KALUZA-KLEIN MODEL FOR THE UNIFICATION OF THE BOSONIC SECTOR OF THE ELECTROWEAK MODEL WITH GRAVITATION

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ABSTRACT:
A fiber bundle treatment for the unification of gravitation with the bosonic sector of the standard electroweak theory is presented, and the full quadratic Lagrangian is given. It is shown that the Higgs and Yang-Mills fields arise naturally from the torsion. The formalism suggests the possibility of obtaining a prediction for the values of the Yang-Mills coupling constants by means of the spontaneous compactification of the base manifold.

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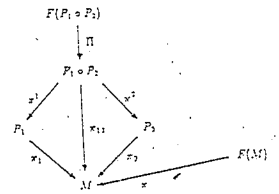
I. The Bundle Framework For $SU(2)\times U(1)$

We shall present here the appropriate fiber bundle formalism for $SU(2)\times U(1)$ which leads to a general G-invariant Kaluza-Klein type Lagrangian which unifies geometrically the bosonic part of the electroweak model with gravitation. To arrive at the appropriate theory requires a new prescription for the law of transformation of the torsion as well as a very careful choice of connections. Due to space restrictions we shall give here only the main ideas; the details will be published elsewhere.

In the specific case of $SU(2)\times U(1)$, the theory requires five different PFB's that are inter-related according to the diagram



Here M denotes an n-dimensional oriented manifold, which we take to be space-time and which acts as the base space of the following PFB's :

1) $\pi : F(M) \longrightarrow M$, is the orthonormal frame bundle of M with group $O(r,s)$. For $u \in F(M)_x$ and the usual basis $\{e_i\}, i = 1,...,n$ of $\Re^n$, we choose an orthonormal frame at $x \in U \subset M$ by means of the linear isomorphism $u : \Re^n \longrightarrow T_xM$, i.e., $u(e_i) = E_i, i = 1,...,n$, are orthonormal vector fields with respect to the metric g on M, defined in a neighborhood U of $x = \pi(u)$ in such a way that the local section $\sigma : U \longrightarrow F(M)$ determined by $E_1,...,E_n$ is tangent to the horizontal subspace of $T_{\sigma(x)}F(M)$ relative to the connection $\theta(g)$. The curvature of this connection $\theta(g) \in \bar{\Lambda}^1(F(M), O(r,s))$ is given by

$$\Omega^{\theta(g)} \equiv D^{\theta(g)}\theta(g) = d\theta(g) + \frac{1}{2}[\theta(g), \theta(g)] \in \bar{\Lambda}^2(F(M), O(r,s)), \tag{1.1}$$

and we can write

$$\Omega^{\theta(g)}(\sigma_* E_i, \sigma_* E_j)(e_k) = R^h{}_{kij}(\sigma(x))e_h. \tag{1.2}$$

Note that for $X_u \in T_uF(M)$ we can define the canonical 1-form $\varphi_M \in \bar{\Lambda}^1(F(M), \Re^n)$ by

$$\varphi_M(X_u) = u^{-1}(\pi_*(X_u)) \in \Re^n. \tag{1.3}$$

In terms of this canonical 1-form the torsion 2-form $\Theta^{\theta(g)}$ of $\theta(g)$ is given by

$$\Theta^{\theta(g)} = D^{\theta(g)}\varphi_M = d\varphi_M + \theta(g)\dot{\wedge}\varphi_M \in \bar{\Lambda}^2(F(M), \Re^n), \tag{1.4}$$

where the quantity $\theta(g)\dot{\wedge}\varphi_M$ is defined by

$$(\theta(g)\dot{\wedge}\varphi_M)(X_u, Y_u) = \theta(g)(X_u)\cdot\varphi_M(Y_u) - \theta(g)(Y_u)\cdot\varphi_M(X_u), \tag{1.5}$$

and the "dot" operation denotes the left action of $O(r,s)$ on $\Re^n$. Furthermore, since $\Theta^{\theta(g)}(\sigma_* \bar{E}_i, \sigma_* \bar{E}_j) \in \Re^n$, we can write

$$\Theta^{\theta(g)}(\sigma_* E_i, \sigma_* E_j) = S^k{}_{ij}(\sigma(x))e_k. \tag{1.6}$$

If we now let $\bar{\varphi}^i_M$ be the 1-forms dual to $\bar{E}_i$, i.e. $\bar{\varphi}^i_M(\bar{E}_j) = \delta^i_j$, then the pull-back with the local section σ of the canonical 1-forms allows us to relate the curvature and torsion tensors in $F(M)$, as given by (1.2) and (1.6), to the corresponding tensors in $T_x(M)$. Indeed, acting with σ^* on (1.1) and using (1.2) we get

$$R^h{}_{kij}(\sigma(x)) = \underline{R}^h{}_{kij}(x). \tag{1.7}$$

Proceeding in a similar fashion with the torsion, we get, from (1.4),

$$S^i{}_{jk}(\sigma(x)) = \underline{S}^i{}_{jk}(x). \tag{1.8}$$

2) $\pi_1 : P_1 \longrightarrow M$, is a PFB with group $G_1 = SU(2)$ and connection $\omega_1 \in \Lambda^1(P_1, \mathcal{G}_1)$, where $\mathcal{G}_1$ is the Lie algebra of G_1. The curvature of the connection ω_1 is

$$\Omega_1 \equiv D^{\omega_1}\omega_1 \equiv d\omega_1 + \frac{1}{2}[\omega_1, \omega_1] \in \bar{\Lambda}^2(P_1, \mathcal{G}_1). \tag{1.9}$$

Note that if ℓ_α $(\alpha = 1, 2, 3)$ is a basis for $\mathcal{G}_1$, we can write $\omega_1 = \omega_1^\alpha \ell_\alpha$, and

$$\Omega_1 = (d\omega_1^\alpha + \frac{1}{2}c^\alpha{}_{\beta\gamma}\omega_1^\beta \wedge \omega_1^\gamma)\ell_\alpha, \tag{1.10}$$

where $c^\alpha{}_{\beta\gamma}$ are the structure constants of G_1. Moreover, if we let $E_1^{(1)}, ..., E_n^{(1)}$ be an orthonormal basis of the horizontal subspace of $T_{p_1}P_1$ relative to ω_1, such that $\pi_{1*}E_i^{(1)} = \bar{E}_i$, and we also let $\bar{\varphi}^1_{(1)}, ..., \bar{\varphi}^n_{(1)}$ be the 1-forms dual to $E_1^{(1)}, ..., E_n^{(1)}$, then we can also write

$$\Omega_1 = \frac{1}{2}(\Omega_1)^\alpha{}_{ij}\ell_\alpha \otimes (\bar{\varphi}^i_{(1)} \wedge \bar{\varphi}^j_{(1)}), \tag{1.11}$$

where $(\Omega_1)^\alpha{}_{ij}$ is a real function in $\pi_1{}^{-1}(U)$.

3) $\pi_2 : P_2 \longrightarrow M$ is a PFB with group $G_2 = U(1)$ and connection $\omega_2 \in \Lambda^1(P_2, \mathcal{G}_2)$, where $\mathcal{G}_2$ is the Lie algebra of G_2. Since $U(1)$ is Abelian, the curvature of the connection ω_2 is

$$\Omega_2 \equiv D^{\omega_2}\omega_2 = d\omega_2 \in \bar{\Lambda}^2(P_2, \mathcal{G}_2). \tag{1.12}$$

Similarly to Ω_1 we can write Ω_2 as

$$\Omega_2 = \frac{1}{2}(\Omega_2)_{ij}\, i \otimes \bar{\varphi}^i_{(2)} \wedge \bar{\varphi}^j_{(2)}, \tag{1.13}$$

where $(\Omega_2)_{ij}$ is a real function defined on $\pi_2{}^{-1}(U)$.

4) $\pi_{12} : P_1 \circ P_2 \longrightarrow M$ is the PFB with group $SU(2) \times U(1)$, obtained by splicing the bundles $\pi_i : P_i \longrightarrow M$. In this way we have that $P_1 \circ P_2 = \{(p_1, p_2) \in P_1 \times P_2 \mid \pi_1(p_1) = \pi_2(p_2)\}$, and that $\pi_{12}(p_1, p_2) = \pi_1(p_1) = \pi_2(p_2)$. It is a simple matter to show that $\pi^{1*}\omega_1 \oplus \pi^{2*}\omega_2$ is a connection for the spliced bundle

$\pi_{12} : P_1 \circ P_2 \longrightarrow M$. We introduce additional structure on $P_1 \circ P_2$ by defining a non-degenerate bundle metric h as follows: Let k_1 and k_2 be Ad-invariant metrics on $\mathcal{G}_1$ and $\mathcal{G}_2$ respectively, and set

$$h = \pi_{12}^{*} g + k_1 \overset{\circ}{\omega}_1 \oplus k_2 \overset{\circ}{\omega}_2 \tag{1.14}$$

where $\overset{\circ}{\omega}_i \equiv \pi^{i*}\omega_i$. Relative to this metric, an orthonormal frame at $(p_1, p_2) \in P_1 \circ P_2$ is given by $\overset{\circ}{E}_1, ..., \overset{\circ}{E}_n, \overset{\circ}{E}_{n+1}, ..., \overset{\circ}{E}_{n+4}$, the horizontal lifts of the orthonormal basis $E_1, ..., E_n$ on (M, g) such that $\pi_{12*} \overset{\circ}{E}_i = E_i$ and $(\overset{\circ}{\omega}_1 \oplus \overset{\circ}{\omega}_2)(\overset{\circ}{E}_i) = 0$, while $\overset{\circ}{E}_{n+\alpha} = \ell_\alpha^* \oplus 0$ $(\alpha = 1, 2, 3)$ and $\overset{\circ}{E}_{n+4} = 0 \oplus \ell_4^*$ are fundamental vertical fields on $P_1 \circ P_2$. With this particular choice of an orthonormal basis on a neighborhood of (p_1, p_2), the calculations in the following sections will simplify considerably. The curvature $\Omega^{\overset{\circ}{\omega}_1 \oplus \overset{\circ}{\omega}_2} \in \bar{\Lambda}^2(P_1 \circ P_2, \mathcal{G}_1 \oplus \mathcal{G}_2)$ of $\overset{\circ}{\omega}_1 \oplus \overset{\circ}{\omega}_2$ is given by

$$\Omega^{\overset{\circ}{\omega}_1 \oplus \overset{\circ}{\omega}_2} \equiv D^{\overset{\circ}{\omega}_1 \oplus \overset{\circ}{\omega}_2}(\overset{\circ}{\omega}_1 \oplus \overset{\circ}{\omega}_2) = \pi^{1*}(\Omega_1^{\omega_1}) \oplus \pi^{2*}(\Omega_2^{\omega_2}) \in \bar{\Lambda}^2(P_1 \circ P_2, \mathcal{G}_1) \oplus \bar{\Lambda}^2(P_1 \circ P_2, \mathcal{G}_2). \tag{1.15}$$

We can write

$$\begin{aligned} \Omega^{\overset{\circ}{\omega}_1 \oplus \overset{\circ}{\omega}_2} &= (\overset{\circ}{\Omega}_1)^\alpha{}_{ij}(\ell_\alpha) \otimes (\pi^{1*}\bar{\varphi}^i_{(1)}) \wedge (\pi^{1*}\bar{\varphi}^j_{(1)}) \oplus (\overset{\circ}{\Omega}_2)_{ij}(i) \otimes (\pi^{2*}\bar{\varphi}^i_{(2)}) \wedge (\pi^{2*}\bar{\varphi}^j_{(2)}) \\ &= [(\overset{\circ}{\Omega}_1)^\alpha{}_{ij}\ell_\alpha \oplus (\overset{\circ}{\Omega}_2)_{ij} i] \otimes \overset{\circ}{\bar{\varphi}}{}^i \wedge \overset{\circ}{\bar{\varphi}}{}^j, \end{aligned} \tag{1.16}$$

where

$$(\overset{\circ}{\Omega}_1)^\alpha{}_{ij} = (\Omega_1)^\alpha{}_{ij} \circ \pi^1, \qquad (\overset{\circ}{\Omega}_2)_{ij} = (\Omega_2)_{ij} \circ \pi^2.$$

The one other construction which appears in our diagram is the orthonormal bundle of frames $\Pi : F(P_1 \circ P_2) \longrightarrow P_1 \circ P_2$, for which $P_1 \circ P_2$ acts as a base manifold. If we let $\theta(h) \in \bar{\Lambda}^1(F(P_1 \circ P_2), O(r+4, s))$ denote a general connection on $F(P_1 \circ P_2)$, we can now choose the vectors $\overset{\circ}{E}_1, ..., \overset{\circ}{E}_{n+4}$ as an orthonormal frame for the horizontal subspace of $T_{(p_1,p_2)}F(P_1 \circ P_2)$ relative to $\theta(h)$. Furthermore, if $\Omega = D^{\theta(h)}\theta(h) \in \bar{\Lambda}^2(F(P_1 \circ P_2), O(r+4, s))$ is the curvature of $\theta(h)$, $\bar{\sigma} : \Pi^{-1}(U) \longrightarrow F(P_1 \circ P_2)$ is a local section determined by the above orthonormal fields, and $\bar{e}_a$ are standard horizontal vectors on $F(P_1 \circ P_2)$ associated with $e_a \in \Re^{n+4}$, then

$$\Omega^{\theta(h)}(\bar{e}_c, \bar{e}_d)(e_b) = R^a{}_{bcd}(\bar{\sigma}(p_1, p_2))e_a. \tag{1.17}$$

If we now write

$$(\Omega^{\bar{\theta}(h)})^a{}_b = \frac{1}{2}\mathcal{R}^a{}_{bcd}(p_1, p_2)\overset{\circ}{\bar{\varphi}}{}^c \wedge \overset{\circ}{\bar{\varphi}}{}^d, \tag{1.18}$$

and make use of (1.17), we get

$$R^a{}_{bcd}(\bar{\sigma}(p_1, p_2)) = \mathcal{R}^a{}_{bcd}(p_1, p_2). \tag{1.19}$$

We can use the local section $\bar{\sigma}$ to define canonical 1-forms $\overset{\circ}{\varphi} \in \bar{\Lambda}^1(F(P_1 \circ P_2), \Re^{n+4})$, and corresponding to $\overset{\circ}{\varphi}$, we have that the torsion 2-form $\overset{\circ}{\Theta} \in \bar{\Lambda}^2(F(P_1 \circ P_2), \Re^{n+4})$ of $\theta(h)$ is given by

$$\overset{\circ}{\Theta}{}^{\theta(h)} \equiv D^{\theta(h)}\,\overset{\circ}{\varphi} = d\,\overset{\circ}{\varphi} + \theta(h) \wedge \overset{\circ}{\varphi}, \tag{1.20}$$

i.e.

$$\overset{\circ}{\Theta}{}^{\theta(h)}(\bar{\sigma}_*\overset{\circ}{E}_c, \bar{\sigma}_*\overset{\circ}{E}_d) = S^a{}_{cd}(\bar{\sigma}(p_1, p_2))e_a, \tag{1.21}$$

from where

$$\overset{\circ}{S}{}^a{}_{cd}(p_1, p_2) = S^a{}_{cd}(\bar{\sigma}(p_1, p_2)). \tag{1.22}$$

With these basic definitions and results, and choosing an appropriate real linear representation of the infinitesimal generators of $SU(2) \times U(1)$, one can compute the components relative to $\overset{\circ}{E}_1, ..., \overset{\circ}{E}_n, \overset{\circ}{E}_{n+1}, ..., \overset{\circ}{E}_{n+4}$ of the curvature and torsion tensors for the metric h on $P_1 \circ P_2$. We shall impose the restriction of vanishing torsion on the base space M, i.e.

$$\underline{S}^i{}_{jk} = 0. \tag{1.23}$$

Furthermore, we take as ansatz the following expression for some of the connection components:

$$\bar{\theta}(h)^{n+A}{}_{n+B} = -(\rho(\overset{\circ}{\ell}_C))^A{}_B\,\overset{\circ}{\bar{\varphi}}{}^{n+C}; \tag{1.24}$$

where the matrices $\rho(\overset{\circ}{\ell}_C)$ constitute the representation of the infinitesimal generators of $SU(2) \times U(1)$ mentioned above.

II. The Unified Lagrangian

Recall that the components of the Riemann tensor on $P_1 \circ P_2$ are related to the connection 1-forms $\bar{\theta}(h) = \bar{\sigma}^* \theta(h)$ by means of (1.18). In matrix notation

$$\frac{1}{2}\mathcal{R}^a{}_{bcd}(p_1, p_2)\overset{\circ}{\varphi}{}^c \wedge \overset{\circ}{\varphi}{}^d = d\bar{\theta}(h)^a{}_b + \bar{\theta}(h)^a{}_c \wedge \bar{\theta}(h)^c{}_b. \tag{2.1}$$

Also, as mentioned previously, the Higgs fields may originate directly from torsion by assuming that the connection $\bar{\theta}(h)$ is semi-symmetric. With this in mind, we make the following additional ansatz

$$S_{ij}{}^{n+A} = \frac{1}{n} g_{ij}\Phi^A, \tag{2.2}$$

and

$$S^{n+\alpha}{}_{ij} = (\overset{\circ}{\Omega}_1)^\alpha{}_{ij}, \qquad S^{n+4}{}_{ij} = (\overset{\circ}{\Omega}_2)_{ij}. \tag{2.3}$$

Evaluating all the components $\mathcal{R}^a{}_{bcd}$ of the curvature tensor for the metric h on $P_1 \circ P_2$ relative to our orthonormal basis $\overset{\circ}{E}_1, ..., \overset{\circ}{E}_{n+4}$, as well as those of the torsion tensor, the Ricci tensor, and the Ricci scalar, we may construct the most general G-invariant Lagrangian density on $P_1 \circ P_2$ up to quadratic terms in these quantities by adding up all the G-invariant terms which can be obtained from them. The result is

$$\begin{aligned}\mathcal{L} = \frac{\sqrt{-g}}{V_I}\{&\alpha_0(\underline{R} - \frac{n-1}{n}\Phi^A\Phi_A + 2) \\ &+ \alpha_1[\frac{(n-1)^2}{n^2}(\Phi^A\Phi_A)^2 - \frac{2(n-1)}{n}\underline{R}\Phi_A\Phi^A - 4\frac{(n-1)}{n}\Phi^A\Phi_A + (\underline{R}+2)^2] \\ &+ \alpha_2[\underline{R}_{ijkm}\underline{R}^{ijkm} - \frac{4}{n^2}\underline{R}(\Phi_A\Phi^A) + \frac{2}{n^3}(n-1)(\Phi_A\Phi^A)^2] \\ &- \alpha_3(\overset{\circ}{\Omega}_1)^\gamma{}_{ij}(\overset{\circ}{\Omega}_1)_\gamma{}^{ij} - \alpha_4(\overset{\circ}{\Omega}_2)_{ij}(\overset{\circ}{\Omega}_2)^{ij} + \alpha_5(D^i\Phi^A)(D_i\Phi_A) \\ &+ \alpha_6[\underline{R}_{ij}\underline{R}^{ij} - \frac{2}{n^2}(n-1)\underline{R}(\Phi_A\Phi^A) + \frac{(n-1)^2}{n^3}(\Phi_a\Phi^A)^2] \\ &+ \alpha_7\frac{n-1}{n}\Phi_A\Phi^A + K\}\end{aligned} \tag{2.4}$$

where V_I is the volume of the n-4 compact "internal" coordinates of the base manifold, and K is a constant that contributes to the cosmological constant. The Lagrangian density (2.4) is a well defined function on the base manifold M, and we can write an action by integrating it over a volume element μ_g on M determined by g and the orientation of M, i.e.

$$I = \int_U \mathcal{L}\ \mu_g, \tag{2.5}$$

where U in an open subset of M with compact closure.

We now turn to the physical interpretation of the curvatures Ω_1 and Ω_2. If we let $(\sigma_1)_u$ and $(\sigma_2)_u$ be local sections $(\sigma_1)_u : M \longrightarrow P_1$, $(\sigma_2)_u : M \longrightarrow P_2$, such that $(\sigma_1)_{u*}E_i \in T_{p_1}P_1$ and $(\sigma_2)_{u*}E_i \in T_{p_2}P_2$, and if we further choose the orthonormal basis at each $x \in U \subset M$ to be a coordinate basis $E_i = \partial_i$, we then have

$$\begin{aligned}(\Omega_1)^\alpha{}_{ij} &= \partial_i((\sigma_1)^*_u\omega^\alpha_1(\partial_j)) - \partial_j((\sigma_1)^*_u\omega^\alpha_1(\partial_i)) + \varepsilon_{\alpha\beta\gamma}((\sigma_1)^*_u\omega^\beta_1(\partial_i))((\sigma_1)^*_u\omega^\gamma_1(\partial_j)) \\ &= g(\partial_i W^\alpha{}_j - \partial_j W^\alpha{}_i + g\varepsilon_{\alpha\beta\gamma}W^\beta{}_i W^\gamma{}_j),\end{aligned} \tag{2.6}$$

$$(\Omega_2)_{ij} = \partial_i((\sigma_2)^*_u(-i\omega_2)(\partial_j)) - \partial_j((\sigma_2)^*_u(-i\omega_2)(\partial_i)) = g'(\partial_i B_j - \partial_j B_i). \tag{2.7}$$

where we have used the definitions

$$gW^\alpha{}_j \equiv ((\sigma_1)^*_u\omega^\alpha_1)(\partial_j), \qquad g'B_j \equiv ((\sigma_2)^*_u(-i\omega_2))(\partial_j), \tag{2.8}$$

and g, g' denote the dimensionless coupling constants for the $SU(2)$ and $U(1)$ factors, respectively. Hence

$$F^\alpha{}_{ij} \equiv \frac{1}{g}(\Omega_1)^\alpha{}_{ij} = \partial_i W^\alpha{}_j - \partial_j W^\alpha{}_i + g\varepsilon_{\alpha\beta\gamma}W^\beta{}_i W^\gamma{}_j, \tag{2.9}$$

$$F_{ij} \equiv \frac{1}{g'}(\Omega_2)_{ij} = \partial_i B_j - \partial_j B_i, \tag{2.10}$$

are the field tensors for the $SU(2)$ and $U(1)$ vector bosons, respectively.

To conclude this section we have only to properly dimension and interpret the parameters which occur in (2.4) in order to bring it into the usual form of Einstein-Cartan gravity coupled to the Yang-Mills and Higgs fields for the electroweak model. After combining terms our action becomes

$$\begin{aligned} I = \frac{1}{V_I} \int \sqrt{-g} \{ & \kappa \underline{R} + \alpha_1 \underline{R}^2 + \alpha_2 \underline{R}_{ijkm} \underline{R}^{ijkm} + \alpha_6 \underline{R}_{ij} \underline{R}^{ij} \\ & - \frac{1}{4} F^{\alpha}{}_{ij} F_{\alpha}{}^{ij} - \frac{1}{4} F_{ij} F^{ij} + \frac{1}{2} (D_i \Phi_A)(D^i \Phi^A) \\ & + \frac{m^2}{2} \Phi_A \Phi^A - \frac{\lambda}{4} (\Phi_A \Phi^A)^2 + \frac{2n}{n-1} \lambda \underline{R} \Phi_A \Phi^A + \kappa \Lambda \} \, d^n x, \end{aligned} \tag{2.11}$$

where we have made the following obvious identifications in order to fix the physical parameters $(\alpha_0 + 4\alpha_1) r^2 = \kappa$ (the proportionality factor in the Einstein-Hilbert Lagrangian), $2\frac{(n-1)}{n}(\alpha_7 r^2 - \kappa) = m^2 > 0$ (square of the mass parameter associated with the Higgs field), $\frac{1-n}{n^3}[\alpha_1 n(n-1) + 2\alpha_2 + \alpha_6(n-1)] = \lambda > 0$ (coupling constant of the self-interaction term of the scalar field), $\Lambda =$ the cosmological constant, $\alpha_3 g^2 = \alpha_4 g'^2 = \frac{1}{4}$, $\alpha_5 = \frac{1}{2}$. Note that the above provides a relation between the parameters α_3 and α_4 and the Weinberg angle. Indeed,

$$tan\theta_W = \frac{g'}{g} = \sqrt{\frac{\alpha_3}{\alpha_4}}, \tag{2.12}$$

The Lagrangian for the scalar fields given above can be related via a unitary transformation to the form in which it usually appears in the electroweak model.