(A) = (A \times N, A \times s)$.

In the other direction, if U has a left adjoint F such that the canonical morphism $F(A \times B) \to (A, 1_A) \times F(B)$ is an isomorphism for all A and B, then $\mathbf{A}$ has a natural numbers object $(N, s) = F(1)$.

Now, assume that $\mathbf{A}$ has a natural numbers object. Given $n : 1 \to N$ we shall define a BDN, $(\ )^{(n)} : \mathrm{Hom}_{\mathbf{A}} \xrightarrow{\cdot} \mathrm{Hom}_{\mathbf{A}}$, as follows. For $f : A \to A$, there exists a unique h such that

$$\begin{tikzcd} A \times 1 \arrow[r, "A \times 0"] \arrow[d, "\cong"'] & A \times N \arrow[r, "A \times s"] \arrow[d, "h"] & A \times N \arrow[d, "h"] \\ A \arrow[r, "1_A"'] & A \arrow[r, "f"'] & A. \end{tikzcd}$$

Let $f^{(n)}$ be the composite

$$A \xrightarrow{\cong} A \times 1 \xrightarrow{A \times n} A \times N \xrightarrow{h} A\,.$$

Proposition 2. *The above definition gives a BDN,* $(\ )^{(n)} : \mathrm{Hom}_{\mathbf{A}} \xrightarrow{\cdot} \mathrm{Hom}_{\mathbf{A}}$.

Proof. Given

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\varphi"'] & A \arrow[d, "\varphi"] \\ B \arrow[r, "g"'] & B \end{tikzcd}$$

let k be the unique morphism satisfying

$$\begin{tikzcd}
B \times 1 \arrow[r, "B \times 0"] \arrow[d, "\cong"'] & B \times N \arrow[r, "B \times s"] \arrow[d, "k"] & B \times N \arrow[d, "k"] \\
B \arrow[r, "1_B"'] & B \arrow[r, "g"'] & B.
\end{tikzcd}$$

Compare the diagrams

$$\begin{tikzcd}
A \times 1 \arrow[r, "A \times 0"] \arrow[d, "\cong"'] & A \times N \arrow[r, "A \times s"] \arrow[d, "h"] & A \times N \arrow[d, "h"] \\
A \arrow[r, "1"] \arrow[d, "\varphi"'] & A \arrow[r, "f"] \arrow[d, "\varphi"] & A \arrow[d, "\varphi"] \\
B \arrow[r, "1"'] & B \arrow[r, "g"'] & B
\end{tikzcd}$$

and

$$\begin{tikzcd}
A \times 1 \arrow[r, "A \times 0"] \arrow[d, "\varphi \times 1"'] & A \times N \arrow[r, "A \times s"] \arrow[d, "\varphi \times N"] & A \times N \arrow[d, "\varphi \times N"] \\
B \times 1 \arrow[r, "B \times 0"] \arrow[d, "\cong"'] & B \times N \arrow[r, "B \times s"] \arrow[d, "k"] & B \times N \arrow[d, "k"] \\
B \arrow[r, "1"'] & B \arrow[r, "g"'] & B
\end{tikzcd}$$

to see that $\varphi \cdot h$ and $k \cdot \varphi \times N$ both satisfy the same recurrence relations and so are equal.

Thus, we have

$$\begin{tikzcd}
A \arrow[r, "\cong"] \arrow[d, "\varphi"'] & A \times 1 \arrow[r, "A \times n"] \arrow[d, "\varphi \times 1"'] & A \times N \arrow[r, "h"] \arrow[d, "\varphi \times N"] & A \arrow[d, "\varphi"] \\
B \arrow[r, "\cong"'] & B \times N \arrow[r, "B \times n"'] & B \times N \arrow[r, "k"'] & B
\end{tikzcd}$$

i.e.

$$\begin{tikzcd} A \arrow[r, "f^{(n)}"] \arrow[d, "\varphi"'] & A \arrow[d, "\varphi"] \\ B \arrow[r, "g^{(n)}"'] & B \end{tikzcd}$$

commutes, therefore $(\ \ )^{(n)}$ is a BDN as claimed. $\square$

As in the proof of Proposition 1, a BDN, $t:\mathrm{Hom}_{\mathbf{A}} \xrightarrow{\cdot} \mathrm{Hom}_{\mathbf{A}}$, gives $n:1 \to N$ as the composite $1 \xrightarrow{0} N \xrightarrow{t(s)} N$. If we start with an $n:1\to N$, we get a BDN, $(\ \ )^{(n)}$, and this gives the same n back. Indeed,

$$t(s) = (N \xrightarrow{\cong} N \times 1 \xrightarrow{N \times n} N \times N \xrightarrow{+} N)$$

so

$$t(s)(0) = (1 \cong 1 \times 1 \xrightarrow{0 \times n} N \times N \xrightarrow{+} N) = n.$$

But starting with a BDN, t, and letting $n = t(s)(0)$ we do not automatically get $t = (\ \ )^{(n)}$. A calculation shows that at some point we need $t(A \times s) = A \times t(s)$. This leads us to the following concept.

2.6. Strong BDNs

Definition. A BDN, $t:\mathrm{Hom}_{\mathbf{A}} \xrightarrow{\cdot} \mathrm{Hom}_{\mathbf{A}}$, is *strong* if for every A and $g:B\to B$ we have $t(A \times g) = A \times t(g)$.

Theorem 1. *Suppose* $\mathbf{A}$ *has a natural numbers object, then the relations* $n = t(s)(0)$ *and* $t = (\ \ )^{(n)}$ *establish a one-to-one correspondence between strong BDNs and elements of* N, $n:1\to N$.

Proof. We have already shown how $n:1\to N$ gives a BDN in Proposition 2. We must show that it is strong. If we apply $A \times (\ \ )$ to the defining diagram for k (same notation as in Proposition 2) we get

$$\begin{tikzcd} A \times B \times 1 \arrow[r, "A \times B \times 0"] \arrow[d, "\cong"'] & A \times B \times N \arrow[r, "A \times B \times s"] \arrow[d, "A \times k"] & A \times B \times N \arrow[d, "A \times k"] \\ A \times B \arrow[r, "1"'] & A \times B \arrow[r, "A \times g"'] & A \times B \end{tikzcd}$$

so

$$(A \times g)^{(n)} = (A \times B \xrightarrow{\cong} A \times B \times 1 \xrightarrow{A \times B \times n} A \times B \times N \xrightarrow{A \times k} A \times B)$$

$$= A \times g^{(n)}.$$

Now starting with a strong BDN, $t:\mathrm{Hom}_{\mathbf{A}} \overset{\cdot\cdot}{\longrightarrow} \mathrm{Hom}_{\mathbf{A}}$, we define $n=t(s)(0)$. Then $f^{(n)}$ is the composite

$$A \xrightarrow{\cong} A\times 1 \xrightarrow{A\times 0} A\times N \xrightarrow{A\times t(s)} A\times N \xrightarrow{h} A,$$

where h satisfies

$$\begin{tikzcd}
A\times 1 \arrow[r, "A\times 0"] \arrow[d, "\cong"'] & A\times N \arrow[r, "A\times s"] \arrow[d, "h"] & A\times N \arrow[d, "h"] \\
A \arrow[r, "1_A"'] & A \arrow[r, "f"'] & A.
\end{tikzcd}$$

By dinaturality we have

$$\begin{tikzcd}
A\times 1 \arrow[r, "A\times 0"] \arrow[d, "\cong"'] & A\times N \arrow[r, "t(A\times s)"] \arrow[d, "h"] & A\times N \arrow[d, "h"] \\
A \arrow[r, "1_A"'] & A \arrow[r, "t(f)"'] & A
\end{tikzcd}$$

and given that $t(A\times s)=A\times t(s)$, we see that $f^{(n)}=t(f)$. $\square$

2.7. *The strength of strength*

Note that the above proof only uses strength in the form $t(A\times s)=A\times t(s)$, so one might wonder whether strength could not be eliminated altogether. In fact, we can reduce it to some simple conditions but we cannot quite eliminate it.

For any endomorphisms f and g we have

$$\begin{tikzcd}
A \arrow[r, "f"] & A \\
A\times B \arrow[u, "p_1"] \arrow[r, "f\times g"] \arrow[d, "p_2"'] & A\times B \arrow[u, "p_1"'] \arrow[d, "p_2"] \\
B \arrow[r, "g"'] & B
\end{tikzcd}
\quad\Rightarrow\quad
\begin{tikzcd}
A \arrow[r, "t(f)"] & A \\
A\times B \arrow[u, "p_1"] \arrow[r, "t(f\times g)"] \arrow[d, "p_2"'] & A\times B \arrow[u, "p_1"'] \arrow[d, "p_2"] \\
B \arrow[r, "t(g)"'] & B
\end{tikzcd}$$

so $t(f\times g)=t(f)\times t(g)$. Thus, t is strong if for every A, $t(1_A)=1_A$, a reasonable condition if we expect to characterize iteration.

Any subobject, S, of 1 only has one endomorphism, 1_S, so $t(1_S) = 1_S$. If the subobjects of 1 form a generating set, then for any A and any morphism $\varphi : S \to A$ we have

$$\begin{tikzcd} S \arrow[r, "1_S"] \arrow[d, "\varphi"'] & S \arrow[d, "\varphi"] \\ A \arrow[r, "1_A"'] & A \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} S \arrow[r, "t(1_S)"] \arrow[d, "\varphi"'] & S \arrow[d, "\varphi"] \\ A \arrow[r, "t(1_A)"'] & A \end{tikzcd}$$

and since the set of all such φ is jointly epic, we must have $t(1_A) = 1_A$. Thus, in this case all BDNs, $t : \mathrm{Hom} \stackrel{..}{\longrightarrow} \mathrm{Hom}$, are strong. This is the case with finite sets, $\mathbf{Set}_0$, the category of sets or any category of sheaves on a topological space, as well as many other categories.

However, not all BDNs are strong, even on such a nice category as $\mathbf{Set}^{\mathbf{N}}$, i.e. sets with an endomorphism. Given $f : (X, \xi) \to (X, \xi)$, define $t(f) = \xi : (X, \xi) \to (X, \xi)$. Then

$$\begin{tikzcd} (X,\xi) \arrow[r, "f"] \arrow[d, "\varphi"'] & (X,\xi) \arrow[d, "\varphi"] \\ (Y,\theta) \arrow[r, "g"'] & (Y,\theta) \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} (X,\xi) \arrow[r, "\xi"] \arrow[d, "\varphi"'] & (X,\xi) \arrow[d, "\varphi"] \\ (Y,\theta) \arrow[r, "\theta"'] & (Y,\theta) \end{tikzcd}$$

so t is a BDN, but $t(1_{(x,\xi)}) = \xi$ which need not be the identity.

There is something mysterious about strong BDNs. It does not seem possible to define them between general functors $F, G : \mathbf{A}^{\mathrm{op}} \times \mathbf{A} \to \mathbf{B}$, even when $\mathbf{B} = \mathbf{Set}$. In order to get a better understanding, we shall reformulate the definition.

2.8. *The category of endomorphisms*

A BDN, $t : \mathrm{Hom}_{\mathbf{A}} \stackrel{..}{\longrightarrow} \mathrm{Hom}_{\mathbf{A}}$, is the same as a functor T over $\mathbf{A}$

$$\begin{tikzcd} E(\mathbf{A}) \arrow[rr, "T"] \arrow[dr, "U"'] & & E(\mathbf{A}) \arrow[dl, "U"] \\ & \mathbf{A} & \end{tikzcd}$$

where $E(\mathbf{A})$ is the category of endomorphisms in $\mathbf{A}$, $\mathbf{A}^{\mathbf{N}}$, introduced in Section 2.5.

Given a BDN, $t : \mathrm{Hom}_{\mathbf{A}} \stackrel{..}{\longrightarrow} \mathrm{Hom}_{\mathbf{A}}$, T is defined by $T(A, \alpha) = (A, t(\alpha))$. For morphisms, $T(\varphi) = \varphi$. The reader can easily check the claim.

Now, if **A** were cartesian closed and had equalizers, then $E(\mathbf{A})$ would be an **A**-enriched category. The **A**-valued hom would be defined to be the equalizer

$$[(A,\alpha),(B,\beta)] \xrightarrow{e} B^A \underset{\beta^A}{\overset{B^\alpha}{\rightrightarrows}} B^A .$$

$U : E(\mathbf{A}) \to \mathbf{A}$ is a strong functor, its strength being given by

$$e : [(A,\alpha),(B,\beta)] \to B^A .$$

As e is monic, we see that U is **A**-faithful.

Proposition 3. *t is a strong BDN if and only if T is a strong functor over* **A**.

Proof. For T to be strong over **A** we must have morphisms st_T such that

$$\begin{tikzcd}
{[(A,\alpha),(B,\beta)]} \arrow[rr, "\mathrm{st}_T"] \arrow[dr, "e"'] & & {[(A,t(\alpha)),(B,t(\beta))]} \arrow[dl, "e'"] \\
& B^A &
\end{tikzcd}$$

Since all the e' are monic, the st_T are unique if they exist, and automatically satisfy the coherence conditions for a strength. Such morphisms exist if and only if for every $\bar{\varphi} : C \to B^A$,

$$B^\alpha \cdot \bar{\varphi} = \beta^A \cdot \bar{\varphi} \;\Rightarrow\; B^{t(\alpha)} \cdot \bar{\varphi} = t(\beta)^A \cdot \bar{\varphi} ,$$

i.e. if and only if for every $\varphi : C \times A \to B$ we have

$$\begin{tikzcd}
C \times A \arrow[r, "C \times \alpha"] \arrow[d, "\varphi"'] & C \times A \arrow[d, "\varphi"] \\
B \arrow[r, "\beta"'] & B
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
C \times A \arrow[r, "C \times t(\alpha)"] \arrow[d, "\varphi"'] & C \times A \arrow[d, "\varphi"] \\
B \arrow[r, "t(\beta)"'] & B.
\end{tikzcd}$$

If t is a strong BDN then, this clearly holds. Conversely, if it holds, then taking $\varphi = 1_{C \times A}$, we see that t is strong. □

$E(\mathbf{A})$ is a tensored **A**-category in the sense that for every C in **A**, there is an object $C \otimes (A,\alpha) = (C \times A, C \times \alpha)$ with the property that morphisms $C \otimes (A,\alpha) \to (B,\beta)$ are in bijection with **A** morphisms

$$C \to [(A,\alpha),(B,\beta)],$$

i.e. the functor $[(A,\alpha),-] : E(\mathbf{A}) \to \mathbf{A}$ has a left adjoint $(\;\;) \otimes (A,\alpha)$. (See [8] for more on tensored categories.)

As each of the functors in an adjoint pair determines the other, it is possible to reformulate the notions of enriched category theory in terms of the tensor rather than as is usually done in terms of the hom. Since $E(\mathbf{A})$ is always tensored regardless of whether it has equalizers or is cartesian closed, and the tensor has already shown up in the proof of Proposition 3, we shall express things in these terms.

3. The monoidal setting

3.1. *Actions of monoidal categories*

It is best, both for conceptual clarity and applicability, to replace our base category $\mathbf{A}$ with a general monoidal category $\mathbf{V}$. By a *(left)* $\mathbf{V}$*-tensored category* we shall mean a category $\mathbf{X}$ with an action of $\mathbf{V}$, $\otimes : \mathbf{V} \times \mathbf{X} \to \mathbf{X}$. This action must be unitary and associative up to coherent isomorphism. We do not assume that $\mathbf{X}$ is a $\mathbf{V}$-category. For example, the tensor on $\mathbf{V}$ gives an action of $\mathbf{V}$ on itself even if $\mathbf{V}$ is not closed. So $\mathbf{V}$ is always a $\mathbf{V}$-tensored category. The category $E(\mathbf{V})$ of endomorphisms in $\mathbf{V}$ is also a $\mathbf{V}$-tensored category via $V \otimes (A, \alpha) = (V \otimes A, V \otimes \alpha)$. If $\mathbf{X}$ and $\mathbf{Y}$ are $\mathbf{V}$-tensored categories, a $\mathbf{V}$-functor $F : \mathbf{X} \to \mathbf{Y}$ is a functor which respects the actions in the following sense: for every X in $\mathbf{X}$ and V in $\mathbf{V}$ we are given a morphism

$$V \otimes FX \to F(V \otimes X).$$

These morphisms must respect the associativity and unity isomorphisms for the actions. For example, the forgetful functor $U : E(\mathbf{V}) \to \mathbf{V}$ is a $\mathbf{V}$-functor with identities

$$V \otimes U(A, \alpha) = V \otimes A = U(V \otimes (A, \alpha))$$

as structure morphisms. If $F, G : \mathbf{X} \to \mathbf{Y}$ are $\mathbf{V}$-functors, a $\mathbf{V}$-natural transformation $t : F \to G$ is a natural transformation such that for every X and V

$$\begin{tikzcd} V \otimes FX \arrow[r, "V \otimes t(X)"] \arrow[d] & V \otimes GX \arrow[d] \\ F(V \otimes X) \arrow[r, "t(V \otimes X)"'] & G(V \otimes X). \end{tikzcd}$$

$\mathbf{V}$-tensored categories, $\mathbf{V}$-functors and $\mathbf{V}$-natural transformations form a 2-category so we have a notion of $\mathbf{V}$-adjointness. If one works through the definition it can be seen that $\mathbf{V}$-functors F and U are $\mathbf{V}$-adjoint if there is a natural bijection

$$\frac{V \otimes FX \to Y}{V \otimes X \to UY}$$

or, what is equivalent, F is left adjoint to U and F preserves the tensor in the sense that the structural morphism

$$V \otimes FX \to F(V \otimes X)$$

is an isomorphism.

If $\mathbf{X}$ is a $\mathbf{V}$-tensored category and for every X in $\mathbf{X}$ the functor $(\ \ )\otimes X : \mathbf{V} \to \mathbf{X}$ has a right adjoint $[X,-] : \mathbf{X} \to \mathbf{V}$, then $\mathbf{X}$ becomes a $\mathbf{V}$-category with hom given by $[X,X'] \in \mathbf{V}$. Actually, in the non-symmetric case, there are two notions of $\mathbf{V}$-category. The one we get here, corresponding to a left action, has composition morphisms

$$[X',X''] \otimes [X,X'] \to [X,X''].$$

We can think that composition is performed in the classical order. In any case $\mathbf{V}$-functors, $\mathbf{V}$-natural transformations and $\mathbf{V}$-adjunctions are all the usual $\mathbf{V}$-category concepts.

For the basic theory of $\mathbf{V}$-categories, the reader is referred to [3, 8]. For more on the use of the tensor as basic notion, one may consult [5, 12, 13].

3.2. NNOs in monoidal categories

A *(right) natural numbers object* in $\mathbf{V}$ is a diagram $I \xrightarrow{0} N \xrightarrow{s} N$ such that for every diagram $A \xrightarrow{f} B \xrightarrow{g} B$ there exists a unique h such that

$$\begin{tikzcd} A \otimes I \arrow[r, "A\otimes 0"] \arrow[d, "\cong"'] & A \otimes N \arrow[r, "A\otimes s"] \arrow[d, "h"] & A \otimes N \arrow[d, "h"] \\ A \arrow[r, "f"'] & B \arrow[r, "g"'] & B. \end{tikzcd}$$

This is a straightforward generalization of the usual notion with 1 replaced by the unit I and $\times$ by $\otimes$. It appeared in [12] in the form of free actions but as far as we know, it was only taken seriously as a natural numbers object in [11], where it was shown that such an N is a commutative comonoid and that all the primitive recursive functions can be defined on it.

Proposition 4. $\mathbf{V}$ *has a natural numbers object if and only if the forgetful functor* $U : E(\mathbf{V}) \to \mathbf{V}$ *has a left* $\mathbf{V}$*-adjoint.*

Proof. Once we note that a left $\mathbf{V}$-adjoint is an ordinary adjoint F with the property that $F(V \otimes W) \cong V \otimes F(W)$, the proof is easy. If $\mathbf{V}$ has an NNO, then $F(V) = (V \otimes N, V \otimes s)$ is a left $\mathbf{V}$-adjoint. If F is a left $\mathbf{V}$-adjoint, then $F(I)$ is an NNO. $\square$

3.3. Strong BDNs revisited

Clearly, the notion of strong BDN makes sense, not only for cartesian $\mathbf{A}$, but also for monoidal $\mathbf{V}$. We consider again $\mathrm{Hom}_{\mathbf{V}} : \mathbf{V}^{\mathrm{op}} \times \mathbf{V} \to \mathbf{Set}$ (*not* into $\mathbf{V}$ even if $\mathbf{V}$ is closed). A BDN $t : \mathrm{Hom}_{\mathbf{V}} \xrightarrow{\cdot\cdot} \mathrm{Hom}_{\mathbf{V}}$ is *strong* if $t(V \otimes f) = V \otimes t(f)$ for all $f : X \to X$ in $\mathbf{V}$. In this generality, the proof of Proposition 3 is much easier.

Proposition 5. *$t : \mathrm{Hom}_{\mathbf{V}} \to \mathrm{Hom}_{\mathbf{V}}$ is a strong BDN if and only if the corresponding $T : E(\mathbf{V}) \to E(\mathbf{V})$ is a $\mathbf{V}$-functor over $\mathbf{V}$.*

Proof. T is a $\mathbf{V}$-functor if and only if there are morphisms $V \otimes T(A,\alpha) \to T(V \otimes (A,\alpha))$ satisfying some compatibilities. T is over $\mathbf{V}$ means $UT = U$ as $\mathbf{V}$-functors. This forces the above morphisms to be identities at the underlying level. Thus, T is a $\mathbf{V}$-functor over $\mathbf{V}$ if and only if $V \otimes T(A,\alpha) = T(V \otimes (A,\alpha))$, i.e. iff $(V \otimes A, V \otimes t(\alpha)) = (V \otimes A, t(V \otimes \alpha))$, i.e. iff $V \otimes t(\alpha) = t(V \otimes \alpha)$. □

One might ask whether strong BDNs, $t : \mathrm{Hom}_{\mathbf{V}} \to \mathrm{Hom}_{\mathbf{V}}$, correspond to natural numbers $n : I \to N$ in the monoidal setting as well. The answer is "yes" and the proof is exactly the same as for Theorem 1. For $f : A \to A$, let h be the unique morphism such that

$$\begin{tikzcd} & A \otimes N \arrow[r, "A \otimes s"] \arrow[d, "h"] & A \otimes N \arrow[d, "h"] \\ A \otimes I \arrow[ur, "A \otimes 0"] \arrow[r, "\rho"'] & A \arrow[r, "f"'] & A \end{tikzcd}$$

and define $f^{(n)}$ to be

$$A \xrightarrow{\rho^{-1}} A \otimes I \xrightarrow{A \otimes n} A \otimes N \xrightarrow{h} A.$$

Theorem 2. *$(\ )^{(n)}$ is a strong BDN $\mathrm{Hom}_{\mathbf{V}} \xrightarrow{\cdot\cdot} \mathrm{Hom}_{\mathbf{V}}$ and establishes a one-to-one correspondence between natural numbers $n : I \to N$ and strong BDNs. The inverse is given by $n = I \xrightarrow{0} N \xrightarrow{t(s)} N$.*

Whether or not $\mathbf{V}$ has a natural numbers object, we can think of a strong BDN, $\mathrm{Hom} \xrightarrow{\cdot\cdot} \mathrm{Hom}$, as iteration by some kind of natural number specially adapted to the category $\mathbf{V}$, much like Church numerals.

Definition. A strong BDN, $t : \mathrm{Hom}_{\mathbf{V}} \xrightarrow{\cdot\cdot} \mathrm{Hom}_{\mathbf{V}}$, will be called a *dinatural number* for $\mathbf{V}$.

Some examples will illustrate how things work in the monoidal case.

3.4. Examples

Example 1. The category of endomorphisms on sets, $E(\mathbf{Set})$, is a Grothendieck topos and as such has a natural numbers object $(\mathbf{N}, 1_{\mathbf{N}})$ with successor s. A natural number $1 \xrightarrow{n} N$ is just an ordinary natural number, and the corresponding strong BDN takes $f:(X,\xi)\to(X,\xi)$ to the nth iterate $f^{(n)}:(X,\xi)\to(X,\xi)$. As we saw in Section 2.7, not all BDNs are strong in $E(\mathbf{Set})$.

The category, $E(\mathbf{Set})$, also has a tensor by virtue of its being M-sets for the commutative monoid $M=(\mathbf{N},+)$. Thus, $(A,\alpha)\otimes(B,\beta)=(C,\gamma)$ where $C=A\times B/\!\sim$ and $\sim$ is the equivalence relation generated by $(\alpha(a),b)\sim(a,\beta(b))$. The effect of γ on an equivalence class $a\otimes b$ is $\gamma(a\otimes b)=\alpha(a)\otimes b$. The unit is $(\mathbf{N},s)$. The forgetful functor $U:E(E(\mathbf{Set}))\to E(\mathbf{Set})$ has a left adjoint $F(A,\alpha)=((A\times\mathbf{N},\alpha\times\mathbf{N}),A\times s)$ which is strong for both cartesian product and $\otimes$. Thus, $E(\mathbf{Set})$ has two NNOs, the cartesian one discussed above $F(1,1)=((\mathbf{N},1),s)$ and the monoidal one $F(\mathbf{N},s)=((\mathbf{N}\times\mathbf{N}, s\times\mathbf{N}),\mathbf{N}\times s)$.

With respect to $\otimes$ a natural number is $(\mathbf{N},s)\to(\mathbf{N}\times\mathbf{N},s\times\mathbf{N})$ which corresponds to a pair (m,n) of ordinary natural numbers. Given $f:(X,\xi)\to(X,\xi)$, we get a unique $h:(X,\xi)\otimes(\mathbf{N}\times\mathbf{N},s\times\mathbf{N})\to(X,\xi)$ making

$$\begin{tikzcd}
 & (X\times\mathbf{N},\ \xi\times\mathbf{N}) \arrow[r, "X\times s"] \arrow[d, "h"] & (X\times\mathbf{N},\ \xi\times\mathbf{N}) \arrow[d, "h"] \\
(X,\xi) \arrow[ur, "X\times 0"] \arrow[dr, "1_X"'] & & \\
 & (X,\xi) \arrow[r, "f"'] & (X,\xi)
\end{tikzcd}$$

commute. h is given by $h(x,p)=f^{(p)}(x)$. Then $f^{(m,n)}(x)=f^{(n)}(\xi^{(m)}(x))$, i.e. $f^{(m,n)}=f^{(n)}\xi^{(m)}$. Thus, the example in Section 2.7 of a BDN which was not strong *is* strong for the $\otimes$. It corresponds to the natural number $(1,0)$ in this case.

In fact, all BDNs are strong with respect to this $\otimes$. Note first of all that the strength condition always holds for the unit object $t(I\otimes f)=I\otimes t(f)$. This is because

$$\begin{tikzcd}
I\otimes B \arrow[r, "I\otimes f"] \arrow[d, "\lambda"'] & 1\otimes B \arrow[d, "\lambda"] \\
B \arrow[r, "f"'] & B
\end{tikzcd}
\quad\Rightarrow\quad
\begin{tikzcd}
I\otimes B \arrow[r, "t(I\otimes f)"] \arrow[d, "\lambda"'] & I\otimes B \arrow[d, "\lambda"] \\
B \arrow[r, "t(f)"'] & B.
\end{tikzcd}$$

Next, if the $\otimes$ has an associated internal hom, so that $(\ \ )\otimes B$ preserves jointly epic families and if the unit is a generator, then for any A there is an epimorphic family

$e_\alpha : I \to A$ and

$$\begin{tikzcd} I \otimes B \arrow[r, "I \otimes f"] \arrow[d, "e_\alpha \otimes B"'] & 1 \otimes B \arrow[d, "e_\alpha \otimes B"] \\ A \otimes B \arrow[r, "A \otimes f"'] & \otimes B \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} I \otimes B \arrow[r, "t(I \otimes f)"] \arrow[d, "e_\alpha \otimes B"'] & I \otimes B \arrow[d, "e_\alpha \otimes B"] \\ B \arrow[r, "t(A \otimes f)"'] & A \otimes B. \end{tikzcd}$$

Comparing this with

$$\begin{tikzcd} I \otimes B \arrow[r, "I \otimes t(f)"] \arrow[d, "e_\alpha \otimes B"'] & 1 \otimes B \arrow[d, "e_\alpha \otimes B"] \\ A \otimes B \arrow[r, "A \otimes t(f)"'] & A \otimes B \end{tikzcd}$$

and using joint epiness of $e_\alpha \otimes B$, we conclude $t(A \otimes f) = A \otimes t(f)$.

In the case of $E(\mathbf{Set})$, the unit for the tensor is $(\mathbf{N}, s)$, which is a generator. There is also an internal hom

$$[(X,\xi),(Y,\theta)] = (\{f : X \to Y \mid \theta f = f\xi\}, \phi),$$

where $\phi(f) = \theta f$. So all BDNs are strong for $\otimes$.

Example 2. In this example, the opposite occurs. Consider the Grothendieck topos $\mathbf{Set} \times \mathbf{Set}$. The subobjects of $(1,1)$ generate, so all BDNs are strong for the cartesian product. An element of N is $(1,1) \to (\mathbf{N},\mathbf{N})$, i.e. a pair of natural numbers (m,n). The associated BDN takes (f,g) to $(f^{(m)}, g^{(n)})$. Every BDN is of this form.

But there is a $\otimes$ in $\mathbf{Set} \times \mathbf{Set}$ other than cartesian product

$$(A,B) \otimes (C,D) = (A \times C,\ A \times D + B \times C)$$

with unit $(1,0)$. It is an easy exercise to check that this is a monoidal closed structure: the internal hom is given by $[(A,B),(C,D)] = (C^A \times D^B, D^A)$. There is also a natural numbers object with respect to the $\otimes$, namely $(N,0)$ with successor $(s, 1_0)$. For this $\otimes$, an element of N is $(1,0) \to (\mathbf{N},0)$, i.e. a single natural number n. The strong BDN which this yields is $(f,g)^{(n)} = (f^{(n)}, g^{(n)})$. So here not all BDNs are strong for $\otimes$.

Example 3. R-**mod** has no natural numbers object for the cartesian product $\oplus$. Actually, $0 \to 0 \to 0$ has the universal property that for every f there exists a unique h such that

$$\begin{tikzcd} & 0 \arrow[r] \arrow[dd, "h"] & 0 \arrow[dd, "h"] \\ 0 \arrow[ur] \arrow[dr] & & \\ & A \arrow[r, "f"'] & A \end{tikzcd}$$

but not for parameters. This dramatically illustrates the uselessness of the universal property without parameters.

There is, however, a natural numbers object for the $\otimes$. It is $R[x]$ with successor $(\ ) \cdot x$, multiplication by x. A morphism $I \to N$ in this case is a linear map $R \to R[x]$, i.e. a polynomial $P(x)$. The corresponding BDN associates to each $f : M \to M$ the linear map $P(f) : M \to M$. Again, R is a generator and there is an internal hom so all BDNs are strong with respect to the $\otimes$. Thus, every function t which takes linear operators to linear operators on the same space with the property that $\varphi f = g\varphi \Rightarrow \varphi t(f) = t(g)\varphi$ is of the form

$$t(f) = r_0 + r_1 f + r_2 f^{(2)} + \cdots + r_n f^{(n)}.$$

Note that we are considering BDNs on

$$\mathrm{Hom}_R : (R\text{-}\mathbf{mod})^{\mathrm{op}} \times R\text{-}\mathbf{mod} \to \mathbf{Set}.$$

If we were to consider BDNs on the enriched hom, into R-**mod**, they would all be of the form $t(f) = rf$, and not related to iteration in any way.

3.5. Strong profunctors

We still have not said how to define *strong* BDNs for arbitrary functors $\mathbf{V}^{\mathrm{op}} \times \mathbf{V} \to \mathbf{Set}$. In order to do this, we must introduce the notion of strong functor $\mathbf{V}^{\mathrm{op}} \times \mathbf{V} \to \mathbf{Set}$. This can be done for arbitrary $\mathbf{V}$-tensored categories $\mathbf{A}, \mathbf{B}$. A functor $F : \mathbf{A}^{\mathrm{op}} \times \mathbf{B} \to \mathbf{Set}$ is *strong* if it comes equipped with strength morphisms for all A, B, V.

$$\mathrm{st}_{A,B,V} : F(A,B) \to F(V \otimes A, V \otimes B)$$

satisfying the following conditions:

(1) $\mathrm{st}_{A,B,V}$ is natural in A and B and dinatural in V,

(2)

$$\begin{tikzcd}
F(A,B) \arrow[rr, "\mathrm{st}_{A,B,I}"] \arrow[dr, "F(I\otimes A,\lambda)"'] & & F(I\otimes A, I\otimes B) \arrow[dl, "F(I\otimes A,\lambda)"] \\
& F(I\otimes A, B) &
\end{tikzcd}$$

(3)

$$\begin{tikzcd}
F(A,B) \arrow[r, "\mathrm{st}_{A,B,W\otimes V}"] \arrow[d, "\mathrm{st}_{A,B,V}"'] & F((W\otimes V)\otimes A, (W\otimes V)\otimes B) \arrow[d, "\cong"] \\
F(V\otimes A, V\otimes B) \arrow[r, "\mathrm{st}_{V\otimes A, V\otimes B, W}"'] & F(W\otimes(V\otimes A), W\otimes(V\otimes B)).
\end{tikzcd}$$

For $x \in F(A,B)$, we denote $\mathrm{st}_{A,B,V}(x)$ by $V \otimes x$.

$\mathrm{Hom}_{\mathbf{V}} : \mathbf{V}^{\mathrm{op}} \times \mathbf{V} \to \mathbf{Set}$, as well as all the functors $\mathrm{Hom}^{L}_{\mathbf{V}}$ introduced in Section 5.1 below, are strong functors. A functor $F : \mathbf{A}^{\mathrm{op}} \times \mathbf{B} \to \mathbf{Set}$ corresponds, via its category of elements to a discrete bifibration

$$\begin{tikzcd}
& El(F) \arrow[dl, "P"'] \arrow[dr, "Q"] & \\
\mathbf{A} & & \mathbf{B}.
\end{tikzcd}$$

If F is strong, then $El(F)$ is a $\mathbf{V}$-tensored category and the functors P and Q preserve the action. This statement is in fact equivalent to F being strong. So our notation in $V \otimes x$ is not a bad one.

If $F, G : \mathbf{A}^{\mathrm{op}} \times \mathbf{A} \to \mathbf{Set}$ are two strong functors, a *strong* BDN, $t : F \to G$ is a BDN such that for every $x \in F(A,A)$

$$V \otimes t(x) = t(V \otimes x).$$

A profunctor $F : \mathbf{B} \nrightarrow \mathbf{A}$ is by definition a functor $F : \mathbf{A}^{\mathrm{op}} \times \mathbf{B} \to \mathbf{Set}$. So we have a definition of strong profunctor between categories with $\mathbf{V}$-action. The identity profunctor $I : \mathbf{A} \nrightarrow \mathbf{A}$ is the hom functor, $\mathrm{Hom}_{\mathbf{A}} : \mathbf{A}^{\mathrm{op}} \times \mathbf{A} \to \mathbf{Set}$, which is strong, the strength being given by

$$\mathrm{st}_{A,B,V} : \mathbf{A}(A,B) \to \mathbf{A}(V \otimes A, V \otimes B),$$

$$A \xrightarrow{f} B \longmapsto V \otimes A \xrightarrow{V \otimes f} V \otimes B.$$

Given profunctors $F : \mathbf{B} \dashrightarrow \mathbf{A}$ and $G : \mathbf{C} \dashrightarrow \mathbf{B}$, their composite $F \otimes G : \mathbf{C} \dashrightarrow \mathbf{A}$ is given by the formula

$$F \otimes G(A,C) = \int^{B} F(A,B) \times G(B,C).$$

If F and G are strong, so is $F \otimes G$. The strength is given by the universal property of the coend

$$\begin{tikzcd}
F \otimes G(A,C) \arrow[r, "\mathrm{st}"] \arrow[d, equal] & F \otimes G(V \otimes A, V \otimes C) \arrow[d, equal] \\
\int^B F(A,B) \times G(B,C) \arrow[r] & \int^B F(V \otimes A, B) \times G(B, V \otimes C) \\
F(A,B) \times G(B,C) \arrow[r, "\mathrm{st} \times \mathrm{st}"'] \arrow[u, "j_B"] & F(V \otimes A, V \otimes B) \times G(V \otimes B, V \otimes C). \arrow[u, "j_{V \otimes B}"']
\end{tikzcd}$$

A functor $U : \mathbf{B} \to \mathbf{A}$ induces two profunctors, $U_* : \mathbf{B} \dashrightarrow \mathbf{A}$ and $U^* : \mathbf{A} \dashrightarrow \mathbf{B}$ such that U^* is right adjoint to U_* in the bicategory **Prof**. They are given by $U_*(A,B) = \mathbf{A}(A, UB)$ and $U^*(B,A) = \mathbf{A}(UB, A)$. The usual arguments with the Yoneda lemma show that a strength for the profunctor U_* is the same as a strength for $U : \mathbf{B} \to \mathbf{A}$ as defined in Section 3.1, i.e. a family of morphisms

$$s_{V,B} : V \otimes U(B) \;\to\; U(V \otimes B)$$

natural in V and B and respecting associativity and unity isomorphisms.

On the other hand, a strength for U^* corresponds to a family of morphisms

$$\tau_{V,B} : U(V \otimes B) \;\to\; V \otimes U(B)$$

with properties similar to the $s_{V,B}$ above. If the adjointness $U_* \dashv U^*$ is also strong, then the $s_{V,B}$ and $t_{V,B}$ are inverse to each other, and U preserves the $\mathbf{V}$-action.

4. The arithmetic of dinatural numbers

All our definitions below are motivated by thinking of a dinatural number as iteration of endomorphisms. Thus, e.g., the law $f^{(m+n)} = f^{(m)} \circ f^{(n)}$ inspired the definition of addition in Section 4.3 below.

4.1. Zero

Define $t(A) : \mathrm{Hom}(A,A) \to \mathrm{Hom}(A,A)$ by the formula $t(A)(f) = 1_A$ for all f.

Proposition 6. *t is a dinatural number.*

Proof. Obvious. □

We denote this dinatural number by $\underline{0}$. We also omit the A when this does not lead to confusion. Thus $\underline{0}(f) = 1_A$.

4.2. Successor

Let $t : \mathrm{Hom} \xrightarrow{\cdot\cdot} \mathrm{Hom}$ be a dinatural number. Define $\sigma(t)(A) : \mathrm{Hom}(A,A) \to \mathrm{Hom}(A,A)$ by the formula $\sigma(t)(A)(f) = f \circ t(A)(f)$.

Proposition 7. *$\sigma(t)$ is a dinatural number.*

Proof. If we have

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\phi"'] & A \arrow[d, "\phi"] \\
B \arrow[r, "g"'] & B
\end{tikzcd}$$

then

$$\begin{tikzcd}
A \arrow[r, "t(A)(f)"] \arrow[d, "\phi"'] & A \arrow[r, "f"] \arrow[d, "\phi"] & A \arrow[d, "\phi"] \\
B \arrow[r, "t(B)(g)"'] & B \arrow[r, "g"'] & B
\end{tikzcd}$$

also commutes, so $\sigma(t)$ is a BDN. Strength follows from commutativity of

$$\begin{tikzcd}
& C\otimes A \arrow[dr, "C\otimes f"] \arrow[dd, equal] & \\
C\otimes A \arrow[ur, "t(C\otimes f)"] \arrow[dd, equal] & & C\otimes A \arrow[dd, equal] \\
& C\otimes A \arrow[dr, "C\otimes f"] & \\
C\otimes A \arrow[ur, "C\otimes t(f)"] \arrow[rr, "C\otimes(f\circ t(f))"'] & & C\otimes A .
\end{tikzcd}$$ □

We call $\sigma(t)$ the *successor* of t. With successor we can define the *standard numerals*. If n is an ordinary natural number, we let $\underline{n} = \sigma(\sigma(\sigma(\cdots(\underline{0})\cdots)))$, where σ is applied n times. Thus, $\underline{n}(f) = f \circ f \circ \cdots \circ f$, n times. In particular, $\underline{1}(f) = f$ for all f.

4.3. Addition

Given t and $u:\mathrm{Hom} \xrightarrow{\cdot\cdot} \mathrm{Hom}$ dinatural numbers, define $t+u$ by $(t+u)(f) = t(f)u(f)$.

Proposition 8. *$t+u$ is a dinatural number.*

Proof. If

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\phi"'] & A \arrow[d, "\phi"] \\ B \arrow[r, "g"'] & B \end{tikzcd}$$

commutes, then so does

$$\begin{tikzcd} A \arrow[r, "u(f)"] \arrow[d, "\phi"'] & A \arrow[r, "t(f)"] \arrow[d, "\phi"] & A \arrow[d, "\phi"] \\ B \arrow[r, "u(g)"'] & B \arrow[r, "t(g)"'] & B. \end{tikzcd}$$

Strength follows from functoriality of $\otimes$: $(t+u)(C \otimes f) = t(C \otimes f)u(C \otimes f) = (C \otimes t(f))(C \otimes u(f)) = C \otimes (t(f)u(f)) = C \otimes ((t+u)(f))$. $\square$

Proposition 9. *Addition of dinatural numbers satisfies the following identities:*

(i) $(t+u)+v = t+(u+v)$,

(ii) $t+u = u+t$,

(iii) $t+\underline{0} = t = \underline{0}+t$,

(iv) $\sigma(t)+u = \sigma(t+u) = t+\sigma(u)$.

Proof. (i) follows from associativity of composition. For (ii), consider the following commutative square:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "f"'] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & A. \end{tikzcd}$$

Dinaturality of t implies that

$$\begin{tikzcd} A \arrow[r, "t(f)"] \arrow[d, "f"'] & A \arrow[d, "f"] \\ A \arrow[r, "t(f)"'] & A \end{tikzcd}$$

commutes, and dinaturality of u implies that

$$\begin{tikzcd} A \arrow[r, "t(f)"] \arrow[d, "u(f)"'] & A \arrow[d, "u(f)"] \\ A \arrow[r, "t(f)"'] & A \end{tikzcd}$$

commutes, too. Thus, $t(f)u(f)=u(f)t(f)$, i.e. $(t+u)(f)=(u+t)(f)$ for all f. Thus $t+u=u+t$. For (iii), $(t+\underline{0})(f)=t(f)\underline{0}(f)=t(f)1_A=t(f)$. Finally, (iv) follows from $(\sigma(t)+u)(f)=\sigma(t)(f)u(f)=ft(f)u(f)=f(t+u)(f)=\sigma(t+u)(f)$. $\square$

It follows from (iii) and (iv) that for standard numerals, addition agrees with the usual; $\underline{m}+\underline{n}=\underline{m+n}$.

4.4. Multiplication

We can also define a multiplication for dinatural numbers although it is not quite satisfactory in the monoidal case. It is given by composition of BDNs.

Let $t,u:\mathrm{Hom}\xrightarrow{\cdot\cdot}\mathrm{Hom}$ be dinatural numbers. Define the *product* of t and u by $(t\cdot u)(f)=t(u(f))$.

Proposition 10. *$t\cdot u$ is a dinatural number.*

Proof. Assume that

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\phi"'] & A \arrow[d, "\phi"] \\ B \arrow[r, "g"'] & B \end{tikzcd}$$

commutes. Then so does

$$\begin{array}{ccc} A & \xrightarrow{u(f)} & A \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi} \\ B & \xrightarrow[u(g)]{} & B \end{array}$$

and, consequently, so does

$$\begin{array}{ccc} A & \xrightarrow{t(u(f))} & A \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi} \\ B & \xrightarrow[t(u(g))]{} & B. \end{array}$$

So $t \cdot u$ is a BDN. Strength is easy: $(t \cdot u)(C \otimes f) = t(u(C \otimes f)) = t(C \otimes u(f)) = C \otimes t(u(f)) = C \otimes (t \cdot u)(f)$. □

Proposition 11. *The product of dinatural numbers satisfies the following identities*:
(i) $(t \cdot u) \cdot v = t \cdot (u \cdot v)$,
(ii) $\underline{1} \cdot t = t = t \cdot \underline{1}$,
(iii) $\underline{0} \cdot t = \underline{0}$,
(iv) $\sigma t \cdot u = t \cdot u + u$,
(v) $(t + u) \cdot v = t \cdot v + u \cdot v$.

Proof. All are immediate consequences of the definitions, but see Section 6.3. □

Note that in this generality, product is not commutative nor do the other versions of (iii)–(v) hold.

Example 3 of Section 3.4 discussed the case of R-**mod** with its usual $\otimes$ (R, a commutative ring). It was seen that all BDNs were strong and that they were in one-to-one correspondence with polynomials $P(x) \in R[x]$. An endomorphism $f : M \to M$ is sent to $P(f) : M \to M$, the linear transformation obtained by substituting f into the polynomial. Directly from the definitions above we can construct Table 1 of correspondences, where t corresponds to $P(x)$ and u to $Q(x)$.

Taking $P(x) = x + 1$ and $Q(x) = x^2$ we see immediately that product is not commutative nor do the other halves of (iii)–(v) hold. We also see that it does not follow from $fg = gf$ that $t(fg) = t(f)t(g)$.

In the cartesian case the other half of (iii) does hold. Indeed, $t \cdot \underline{0}(f) = t(\underline{0}(f)) = t(1_A) = 1_A = \underline{0}(f)$, by strength of t. However, the other properties do not. To see this,

Table 1

BDNs	Polynomials
$\underline{0}$	1(constant)
$\underline{1}$	x
$\sigma(t)$	$xP(x)$
$t+u$	$P(x)Q(x)$
$t \cdot u$	$P(Q(x))$

note that if a polynomial $P(x)$ has the property that $P(1)=1$, then the BDN it defines is strong for $\oplus$ on R-**mod**, so if we replace P above by $P(x)=2x-1$ and keep the same Q, we get counter-examples to the other halves of (iv) and (v) as well as commutativity of multiplication.

Still, (iii) and (iv) are enough to show that multiplication of standard numerals is the usual one, $\underline{m} \cdot \underline{n} = \underline{mn}$.

5. Families of dinatural numbers

5.1. Families of dinatural numbers

Our experience with topos theory has taught us that it is not sufficient to consider elements as morphisms $1 \to X$, but that we should also consider generalized elements $L \to X$ and these are, of course, enough to determine X.

Let **V** be an arbitrary monoidal category with natural numbers object $(N, 0, s)$, and let L be an arbitrary object of **V**. A morphism $n : L \to N$ can be viewed as a family of natural numbers indexed by L, $n = \langle n_\lambda \rangle_{\lambda \in L}$. Given $f : A \to A$ there is a unique h such that

$$\begin{tikzcd}
 & A\otimes N \arrow[rr, "A\otimes s"] \arrow[dd, "h"] & & A\otimes N \arrow[dd, "h"] \\
A\otimes I \arrow[ur, "A\otimes 0"] \arrow[dr, "\rho"'] & & & \\
 & A \arrow[rr, "f"'] & & A .
\end{tikzcd}$$

We now define

$$f^{(n)} = (A \otimes L \xrightarrow{A\otimes n} A \otimes N \xrightarrow{h} A).$$

We may think of $f^{(n)}$ as an L-family of iterates, $\langle f^{(n_\lambda)} \rangle_{\lambda \in L}$.

Let us define a functor $\mathrm{Hom}^L_{\mathbf{V}} : \mathbf{V}^{\mathrm{op}} \times \mathbf{V} \to \mathbf{Set}$ by $\mathrm{Hom}^L_{\mathbf{V}}(A,B) = \mathrm{Hom}_{\mathbf{V}}(A \otimes L, B)$, the set of L-families of morphisms from A to B. A BDN, $t : \mathrm{Hom}_{\mathbf{V}} \overset{\cdot}{\longrightarrow} \mathrm{Hom}^L_{\mathbf{V}}$, is said to be *strong* if for every A and every $g : B \to B$,

$$\begin{tikzcd}
(A \otimes B) \otimes L \arrow[r, "t(A \otimes g)"] \arrow[d, "\alpha"'] & A \otimes B \arrow[d, "1"] \\
A \otimes (B \otimes L) \arrow[r, "uA \otimes t(g)"'] & A \otimes B
\end{tikzcd}$$

commutes. Such a strong BDN is called an *L-family of dinatural numbers.*

Theorem 3. *For any $n : L \to N$, $(\ )^{(n)}$ is an L-family of dinatural numbers, and every L-family of dinatural numbers, $t : \mathrm{Hom}_{\mathbf{V}} \overset{\cdot}{\longrightarrow} \mathrm{Hom}^L_{\mathbf{V}}$, is of this form for a unique $n : L \to N$.*

Proof. Let

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\phi"'] & A \arrow[d, "\phi"] \\
B \arrow[r, "g"'] & B.
\end{tikzcd}$$

Compare the diagrams

$$\begin{tikzcd}
 & A \otimes N \arrow[r, "A \otimes s"] \arrow[dd, "\phi \otimes N"] & A \otimes N \arrow[dd, "\phi \otimes N"] \\
A \otimes I \arrow[ur, "A \otimes 0"] \arrow[dd, "\phi \otimes I"'] & & \\
 & B \otimes N \arrow[r, "B \otimes s"] \arrow[dd, "k"] & B \otimes N \arrow[dd, "k"] \\
B \otimes I \arrow[ur, "B \otimes 0"] \arrow[dr, "\rho"'] & & \\
 & B \arrow[r, "g"'] & B
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & A \otimes N \arrow[r, "A \otimes s"] \arrow[dd, "h"] & A \otimes N \arrow[dd, "h"] \\
A \otimes I \arrow[ur, "A \otimes 0"] \arrow[dr, "\rho"'] \arrow[dd, "\phi \otimes I"'] & & \\
 & A \arrow[r, "f"] \arrow[dd, "\phi"] & A \arrow[dd, "\phi"] \\
B \otimes I \arrow[dr, "\rho"'] & & \\
 & B \arrow[r, "g"'] & B
\end{tikzcd}$$

to conclude that the right square in

$$\begin{tikzcd}
A \otimes L \arrow[r, "A\otimes n"] \arrow[d, "\phi\otimes L"'] & A \otimes N \arrow[r, "h"] \arrow[d, "\phi\otimes N"] & A \arrow[d, "\phi"] \\
B \otimes L \arrow[r, "B\otimes n"'] & B \otimes N \arrow[r, "k"'] & B
\end{tikzcd}$$

commutes. (The left one obviously commutes by functoriality.) Thus $(\ )^{(n)}$ is a BDN.

To see that it is strong, tensor the defining diagram for k by A

$$\begin{tikzcd}
 & A \otimes B \otimes N \arrow[r, "A \otimes B \otimes s"] \arrow[dd, "A \otimes k"] & A \otimes B \otimes N \arrow[dd, "A \otimes k"] \\
A \otimes B \otimes I \arrow[ur, "A \otimes B \otimes 0"] \arrow[dr, "\rho"'] & & \\
 & A \otimes B \arrow[r, "A \otimes g"'] & A \otimes B
\end{tikzcd} .$$

So

$$\begin{aligned} t(A \otimes g) &= (A \otimes B \otimes L \xrightarrow{A\otimes B\otimes n} A \otimes B \otimes N \xrightarrow{A\otimes k} A \otimes B) \\ &= A \otimes t(g). \end{aligned}$$

Now, given a BDN, $t : \mathrm{Hom}_{\mathbf{V}} \xrightarrow{\cdot} \mathrm{Hom}_{\mathbf{V}}^{L}$, define

$$n = (L \xrightarrow{\lambda^{-1}} I \otimes L \xrightarrow{0\otimes L} N \otimes L \xrightarrow{t(s)} N).$$

If $t=(\ \ )^{(m)}$ for $m:L\to N$, then the corresponding n is obtained as follows. First note that

$$\begin{tikzcd}
& N\otimes N \arrow[r, "N\otimes s"] \arrow[dd, "+"] & N\otimes N \arrow[dd, "+"] \\
N\otimes I \arrow[ur, "N\otimes 0"] \arrow[dr, "\rho"'] & & \\
& N \arrow[r, "s"'] & N
\end{tikzcd}\ .$$

So $t(s)=(N\otimes L \xrightarrow{N\otimes m} N\otimes N \xrightarrow{+} N)$. Then

$$\begin{aligned}
n &= (L \xrightarrow{\lambda^{-1}} I\otimes L \xrightarrow{0\otimes L} N\otimes L \xrightarrow{N\otimes m} N\otimes N \xrightarrow{+} N) \\
&= (L \xrightarrow{\lambda^{-1}} I\otimes L \xrightarrow{I\otimes m} I\otimes N \xrightarrow{0\otimes N} N\otimes N \xrightarrow{+} N) \\
&= (L \xrightarrow{\lambda^{-1}} I\otimes L \xrightarrow{I\otimes m} I\otimes N \xrightarrow{\lambda} N) \\
&= m.
\end{aligned}$$

Now, starting with a strong BDN, $t:\mathrm{Hom}_{\mathbf{V}}\to\mathrm{Hom}_{\mathbf{V}}^{L}$, we construct $n=t(s)\cdot 0\otimes L\cdot\lambda^{-1}$. For any $f:A\to A$ we have h such that

$$\begin{tikzcd}
& A\otimes N \arrow[r, "A\otimes s"] \arrow[dd, "h"] & A\otimes N \arrow[dd, "h"] \\
A\otimes I \arrow[ur, "A\otimes 0"] \arrow[dr, "\rho"'] & & \\
& A \arrow[r, "f"'] & A
\end{tikzcd}$$

so we also have

$$\begin{tikzcd}
& A\otimes N\otimes L \arrow[r, "t(A\otimes s)"] \arrow[dd, "h\otimes L"] & A\otimes N \arrow[dd, "h"] \\
A\otimes I\otimes L \arrow[ur, "A\otimes 0\otimes L"] \arrow[dr, "\rho\otimes L=A\otimes\lambda"'] & & \\
& A\otimes L \arrow[r, "t(f)"'] & A
\end{tikzcd}\ .$$

Thus,

$$\begin{aligned}
t(f) &= h\cdot t(A\otimes s)\cdot A\otimes 0\otimes N\cdot A\otimes\lambda^{-1} \\
&= h\cdot A\otimes(t(s)\cdot 0\otimes N\cdot\lambda^{-1}) \\
&= h\cdot A\otimes n \\
&= f^{(n)}. \qquad \square
\end{aligned}$$

5.2. The functor $\mathcal{N}$

Even if $\mathbf{V}$ does not have a natural numbers object we can still consider the set $\mathcal{N}(L)$ of all L-families of dinatural numbers, $t : \mathrm{Hom}_{\mathbf{V}} \to \mathrm{Hom}_{\mathbf{V}}^{L}$.

Remark. Actually, $\mathcal{N}(L)$ could be a proper class, although this is not really relevant to our discussion, as we could equally well consider $\mathcal{N} : \mathbf{V}^{\mathrm{op}} \to \mathbf{Set}$ the category of sets in the next universe.

However, let $\mathbf{V}$ be the category whose objects are those sequences of sets $\langle A_\kappa \rangle$ indexed by the ordinals, for which there exists an ordinal κ_0 such that $A_\kappa = 1$ for all $\kappa \geq \kappa_0$. A morphism $\langle A_\kappa \rangle \to \langle B_\kappa \rangle$ is a sequence of functions $\langle f_\kappa : A_\kappa \to B_\kappa \rangle$. $\mathbf{V}$ is a legitimate cartesian category (in fact cartesian closed) generated by its subobjects of 1. Each ordinal sequence of natural numbers $\langle n_\lambda \rangle$ gives a strong BDN, $t : \mathrm{Hom}_{\mathbf{V}} \stackrel{\cdot}{\longrightarrow} \mathrm{Hom}_{\mathbf{V}}$, by $t(\langle f_\lambda \rangle) = \langle f_\lambda^{n_\lambda} \rangle$, and these are all distinct. Thus $\mathcal{N}(1)$ is certainly not a set here.

Should we wish to impose conditions on $\mathbf{V}$ to insure that each $\mathcal{N}(L)$ is a (small) set, then accessibility (see [10]) would be enough and would cover all the examples we have in mind. Indeed, if $\mathbf{V}$ is accessible, then the category of endomorphisms, $E(\mathbf{V})$, is also accessible as is the forgetful functor $U : E(\mathbf{V}) \to \mathbf{V}$. This means that there is a *set* of endomorphisms $\mathscr{E}_\kappa$ such that every endomorphism, $f : A \to A$, is a κ-filtered colimit in $E(\mathbf{V})$ of $f_i : A_i \to A_i$ in $\mathscr{E}_\kappa$. Furthermore, the forgetful functor U preserves this colimit, i.e. $\varinjlim A_i \cong A$. Then, as

$$\begin{tikzcd} A_i \arrow[r, "f_i"] \arrow[d, "j_i"'] & A_i \arrow[d, "j_i"] \\ A \arrow[r, "f"'] & A \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} A_i \otimes L \arrow[r, "t(f_i)"] \arrow[d, "j_i \otimes L"'] & A_i \arrow[d, "j_i"] \\ A \otimes L \arrow[r, "t(f)"'] & A \end{tikzcd}$$

we see that $t(f) = \varinjlim t(f_i)$ so that t is determined by its values on f_i, of which there is but a set. So there can only be a set of BDNs, $\mathrm{Hom}_{\mathbf{V}} \to \mathrm{Hom}_{\mathbf{V}}^{L}$. (Note that $(\ ) \otimes L$ preserving κ-filtered colimits is part of the definition of accessible monoidal category.)

Proposition 12. *$\mathcal{N}(L)$ is the object part of a functor $\mathbf{V}^{\mathrm{op}} \to \mathbf{Set}$, which takes any colimit in $\mathbf{V}$ which is preserved by all the functors $V \otimes (\ \ )$ to a limit in $\mathbf{Set}$.*

Proof. A morphism $l : L' \to L$ gives a strong natural transformation

$$l^* : \mathrm{Hom}_{\mathbf{V}}^{L} \to \mathrm{Hom}_{\mathbf{V}}^{L'}$$

defined by $l^*(A \otimes L \xrightarrow{f} B) = (A \otimes L' \xrightarrow{A \otimes l} A \otimes L \xrightarrow{f} B)$. A family of dinatural numbers composed with a strong natural transformation such as this, gives again a family of dinatural numbers. Thus $\mathcal{N}(l)(t) = l^* \circ t$.

Let $j_\alpha : L_\alpha \to L$ be a colimit cocone. Given a family of dinatural numbers, $t : \mathrm{Hom}_{\mathbf{V}} \to \mathrm{Hom}_{\mathbf{V}}^{L}$, we get a compatible family $\langle t_\alpha \rangle$ of dinatural numbers in the canonical way. For $f : A \to A$ we have

$$\begin{tikzcd} A \otimes L \arrow[dr, "t(f)"] & \\ & A \\ A \otimes L_\alpha \arrow[uu, "A \otimes j_\alpha"] \arrow[ur, "t_\alpha(f)"'] & \end{tikzcd}$$

For a compatible family $\langle t_\alpha \rangle$ we define $t(f)$ to be the unique morphism making the above diagram commute, which exists because $\langle A \otimes L_\alpha \to A \otimes L \rangle_\alpha$ is a colimit cocone. That the t is a BDN follows from the fact that the $A \otimes j_\alpha$ are jointly epic and the t_α are BDNs. Strength of t follows for the same reason. □

Remark. The condition that a colimit is preserved by the functors $V \otimes (\ )$ is a perfectly natural one in our situation. It simply says that the colimit is strong.

It follows from this proposition that $\mathcal{N}$ has every chance of being representable. Theorem 3 says that if $\mathbf{V}$ has a natural numbers object, then $\mathcal{N} \cong \mathbf{V}(-, N)$. The following theorem is the converse.

Theorem 4. *If $\mathcal{N}$ is representable, the representing object is a natural numbers object.*

Proof. Suppose $\mathcal{N} \cong \mathbf{V}(-, N)$. Then there exists a universal element $h \in \mathcal{N}(N)$ with the property that for every strong BDN, $t : \mathrm{Hom}_{\mathbf{V}} \dot{\to} \mathrm{Hom}_{\mathbf{V}}^{L}$, there exists a unique $n : L \to N$ such that

$$\begin{tikzcd} & \mathrm{Hom}_{\mathbf{V}} \arrow[dl, "h"'] \arrow[dr, "t"] & \\ \mathrm{Hom}_{\mathbf{V}}^{N} \arrow[rr, "n^*"'] & & \mathrm{Hom}_{\mathbf{V}}^{L}. \end{tikzcd}$$

Thus, for every $f : A \to A$,

$$\begin{tikzcd} A \otimes L \arrow[rr, "A \otimes n"] \arrow[dr, "t(f)"'] & & A \otimes N \arrow[dl, "h(f)"] \\ & A & \end{tikzcd}$$

Define $\sigma : \mathcal{N} \to \mathcal{N}$ by

$$\sigma(t)(f) = (A \otimes L \xrightarrow{t(f)} A \xrightarrow{f} A).$$

$\sigma(t)$ is a strong BDN, $\mathrm{Hom}_{\mathbf{V}} \to \mathrm{Hom}_{\mathbf{V}}^{L}$, and σ is natural in L. Now $\sigma(h) : \mathrm{Hom}_{\mathbf{V}} \to \mathrm{Hom}_{\mathbf{V}}^{L}$, so there exists a unique $s : N \to N$ such that

$$\begin{tikzcd}
 & \mathrm{Hom}_{\mathbf{V}} \arrow[dl, "h"'] \arrow[dr, "\sigma(h)"] & \\
\mathrm{Hom}_{\mathbf{V}}^{N} \arrow[rr, "s^*"'] & & \mathrm{Hom}_{\mathbf{V}}^{N}.
\end{tikzcd}$$

So for every $f : A \to A$, we have

$$\begin{tikzcd}
A \otimes N \arrow[r, "A \otimes s"] \arrow[d, "h(f)"'] & A \otimes N \arrow[d, "h(f)"] \\
A \arrow[r, "f"'] & A.
\end{tikzcd} \tag{1}$$

Also define $\imath : \mathrm{Hom}_{\mathbf{V}} \to \mathrm{Hom}_{\mathbf{V}}^{I}$ by

$$\imath(f) = (\rho : A \otimes I \to A).$$

Again, there exists a unique morphism $0 : I \to N$ such that

$$\begin{tikzcd}
A \otimes I \arrow[rr, "A \otimes 0"] \arrow[dr, "\rho"'] & & A \otimes N \arrow[dl, "h(f)"] \\
 & A & .
\end{tikzcd} \tag{2}$$

Given $B \xrightarrow{g} A \xrightarrow{f} A$ we have

$$\begin{tikzcd}
B \otimes I \arrow[rr, "B \otimes 0"] \arrow[dr, "g \otimes I"] \arrow[dd, "\rho"'] & & B \otimes N \arrow[r, "B \otimes s"] \arrow[d, "g \otimes N"] & B \otimes N \arrow[d, "g \otimes N"] \\
 & A \otimes I \arrow[r, "A \otimes 0"] \arrow[dr, "\rho"] & A \otimes N \arrow[r, "A \otimes s"] \arrow[d, "h(f)"] & A \otimes N \arrow[d, "h(f)"] \\
B \arrow[rr, "g"'] & & A \arrow[r, "f"'] & A
\end{tikzcd}$$

so $h(f)(g \otimes N)$ gives a fill-in for our natural numbers candidate. We must show uniqueness. Let k be such that

$$\begin{tikzcd} B \otimes I \arrow[r, "B \otimes 0"] \arrow[d, "\rho"'] & B \otimes N \arrow[r, "B \otimes s"] \arrow[d, "k"] & B \otimes N \arrow[d, "k"] \\ B \arrow[r, "g"'] & A \arrow[r, "f"'] & A. \end{tikzcd}$$

Apply h to the right square to get

$$\begin{tikzcd} B \otimes I \otimes N \arrow[r, "B \otimes 0 \otimes N"] \arrow[d, "\rho \otimes N"'] & B \otimes N \otimes N \arrow[r, "B \otimes h(s)"] \arrow[d, "k \otimes N"] & B \otimes N \arrow[d, "k"] \\ B \otimes N \arrow[r, "g \otimes N"'] & A \otimes N \arrow[r, "h(f)"'] & A. \end{tikzcd}$$

Let

$$x = \left(N \xrightarrow{\lambda^*} I \otimes N \xrightarrow{0 \otimes N} N \otimes N \xrightarrow{h(s)} N\right),$$

and compute

$$x^* \circ h : \mathrm{Hom}_{\mathbf{V}} \to \mathrm{Hom}_{\mathbf{V}}^N$$

at an arbitrary $c : C \to C$.

$$\begin{tikzcd} C \otimes N \arrow[r, "C \otimes \lambda^{-1}"] \arrow[drr, "1"'] & C \otimes I \otimes N \arrow[r, "C \otimes 0 \otimes N"] \arrow[dr, "\rho \otimes N"] & C \otimes N \otimes N \arrow[r, "C \otimes h(s)"] \arrow[d, "h(c) \otimes N"] & C \otimes N \arrow[d, "h(c)"] \\ & & C \otimes N \arrow[r, "h(c)"'] & C \end{tikzcd}.$$

The square commutes because it is h applied to (1) with c replacing f. The triangle in the middle is an instance of (2) tensored by C, and the triangle on the left is one of the coherence conditions for monoidal categories. Thus, $x^* \circ h = h$ so $x = 1_N$ by the uniqueness property of h. It follows from (3) that $k = h(f)(g \otimes N)$, using once again that $\rho \otimes N = B \otimes \lambda^{-1}$. Thus, N is a natural numbers object. $\square$

This theorem together with its converse, Theorem 3, gives a universal property for a natural numbers object as a right representor. The (generalized) elements of N are families of dinatural numbers.

5.3. Arithmetic on families of dinatural numbers

There is a certain amount of arithmetic that can be done on the elements of $\mathcal{N}$ in this generality. The operations we introduce are the natural extension of those of Sections 4.1–4.3 to families of BDNs.

First there is the *successor* already introduced in the previous proposition. For $t \in \mathcal{N}(L)$ and $f: A \to A$, let $\sigma(t) = f \circ t(f)$

$$\sigma(t) : (A \xrightarrow{f} A) \mapsto (A \otimes L \xrightarrow{t(f)} A \xrightarrow{f} A).$$

There is also a *zero element* $\underline{0}$ which should satisfy $\underline{0}(f) = 1_A$. In fact we must define $\underline{0}(f) = (\rho : A \otimes I \to A)$. This is a dinatural number so $\underline{0} \in \mathcal{N}(I)$.

Successor and $\underline{0}$ allow us to define all *numerals* $\underline{n} = \sigma(\sigma(\cdots \sigma(\underline{0}) \cdots))$, n times. Then $\underline{n}(f) = f \circ f \circ \cdots \circ f \circ \rho : A \otimes I \to A$. Except in degenerate cases these are all distinct. For example, if $\otimes$ is cartesian product and our category is not a poset, then they are distinct. Suppose $g, h : B \to A$ are distinct morphisms, then if we let $f = \langle p_2, p_3, \ldots, p_n, p_1 \rangle : A^n \to A^n$, $\underline{n}(f) = 1_{A^n}$ but $\underline{k}(f) \neq 1_{A^n}$ for $0 < k < n$. To see this consider $\langle g, h, h, \ldots, h \rangle : B \to A^n$. Then $\underline{k}(f) \circ \langle g, h, \ldots, h \rangle \circ p_1 = h \neq g = \langle g, h, \ldots, h \rangle \circ p_1$. It is not clear what happens in the monoidal case.

Addition was defined in Section 4.3 by the formula $(t+u)(f) = t(f)u(f)$, but we must also define it with parameters. Thus, let $t \in \mathcal{N}(L)$ and $u \in \mathcal{N}(L')$. Given $f: A \to A$,

$$(t+u)(f) = (A \otimes L' \otimes L \xrightarrow{u(f) \otimes L} A \otimes L \xrightarrow{t(f)} A).$$

$t + u$ is easily seen to be a family of dinatural numbers, $\mathrm{Hom}_{\mathbf{V}} \to \mathrm{Hom}_{\mathbf{V}}^{L' \otimes L}$.

It is easily verified that

$$t + \sigma(u) = \sigma(t) + u = \sigma(t+u).$$

Now

$$(\underline{0} + t)(f) = A \otimes L \otimes I \xrightarrow{t(f) \otimes I} A \otimes I \xrightarrow{\rho} A$$

and

$$(t + \underline{0})(f) = A \otimes I \otimes L \xrightarrow{\rho \otimes L} A \otimes L \xrightarrow{t(f)} A,$$

which are essentially equal to $t(f)$, if we make the identifications $A \otimes L \otimes I = A \otimes L = A \otimes I \otimes L$. From these laws we conclude that, on standard numerals, $+$ agrees with the usual one.

In Section 4.3 we gave a proof that addition of dinatural numbers is commutative. But again, we must show commutativity for families of dinatural numbers, not just simple ones. With parameters, the argument would go as follows.

Let $t \in \mathcal{N}(L)$ and $u \in \mathcal{N}(L')$. Then for any $f : A \to A$, we apply t to the diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "f"'] & A \arrow[d, "f"] \\
A \arrow[r, "f"'] & A
\end{tikzcd}$$

to get

$$\begin{tikzcd}
A \otimes L \arrow[r, "t(f)"] \arrow[d, "f \otimes L"'] & A \arrow[d, "f"] \\
A \arrow[r, "t(f)"'] & A
\end{tikzcd}$$

and then apply u

$$\begin{tikzcd}
A \otimes L \otimes L' \arrow[r, "t(f) \otimes L'"] \arrow[d, "u(f \otimes L)"'] & A \otimes L' \arrow[d, "u(f)"] \\
A \otimes L \arrow[r, "t(f)"'] & A.
\end{tikzcd} \tag{$*$}$$

If $\otimes$ is symmetric, with symmetry isomorphisms γ, then

$$\begin{tikzcd}
A \otimes L' \otimes L \arrow[dr, "u(f) \otimes L"] \arrow[dd, "A \otimes \gamma"'] & \\
& A \otimes L \\
A \otimes L \otimes L' \arrow[ur, "u(f \otimes L)"'] &
\end{tikzcd} .$$

Indeed,

$$\begin{tikzcd}
A \otimes L' \otimes L \arrow[rr, "u(f) \otimes L"] \arrow[dr, "\gamma"] \arrow[dd, "A \otimes \gamma"'] & & A \otimes L \arrow[dr, "\gamma"] & \\
& L \otimes A \otimes L' \arrow[rr, "L \otimes u(f)", "= u(L \otimes f)"'] \arrow[dl, "\gamma \otimes L'"'] & & L \otimes A \arrow[dl, "\gamma"] \\
A \otimes L \otimes L' \arrow[rr, "u(f \otimes L)"'] & & A \otimes L &
\end{tikzcd} .$$

By the same token

$$\begin{tikzcd}
A \otimes L' \otimes L \arrow[d, "A\otimes\gamma"'] \arrow[dr, "t(f\otimes L')"] & \\
& A \otimes L' \\
A \otimes L \otimes L' \arrow[ur, "t(f)\otimes L'"'] &
\end{tikzcd}$$

Thus, if we precede $(*)$ by $A \otimes \gamma$, we get a similar diagram in which u and t are interchanged. It is in this sense that addition is commutative.

So we need to assume that $\otimes$ is symmetric and we only get commutativity up to the symmetry isomorphism. This is not a problem once things are set up properly, which we now do.

5.4. *The convolution tensor on* $\mathbf{Set}^{\mathbf{V}^{op}}$

From [5] we know that $\mathbf{Set}^{\mathbf{V}^{op}}$ has a convolution $\otimes$. For $\mathscr{F}, \mathscr{G} \in \mathbf{Set}^{\mathbf{V}^{op}}$

$$\begin{aligned}
\mathscr{F} \otimes \mathscr{G}(V) &= \varinjlim_{V \to V_1 \otimes V_2} \mathscr{F}(V_1) \times \mathscr{G}(V_2) \\
&\cong \int^{V_1, V_2} \mathscr{F}(V_1) \times \mathbf{V}(V, V_1 \otimes V_2) \times \mathscr{G}(V_2) \\
&\cong \varinjlim_{x \in \mathscr{F}(V_1)} \; \varinjlim_{y \in \mathscr{G}(V_2)} \mathbf{V}(V, V_1 \otimes V_2).
\end{aligned}$$

This indeed gives a tensor with unit $\mathscr{I} = \mathbf{V}(-, I)$. $\mathbf{Set}^{\mathbf{V}^{op}}$ is biclosed with this tensor, i.e. the functors $\mathscr{F} \otimes (\ \ )$ and $(\ \ ) \otimes \mathscr{G}$ have right adjoints. The Yoneda embedding preserves the $\otimes$; $\mathbf{V}(-, V_1) \otimes \mathbf{V}(-, V_2) \cong \mathbf{V}(-, V_1 \otimes V_2)$. If the $\otimes$ on $\mathbf{V}$ is symmetric, so is this new extended one. If the tensor on $\mathbf{V}$ is the cartesian product so is the convolution product. Indeed, in this case,

$$\begin{aligned}
\mathscr{F} \otimes \mathscr{G}(V) &= \int^{V_1, V_2} \mathscr{F}(V_1) \times \mathbf{V}(V, V_1 \times V_2) \times \mathscr{G}(V_2) \\
&\cong \int^{V_1, V_2} \mathscr{F}(V_1) \times \mathbf{V}(V, V_1) \times \mathbf{V}(V, V_2) \times \mathscr{G}(V_2) \\
&\cong \left(\int^{V_1} \mathscr{F}(V_1) \times \mathbf{V}(V, V_1) \right) \times \left(\int^{V_2} \mathbf{V}(V, V_2) \times \mathscr{G}(V_2) \right) \\
&\cong \mathscr{F}(V) \times \mathscr{G}(V).
\end{aligned}$$

5.5. Operations on $\mathscr{N}$

Since $\mathscr{F} \otimes \mathscr{G}$ is defined as a coend, morphisms $\mathscr{F} \otimes \mathscr{G} \to \mathscr{H}$ are easily described; they are families of functions

$$\phi_{V_1,V,V_2} : \mathscr{F}(V_1) \times \mathbf{V}(V, V_1 \otimes V_2) \times \mathscr{G}(V_2) \to \mathscr{H}(V)$$

natural in V and dinatural in V_1 and V_2.

Using the Yoneda lemma we see that they are the same as families

$$\psi_{V_1,V_2} : \mathscr{F}(V_1) \times \mathscr{G}(V_2) \to \mathscr{H}(V_1 \otimes V_2)$$

natural in V_1 and V_2. This is precisely the sort of thing our addition was. Thus we summarize.

Theorem 5. *The set $\mathscr{N}(L)$ of L-families of dinatural numbers*

$$\mathrm{Hom}_{\mathbf{V}} \xrightarrow{\,\cdot\cdot\,} \mathrm{Hom}_{\mathbf{V}}^{L}$$

defines an object $\mathscr{N}$ of $\mathbf{Set}^{\mathbf{V}^{\mathrm{op}}}$. In $\mathbf{Set}^{\mathbf{V}^{\mathrm{op}}}$ we have morphisms $\underline{0} : \mathscr{I} \to \mathscr{N}$, $\sigma : \mathscr{N} \to \mathscr{N}$ and $+ : \mathscr{N} \otimes \mathscr{N} \to \mathscr{N}$.

(1) *$+$ satisfies the following conditions with respect to $\underline{0}$ and σ:*

$$\begin{tikzcd}
 & \mathscr{N} \otimes \mathscr{N} \arrow[r, "\sigma \otimes \mathscr{N}"] \arrow[d, "+"] & \mathscr{N} \otimes \mathscr{N} \arrow[d, "+"] \\
\mathscr{I} \otimes \mathscr{N} \arrow[ur, "\underline{0} \otimes \mathscr{N}"] \arrow[r, "\lambda"'] & \mathscr{N} \arrow[r, "\sigma"'] & \mathscr{N}
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & \mathscr{N} \otimes \mathscr{N} \arrow[r, "\mathscr{N} \otimes \sigma"] \arrow[d, "+"] & \mathscr{N} \otimes \mathscr{N} \arrow[d, "+"] \\
\mathscr{N} \otimes \mathscr{I} \arrow[ur, "\mathscr{N} \otimes \underline{0}"] \arrow[r, "\rho"'] & \mathscr{N} \arrow[r, "\sigma"'] & \mathscr{N}
\end{tikzcd} .$$

(2) *$\mathscr{N}$ with $\underline{0}$ and $+$ is a monoid.*

(3) *If $\otimes$ is symmetric, then $\mathscr{N}$ is a commutative monoid.*

Proof. Everything stated here is a reformulation of what was proved in Section 5.3, except for associativity of addition. Given t, u, v

$$((t+u)+v)(f) = A \otimes L'' \otimes L' \otimes L \xrightarrow{v(f)\otimes(L'\otimes L)} A \otimes L' \otimes L \xrightarrow{(t+u)(f)} A$$
$$= A \otimes L'' \otimes L' \otimes L \xrightarrow{v(f)\otimes(L'\otimes L)} A \otimes L' \otimes L \xrightarrow{u(f)\otimes L} A \otimes L \xrightarrow{t(f)} A$$

and

$$(t+(u+v))(f) = A \otimes L'' \otimes L' \otimes L \xrightarrow{(u+v)(f)\otimes L} A \otimes L \xrightarrow{t(f)} A$$
$$= A \otimes L'' \otimes L' \otimes L \xrightarrow{v(f)\otimes L'\otimes L} A \otimes L' \otimes L \xrightarrow{u(f)\otimes L} A \otimes L \xrightarrow{t(f)} A .$$

So $(t+u)+v = t+(u+v)$. $\square$

Remark. One might think that $\mathscr{N}$ is a natural numbers object in $\mathbf{Set}^{\mathbf{V}^{\mathrm{op}}}$, but it is not. In fact, as $\mathbf{Set}^{\mathbf{V}^{\mathrm{op}}}$ has countable coproducts (indeed, all colimits) preserved by $\mathscr{F} \otimes (\ )$, the countable coproduct $\sum_{\aleph_0} \mathscr{I}$ is the NNO, and this is not usually the same as $\mathscr{N}$ (in particular, if $\mathbf{V}$ has a NNO then $\sum_{\aleph_0} \mathscr{I} \ncong \mathbf{V}(-, N)$).

5.6. Strong profunctors revisited

Because $\mathbf{V}$ is embedded in $\mathbf{Set}^{\mathbf{V}^{\mathrm{op}}}$ via the Yoneda functor, we get an action of $\mathbf{V}$ on $\mathbf{Set}^{\mathbf{V}^{\mathrm{op}}}$; $V \otimes \mathscr{F} = \mathbf{V}(-, V) \otimes \mathscr{F}$. Thus,

$$(V \otimes \mathscr{F})(A) = \varinjlim_{v : V_1 \to V} \ \varinjlim_{x \in \mathscr{F}(V_2)} \mathbf{V}(A, V_1 \otimes V_2)$$
$$\cong \varinjlim_{x \in \mathscr{F}(V_2)} \mathbf{V}(A, V \otimes V_2).$$

In fact, we see that this formula works for $\mathbf{Set}^{\mathbf{A}^{\mathrm{op}}}$ where $\mathbf{A}$ is a category equipped with a $\mathbf{V}$-action. $(V \otimes \mathscr{F})(A) = \varinjlim_{x \in \mathscr{F}(A')} \mathbf{A}(A, V \otimes A')$.

Now let $\mathbf{B}$ be another category with $\mathbf{V}$ action and $F : \mathbf{A}^{\mathrm{op}} \times \mathbf{B} \to \mathbf{Set}$, a profunctor $\mathbf{B} \nrightarrow \mathbf{A}$. F corresponds to a functor $\mathbf{F} : \mathbf{B} \to \mathbf{Set}^{\mathbf{A}^{\mathrm{op}}}$. Then to say that $\mathbf{F}$ is a $\mathbf{V}$-functor means that it is equipped with strength morphisms

$$s_{V,B} : V \otimes \mathbf{F}(B) \to \mathbf{F}(V \otimes B)$$

satisfying the obvious compatibilities. This means that for every A' we are given

$$s_{V,B,A'} : \varinjlim_{x \in F(A,B)} \mathbf{A}(A', V \otimes A) \to F(A', V \otimes B),$$

i.e. a compatible family

$$\langle \sigma_x : \mathbf{A}(A', V \otimes A) \to F(A', V \otimes B) \rangle_{x \in F(A,B)}.$$

Each of the σ_x is natural in A, so by Yoneda they correspond to elements of $F(V\otimes A, V\otimes B)$, one for each $x\in F(A,B)$. Thus, the strength corresponds to $\mathrm{st}_{A,B,V}: F(A,B)\to F(V\otimes A, V\otimes B)$ and it is easily checked that the compatibilities for the strength s translate to the ones given in Section 3.5 for a strong profunctor. So what seemed like an ad hoc notion, invented specifically to make sense of strong BDNs, is now seen to be perfectly natural.

5.7. *The problem with products*

We have already seen that commutativity of addition required that the tensor be symmetric. In order to get a proper theory of multiplication of dinaturals we will have to further specialize our tensor to be the cartesian product. The problem is this. We would like multiplication to be a morphism

$$\cdot : \mathcal{N}\otimes\mathcal{N} \to \mathcal{N}.$$

This would require, not only multiplication of dinatural transformations but of families of such. Thus, given $t\in\mathcal{N}(L)$ and $u\in\mathcal{N}(L')$, we must produce $t\cdot u\in\mathcal{N}(L'\otimes L)$. Given an endomorphism $f:A\to A$, we get $t(f): A\otimes L\to A$ but we do not know how to apply u to this. Even for standard numerals we do not. If $g: A\otimes L\to A$, and we wish to compose it with itself, the best we can do is

$$g^{(2)} = (A\otimes L\otimes L \xrightarrow{g\otimes L} A\otimes L \xrightarrow{g} A).$$

Similarly, $g^{(3)}: A\otimes L\otimes L\otimes L\to A$, etc. Thus, it would appear that $u(g)$ should be a morphism $A\otimes u(L)\to A$, but we have no idea what the u-fold tensor, $u(L)$, of L might even be.

However, in the cartesian case it does work. There, we can use the diagonal $\delta: L\to L\times L$ to reduce an $L\times L$-family to an L-family and so keep things under control. Thus, if $g: A\times L\to A$, then

$$g^{(2)} = (A\times L \xrightarrow{A\times\delta} A\times L\times L \xrightarrow{g\times L} A\times L \xrightarrow{g} A).$$

In terms of subscripts, given a family $\langle g_\lambda : A\to A\rangle_{\lambda\in L}$, if we wish to compose it with itself we have two options, either

$$\langle g_\lambda\circ g_\mu : A\to A\rangle_{(\lambda,\mu)\in L\times L}$$

or

$$\langle g_\lambda\circ g_\lambda : A\to A\rangle_{\lambda\in L}.$$

The first is more general and is the only choice in the monoidal case, whereas the second, which uses the diagonal to duplicate λ, is only available in the cartesian situation but is better behaved.

Thus, from now on, we shall work solely with the cartesian product.

6. Back to the cartesian case

6.1. BDNs applied to families

Assume that **A** is a cartesian category. It will be useful for us to upgrade our result of Section 2.7 characterizing strong BDNs as those preserving identities to BDNs, $\mathrm{Hom}^K \xrightarrow{\cdot} \mathrm{Hom}^L$.

Proposition 13. *A BDN, $t : \mathrm{Hom}^K \xrightarrow{\cdot} \mathrm{Hom}^L$, is strong if and only if for every C, if $p_1 : C \times K \to C$ is the first projection, then $t(p_1) = p_1 : C \times L \to C$.*

Proof. Assume t is strong. For the unique morphism $\tau : K \to 1$, $t(\tau) = \tau : L \to 1$. Thus, $t(p_1) = t(C \times \tau) = C \times t(\tau) = C \times \tau = p_1$.

Conversely, assume that t preserves projections. Then for any $f : A \times K \to A$ and C we have commutative diagrams

$$\begin{tikzcd}
C \times K \arrow[r, "p_1"] & C \\
C \times A \times K \arrow[u, "p_{13}"] \arrow[r, "C \times f"] \arrow[d, "p_{23}"'] & C \times A \arrow[u, "p_1"'] \arrow[d, "p_2"] \\
A \times K \arrow[r, "f"'] & A
\end{tikzcd}$$

so, as t is a BDN, we also have

$$\begin{tikzcd}
C \times L \arrow[r, "t(p_1)"] & C \\
C \times A \times L \arrow[u, "p_{13}"] \arrow[r, "t(C \times f)"] \arrow[d, "p_{23}"'] & C \times A \arrow[u, "p_1"'] \arrow[d, "p_2"] \\
A \times L \arrow[r, "t(f)"'] & A
\end{tikzcd}$$

and as $t(p_1) = p_1$, it follows that $t(C \times f) = C \times t(f)$, i.e. t is strong. $\square$

Remark. For BDNs of the form $t : \mathrm{Hom} \to \mathrm{Hom}^L$, the preservation of projections takes the form $t(1_C) = p_1 : C \times L \to C$.

In the cartesian case, not only can BDNs be applied to endomorphisms but also to families of endomorphisms $f : A \times K \to A$.

Proposition 14. *Any strong BDN, $t : \mathrm{Hom} \to \mathrm{Hom}^L$ induces, for each K, a strong BDN, $t^K : \mathrm{Hom}^K \to \mathrm{Hom}^{K\times L}$. For each $k : J \to K$, the following square of BDNs commutes*:

$$\begin{tikzcd} \mathrm{Hom}^K \arrow[r, "t^K"] \arrow[d, "k^*"'] & \mathrm{Hom}^{K\times L} \arrow[d, "(k\times L)^*"] \\ \mathrm{Hom}^J \arrow[r, "t^J"'] & \mathrm{Hom}^{J\times L}. \end{tikzcd}$$

Proof. Given $f : A \times K \to A$, we construct the endomorphism $\langle f, p_2\rangle : A \times K \to A \times K$ to which t can be applied. Then t^K is defined by

$$t^K(f) = (A \times K \times L \xrightarrow{t(\langle f,p_2\rangle)} A \times K \xrightarrow{p_1} A).$$

To show that t^K is a BDN, assume

$$\begin{tikzcd} A\times K \arrow[r, "f"] \arrow[d, "\phi\times K"'] & A \arrow[d, "\phi"] \\ B\times K \arrow[r, "g"'] & B \end{tikzcd}$$

commutes. Then so does

$$\begin{tikzcd} A\times K \arrow[r, "\langle f,p_2\rangle"] \arrow[d, "\phi\times K"'] & A\times K \arrow[d, "\phi\times K"] \\ B\times K \arrow[r, "\langle g,p_2\rangle"'] & B\times K \end{tikzcd}$$

and, because t is a BDN, so does

$$\begin{tikzcd} A\times K\times L \arrow[r, "t(\langle f,p_2\rangle)"] \arrow[d, "\phi\times K\times L"'] & A\times K \arrow[r, "p_1"] \arrow[d, "\phi\times K"] & A \arrow[d, "\phi"] \\ B\times K\times L \arrow[r, "t(\langle g,p_2\rangle)"'] & B\times K \arrow[r, "p_1"'] & B. \end{tikzcd}$$

Thus t^K is a BDN.

To show that t^K is strong, let $p_1 : C \times K \to C$ be the projection. Then

$$\begin{aligned} t^K(p_1) &= (C \times K \times L \xrightarrow{t(\langle p_1, p_2 \rangle)} C \times K \xrightarrow{p_1} C) \\ &= (C \times K \times L \xrightarrow{t(1_{C\times K})} C \times K \xrightarrow{p_1} C) \\ &= (C \times K \times L \xrightarrow{p_{12}} C \times K \xrightarrow{p_1} C) \\ &= p_1 : C \times K \times L \to C. \end{aligned}$$

Finally, for $k : J \to K$, the BDN, $k^* : \mathrm{Hom}^K \to \mathrm{Hom}^J$ was defined in Section 5.2 by

$$k^*(f) = (A \times J \xrightarrow{A\times k} A \times K \xrightarrow{f} A).$$

Now

$$\begin{tikzcd} A \times J \arrow[r, "\langle f\cdot A\times k, p_2\rangle"] \arrow[d, "A\times k"'] & A \times J \arrow[d, "A\times k"] \\ A \times K \arrow[r, "\langle f, p_2\rangle"'] & A \times K \end{tikzcd}$$

commutes and therefore so does

$$\begin{tikzcd} A \times J \times L \arrow[r, "t(\langle f\cdot A\times k, p_2\rangle)"] \arrow[d, "A\times k\times L"'] & A \times J \arrow[r, "p_1"] \arrow[d, "A\times k"] & A \arrow[d, "1_A"] \\ A \times K \times L \arrow[r, "t(\langle f, p_2\rangle)"'] & A \times K \arrow[r, "p_1"'] & A. \end{tikzcd}$$

The top is $t^J(k^*(f))$ and the other composite is $(k \times L)^* t^K(f)$. □

Remark. This result does not hold in the monoidal case, thus reinforcing our argument of Section 5.7. For example, let **Vect** be the monoidal category of k-vector spaces. **Vect** has a natural numbers object $k[x]$. So a strong BDN, $t : \mathrm{Hom} \to \mathrm{Hom}$, is evaluation at a polynomial $P \in k[x]$. Consider $P(x) = x^2$ and t the corresponding BDN, $\mathrm{Hom} \to \mathrm{Hom}$. Can t be extended to a BDN, $t^L : \mathrm{Hom}^L \to \mathrm{Hom}^L$ in such a way that

$$\begin{tikzcd} \mathrm{Hom}^L \arrow[r, "t^L"] \arrow[d, "l^*"'] & \mathrm{Hom}^L \arrow[d, "l^*"] \\ \mathrm{Hom} \arrow[r, "t"'] & \mathrm{Hom} \end{tikzcd}$$

commutes for all l? This means that if $f: V \otimes L \to V$, then $t^L(f): V \otimes L \to V$ has the property that $t^L(f)(v \otimes l) = f(f(v \otimes l) \otimes l)$ for all $v \in V$, $l \in L$. But the right side of this equation is not even linear in l:

$$f(f(v \otimes \alpha l) \otimes \alpha l) = \alpha^2 f(f(v \otimes l) \otimes l).$$

Now that we have established Proposition 14, we can define multiplication of families of dinatural numbers. Given $t: \mathrm{hom} \xrightarrow{\cdot\cdot} \mathrm{hom}^L$ and $u: \mathrm{Hom} \xrightarrow{\cdot\cdot} \mathrm{Hom}^K$, we define $t \cdot u: \mathrm{Hom} \xrightarrow{\cdot\cdot} \mathrm{Hom}^{K \times L}$ by the formula $(t \cdot u)(f) = t^K(u(f))$, i.e. $t \cdot u$ is the composite of BDNs

$$\mathrm{Hom} \xrightarrow{u} \mathrm{Hom}^K \xrightarrow{t^K} \mathrm{Hom}^{K \times L}.$$

By Proposition 14, $t \cdot u$ is a family of dinatural numbers if t and u are. It is the natural extension of the definition of Section 4.4 to families. Thus, we get a function

$$\cdot : \mathcal{N}(L) \times \mathcal{N}(K) \to \mathcal{N}(K \times L).$$

Proposition 15. *The function* $\cdot : \mathcal{N}(L) \times \mathcal{N}(K) \to \mathcal{N}(K \times L)$ *is natural in* K *and* L.

Proof. Let $t \in \mathcal{N}(L)$, $u \in \mathcal{N}(K)$ and $k: K' \to K$, $l: L' \to L$ be morphisms of $\mathbf{A}$. We have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom} \arrow[d, "u"'] \arrow[dr, "t \cdot u"] & & \\
\mathrm{Hom}^K \arrow[r, "t^K"] \arrow[d, "k^*"'] & \mathrm{Hom}^{K \times L} \arrow[d, "(K \times L)^*"'] \arrow[dr, "(K \times L)^*"] & \\
\mathrm{Hom}^{K'} \arrow[r, "t^{K'}"'] & \mathrm{Hom}^{K' \times L} \arrow[r, "(K' \times l)^*"'] & \mathrm{Hom}^{K' \times L'}.
\end{tikzcd}$$

The left-hand side is $k^* u = \mathcal{N}(k)(u)$ and the top composite is $(k \times l)^* t \cdot u = \mathcal{N}(K \times L)(t \cdot u)$. We claim that the bottom morphism $(K' \times l)^* t^{K'}$ is $\mathcal{N}(l)(t)^{K'}$. Indeed,

$$\mathcal{N}(l)(t)^{K'}(A \times K' \xrightarrow{f} A)$$

$$= A \times K' \times L' \xrightarrow{\mathcal{N}(l)(t)(\langle f, p_2 \rangle)} A \times K' \xrightarrow{p_1} A$$

$$= A \times K' \times L' \xrightarrow{l^* t(\langle f, p_2 \rangle)} A \times K' \xrightarrow{p_1} A$$

$$= A \times K' \times L' \xrightarrow{A \times K' \times l} A \times K' \times L \xrightarrow{t(\langle f, p_2 \rangle)} A \times K' \xrightarrow{p_1} A$$

$$= (K' \times l)^* t^{K'}(f).$$

Thus, from the diagram,

$$\begin{aligned}\mathcal{N}(k\times l)(t\cdot u) &= (k\times l)^*t\cdot u\\ &= (K'\times l)^*t^{K'}k^*u\\ &= \mathcal{N}(l)(t)^{K'}\mathcal{N}(k)(u)\\ &= \mathcal{N}(l)(t)\cdot\mathcal{N}(k)(u). \qquad \square\end{aligned}$$

However, the proofs of the properties of multiplication, such as associativity or distributivity, become cumbersome if we have to take care of different kinds of parameters J, K, L, etc. As mentioned in Section 5.7, in the cartesian case the diagonal and projection morphisms can be used to reduce doubly indexed families to singly indexed ones and so get multiplication (and addition) as natural transformations $\mathcal{N}(L)\times\mathcal{N}(L)\to\mathcal{N}(L)$. This simplifies matters considerably.

6.2. Controlling families

Proposition 16. *Families of functions $\phi(K,L):\mathcal{N}(L)\times\mathcal{N}(K)\to\mathcal{N}(K\times L)$ natural in K and L are in bijection with families $\psi(L):\mathcal{N}(L)\times\mathcal{N}(L)\to\mathcal{N}(L)$ natural in L.*

Proof. This follows from the general 2-categorical fact that an adjoint pair $U:\mathbf{B}\to\mathbf{A}$, $F:\mathbf{A}\to\mathbf{B}$ with $F\dashv U$, induces an adjoint pair $\mathbf{C}^{F^{\mathrm{op}}}\dashv\mathbf{C}^{U^{\mathrm{op}}}:\mathbf{C}^{\mathbf{A}^{\mathrm{op}}}\to\mathbf{C}^{\mathbf{B}^{\mathrm{op}}}$. We take $U=\times:\mathbf{A}\times\mathbf{A}\to\mathbf{A}$ and $F=\Delta:\mathbf{A}\to\mathbf{A}\times\mathbf{A}$. Let $\mathcal{M}$ in $\mathbf{Set}^{(\mathbf{A}\times\mathbf{A})^{\mathrm{op}}}$ be defined by $\mathcal{M}(K,L)=\mathcal{N}(L)\times\mathcal{N}(K)$. Then we have the bijection

$$\frac{\phi:\mathcal{M}\to\mathcal{N}\circ\times}{\psi:\mathcal{M}\circ\Delta\to\mathcal{N}}. \qquad \square$$

Given ϕ as above, the ψ which corresponds to it is given by

$$\psi(L)=\mathcal{N}(L)\times\mathcal{N}(L)\xrightarrow{\phi(L,L)}\mathcal{N}(L\times L)\xrightarrow{\mathcal{N}(\delta)}\mathcal{N}(L).$$

We have already defined addition (in Section 4.3) and multiplication (in Section 6.1) as operations on doubly indexed families. We now translate these definitions into singly indexed operations. This controls the complexity of the calculations. We use the same notation, $+$ and $\cdot$, as before. This should not cause confusion.

Given $t,u:\mathrm{Hom}\xrightarrow{\cdot}\mathrm{Hom}^L$, $t+u:\mathrm{Hom}\xrightarrow{\cdot}\mathrm{Hom}^L$ is given by

$$(t+u)(f)=A\times L\xrightarrow{A\times\delta}A\times L\times L\xrightarrow{u(f)\times L}A\times L\xrightarrow{t(f)}A$$

and $t\cdot u:\mathrm{Hom}\xrightarrow{\cdot}\mathrm{Hom}^L$ by

$$\begin{aligned}(t\cdot u)(f) &= A\times L\xrightarrow{A\times\delta}A\times L\times L\xrightarrow{t^L(u(f))}A\\ &= A\times L\xrightarrow{A\times\delta}A\times L\times L\xrightarrow{t(\langle u(f),p_2\rangle)}A\times L\xrightarrow{p_1}A.\end{aligned}$$

Constants $\underline{0}$, $\underline{1}$, $\underline{n}$, must similarly be interpreted as natural transformations $1 \to \mathcal{N}$. At L, $\underline{0}(L) : 1(L) \to \mathcal{N}(L)$ is the element of $\mathcal{N}(L)$ given by $\underline{0}(f) = A \times L \xrightarrow{p_1} A$, $\underline{1}(f) = A \times L \xrightarrow{p_1} A \xrightarrow{t} A$, and in general $\underline{n}(f) = A \times L \xrightarrow{p_1} A \xrightarrow{f^{(n)}} A$.

Successor σ is already a natural transformation $\mathcal{N} \to \mathcal{N}$, so now everything is taking place in the functor category $\mathbf{Set}^{\mathbf{A}^{op}}$.

The properties of addition, successor and zero, expressed in Proposition 9 still hold in the present context by transport of structure.

6.3. Multiplication

We now study the properties of multiplication. Before stating our theorem, it will be useful to establish the following lemma expressing how strength interacts with symmetry.

Lemma 1. *Let $u : \mathrm{Hom} \xrightarrow{\sim} \mathrm{Hom}^L$ be a strong BDN, $f : A \to A$ an endomorphism and C an object of $\mathbf{A}$; then the following diagram commutes:*

$$\begin{tikzcd} A \times C \times L \arrow[d, "\sigma_{23}"'] \arrow[dr, "u(f \times C)"] & \\ A \times L \times C \arrow[r, "u(f) \times C"'] & A \otimes C \end{tikzcd}$$

where σ_{23} is the "twist" morphism.

Proof. Consider the commutative diagram

$$\begin{tikzcd} A \times C \arrow[r, "f \times C"] \arrow[d, "\sigma_{12}"'] & A \times C \arrow[d, "\sigma_{12}"] \\ C \times A \arrow[r, "C \times f"'] & C \times A. \end{tikzcd}$$

As t is a BDN,

$$\begin{tikzcd} A \times C \times L \arrow[r, "t(f \times C)"] \arrow[d, "\sigma_{12} \times L"'] & A \times C \arrow[d, "\sigma_{12}"] \\ C \times A \times L \arrow[r, "t(C \times f)"'] & C \times A \end{tikzcd}$$

also commutes. The diagram

$$\begin{tikzcd} C \times A \times L \arrow[r, "C\times t(f)"] \arrow[d, "\sigma_{123}"'] & C \times A \arrow[d, "\sigma_{12}"] \\ A \times L \times C \arrow[r, "t(f)\times C"'] & A \times C \end{tikzcd}$$

also commutes and as t is strong, $t(C \times f) = C \times t(f)$, so we can paste the two squares above and thus obtain the commutativity asserted in the lemma. □

Theorem 6. *Multiplication of families of dinatural numbers satisfies*

(i) $(t \cdot u) \cdot v = t \cdot (u \cdot v)$,

(ii) $\underline{1} \cdot t = t = t \cdot \underline{1}$,

(iii) $\underline{0} \cdot t = \underline{0} = t \cdot \underline{0}$,

(iv) $\sigma t \cdot u = t \cdot u + u$,

(v) $(t + u) \cdot v = t \cdot v + u \cdot v$.

Proof. Let $t, u, v : \mathrm{Hom} \xrightarrow{\cdot\cdot} \mathrm{Hom}^L$ be L-families of dinatural numbers and $f : A \to A$ an endomorphism.

(i)

$$(t \cdot (u \cdot v))(f) = A \times L \xrightarrow{A\times\delta} A \times L \times L \xrightarrow{t(\langle (u\cdot v)(f), p_2\rangle)} A \times L \xrightarrow{p_1} A$$

and a simple calculation shows that

$$((t \cdot u) \cdot v)(f) = A \times L \xrightarrow{A\times\delta} A \times L \times L \times L \xrightarrow{t(\langle u(\langle v(f), p_2\rangle), p_3\rangle)} A \times L \times L \xrightarrow{p_1} A.$$

Consider the diagram

$$\begin{tikzcd} A \times L \arrow[r, "\langle (u\cdot v)(f), p_2\rangle"] \arrow[d, "A\times\delta"'] & A \times L \arrow[d, "A\times\delta"] \\ A \times L \times L \arrow[r, "\langle u(\langle v(f), p_2\rangle), p_3\rangle"'] & A \times L \times L. \end{tikzcd} \qquad (*)$$

When followed by p_1, it becomes

$$\begin{tikzcd} A \times L \arrow[r, "(u\cdot v)(f)"] \arrow[d, "A\times\delta"'] & A \\ A \times L \times L \arrow[r, "u(\langle v(f), p_2\rangle)"'] & A \times L \arrow[u, "p_1"'] \end{tikzcd}$$

which commutes by definition of $u \cdot v$. When $(*)$ is followed by p_2 we get

$$\begin{tikzcd}
A \times L \arrow[r, "p_2"] \arrow[d, "A\times\delta"'] & L \\
A \times L \times L \arrow[r, "u(\langle v(f), p_2\rangle)"'] & A \times L. \arrow[u, "p_2"']
\end{tikzcd} \tag{$**$}$$

Now

$$\begin{tikzcd}
L \arrow[r, "1_L"] & L & & L \times L \arrow[r, "u(1_L)"] & L \\
A \times L \arrow[u, "p_2"] \arrow[r, "\langle u(f), p_2\rangle"'] & A \times L \arrow[u, "p_2"'] & \Rightarrow & A \times L \times L \arrow[u, "p_2 \times L"] \arrow[r, "u(\langle v(f), p_2\rangle)"'] & A \times L \arrow[u, "p_2"']
\end{tikzcd}$$

but strength of u says that $u(1_L) = p_2$, so $p_2 u(\langle v(f), p_2\rangle) = p_3$ so $(**)$ commutes.

Finally, when we follow $(*)$ by p_3, we get p_2 for both composites. Thus $(*)$ commutes.

Now apply t to the top and bottom of $(*)$ to get

$$\begin{tikzcd}
 & A \times L \times L \arrow[rr, "t(\langle (u\cdot v)(f), p_2\rangle)"] \arrow[d, "A\times\delta\times L"] & & A \times L \arrow[dr, "p_1"] \arrow[d, "A\times\delta"'] & \\
A \times L \arrow[ur, "A\times\delta"] \arrow[dr, "A\times\delta"'] & & & & A \\
 & A \times L \times L \times L \arrow[rr, "t(\langle u(\langle v(f), p_2\rangle), p_3\rangle)"'] & & A \times L \times L \arrow[ur, "p_1"'] &
\end{tikzcd}$$

Thus $t \cdot (u \cdot v) = (t \cdot u) \cdot v$.

(ii)

$$\begin{tikzcd}
 & A \times L \times L \arrow[r, "\underline{1}(\langle t(f), p_2\rangle)"] \arrow[d, "p_{12}"] & A \times L \arrow[d, "p_1"] \\
A \times L \arrow[ur, "A\times\delta"] \arrow[dr, "1_{A\times L}"'] & & \\
 & A \times L \arrow[r, "t(f)"'] \arrow[uur, "\langle t(f), p_2\rangle"'] & A
\end{tikzcd}$$

commutes, so $\underline{1} \cdot t = t$. Also

$$t \cdot \underline{1}(f) = A \times L \xrightarrow{A\times\delta} A \times L \times L \xrightarrow{t(\langle \underline{1}(f), p_2\rangle)} A \times L \xrightarrow{p_1} A$$

which is equal to the common composite of the commutative diagram

$$\begin{tikzcd} & A \times L \times L \arrow[r, "t(f\times L)"] \arrow[d, "p_1 \times L"] & A \times L \arrow[d, "p_1"] \\ A \times L \arrow[ur, "A\times\delta"] \arrow[dr, "1_{A\times L}"'] & & \\ & A \times L \arrow[r, "t(f)"'] & A \end{tikzcd}$$

i.e. $t(f)$. Therefore $t \cdot \underline{1} = t$.

(iii)

$$\begin{aligned} \underline{0} \cdot t(f) &= A \times L \xrightarrow{A\times\delta} A \times L \times L \xrightarrow{\underline{0}(\langle t(f), p_2\rangle)} A \times L \xrightarrow{p_1} A \\ &= A \times L \xrightarrow{A\times\delta} A \times L \times L \xrightarrow{p_{12}} A \times L \xrightarrow{p_1} A \\ &= A \times L \xrightarrow{p_1} A = \underline{0}(f). \end{aligned}$$

So $\underline{0} \cdot t = \underline{0}$

$$\begin{aligned} \underline{t} \cdot \underline{0}(f) &= A \times L \xrightarrow{A\times\delta} A \times L \times L \xrightarrow{t(\langle \underline{0}(f), p_2\rangle)} A \times L \xrightarrow{p_1} A \\ &= A \times L \xrightarrow{A\times\delta} A \times L \times L \xrightarrow{t(\langle p_1, p_2\rangle)} A \times L \xrightarrow{p_1} A \\ &= A \times L \xrightarrow{A\times\delta} A \times L \times L \xrightarrow{t(1_{A\times L})} A \times L \xrightarrow{p_1} A \\ &= A \times L \xrightarrow{A\times\delta} A \times L \times L \xrightarrow{p_{12}} A \times L \xrightarrow{p_1} A \\ &= A \times L \xrightarrow{p_1} A = \underline{0}(f). \end{aligned}$$

So $t \cdot \underline{0} = \underline{0}$.

(iv) $(t + \underline{1})(f)$ is the common composite in the following commutative diagram:

$$\begin{tikzcd} A \times L \arrow[r, "A\times\delta"] \arrow[dr, "1_{A\times L}"'] & A \times L \times L \arrow[r, "t(f)\times L"] \arrow[d, "p_{12}"] & A \times L \arrow[d, "p_1"] \\ & A \times L \arrow[r, "t(f)"] \arrow[dr, "\sigma(t)(f)"'] & A \arrow[d, "f"] \\ & & A. \end{tikzcd}$$

So $\sigma(t) = t + \underline{1}$. Thus, (iv) will follow from (v) and (ii).

(v) $((t+u)\cdot v)(f)$ is the composite of the top and the right sides of

$$\begin{tikzcd}
A\times L \arrow[rrr, "A\times\delta"] \arrow[ddr, "A\times\delta"] \arrow[ddddd, "A\times\delta"'] & & & A\times L\times L \arrow[rr, "(t+u)(\langle v(f),p_2\rangle)"] \arrow[d, "A\times L\times\delta"] & & A\times L \arrow[ddd, equal] \\
 & & & A\times L\times L\times L \arrow[dd, "u(\langle v(f),p_2\rangle)\times L"] & (2) & \\
 & A\times L\times L\times L \arrow[urr, "A\times L\times\sigma"] \arrow[drr, "u(\langle v(f),p_2\rangle\times L)"'] & (1) & & & \\
 & & & A\times L\times L \arrow[rr, "t(\langle v(f),p_2\rangle)"'] & & A\times L \arrow[dd, "p_1"] \\
 & & (4) & & (3) & \\
A\times L\times L \arrow[uuur, "A\times\delta\times L"] \arrow[rrr, "u(\langle v(f),p_2\rangle)"'] & & & A\times L \arrow[uu, "A\times\delta"] \arrow[rr, "(t\cdot v)(f)"'] & & A,
\end{tikzcd}$$

and $(t\cdot v+u\cdot v)(f)$ is the composite of the top and right sides of

$$\begin{tikzcd}
A\times L \arrow[r, "A\times\delta"] \arrow[dddd, "A\times\delta"'] & A\times L\times L \arrow[r, "A\times\delta\times L"] & A\times L\times L\times L \arrow[rr, "u(\langle v(f),p_2\rangle)\times L"] \arrow[dd, "A\times L\times\sigma"'] & & A\times L\times L \arrow[dd, "p_1\times L"] \\
 & & & (1) & \\
 & & A\times L\times L\times L \arrow[uurr, "u(\langle v(f),p_2\rangle)\times L)"] & & A\times L \arrow[dd, "(t\cdot v)(f)"] \\
 & & (4) & & \\
A\times L\times L \arrow[uurr, "A\times\delta\times L"] \arrow[rrr, "u(\langle v(f),p_2\rangle)"'] & & & A\times L \arrow[uuuur, "A\times\delta"] \arrow[r, "(t\cdot v)(f)"'] & A\ .
\end{tikzcd}$$

So $((t+u)\cdot v)(f)=(t\cdot v+u\cdot v)(f)$ once we remark that (1) commutes by the lemma, (2) by definition of $t+u$, (3) by definition of $t\cdot v$, and (4) by Barr dinaturality of u applied to the commutative diagram

$$\begin{tikzcd}
A\times L\times L \arrow[r, "\langle v(f),p_2\rangle\times L"] & A\times L\times L \\
A\times L \arrow[u, "A\times\delta"] \arrow[r, "\langle v(f),p_2\rangle"'] & A\times L \arrow[u, "A\times\delta"']
\end{tikzcd}$$

$\square$

Remark. The proof of part (i) is much longer than one would expect. After all, for $L=1$, multiplication of dinatural numbers is just composition, which is clearly associative. The complication occurs when families of dinatural numbers are introduced.

A conceptual simplification can be obtained by the introduction of the categories $\mathbf{A}[L]$. The objects of $\mathbf{A}[L]$ are the same as those of $\mathbf{A}$ but a morphism $f : A \to A'$ in $\mathbf{A}[L]$ is an L-family of morphisms from A to A', i.e. a morphism $A \times L \to A'$ in $\mathbf{A}$. Identities are given by projections $p_1 : A \times L \to A$ and composition of g with f by

$$A \times L \xrightarrow{A \times \delta} A \times L \times L \xrightarrow{f \times L} A \times L \xrightarrow{g} A.$$

These identities and compositions occur everywhere in the above calculations.

This is a well-known construction. It is nothing but the Kleisly category for the comonad $(\ )\times L$ on $\mathbf{A}$. It is also discussed in [9] where it is viewed as the result of adjoining an indeterminate of the form $x : 1 \to L$ to the category $\mathbf{A}$.

It can be shown that strong BDNs, $t : \mathrm{Hom}_{\mathbf{A}} \xrightarrow{\dot{-}} \mathrm{Hom}_{\mathbf{A}}^{L}$, are in bijection with strong BDNs, $u : \mathrm{Hom}_{\mathbf{A}[L]} \xrightarrow{\dot{-}} \mathrm{Hom}_{\mathbf{A}[L]}$ and that multiplication corresponds to composition, but the calculations are similar to the above but more complicated. For this reason we decided to give a direct proof of (i). However, further work in this direction will surely involve the categories $\mathbf{A}[L]$.

6.4. Exponentials

In this section, we further explore the relationship between the categorical structure on $\mathbf{A}$ and the arithmetic of dinatural numbers. We shall show that if $\mathbf{A}$ is cartesian closed and has pullbacks then we can define the exponential of two dinatural numbers and that this exponential has nice properties. Our discussion below is still preliminary. An extensive study of exponentiation must await a future work.

Let $\mathbf{A}$ be cartesian closed with internal hom $[\ ,\ ] : \mathbf{A}^{\mathrm{op}} \times \mathbf{A} \to \mathbf{A}$. Further assume that $\mathbf{A}$ has pullbacks. We shall show that dinatural numbers can be internalized and, therefore, that we can apply BDNs to them.

Let $t : \mathrm{Hom} \xrightarrow{\dot{-}} \mathrm{Hom}^{L}$ be an L-family of dinatural numbers. Then by proposition 14, t can be extended to strong BDNs, $t^{K} : \mathrm{Hom}^{K} \xrightarrow{\dot{-}} \mathrm{Hom}^{K \times L}$, naturally in K. Thus, for each A, we have

$$t_A^K : \mathrm{Hom}(A \times K, A) \to \mathrm{Hom}(A \times K \times L, A)$$

which gives, by cartesian closedness, morphisms

$$\mathrm{Hom}(K, [A,A]) \to \mathrm{Hom}(K \times L, [A,A])$$

natural in K. So, by the Yoneda lemma, we get

$$\widehat{t}_A : [A,A] \times L \to [A,A].$$

For $f : A \times K \to A$ let us denote the corresponding morphism $K \to [A,A]$ by $\ulcorner f \urcorner$. Now the basic property of $\widehat{t}$ can be expressed by

$$\begin{tikzcd}
K \times L \arrow[rr, "\ulcorner f \urcorner \times L"] \arrow[dr, "\ulcorner t^K(f) \urcorner"'] & & {[A,A] \times L} \arrow[dl, "\hat{t}_A"] \\
& {[A,A].} &
\end{tikzcd}$$

Proposition 17. $\widehat{t} : [\,,\,] \times L \to [\,,\,]$ *is a strong BDN.*

Proof. To say that $\widehat{t}$ is a BDN means that for any morphism $\phi : A \to B$, the following hexagon commutes:

$$\begin{tikzcd}
& {[A,A] \times L} \arrow[r, "\hat{t}_A"] & {[A,A]} \arrow[dr, "{[A,\phi]}"] & \\
P \arrow[ur] \arrow[dr] & & & {[A,B]} \\
& {[B,B] \times L} \arrow[r, "\hat{t}_B"'] & {[B,B]} \arrow[ur, "{[\phi,B]}"'] &
\end{tikzcd} \qquad (*)$$

where P is the pullback of $[A,\phi] \times L$ and $[\phi,B] \times L$. But as **A** is cartesian closed, P is isomorphic to $Q \times L$ where Q is given by the pullback

$$\begin{tikzcd}
& {[A,A]} \arrow[dr, "{[A,\phi]}"] & \\
Q \arrow[ur] \arrow[dr] & & {[A,B]} \\
& {[B,B]} \arrow[ur, "{[\phi,B]}"'] &
\end{tikzcd} \quad .$$

To show that $(*)$ commutes, it is sufficient to test it on morphisms of the form $q \times L : K \times L \to Q \times L$. But a morphism $q : K \to Q$ corresponds to a pair $\ulcorner f \urcorner$, $\ulcorner g \urcorner$ making

$$\begin{tikzcd}
& {[A,A]} \arrow[dr, "{[A,\phi]}"] & \\
K \arrow[ur, "\ulcorner f \urcorner"] \arrow[dr, "\ulcorner g \urcorner"'] & & {[A,B]} \\
& {[B,B]} \arrow[ur, "{[\phi,B]}"'] &
\end{tikzcd}$$

commute, i.e. a diagram

$$\begin{tikzcd} A \times K \arrow[r, "f"] \arrow[d, "\phi \times K"'] & A \arrow[d, "\phi"] \\ B \times K \arrow[r, "g"'] & B. \end{tikzcd}$$

Dinaturality of t^K implies that

$$\begin{tikzcd} A \times K \times L \arrow[r, "t^K(f)"] \arrow[d, "\phi \times K \times L"'] & A \arrow[d, "\phi"] \\ B \times K \times L \arrow[r, "t^K(g)"'] & B \end{tikzcd}$$

also commutes, which means that $(*)$ commutes when preceded by $q \times L$. Thus $(*)$ commutes.

By $\hat{t}$ *strong* we mean that for every A and C,

$$\begin{tikzcd} {[A,A] \times L} \arrow[r, "\hat{t}_A"] \arrow[d, "\mathrm{st} \times L"'] & {[A,A]} \arrow[d, "\mathrm{st}"] \\ {[C \times A, C \times A] \times L} \arrow[r, "\hat{t}_{C \times A}"'] & {[C \times A, C \times A]} \end{tikzcd}$$

commutes, where st is the strength for the functor $C \times (\ )$. Let $\ulcorner f \urcorner : K \to [A,A]$ correspond to $\ulcorner f \urcorner : A \times K \to A$. Preceding $(**)$ by $\ulcorner f \urcorner \times L$, we see that the top composite is $C \times t^K(f) : C \times A \times K \times L \to C \times A$, whereas the bottom is $t^K(C \times f)$. Strength of t^K says that these are equal. $\square$

Conversely, a strong BDN, $u : [\,,\,] \times L \xrightarrow{\cdot\cdot} [\,,\,]$ gives a strong BDN, $\check{u} : \mathrm{Hom} \xrightarrow{\cdot\cdot} \mathrm{Hom}^L$ as follows. For $f : A \to A$ we get

$$1 \times L \xrightarrow{\ulcorner f \urcorner \times L} [A,A] \times L \xrightarrow{u_A} [A,A]$$

which corresponds to $\check{u}(f) : A \times L \to A$. Thus $\ulcorner \check{u}(f) \urcorner = u_A \cdot \ulcorner f \urcorner \times L$. That $\check{u}$ is a strong BDN is straightforward.

Proposition 18. *If* $t : \mathrm{Hom} \xrightarrow{\cdot\cdot} \mathrm{Hom}^L$ *is a strong BDN, then* $\check{\hat{t}} = t$.

Proof. $\ulcorner \check{\hat{t}}(f) \urcorner = \hat{t}_A \cdot \ulcorner f \urcorner \times L = \ulcorner t(f) \urcorner$. $\square$

However, it is not true that $\widehat{\check{u}} = u$ for all u as the following example shows. Consider the topos $\mathbf{Set}^{\mathbf{Z}}$ of $\mathbf{Z}$-sets. An object may be viewed as a pair (X, ξ) where X is a set and $\xi : X \to X$ a bijection. $\mathbf{Set}^{\mathbf{Z}}$ has a natural numbers object, $((\mathbf{N}, 1_{\mathbf{N}}), s)$ so strong BDNs, $t : \mathrm{Hom} \stackrel{\cdot\cdot}{\longrightarrow} \mathrm{Hom}$, correspond to morphisms $1 \to (\mathbf{N}, 1_{\mathbf{N}})$, i.e. ordinary natural numbers. So all dinatural numbers are standard. The internal hom in $\mathbf{Set}^{\mathbf{Z}}$ is given by $[(X,\xi),(Y,\theta)] = (W, \omega)$ where W is the set of *all* functions $f : X \to Y$ and $\omega(f) = \theta \circ f \circ \xi^{-1}$. Define

$$\begin{array}{rcl} u_{(X,\xi)} : [(X,\xi),(X,\xi)] & \to & [(X,\xi),(X,\xi)] \\ f : X \to X & \longmapsto & \xi \circ f \circ \xi^{-1}. \end{array}$$

It is straightforward to show that u is a strong BDN, $u : [\ ,\ ] \stackrel{\cdot\cdot}{\longrightarrow} [\ ,\ ]$. But u does not correspond to a standard dinatural number. In fact, $\check{u} : \mathrm{Hom} \stackrel{\cdot\cdot}{\longrightarrow} \mathrm{Hom}$ is the identity (i.e. $\underline{1}$) so $\widehat{\check{u}}_A = 1_{[A,A]} \neq u_A$.

Thus, there are possibly more internal BDNs than external ones. We still maintain that it is the external ones that correspond to natural numbers.

For the remainder of the section, we restrict our attention to single dinatural numbers rather than families. Everything should work for families but the calculations are somewhat involved and have not yet been checked in every detail.

Proposition 19. *Let* $u : [\,,\,] \stackrel{\cdot\cdot}{\longrightarrow} [\,,\,]$ *be a strong BDN and* $t : \mathrm{Hom} \stackrel{\cdot\cdot}{\longrightarrow} \mathrm{Hom}$ *a BDN. Then* $t(u_A) : [A,A] \to [A,A]$ *defines a strong BDN.*

Proof. That u is a BDN means that the hexagon

$$\begin{tikzcd}
 & {[A,A]} \arrow[rr, "u_A"] & & {[A,A]} \arrow[dr, "{[A,\phi]}"] & \\
Q \arrow[ur, "q_1"] \arrow[dr, "q_2"'] \arrow[rr, "\tau"] & & Q \arrow[ur, "q_1"] \arrow[dr, "q_2"'] & & {[A,B]} \\
 & {[B,B]} \arrow[rr, "u_B"'] & & {[B,B]} \arrow[ur, "{[\phi,B]}"'] &
\end{tikzcd}$$

commutes for every ϕ. Q is the pullback of $[A,\phi]$ and $[\phi,B]$ so there exists a fill-in τ as in the diagram. As t is a BDN we get a commutative diagram

$$\begin{tikzcd}
 & {[A,A]} \arrow[rr, "t(u_A)"] & & {[A,A]} \arrow[dr, "{[A,\phi]}"] & \\
Q \arrow[ur, "q_1"] \arrow[dr, "q_2"'] \arrow[rr, "t(\tau)"] & & Q \arrow[ur, "q_1"] \arrow[dr, "q_2"'] & & {[A,B]} \\
 & {[B,B]} \arrow[rr, "t(u_B)"'] & & {[B,B]} \arrow[ur, "{[\phi,B]}"'] &
\end{tikzcd}$$

so $t(u_A)$ is a BDN.

An application of t to the diagram

$$\begin{tikzcd}
{[A,A]} \arrow[r, "u_A"] \arrow[d, "\mathrm{st}"'] & {[A,A]} \arrow[d, "\mathrm{st}"] \\
{[C\times A, C\times A]} \arrow[r, "u_{C\times A}"'] & {[C\times A, C\times A]}
\end{tikzcd}$$

shows immediately that $t(u_A)$ is strong. $\square$

Definition. Let $t, u : \mathrm{Hom} \xrightarrow{\cdot\cdot} \mathrm{Hom}$ be dinatural numbers. The *exponential* t^u is defined by $t^u = (u(\widehat{t}))\check{}$. Thus, for $f : A \to A$, we have

$$\ulcorner t^u(f) \urcorner = u(\widehat{t_A}) \circ \ulcorner f \urcorner.$$

Proposition 20. *Exponentiation of dinatural numbers has the following properties:*
(i) $t^{\underline{0}} = \underline{1}$,
(ii) $t^{\underline{1}} = t$,
(iii) $t^{\sigma(u)} = t^u \cdot t$,
(iv) $t^{(u+v)} = t^u \cdot t^v$.

Proof. Let $f : A \to A$.
(i) $\ulcorner t^{\underline{0}}(f) \urcorner = \underline{0}(\widehat{t_A}) \circ \ulcorner f \urcorner = 1_{[A,A]} \circ \ulcorner f \urcorner = \ulcorner f \urcorner$. Thus $t^{\underline{0}}(f) = f$ for all f, i.e. $t^{\underline{0}} = \underline{1}$.
(ii) $\ulcorner t^{\underline{1}}(f) \urcorner = \underline{1}(\widehat{t_A}) \circ \ulcorner f \urcorner = \widehat{t_A} \circ \ulcorner f \urcorner = \ulcorner t(f) \urcorner$. Thus $t^{\underline{1}}(f) = t(f)$ for all f.
(iii) Follows from (ii) and (iv) and $\sigma(u) = u + \underline{1}$.
(iv)

$$\begin{aligned}
\ulcorner t^{(u+v)}(f) \urcorner &= (u+v)(\widehat{t_A}) \circ \ulcorner f \urcorner \\
&= u(\widehat{t_A}) \circ v(\widehat{t_A}) \circ \ulcorner f \urcorner \\
&= u(\widehat{t_A}) \circ \ulcorner t^v(f) \urcorner \\
&= \ulcorner t^v(f) \urcorner \\
&= \ulcorner (t^u \cdot t^v)(f) \urcorner.
\end{aligned}$$

So $t^{(u+v)}(f) = (t^u \cdot t^v)(f)$ for all f. $\square$

Properties (i) and (iii) insure that if u is a standard numeral $\underline{n}$, then

$$t^{\underline{n}} = t \cdot t \cdot t \cdots \cdot t \quad (n \text{ times}).$$

In particular, $\underline{m}^{\underline{n}} = \underline{m^n}$.

Remark. We do not know if the identity $(t^u)^v = t^{(v \cdot u)}$ holds in general.

7. Examples

7.1. When $\mathbf{A}$ has a natural numbers object

When $\mathbf{A}$ has a natural numbers object (N, s), then Theorem 3 says that L-families of dinatural numbers, $t : \mathrm{Hom} \xrightarrow{\cdot\cdot} \mathrm{Hom}^L$, correspond exactly to morphisms $L \to N$. Thus $\mathcal{N} \cong \mathbf{A}(-, N)$. Successor and $\underline{0}$ for $\mathcal{N}$ correspond under this isomorphism to $\mathbf{A}(-, 0) : \mathbf{A}(-, I) \to \mathbf{A}(-, N)$ and $\mathbf{A}(-, s) : \mathbf{A}(-, N) \to \mathbf{A}(-, N)$, respectively. Theorem 5 then shows that addition corresponds to $\mathbf{A}(-, +)$ as it satisfies the corresponding recurrence relation.

Multiplication is not defined as a natural transformation into $\mathcal{N}$ in the monoidal case, but in the cartesian case, Theorem 6 shows that multiplication does correspond to the usual one $N \times N \to N$. Indeed, the internal definition of multiplication is as the unique morphism $\cdot : N \times N \to N$ which fits into the diagram

$$\begin{tikzcd}
 & N \times N \arrow[r, "s \times N"] \arrow[d, "(\cdot, p_2)"] & N \times N \arrow[d, "(\cdot, p_2)"] \\
1 \times N \arrow[ur, "0 \times N"] \arrow[r, "0 \times N"'] & N \times N \arrow[r, "(+, p_2)"'] & N \times N
\end{tikzcd}$$

If we apply $\mathbf{A}(L, -)$ to this diagram, we see that its commutativity is equivalent to properties (iii) and (iv) of Theorem 6.

In order to see that the exponentiation defined in Section 6.4 coincides with the one defined internally by recursion, we must extend our definition to include families. Indeed, the internal exponential is defined to be the unique morphism $\exp : N \times N \to N$ which fits in the commutative diagram

$$\begin{tikzcd}
 & N \times N \arrow[r, "s \times N"] \arrow[d, "\langle \exp, p_2 \rangle"] & N \times N \arrow[d, "\langle \exp, p_2 \rangle"] \\
1 \times N \arrow[ur, "0 \times N"] \arrow[r, "0 \times N"'] & N \times N \arrow[r, "\langle \cdot, p_2 \rangle"'] & N \times N
\end{tikzcd}$$

and that is all that we know about exp. So to show that our definition of 6.4 agrees with this one we should show that a similar diagram commutes for $\mathcal{N}$ which would require a definition of exponential for L-families of dinatural numbers. However, we can get the result by introducing families only in the exponent and this makes the calculations considerably easier. Let us fix $n : 1 \to N$. Then we can define $n^{(\ )} : N \to N$

recursively by

$$\begin{tikzcd}
 & N \arrow[r, "s"] \arrow[d, "n^{(\ )}"] & N \arrow[d, "n^{(\ )}"] \\
1 \arrow[ur, "0"] \arrow[dr, "1"'] & & \\
 & N \arrow[r, "(\ )\cdot n"'] & N
\end{tikzcd}$$

where $(\ )\cdot n = (N \cong N\times 1 \xrightarrow{N\times n} N\times N \xrightarrow{\cdot} N)$. It is easily seen that $n^{(\ )} = \exp(-, n)$.

Now we define t^u for $t:\mathrm{Hom} \xrightarrow{..} \mathrm{Hom}$ a single BDN and $u:\mathrm{Hom} \xrightarrow{..} \mathrm{Hom}^L$ an L-family. $t^u = (u(\widehat{t}))^{\check{}}:\mathrm{Hom} \xrightarrow{..} \mathrm{Hom}^L$. Thus we get a function $t^{(\ )}:\mathcal{N}(L)\to\mathcal{N}(L)$ for each L.

Proposition 21. *$t^{(\ )}:\mathcal{N}(L)\to\mathcal{N}(L)$ is natural in L, i.e. if $l:K\to L$ then $l^*(t^u) = t^{l^*(u)}$.*

Proof. If $f:A\to A$, then $\ulcorner l^*(t^u)(f)\urcorner$ is given by the composite

$$1\times K \xrightarrow{1\times l} 1\times L \xrightarrow{\ulcorner f\urcorner\times L} [A,A]\times L \xrightarrow{u(\widehat{t_A})} [A,A],$$

whereas $\ulcorner t^{l^*(u)}(f)\urcorner$ is given by

$$1\times K \xrightarrow{\ulcorner f\urcorner\times K} [A,A]\times K \xrightarrow{[A,A]\times l} [A,A]\times L \xrightarrow{u(\widehat{t_A})} [A,A].$$

The two composites are obviously equal. $\square$

If $t:\mathrm{Hom} \xrightarrow{..} \mathrm{Hom}$ and $u,v:\mathrm{Hom} \xrightarrow{..} \mathrm{Hom}$ are strong BDNs, then all of the identities of Proposition 20 still hold, and the proofs are basically the same once we note that the effect of t^u on morphisms is given by $\ulcorner t^u(f)\urcorner = u(\widehat{t})\circ\ulcorner f\urcorner\times L$, which fact was used in the preceding proof.

The natural transformation $t^{(\ )}:\mathcal{N}\to\mathcal{N}$ induces a morphism $e:N\to N$ and if $n:1\to N$ is the natural number corresponding to t, then properties (i) and (iii) of Proposition 20 show that e satisfies the recursion data for $n^{(\ )}$. Thus, the exponential defined in Section 6.4, and improved here, agree with the usual one defined by recursion.

7.2. *Dinaturals for finite sets*

We wish to study strong dinatural transformations $\mathrm{Hom} \xrightarrow{..} \mathrm{Hom}$ on the category of finite sets. From the discussion of Section 2.7, all dinaturals are strong for finite sets. Also, because $\mathrm{Hom}^L(A,B) = \prod \mathrm{Hom}(A,B)$, an L-family of dinaturals is just L independent dinaturals. So it will be sufficient to understand BDNs, $\mathrm{Hom} \xrightarrow{..} \mathrm{Hom}$.

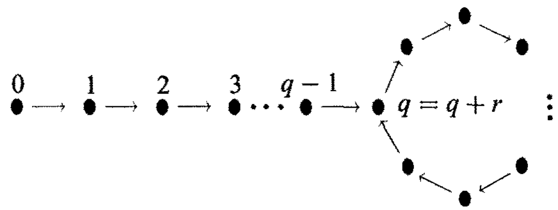
Consider $\mathbf{N}$ as a universal algebra with one nullary operation, 0, and one unary operation s. The finite quotients of $\mathbf{N}$ are all of the form

$$A_{q,r} = \mathbf{N}/q \equiv q + r,$$

where q, $r \in \mathbf{N}$ and $r \neq 0$. By $q \equiv q + r$ we mean the congruence generated by setting $q \equiv q + r$. Explicitly,

$$m \equiv n \Leftrightarrow \begin{cases} m = n \text{ or} \\ m,n \geq q \text{ and } r \mid m - n. \end{cases}$$

We may picture $A_{q,r}$ as



As homomorphisms $A_{q,r} \to A_{q',r'}$ must preserve 0 and s, there can be at most one and there is one if and only if $q' \leq q$ and $r' \mid r$. Thus, we have a directed diagram of finite algebras indexed by the poset $\mathbf{N} \times \mathbf{N}^*$ of all (q, r) as above. Define $\widehat{\mathbf{N}} = \varprojlim_{q,r} A_{q,r}$. Note that the congruence $q \equiv q + r$ is also a congruence for addition and multiplication so that each $A_{q,r}$ has these operations, given by $[a]+[b] = [a+b]$ and $[a]\cdot[b] = [ab]$. When there is a homomorphism $\phi : A_{q,r} \to A_{q',r'}$ it is given by $\phi([a]) = [a]$, so it preserves $+$ and $\cdot$. Consequently, $\widehat{\mathbf{N}}$ has $+$ and $\cdot$ satisfying the usual properties, i.e. $\widehat{\mathbf{N}}$ is a commutative *rig* (i.e. commutative semi-ring with 1).

Theorem 7. *There is an isomorphism of rigs between* $\mathcal{N}(1)$ *and* $\widehat{\mathbf{N}}$.

Proof. Let $t \in \mathcal{N}(1)$, i.e. $t : \mathrm{Hom} \xrightarrow{\sim} \mathrm{Hom}$. Apply t to the successor $s : A_{q,r} \to A_{q,r}$ to get $t(s) : A_{q,r} \to A_{q,r}$. Let $t(s)[0] = [n_{q,r}]$. If $\phi : A_{q,r} \to A_{q',r'}$ is a homomorphism, then

$$\begin{tikzcd} A_{q,r} \arrow[r, "s"] \arrow[d, "\phi"'] & A_{q,r} \arrow[d, "\phi"] \\ A_{q',r'} \arrow[r, "s"'] & A_{q',r'} \end{tikzcd}$$

commutes, so applying t to top and bottom and evaluating at [0] we see that $\phi([n_{q,r}]) = [n_{q',r'}]$. Thus the family $\langle [n_{q,r}] \rangle$ is an element of $\widehat{\mathbf{N}}$.

Conversely, given an element $\langle [n_{q,r}] \rangle_{q,r}$ of $\widehat{\mathbf{N}}$ we can define $t : \mathrm{Hom} \xrightarrow{\sim} \mathrm{Hom}$ as follows. If $f : A \to A$ with A finite, then there exist q, $r \in \mathbf{N}$ such that $f^{(q)} = f^{(q+r)}$, with $r \neq 0$. Define $t(f) = f^{(n_{q,r})}$. If $q \leq q'$ and $r \mid r'$ then $f^{(q)} = f^{(q+r)} \Rightarrow f^{(q')} = f^{(q'+r')}$

so t is well defined. Also, by directness of our poset $\mathbf{N} \times \mathbf{N}^*$, given $g : B \to B$ we can find a pair (q, r) for which $t(f) = f^{(n_{q,r})}$ and $t(g) = g^{(n_{q,r})}$, so as far as f and g are concerned, t is iteration by a fixed integer and therefore is a BDN.

Note that for $s : A_{q,r} \to A_{q,r}$ we have $s^{(q)} = s^{(q+r)}$, so if we start with $\langle [n_{q,r}] \rangle_{q,r} \in \widehat{\mathbf{N}}$ and construct t as above, then $t(s)[0] = s^{(n_{q,r})}[0] = [n_{q,r}]$. Thus, we get back the same element of $\widehat{\mathbf{N}}$.

On the other hand, let us start with $t \in \mathcal{N}(1)$ and let $[n_{q,r}] = t(s)[0]$, and then construct a new BDN, u, from $\langle [n_{q,r}] \rangle_{q,r}$. For $f : A \to A$ there is qr such that $f^{(q)} = f^{(q+r)}$. For $a \in A$ define $\phi : A_{q,r} \to A$ by $\phi([n]) = f^{(n)}(a)$, which is indeed well defined. Now

$$\begin{tikzcd} A_{q,r} \arrow[r, "s"] \arrow[d, "\phi"'] & A_{q,r} \arrow[d, "\phi"] \\ A \arrow[r, "f"'] & A \end{tikzcd}$$

commutes, so

$$\begin{tikzcd} A_{q,r} \arrow[r, "t(s)"] \arrow[d, "\phi"'] & A_{q,r} \arrow[d, "\phi"] \\ A \arrow[r, "t(f)"'] & A \end{tikzcd}$$

does also. Thus, $t(f)\phi([0]) = t(f)(a)$ is equal to $\phi t(s)([0]) = \phi([n_{q,r}]) = f^{(n_{q,r})}(a) = u(f)(a)$. Thus $u = t$. This shows that we have a bijection $\mathcal{N}(1) \cong \widehat{\mathbf{N}}$:

$\underline{0}(s)[0] = 1_{A_{q,r}}[0] = [0]$, so $\langle [0] \rangle$ corresponds to $\underline{0}$.
$\underline{1}(s)[0] = s[0] = [1]$, so $\langle [1] \rangle$ corresponds to $\underline{1}$.

In order to see that addition and multiplication are preserved first note that if $f : A_{p,q} \to A_{p,q}$ commutes with s and if $f[0] = [n]$ then $f[m] = [m + n]$. Also note that if f commutes with s then so does $t(f)$ for any BDN, t.

Let $t(s)[0] = [m_{q,r}]$ and $u(s)[0] = [n_{q,r}]$. Then

$$(t + u)(s)[0] = t(s) \cdot u(s)[0] = t(s)[n_{q,r}] = [m_{q,r} + n_{q,r}].$$

So addition is preserved.

As $u(s) : A_{q,r} \to A_{q,r}$ commutes with s, it satisfies $u(s)^{(q)} = u(s)^{(q+r)}$. Thus

$$\begin{aligned}(t \cdot u)(s)[0] &= t(u(s))[0] \\ &= u(s)^{(m_{q,r})}[0] \\ &= u(s) \circ u(s) \circ \cdots \circ u(s)[0]\end{aligned}$$

$$= [n_{q,r} + n_{q,r} + \cdots + n_{q,r}]$$
$$= [m_{q,r} \cdot n_{q,r}].$$

So multiplication is also preserved. $\square$

The congruence $q \equiv q + r$ is not in general a congruence for exponentiation. For example, in $A_{2,3}$, $2 \equiv 5$ but $2^2 = 4 \not\equiv 32 = 2^5$. But as $\mathcal{N}(1)$ has exponentiation so does $\widehat{\mathbf{N}}$, by transport of structure. Let us examine how this works.

Let $\langle [e_{q,r}] \rangle = \langle [m_{q,r}] \rangle^{\langle [n_{q,r}] \rangle}$ and let t and u correspond to $\langle [m_{q,r}] \rangle$ and $\langle [n_{q,r}] \rangle$, respectively. Then $t_A : \mathrm{Hom}(A, A) \to \mathrm{Hom}(A, A)$ is already internal so $\widehat{t_A} = t_A$. The exponential t^u is given by externalizing $u(\widehat{t_A})$ which is already external. To find the corresponding element of $\widehat{\mathbf{N}}$ we must apply it to $s : A_{q,r} \to A_{q,r}$. Thus, we consider $\widehat{t} = \widehat{t_{A_{q,r}}} : \mathrm{Hom}(A_{q,r}, A_{q,r}) \to \mathrm{Hom}(A_{q,r}, A_{q,r})$ and find $\overline{q}, \overline{r}$ such that $\widehat{t}^{(\overline{q})} = \widehat{t}^{(\overline{q}+\overline{r})}$. Then

$$\begin{aligned} u(\widehat{t})(s)[0] &= \widehat{t}^{(n_{\overline{q},\overline{r}})}(s)[0] \\ &= \widehat{t} \circ \widehat{t} \circ \widehat{t} \circ \cdots \circ \widehat{t}(s)[0] \\ &= [m_{q,r} \cdot m_{q,r} \cdot \cdots \cdot m_{q,r}] \\ &= [m_{q,r}^{n_{\overline{q},\overline{r}}}]. \end{aligned}$$

Thus $[e_{q,r}] = [m_{q,r}^{n_{\overline{q},\overline{r}}}]$ where $\overline{q}$, $\overline{r}$ are as above.

Thus, exponentiation in $\widehat{\mathbf{N}}$ is not componentwise. The class $[n_{q,r}]$ has many representatives $\overline{n}$ and $n_{\overline{q},\overline{r}}$ is one of them, i.e. $[n_{\overline{q},\overline{r}}] = [n_{q,r}]$ in $A_{q,r}$. As the congruence $q \equiv q + r$ does not respect exponentiation, the classes $[m_{q,r}^{\overline{n}}]$ are not all the same. But as the above discussion shows, it is possible to choose the representative correctly.

It may be interesting to note the analogy with presheaf categories where sums and products are performed componentwise but exponentiation is not.

We can get a better understanding of $\widehat{\mathbf{N}}$ by considering $B_k = A_{k!,k!}$. There is always a morphism $\phi : B_{k+1} \to B_k$ and this gives an initial subdiagram of $\langle A_{q,r} \rangle$. Thus $\widehat{\mathbf{N}} \cong \varprojlim_k B_k$.

An element of $\widehat{\mathbf{N}}$ can thus be considered as a singly indexed family of natural numbers $\langle [n_k] \rangle_k$ which are compatible in the sense that for each k, either $n_{k+1} = n_k$ or they are both greater than $k!$ and $k! \mid (n_{k+1} - n_k)$. Thus, if we are trying to build an element of $\widehat{\mathbf{N}}$ recursively and we have the first k members

$$\langle [n_0], [n_1], [n_2], \ldots, [n_k], \rangle$$

then, if $n_k < k!$ we have only one choice for $[n_{k+1}]$, namely $n_{k+1} = n_k$ and so on for the remaining members. This corresponds to the standard numeral $\underline{n}_k$. On the other hand if $k! \le n_k < 2 \cdot k!$, then we have $k + 1$ choices for $[n_{k+1}]$, namely $[n_k], [n_k + k!], [n_k + 2 \cdot k!], \ldots, [n_k + k \cdot k!]$. Of these, the first is one that becomes constant and the others admit $k + 2$ choices at the next stage. Thus, we can see that apart from the standard numerals we have an uncountable set of dinatural numbers. In fact, $\widehat{\mathbf{N}}$ may be identified with $\mathbf{N} \dot{\cup} \widehat{\mathbf{Z}}$ where $\widehat{\mathbf{Z}}$ is the set of adic numbers $\widehat{\mathbf{Z}} = \varprojlim_{n \neq 0} \mathbf{Z}/(n)$. To $\langle [m_k] \rangle \in \widehat{\mathbf{Z}}$ we associate the dinatural number $\langle [k! + m_k] \rangle$.

For instance, corresponding to 0 in $\widehat{\mathbf{Z}}$, we have the dinatural number $\omega = \langle [k!] \rangle$. $\omega + \omega = \omega \cdot \omega = \omega$ but $\omega + 1 \neq \omega$. This ω is the dinatural number used in Peter Johnstone's example of Section 2.2. Our example was $\langle [k!!] \rangle$ where $k!! = 1!2! \cdots k!$. But in B_k, $[k!] = [m \cdot k!]$ for any $m > 0$, so our example is also ω.

There is a result of Peter Hoffman (see [4]) that the congruence $q \equiv q + r$ respects exponentiation if and only if for every prime p,

$$p^k \,|\, r \;\Rightarrow\; k \leq q,$$

$$p \,|\, r \;\Rightarrow\; (p-1) \,|\, r.$$

If $q = r = n!$, then both of these conditions hold so that exponentiation *is* defined on our algebras B_n. Furthermore the transition morphisms $\phi : B_{n+1} \to B_n$ obviously preserve it, thus in $\widehat{\mathbf{N}}$, exponentiation is componentwise on the B_n.

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