Anomaly-Free Effective Constraints in Loop Quantum Gravity

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Why effective equations in LQG?

- Dynamics is not yet under control (Hamiltonian operator or propagator, quantization ambiguities)
- Even in symmetry reduced models the analysis of the dynamical difference equations has proved difficult to deal with
- Effective classical systems are easier to work with. They have been used successfully in QFT
- Use effective equations (which are in closer contact with classical space-time notions) to explore possible implications from corrections expected from a loop quantization:
  - Inverse triad corrections
  - Holonomy corrections
  - Quantum back reaction effects

Strategy/Goals

Complete derivation of effective equations in LQG is out of reach, but we can extract information by imposing consistency conditions, particularly anomaly-freedom to derive candidate effective equations. This serves several purposes:

- To do a quick consistency check and to get insights of the full model of LQG (anomaly problem, semiclassical limit),
- to check for robustness of the predictions of homogeneous models in LQC,
- to obtain physically relevant information arising from the specific quantum corrections (phenomenology), and maybe even tackle some more fundamental questions...

Unless effective equations are derived rigorously, their predictions should be taken with care!

Outline

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Geometrical Formulation of Quantum Mechanics

Quantum Mechanics may be formulated in the language of symplectic geometry in close analogy with classical mechanics.The Hilbert space of quantum states may be seen as a manifold: quantum phase space (QPS) with a symplectic structure inherited from the inner product $\langle \cdot, \cdot \rangle$. Hermitian operators define observables or functions on quantum phase space through expectation values. Specifically:

- Hilbert space $\mathcal{H}$ can be seen as a real vector space (with complex structure).
- $\langle \Phi, \Psi \rangle = \frac{1}{2\hbar} g(\Phi, \Psi) + \frac{i}{2\hbar} \Omega(\Phi, \Psi)$
- Hermitian operators $\widehat{F}$ define functions on $\mathcal{H}$:

$$F := \langle \widehat{F} \rangle, \qquad F(\Psi) = \langle \Psi, \widehat{F} \Psi \rangle$$

- Associated Hamiltonian vector field "$X_F = \Omega^{ab} dF_a$":

$$X_F(\Psi) = \frac{1}{i\hbar}\widehat{F}\Psi$$

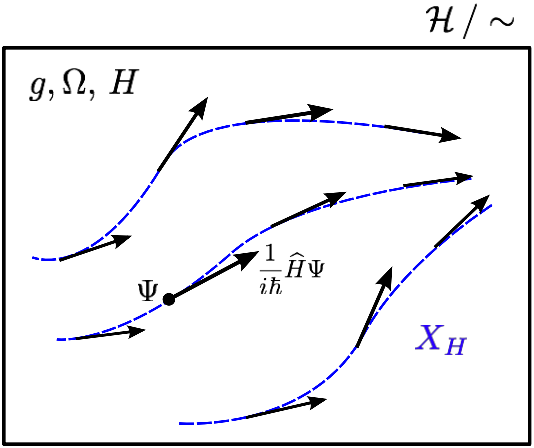
- Poisson Bracket for $F = \langle\widehat{F}\rangle$, $G = \langle\widehat{G}\rangle$:

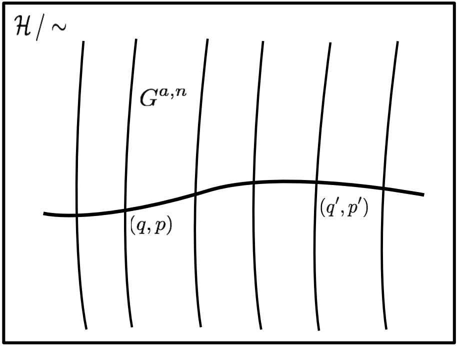
$$\{F, G\} := \frac{1}{i\hbar}\langle[\widehat{F}, \widehat{G}]\rangle$$

- For physical Hamiltonian $\widehat{H}$, its flow X_H gives Schrödinger equation: $\frac{d\Psi}{dt} = X_H(\Psi) = \frac{1}{i\hbar}\widehat{H}\Psi$
- For a general observable F:

$$\dot{F} = \frac{d}{dt}\langle\widehat{F}\rangle = \frac{1}{i\hbar}\langle[\widehat{F}, \widehat{H}]\rangle = \{F, H\}$$

Geometrical Formulation of Quantum Mechanics





- Particularly for basic operators $\hat{q}$ and $\hat{p}$ with canonical commutation relations: $\{q,p\} := \frac{1}{i\hbar}\langle[\hat{q},\hat{p}]\rangle = 1$. The variables $q := \langle\hat{q}\rangle$, $p := \langle\hat{p}\rangle$ define a fiber bundle structure on quantum phase space, and we may embed the classical phase space as a cross section.

- We may put "coordinates" on QPS adapated to these structures, using expectation values of basic operators for fibers and higher moments to identify points on each fiber:

$$G^{a,n} = \langle(\hat{q}-\langle\hat{q}\rangle)^{n-a}(\hat{p}-\langle\hat{p}\rangle)^{a}\rangle_{\rm Weyl}\,, \qquad 1 < n \in \mathbb{N},\ 0 \leq a \leq n$$

- Quantum Hamiltonian

$$\begin{aligned} H^Q(q,p,G^{a,n}) :&= \langle H(\hat{q},\hat{p})\rangle_{\mathrm{Weyl}} \\ &= \sum_{n=0}^{\infty}\sum_{a=0}^{n}\frac{1}{(n-a)!a!}\frac{\partial^n H(q,p)}{\partial q^{n-a}\partial p^a}G^{a,n} \end{aligned}$$

-

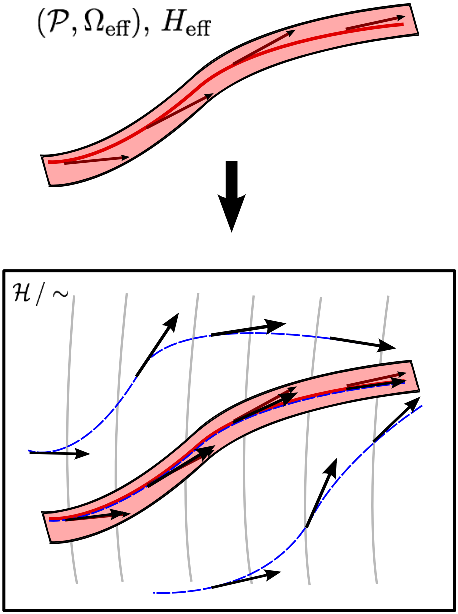
$$\dot{q} = \{q, H^Q\}, \quad \dot{p} = \{p, H^Q\},$$
$$\dot{G}^{a,n} = \{G^{a,n}, H^Q\}$$

Effective Equations

- Infinite number of coupled ODE's. Use approximations to
  - Truncate system to a finite number of equations and degrees of freedom (ignore higher moments quantum back reaction),
  - or decouple equations (no quantum back reaction)
- Different ways to derive/obtain this reduced set of effective classical equations (with the same number of classical degrees of freedom or higher). Use semiclassical or coherent states, $\hbar$-expansions, addiabatic approximation, etc.

Effective Classical Hamiltonian Systems

Symplectic or Poisson manifold $\mathcal{P}$ (submanifold of QPS) with Hamitonian H_{eff}, whose flow $X_{H_{\text{eff}}}$ 'approximates' in some sense the quantum unitary flow X_{HQ}


$(\mathcal{P}, \Omega_{\text{eff}}),\ H_{\text{eff}}$
$\mathcal{H} / \sim$

- Simple systems: linear systems (harmonic oscillator), anharmonic oscillator.
- Successfully used in Loop Quantum Cosmology. Classical equations containing corrections from LQG (some more or less rigorous derivations, "holonomization": $c \rightarrow \frac{\sin \delta c}{\delta}$, etc.), more later today ...

Effective Constraints

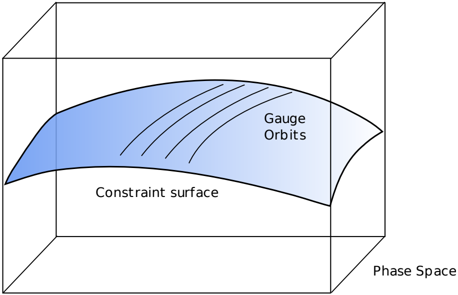


Figure : Classically, first class constraints restrict dynamics to the submanifold of phase space where the constraints vanish. Their Hamiltonian flow is tangential to this 'constraint surface' and determines the 'gauge orbits'.

What about?

$$\widehat{C}_i|\Psi\rangle = 0$$

Proposals

- $< \widehat{C}_i^{\,2} >= 0$ for $\mathcal{H}_{\text{Phys}} \subset \mathcal{H}$
- Infinite tower of constraints: $C_{i,f} :=< f(\hat{q},\hat{p})\widehat{C}_i >= 0$

Outline

Anomaly-free Effective constrained systems

Bypass complications, study particular or symmetry-reduced models and ask a more basic question:

Is there a consistent (first-class) effective system with constraints implementing (some of) the quantum corrections expected from LQG?

In other words, can we "deform" the H and D constraints (insert corrections) and still get a closed algebra?

Why Anomaly Freedom?

Role of constraints

- Restrict initial values of the fields to those which make the constraints vanish
- Generate gauge transformations which in the case of GR coincide with coordinate transformations
- Provide equations of motion for the fields in any coordinate time parameter.

Consistency: constraints must be preserved under the time evolution they generate. Guaranteed if they generate a closed Poisson algebra. Quantum corrections to constraints cannot appear in arbitrary forms, but must be restricted so that the deformed Poisson algebra closes.

Implementation

- Parameterize our ignorance with general correction functions $\alpha_i[A, E]$ i.e. substitute

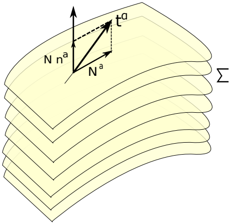
$$H_{\text{class}} \to H_{\text{eff}}^{\alpha_i}$$

- Imposing anomaly freedom, $\{H, H\}$ and $\{H, D\}$ brackets give consistency conditions for the corrections (PDE's for the α_i's)
- Consistent deformations exist. Different models show generic form

$$\{H_{\text{eff}}^{\alpha_i}[M], H_{\text{eff}}^{\alpha_i}[N]\} = D[\beta\, q^{ab}(M\partial_b N - N\partial_b M)]$$

- Extract information from the consistent effective equations

Hypersurface Deformation Algebra



$$\{H[M], H[N]\} = D[q^{ab}(M\partial_b N - N\partial_b M)]$$
$$\{H[N], D[N^a]\} = -H[N^a\partial_a N]$$
$$\{D[N^a], D[M^b]\} = D[[N^a, M^b]]$$

This algebra is a fundamental object, encoding not only the gauge symmetries of Einstein's theory but the structure of spacetime. The Poisson bracket relations express the fact that dynamics takes place on spacelike hypersurfaces embedded in a pseudo-Riemannian spacetime (Hojman,Teitelboim,Kuchar). In a Hamiltonian formulation the dynamics of a general field are obtained by prescribing the field on a spacelike hypersurface and then deforming this hypersurface through spacetime.

Hypersurface Deformation Algebra

- The deformations of hypersurfaces in a pseudo-Riemannian spacetime observe a simple geometrical pattern, and any dynamics taking place on such a spacetime must reflect the structure of this pattern. The closing relations ensure that consecutive deformations of hypersurface embeddings result in the same final embedding.

Hypersurface Deformation Algebra

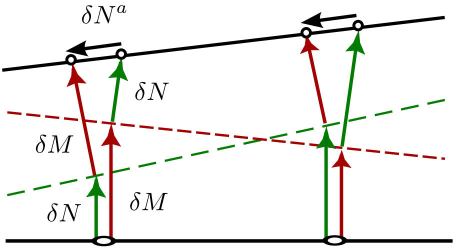


Figure : The commutator of the generators of 'pure deformations' or 'translations': $[\mathcal{H}_{\delta M}, \mathcal{H}_{\delta N}] = \mathcal{D}_{\delta N^a}$. The 'stretching' $\delta N^a = q^{ab}(\delta M \partial_b \delta N - \delta N \partial_b \delta M)$ is needed to compensate reverse order of the two 'translations' δM and δN.

Hypersurface Deformation Algebra

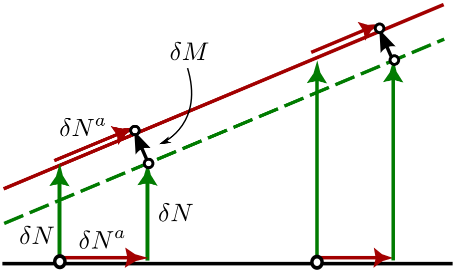


Figure : The commutator of a 'pure deformation' and a 'stretching' or spatial diffeomorphism: $[\mathcal{H}_{\delta N}, \mathcal{D}_{\delta N^a}] = -\mathcal{H}_{\delta M}$. A 'translation' $\delta M = -\delta N^a \partial_a \delta N$ compensates for reversing the order of a 'translation' δN and a 'stretching' δN^a.

Deformed space-time structures????

$$\{H_{\text{eff}}^{\alpha_i}[M], H_{\text{eff}}^{\alpha_i}[N]\} = D[\beta\, q^{ab}(M\partial_b N - N\partial_b M)]$$

Two cases $\beta = 1$ or $\beta \neq 1$ and two options:

- Discard the $\beta \neq 1$ solutions as nonsense!
- Take this deformed algebras as hints of some emergent modified structure of space-time??

Outline

Spherically Symmetric Ashtekar Variables

Adapted Spatial Coords. (x, ϑ, φ)

$$A_x(x),\ K_\varphi(x),\ \eta(x)$$

$$E^x(x),\ E^\varphi(x),\ P^\eta(x)$$

$$\{A_x(x), \frac{1}{2\gamma}E^x(y)\} = \{K_\varphi(x), E^\varphi(y)\} = \{\eta(x), \frac{1}{2\gamma}P^\eta(y)\} = G\delta(x,y)$$

- Metric

$$dq^2 = \frac{E^{\varphi\,2}}{|E^x|}dx^2 + |E^x|d\Omega^2$$

- Angular part of Spin Connection

$$\Gamma_\varphi = -\frac{E^{x\,\prime}}{2E^\varphi}$$

Constraints

Gauss Constraint

$$G_{\rm grav}[\lambda] = \frac{1}{2G\gamma}\int dx\, \lambda(E^{x\,\prime} + P^\eta)$$

Diffeomorphism Constraint

$$D_{\rm grav}[N^x] = \frac{1}{2G}\int dx\, N^x(2E^\varphi K_\varphi' - K_x E^{x\,\prime} + \frac{1}{\gamma}\eta'(E^{x\,\prime} + P^\eta))$$

Hamiltonian Constraint

$$H_{\rm grav}[N] = -\frac{1}{2G}\int dx\, N|E^x|^{-\frac{1}{2}}(K_\varphi^2 E^\varphi + 2K_\varphi K_x E^x + (1-\Gamma_\varphi^2)E^\varphi + 2\Gamma_\varphi' E^x)$$

Modified Effective Hamiltonian

$$H^{\mathrm{eff}}_{\mathrm{grav}}[N] = -\frac{1}{2G}\int \mathrm{d}x\, N\Big(\alpha\,|E^x|^{-\frac{1}{2}}E^\varphi f_1 + 2s\bar{\alpha}\,|E^x|^{\frac{1}{2}} f_2 + \alpha\,|E^x|^{-\frac{1}{2}}E^\varphi - \alpha_\Gamma\,|E^x|^{-\frac{1}{2}}E^\varphi\Gamma_\varphi^2 + 2s\bar{\alpha}_\Gamma\,|E^x|^{\frac{1}{2}}\Gamma_\varphi'\Big)\,.$$

Correction functions α's depending on triad variables $[E^x, E^\varphi]$, $f_1[K_\varphi, E^x, E^\varphi]$ and $f_2[A_x + \eta', K_\varphi, E^x, E^\varphi]$.
Classically $f_1 = K_\varphi^2,\ f_2 = K_\varphi(A_x + \eta')$. $\alpha = \bar{\alpha} = \alpha_\Gamma = \bar{\alpha}_\Gamma = 1$

Inverse Triad Corrections

$$
\begin{aligned}
\{H^{\mathrm{eff}}_{\mathrm{grav}}[M], H^{\mathrm{eff}}_{\mathrm{grav}}[N]\} = {} & D_{\mathrm{grav}}[\bar{\alpha}\bar{\alpha}_\Gamma |E^x|(E^\varphi)^{-2}(MN' - NM')] \\
& - G_{\mathrm{grav}}[\bar{\alpha}\bar{\alpha}_\Gamma |E^x|(E^\varphi)^{-2}(NM' - MN')\eta'] \\
& + \frac{1}{2G}\int \mathrm{d}x\,(MN' - NM')(\bar{\alpha}\alpha_\Gamma - \alpha\bar{\alpha}_\Gamma)\frac{sK_\varphi (E^x)'}{E^\varphi} \\
& + \frac{1}{2G}\int \mathrm{d}x\,(MN' - NM')(\bar{\alpha}'\bar{\alpha}_\Gamma - \bar{\alpha}\bar{\alpha}'_\Gamma)\frac{2K_\varphi |E^x|}{E^\varphi}\,.
\end{aligned}
$$

gives

$$
\bar{\alpha}\alpha_\Gamma - \alpha\bar{\alpha}_\Gamma - 2E^x\left(\bar{\alpha}_\Gamma \frac{\partial\bar{\alpha}}{\partial E^x} - \bar{\alpha}\frac{\partial\bar{\alpha}_\Gamma}{\partial E^x}\right) = 0
$$

Inverse Triad Corrections

$$
\begin{aligned}
\{H^{\mathrm{eff}}_{\mathrm{grav}}[N], D_{\mathrm{grav}}[N^x]\} &= -H^{Q}_{\mathrm{grav}}[N^x N'] \\
&\quad -\frac{1}{2G}\int \mathrm{d}x\, N(N^x)' E^\varphi \Big(\frac{\partial \alpha}{\partial E^\varphi} |E^x|^{-\frac{1}{2}} K_\varphi^2 E^\varphi + 2s \frac{\partial \bar{\alpha}}{\partial E^\varphi} K_\varphi K_x |E^x|^{\frac{1}{2}} \\
&\quad + \frac{\partial \alpha}{\partial E^\varphi} |E^x|^{-\frac{1}{2}} E^\varphi - \frac{\partial \alpha}{\partial E^\varphi} |E^x|^{-\frac{1}{2}} \Gamma_\varphi^2 E^\varphi + 2s \frac{\partial \bar{\alpha}}{\partial E^\varphi} \Gamma_\varphi' |E^x|^{\frac{1}{2}} \Big) \,.
\end{aligned}
$$

gives

$$
\alpha^{-1} \frac{\partial \alpha}{\partial E^\varphi} = \alpha_\Gamma^{-1} \frac{\partial \alpha_\Gamma}{\partial E^\varphi} = \bar{\alpha}_\Gamma^{-1} \frac{\partial \bar{\alpha}_\Gamma}{\partial E^\varphi}
$$

'Holonomized' version

$$H^{\text{eff}}_{\text{grav}}[N] = -\frac{1}{2G}\int dx\, N|E^x|^{-\frac{1}{2}}\left[\frac{\sin^2(\delta\gamma K_\varphi)}{\delta^2\gamma^2}E^\varphi + 2\frac{\sin(2\delta\gamma K_\varphi)}{2\delta\gamma}K_x E^x \right.$$
$$\left. + (1-\Gamma_\varphi^2)E^\varphi + 2\Gamma'_\varphi E^x\right]$$

$$\{H^{\text{eff}}_{\text{grav}}[M], H^{\text{eff}}_{\text{grav}}[N]\} = D_{\text{grav}}[\cos(2\delta\gamma K_\varphi)|E^x|(E^\varphi)^{-2}(MN' - NM')]$$
$$- G_{\text{grav}}[\cos(2\delta\gamma K_\varphi)|E^x|(E^\varphi)^{-2}(NM' - MN')\eta']$$

$$\{H^{\text{eff}}_{\text{grav}}[N], D_{\text{grav}}[N^x]\} = -H^{Q}_{\text{grav}}[N^x N']$$

Outline

Conclusions and Remarks

- Effective equations are a powerful tool that allows to extract information from the underlaying quantum theory
- The geometrical formulation of effective systems is very useful for canonical quantum systems, particularly LQG
- Rigorous derivations of effective equations for more interesting systems are lacking but we can still impose some consistency conditions like anomaly-freedom to gain some insight.
- Consistent anomaly-free deformations of the classical first class constraint algebra exist (incorporating both possible inverse triad and some holonomy effects).
- Even with these simple approximations lessons can be learned:
  - Phenomenology: Bounces not obvious in inhomogeneous situations, obstructions for homogeneous solutions, etc.
  - Interpretation of effective "geometries" requires more careful analysis.

...I think these are good reminders that we have a long way to go...
THANKS!