Aportaciones Matemáticas
Notas de Investigación 8 (1992) 49-54

Yang-Mills Connections over Homogeneous Spaces

Jesús Muciño+, Marcos Rosenbaum*, Raymundo Bautista+

ABSTRACT

In this note we describe a procedure for obtaining explicit Yang-Mills connections in principal fiber bundles P, with structural group G, over an homogeneous space M. We use connections which are invariant under a Lie group action in P. Explicit solutions over $M = S^2 \times S^2$, where $G = SU(2)$ are given, and their second Chern number computed.

1.1 Yang-Mills connections and invariant connections

We recall some facts about Yang-Mills theory (see [4]).

Let P be a principal G bundle, $P \to M$, (all the objects will be smooth), and $\mathcal{A}(P)$ the space of smooth connections in P: Given $A \in \mathcal{A}(P)$ and having fixed a Riemannian metric in M, the Yang-Mills functional measures the total curvature of connection A as

$$YM(A) = \frac{1}{8\pi^2} \int_M |F_A|^2 \, d\nu.$$

Here F_A is the curvature of the connection A, $|\ |$ is the natural norm on the $L(G)$-valued differential forms ($L(G)$ denotes the Lie algebra of G), and $d\nu$ is the volume form in M. If $G = SU(2)$ the Yang-Mills functional takes values in $[|n|, \infty)$ for any connection $A \in \mathcal{A}(P)$, where the integer n is the second Chern number of P. The connections A over P such that the Yang-Mills functional takes on A the minimal value $|n|$ are called multi-instantons
(for $n \geq 0$) or multi-anti-instantons (for $n < 0$): These minimal points of the Yang-Mills functional are precisely the set of connections whose curvature are self-dual (or anti-self-dual): The Hodge star operator $* : \Lambda^k(T^*M) \to \Lambda^{4-k}(T^*M)$ is uniquely defined by the requirement that for each k-form η; $\eta \wedge *\eta = (\eta, \eta) d\nu$, where $(\ ,\ )$ is the scalar

Proceedings of Spaces of Geometries Workshop, CIMAT, Guanajuato, Mexico 20-24 April 1992

product in $\Lambda^k(T^*M)$ induced by the Riemannian metric in M, and $d\nu$ is the volume form in M. We know that a connection A lies in the set of minimal points of the Yang-Mills functional if and only if the curvature of A satisfies

$$F_A = \pm * F_A,$$

where the signs correspond to $\pm n > 0$. The connection is self-dual (anti-self-dual) if the sign $+$ (resp. $-$) occurs.

The above equation are called the (anti) self-duality equations. In local coordinates the (anti) self-duality equations give a system of partial differential equations for the connections $\{A\}$ in P: The critical points of the Yang-Mills functional in the space $\mathcal{A}(P)$ of smooth connections in P, are called Yang-Mills connections. Solutions of the self-duality equations (anti-self-duality) give Yang-Mills connections.

We are interested in finding explicit solutions to the (anti) self-duality equations.

We use the theory of invariant connections under the accion of a Lie group in P. A study of such connections was carried out by Wang (see [11] and [3] vol. I, p. 103). These connections are also used by M. Itoh and T. Laquer in the Yang-Mills framework (see [5], [6]). Our contribution consists in the explicit study of $(SU(2) \times SU(2))$-invariant (anti)-self-dual connections for the case the base manifold M is $S^2 \times S^2$ and the derivation of the corresponding Chern numbers (see [1] and [2]).

For this purpose, let S be a Lie group, and S a fixed smooth transitive action $\phi : S \times P \to P$. Write $M = S/J$ as an homogeneous space, where J is the isotropy group of the associated action of S in M. Given Lie group homomorphism $\mu : J \to G$ (with G the structural group of) there exist an associated G-principal fiber bundle P such that S acts on P in such way that their action in P projects to the natural action of S in M.

Let now $A(w)$ be a connection in P given by a $L(G)$-valued one-form w. We say that $A(w)$ is invariant under the action of S in P (or simply S-invariant), when $s_* w = w$ for each diffeomorphism $s : P \to P$ with $s \in S$. This means that the action of S leaves invariant the horizontal spaces of $A(w)$ in P.

According to Wang's Theorem [11], [3], there is bijective correspondence between S-invariant connections and the linear transformations $\Lambda : L(S) \to L(G)$ of the associated Lie algebras, such that the following conditions are satisfied:

(W_1) $$\Lambda(Y) = \mu_*(Y) \qquad \text{for} \qquad Y \in L(J).$$

(W_2) $$\begin{array}{ccc} L(S) & \xrightarrow{\Lambda} & L(G) \\ \Big\downarrow{\scriptstyle a\delta_j} & & \Big\downarrow{\scriptstyle a\delta_{\mu(j)}} \\ L(S) & \xrightarrow{\Lambda} & L(G) \end{array}$$

where $j \in J$.

At a point $p \in P$, the correspondence is given by:

$$w : T_pP \to L(G)$$

$$w(\tilde{Y}) \longmapsto \mu_*(Y_{L(J)}) + \Lambda(Y_{\mathcal{M}})$$

where $\mathcal{M}$ is linear subspace of $L(S)$ such that

$$L(S) = L(J) \oplus \mathcal{M} \qquad , \qquad Y = Y_{L(J)} + Y_{\mathcal{M}} \in L(S),$$

and $\tilde{Y} \in T_pP$ is the tangent vector associated to Y under the action of S in P.

Wang also gives a formula for the curvature F_w of the connection $A(w)$ in terms of these linear maps as

$$F_w(\tilde{X}, \tilde{Y}) = [\mu_*(X_{L(J)}) + \Lambda(X_{\mathcal{M}}), \mu_*(Y_{L(J)}) + \Lambda(Y_{\mathcal{M}})]$$

$$-\mu_*([X,Y])_{L(J)} - \Lambda([X,Y])_{\mathcal{M}},$$

for $X, Y \in L(S)$, and [,] the usual bracket in the Lie algebras.

The key observation is that: If we now impose the (anti) self-duality conditions $F_w = *\pm F_w$ to an S-invariant connection $A(w)$, then the problem reduces to solving a set of algebraic equations for Λ.

1.2 The case $M = S^2 \times S^2$

We apply the above ideas the concrete case of a base manifold for the bundle P given by the homogeneous space $M = S^2 \times S^2$, where the simetry group is $S = SU(2) \times SU(2)$ and the isotropy group is $J = U(1) \times U(1)$. $S^2 \times S^2$ can be seen as the non singular quadric $\{z_1^2 + z_2^2 + z_3^2 = 0\}$ in $\mathbf{CP}^3$, the three dimensional complex projective space. The existence of (anti) instantons in this case follows from the work of S. Soberón-Chávez [9], on the classification of stable complex vector bundles of rank two (in the sense of Mumford-Takemoto), in particular, from the computation of the number of parameters of the moduli spaces for these bundles, and from the theorem by S. K.Donalson [4] which gives a correspondence between stable complex vector bundles of rank two and solutions to the (anti) self-duality equations (see [4]).

Given the maps $\mu : J = U(X) \times U(1) \to SU(2) = G$ by $\mu(j_1 j_2) = (j_1)^m (j_2)^n$, where m, n are integers and $(j_1)^m (j_2)^n$ is the usual product of powers in a subgroup $U(1)$ of $SU(2)$, we have an associated principal $SU(2)$-bundle $P_{(m,n)}$ over $S^2 \times S^2$, with an action of $S = SU(2) \times SU(2)$. Introduce the local coordinates $(y_1, \ldots, y_4)$ in $S^2 \times S^2$, given by two copies of the stereographic proyection $\mathbf{R}^2 \hookrightarrow S^2$.

Our result is:

Theorem ([1], [2]): There exist in the $SU(2)$-bundles $P_{(m,m)}$ over $S^2 \times S^2$, $(SU(2) \times SU(2))$-invariant connections $A(m)$ such that:

i) The linear homomorphism $\Lambda : SU(2) \oplus SU(2) \to SU(2)$ is given by $\Lambda(Y_1) = \Lambda(Y_4) = mX_1$ and $\Lambda(Y_j) = 0, \quad j = 2, 3, 5, 6$, where $\{Y_1, \ldots, Y_6\}$ is a basis in the domain and $\{X_1, X_2, X_3\}$ is a basis in the codomain.

ii) The curvature in the local coordinates $(y_1, \ldots, y_4)$ of $S^2 \times S^2$ are:

$$F_{A(m)} = \frac{4mX_1}{(1 + y_1^2 + y_2^2)^2} dy_1 \wedge dy_2 + \frac{4mX_1}{(1 + y_1^2 + y_2^2)^2} dy_3 \wedge dy_4$$

where m is an integer and $X_1 \in SU(2)$ as above.

iii) These connections are self-dual ($m \geq 0$) or anti-self-dual ($m < 0$).

iv) The second Chern numbers of the bundles $P_{(m,m)}$ are $2m^2$.

Some Remarks.

- Here we use the cannonical product metric of constant Gaussian curvature one in each factor of $S^2 \times S^2$. With this metric $S^2 \times S^2$ is not conformaly equivalent to $\mathbf{R}^4$ with their flat metric.
- Between all the $(SU(2) \times SU(2))$-invariant connections over $S^2 \times S^2$ (wich are in fact continuous families [2]), only one for each Chern number is a Yang-Mills connection.
- Note that, since $S^2 \times S^2$ admits an orientation reversing isometry, the self-dual and the anti-self-dual connections are equivalent.

1.3 Futher Comments

The method can be applied to other homogeneous spaces (see [6]). However, a more interesting applications of this type of tecniques might be their extension to non-transitive actions of a Lie group in the base manifold M. In this case we have a foliation (probably with singularities) of M by homogeneous spaces of lower dimension, and a smooth collection of invariant connections on the leaves of the foliation, wich are parametrized by smooth transversal of the foliation. Under this construction the problem of finding Yang-Mills connections becomes one of solving a set of differential equations in the transversal parameters. This procedure is also useful in dealing with non-minimal Yang-Mills connections (see [10], [8], [7]).

References

- R. Bautista, J. Muciño, M. Rosenbaum: $SU(2)$-Multi-instantons over $S^2 \times S^2$. To appear in Letters in Mathematical Physics.
- R. Bautista, J. Muciño, E. Nahmad, M. Rosenbaum: Clasification of Gauge-related invariant connections. To appear in Reviews in Mathematical Physics.
- S. Kobayashi, K. Nomizu: Fundations of Differential Geometry, Vol. I. Wiley-Intersience (1969).
- S. K. Donaldson, P. B. Kronheimer: The Geometry of Four-manifolds. Claredon Press. Oxford (1990).
- M. Itoh: Invariant connections and Yang-Mills solutions. Trans. of the Amer. Math. Soc. Vol. 267 (1), p. 229-236 (1981).
- H. T. Laquer: Stability properties of the Yang-Mills functional near the canonical connection. Michigan Math. J. (31), p. 139-159 (1984).
- T. H. Parker: Non minimal Yang-Mills fields and dynamics. Inv. Math. 107, p. 397-420 (1992).
- L. Sadun, J. Segert: Non-self-dual Yang-Mills connections with nonzero Chern number. Bulletin of the Amer. Math. Soc. Vol. 24 (1), p. 163-170 (1991).
- S. Soberón-Chávez: Rank 2 vector bundles over a complex quadric surface. Quart. J. math. Oxford (2), 36 p. 159-172 (1985).
- H. Urakawa: Equivariant theory of Yang-Mills connections over Riemannian manifolds of cohomogeneity one. Indiana Univ. Math. J. (37), p. 753-788 (1988).
- H. C. Wang: On invariant connections over a principal fibre bundle. Nagoya Math. J. 13, p. 1-19 (1958).

+ Instituto de Matemáticas, UNAM

* Instituto de Ciencias Nucleares, UNAM

Coyoacán, México, D.F. 04510

Apartado Postal 70-543
MEXICO