Topology and geometry of the hyperplane section

Gian Pietro Pirola
Pavia University

The lectures will be concerned about the geometry and the topology of the hyperplane sections of an algebraic variety. In particular we will discuss the Bertini and Lefschetz theorems, we will focus on classical and new applications.

1 Bertini theorem

1.1 Connectedness and irreducibility

Let $X \subset \mathbb{P}^N$ be a complex projective variety of dimensione $n \geq 2$. The classical Bertini theorem (see [1] and [2]) gives:

i) $Y = H \cap X$ is connected;

ii) let X_s be the locus of smooth points of X, if H is general then $Y_0 = X_s \cap H$ is smooth.

In particular we discuss a new proof of i), given in [3], that reduced to the surface case, that is the case $n = 2$. The argument is algebraic and descends from the Hodge Index Theorem.

1.2 Fulton and Hansen Theorem

We prove the Fulton and Hansen Theorem (see [1]) and illustrate some applications.

2 Lefschetz theory

2.1 Classical theory

We would like to give an introduction to the Lefschetz hyperplane theorem and the monodromy action on a Lefschetz pencil (cf. Chapters I–III in volume II of [4]).

2.2 Surfaces in $\mathbb{P}^3$

We apply Lefschetz' theory to obtain the classical proof of the Noether-Lefschetz theorem [4], and some recent results [5] on the rational functions field of a very general surface of degree ≥ 5 of $\mathbb{P}^3$.

References

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[2] R. Lazarsfeld. *Positivity in algebraic geometry. I,II,*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A, (48) and (49), Springer-Verlag, Berlin, 2004.

[3] D. Martinelli, J. Naranjo and G. Pirola. *A Generalization of Bertini theorem via Numerical connectedness of surfaces.* to appear on Adv. in Geometry.

[4] C. Voisin *Hodge Theory and Complex Algebraic Geometry, I and II.* Cambridge studies in advanced mathematics 76 (2002).

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