Yang-Mills Connections over Homogeneous Spaces

Jesús Muciño*, Marcos Rosenbaum*, Raymundo Bautista*

ABSTRACT

In this note we describe a procedure for obtaining explicit Yang-Mills connections in principal fiber bundles $P$, with structural group $G$, over an homogeneous space $M$. We use connections which are invariant under a Lie group action in $P$. Explicit solutions over $M = S^2 \times S^2$, where $G = SU(2)$ are given, and their second Chern number computed.

1.1 Yang-Mills connections and invariant connections

We recall some facts about Yang-Mills theory (see [4]).

Let $P$ be a principal $G$ bundle, $P \to M$, (all the objects will be smooth), and $\mathcal{A}(P)$ the space of smooth connections in $P$: Given $A \in \mathcal{A}(P)$ and having fixed a Riemannian metric in $M$, the Yang-Mills functional measures the total curvature of connection $A$ as

$$YM(A) = \frac{1}{8\pi^2} \int_M |F_A|^2 \, d\nu.$$

Here $F_A$ is the curvature of the connection $A$, $\| \cdot \|$ is the natural norm on the $L(G)$-valued differential forms ($L(G)$ denotes the Lie algebra of $G$), and $d\nu$ is the volume form in $M$. If $G = SU(2)$ the Yang-Mills functional takes values in $[|n|, \infty)$ for any connection $A \in \mathcal{A}(P)$, where the integer $n$ is the second Chern number of $P$. The connections $A$ over $P$ such that the Yang-Mills functional takes on $A$ the minimal value $|n|$ are called multi-instantons (for $n \geq 0$) or multi-anti-instantons (for $n < 0$): These minimal points of the Yang-Mills functional are precisely the set of connections whose curvature are self-dual (or anti-self-dual): The Hodge star operator $\ast : \Lambda^k(T^*M) \to \Lambda^{4-k}(T^*M)$ is uniquely defined by the requirement that for each $k$-form $\eta$, $\eta \wedge \ast \eta = (\eta, \eta) d\nu$, where $(\ , \ )$ is the scalar
product in $\Lambda^k(T^*M)$ induced by the Riemannian metric in $M$, and $d\nu$ is the volume form in $M$. We know that a connection $A$ lies in the set of minimal points of the Yang-Mills functional if and only if the curvature of $A$ satisfies

$$F_A = \pm * F_A,$$

where the signs correspond to $\pm n > 0$. The connection is self-dual (anti-self-dual) if the sign $+$ (resp. $-$) occurs.

The above equation are called the (anti) self-duality equations. In local coordinates the (anti) self-duality equations give a system of partial differential equations for the connections $\{A\}$ in $P$: The critical points of the Yang-Mills functional in the space $A(P)$ of smooth connections in $P$, are called Yang-Mills connections. Solutions of the self-duality equations (anti-self-duality) give Yang-Mills connections.

We are interested in finding explicit solutions to the (anti) self-duality equations.

We use the theory of invariant connections under the action of a Lie group in $P$. A study of such connections was carried out by Wang (see [11] and [3] vol. I, p. 103). These connections are also used by M. Itoh and T. Laquer in the Yang-Mills framework (see [5], [6]). Our contribution consists in the explicit study of $(SU(2) \times SU(2))$-invariant (anti)-self-dual connections for the case the base manifold $M$ is $S^2 \times S^2$ and the derivation of the corresponding Chern numbers (see [1] and [2]).

For this purpose, let $S$ be a Lie group, and $S$ a fixed smooth transitive action $\phi : S \times P \to P$. Write $M = S/J$ as an homogeneous space, where $J$ is the isotropy group of the associated action of $S$ in $M$. Given Lie group homomorphism $\mu : J \to G$ (with $G$ the structural group of) there exist an associated $G$-principal fiber bundle $P$ such that $S$ acts on $P$ in such way that their action in $P$ projects to the natural action of $S$ in $M$.

Let now $A(w)$ be a connection in $P$ given by a $L(G)$-valued one-form $w$. We say that $A(w)$ is invariant under the action of $S$ in $P$ (or simply $S$-invariant), when $s_* w = w$ for each diffeomorphism $s : P \to P$ with $s \in S$. This means that the action of $S$ leaves invariant the horizontal spaces of $A(w)$ in $P$.

According to Wang's Theorem [11], [3], there is bijective correspondence between $S$-invariant connections and the linear transformations $\Lambda : L(S) \to L(G)$ of the associated Lie algebras, such that the following conditions are satisfied:

$$(W_1) \quad \Lambda(Y) = \mu_* (Y) \quad \text{for} \quad Y \in L(J).$$

$$L(S) \xrightarrow{\Lambda} L(G)$$

$$(W_2) \quad \begin{array}{ccc}
L(S) & \xrightarrow{\Lambda} & L(G) \\
\downarrow & & \downarrow \\
L(S) & \xrightarrow{\Lambda} & L(G)
\end{array}$$

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where \( j \in J \).

At a point \( p \in P \), the correspondence is given by:

\[
 w : T_pP \to L(G)
\]

\[
 w(\tilde{Y}) \mapsto \mu_*(Y_{L(J)}) + \Lambda(Y_M)
\]

where \( M \) is linear subspace of \( L(S) \) such that

\[
 L(S) = L(J) \oplus M, \quad Y = Y_{L(J)} + Y_M \in L(S),
\]

and \( \tilde{Y} \in T_pP \) is the tangent vector associated to \( Y \) under the action of \( S \) in \( P \).

Wang also gives a formula for the curvature \( F_w \) of the connection \( A(w) \) in terms of these linear maps as

\[
 F_w(\tilde{X}, \tilde{Y}) = [\mu_*(X_{L(J)}) + \Lambda(X_M), \mu_*(Y_{L(J)}) + \Lambda(Y_M)]
\]

\[
 -\mu_*(\{X, Y]\}_{L(J)} + \Lambda([X, Y])_M,
\]

for \( X, Y \in L(S) \), and \( [\ , \ ] \) the usual bracket in the Lie algebras.

The key observation is that: If we now impose the (anti) self-duality conditions \( F_w = \pm F_w \)
to an \( S \)-invariant connection \( A(w) \), then the problem reduces to solving a set of algebraic
equations for \( \Lambda \).

### 1.2 The case \( M = S^2 \times S^2 \)

We apply the above ideas the concrete case of a base manifold for the bundle \( P \) given by
the homogeneous space \( M = S^2 \times S^2 \), where the symmetry group is \( S = SU(2) \times SU(2) \)
and the isotropy group is \( J = U(1) \times U(1) \). \( S^2 \times S^2 \) can be seen as the non singular
quadric \( \{z_1^2 + z_2^2 + z_3^2 = 0\} \) in \( \mathbb{CP}^3 \), the three dimensional complex projective space. The
existence of (anti) instantons in this case follows from the work of S. Soberón-Chávez [9], on the classification of stable complex vector bundles of rank two (in the sense of Mumford-Takemoto), in particular, from the computation of the number of parameters of the moduli spaces for these bundles, and from the theorem by S. K. Donaldson [4] which gives a correspondence between stable complex vector bundles of rank two and solutions to the (anti) self-duality equations (see [4]).

Given the maps \( \mu : J = U(X) \times U(1) \to SU(2) = G \) by \( \mu(j_1j_2) = (j_1)^m(j_2)^n \), where
\( m, n \) are integers and \( (j_1)^m(j_2)^n \) is the usual product of powers in a subgroup \( U(1) \) of \( SU(2) \), we have an associated principal \( SU(2) \)-bundle \( P_{(m,n)} \) over \( S^2 \times S^2 \), with an action
of $S = SU(2) \times SU(2)$. Introduce the local coordinates $(y_1, \ldots, y_4)$ in $S^2 \times S^2$, given by two copies of the stereographic projection $\mathbb{R}^2 \hookrightarrow S^2$.

Our result is:

**Theorem** ([1], [2]): There exist in the $SU(2)$-bundles $P_{(m,m)}$ over $S^2 \times S^2$, ($SU(2) \times SU(2)$)-invariant connections $A(m)$ such that:

i) The linear homomorphism $\Lambda : SU(2) \oplus SU(2) \to SU(2)$ is given by $\Lambda(Y_1) = \Lambda(Y_4) = mX_1$ and $\Lambda(Y_j) = 0$, $j = 2, 3, 5, 6$, where $\{Y_1, \ldots, Y_6\}$ is a basis in the domain and $\{X_1, X_2, X_3\}$ is a basis in the codomain.

ii) The curvature in the local coordinates $(y_1, \ldots, y_4)$ of $S^2 \times S^2$ are:

$$F_{A(m)} = \frac{4mX_1}{(1 + y_1^2 + y_2^2)^2} dy_1 \wedge dy_2 + \frac{4mX_1}{(1 + y_1^2 + y_3^2)^2} dy_3 \wedge dy_4$$

where $m$ is an integer and $X_1 \in SU(2)$ as above.

iii) These connections are self-dual ($m \geq 0$) or anti-self-dual ($m < 0$).

iv) The second Chern numbers of the bundles $P_{(m,m)}$ are $2m^2$.

Some Remarks.

- Here we use the canonical product metric of constant Gaussian curvature one in each factor of $S^2 \times S^2$. With this metric $S^2 \times S^2$ is not conformaly equivalent to $\mathbb{R}^4$ with their flat metric.

- Between all the ($SU(2) \times SU(2)$)-invariant connections over $S^2 \times S^2$ (wich are in fact continuous families [2]), only one for each Chern number is a Yang-Mills connection.

- Note that, since $S^2 \times S^2$ admits an orientation reversing isometry, the self-dual and the anti-self-dual connections are equivalent.

### 1.3 Further Comments

The method can be applied to other homogeneous spaces (see [6]). However, a more interesting applications of this type of techniques might be their extension to non-transitive actions of a Lie group in the base manifold $M$. In this case we have a foliation (probably
with singularities) of $M$ by homogeneous spaces of lower dimension, and a smooth collection of invariant connections on the leaves of the foliation, which are parametrized by smooth transversal of the foliation. Under this construction the problem of finding Yang-Mills connections becomes one of solving a set of differential equations in the transversal parameters. This procedure is also useful in dealing with non-minimal Yang-Mills connections (see [10], [8], [7]).

References

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+ Instituto de Matemáticas, UNAM
* Instituto de Ciencias Nucleares, UNAM
Coyoacán, México, D.F. 04510

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