COLLOQUIUM MATHEMATICUM

VOL. 140 2015 NO. 2

VECTOR FIELDS FROM LOCALLY INVERTIBLE POLYNOMIAL MAPS IN $\mathbb{C}^n$

BY

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Abstract. Let $(F_1, \dots, F_n) : \mathbb{C}^n \to \mathbb{C}^n$ be a locally invertible polynomial map. We consider the canonical pull-back vector fields under this map, denoted by $\partial/\partial F_1, \dots, \partial/\partial F_n$. Our main result is the following: if $n-1$ of the vector fields $\partial/\partial F_j$ have complete holomorphic flows along the typical fibers of the submersion $(F_1, \dots, F_{j-1}, F_{j+1}, \dots, F_n)$, then the inverse map exists. Several equivalent versions of this main hypothesis are given.

1. Introduction and statement of results. We consider n-webs of polynomial vector fields in $\mathbb{C}^n$ which can be obtained from the euclidean n-web $\mathcal{W}$ in $\mathbb{C}^n$ by pull-back under a polynomial map

$$F = (F_1, \dots, F_n) : \mathbb{C}^n \to \mathbb{C}^n \quad \text{with} \quad \det(DF) = 1. \tag{1.1}$$

Recall that the Jacobian Conjecture in $\mathbb{C}^n$ asserts the existence of the inverse map F^{-1}. Each of the polynomial vector fields

$$\frac{\partial}{\partial F_i} = (F_1, \dots, F_n)^* \frac{\partial}{\partial w_i}, \quad i = 1, \dots, n, \tag{1.2}$$

has a restriction to the fibers $\mathcal{A}_{i,c} = (F_1, \dots, \widehat{F_i}, \dots, F_n)^{-1}(c)$ of the submersion; as usual, $\widehat{\ }$ over the ith coordinate indicates that it is omitted.

It is a classical result that the following assertions are equivalent (see [MO87], [Me92], [Cam97] and [Bus03]):

- The inverse map exists.
- $\partial/\partial F_1, \dots, \partial/\partial F_n$ are complete, i.e. their flows are defined for all complex times $t \in \mathbb{C}$ at every initial condition $p \in \mathbb{C}^n$.
- The web of affine curves $\{\mathcal{A}_{1,c}, \dots, \mathcal{A}_{n,c}\}$ is topologically trivial, i.e. every $\mathcal{A}_{i,c}$ is biholomorphic to $\mathbb{C}$.

The map F produces a collection of pairs

$$\{(\mathcal{A}_{i,c}, \partial/\partial F_i) \mid i = 1, \dots, n, \ c \in \mathbb{C}^{n-1}\}. \tag{1.3}$$

2010 *Mathematics Subject Classification*: Primary 14R15; Secondary 37F75, 32M17, 32M25.

Key words and phrases: holomorphic foliations, Jacobian conjecture, non-singular complex polynomial vector fields.

DOI: 10.4064/cm140-2-4

Looking at the foliations $\mathcal{F}_i = \{A_{i,c}\}$, the last point has many facets, very roughly speaking: every $\mathcal{F}_i$ has trivial monodromy, its global Ehresmann connections are well-defined, no atypical fibers appear in all the submersions $(F_1, \dots, \widehat{F}_i, \dots, F_n)$. By studying this, we can deduce:

MAIN THEOREM. *Let $F = (F_1, \dots, F_n) : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map as in (1.1). If $\partial/\partial F_2, \dots, \partial/\partial F_n$ are complete on the typical fibers $\mathcal{A}_{2,c}, \dots, \mathcal{A}_{n,c}$ of $(F_1, \dots, \widehat{F}_j, \dots, F_n)$, $j = 2, \dots, n$, then F^{-1} exists.*

The proof of the main theorem is in two stages. In Lemma 4, we show that the completeness on typical fibers implies the same property on all the fibers $\mathcal{A}_{2,c}, \dots, \mathcal{A}_{n,c}$. Secondly in Theorem 1, we consider a global Ehresmann conection in the directions of $\partial/\partial F_2, \dots, \partial/\partial F_n$ to get the result. Furthermore, in Theorem 1, several equivalences of the completeness hypothesis are described.

The invertibility of F has been considered from many points of view (see [Ess00]). We start mainly from the algebraic point of view of [A77], [NS83]. For $n = 2$, invertibility from completeness in just one pair $(\mathcal{A}_{2,c}, \partial/\partial F_2)$ follows from the Abhyankar–Moh–Suzuki Theorem (see [Dru91], [Cam97] and the references therein, as well as [Dun08]). Actually, our study uses Riemann surfaces ideas and several complex variables.

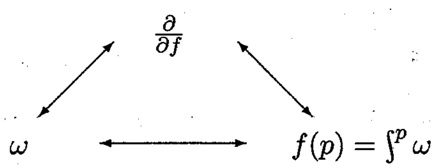
The content of the work is as follows. In Section 2 we study the pull-back vector fields on Riemann surfaces from meromorphic maps. Section 3 contains the study of the pairs (1.3). The proof of the main result is in Section 4.

2. Meromorphic maps and vector fields on compact Riemann surfaces. Let $\mathbb{CP}^1 = \mathbb{C}_w \cup \{\infty\}$ be the projective line, with affine coordinate w. The vector field $\partial/\partial w$ induces a holomorphic vector field in $\mathbb{CP}^1$ having a double zero at $\infty \in \mathbb{CP}^1$. Let $\mathcal{L}$ be a compact Riemann surface.

LEMMA 1. *Let $f : \mathcal{L} \to \mathbb{CP}^1$ be a non-constant meromorphic function. The non-identically zero meromorphic vector field*

$$\frac{\partial}{\partial f} := f^* \left(\frac{\partial}{\partial w} \right)$$

is well-defined on $\mathcal{L}$. Moreover, f has a canonically associated meromorphic one-form ω such that the diagram



commutes. $\partial/\partial f$ and ω are non-identically zero.

(b)$\Leftrightarrow$(e). We assume (b), thus we use the geometry of the set of asymptotic values as in the proof of (a)$\Leftrightarrow$(b): each $\mathcal{A}_{1,c}$ can be pushed by the Ehresmann connection of $\{\partial/\partial F_2, \ldots, \partial/\partial F_n\}$ for every time. Thus, $(F_2, \ldots, F_n) : \mathbb{C}^n_z \to \mathbb{C}^{n-1}_w$ determines a holomorphically trivial fiber bundle. For the converse assertion, if the fiber bundle determined by $(F_2, \ldots, F_n)$ as in the line above is topologically trivial, then the fundamental group of the fiber $\mathcal{A}_{1,c}$ is trivial and $\partial/\partial F_1$ is complete. Therefore (b) is true.

(b)$\Leftrightarrow$(f). Using (b) as hypothesis, $(F_2, \ldots, F_n)$ determines a holomorphically trivial fiber bundle with fiber $\mathbb{C}^{n-1}$, base $\mathcal{A}_{1,c}$ and total space biholomorphic to $\mathbb{C}^n_z$, as in (4.2). For topological reasons, $\mathcal{A}_{1,c}$ is a complex line. The degree of F equals the degree of $F_{1,c} : \mathcal{A}_{1,c} \to \mathbb{C}_{1,c}$ (because $\mathcal{A}_{1,c}$ is a typical fiber), and $F_{1,c}$ is a biholomorphism. Hence, the degree of F is one.

Assume (f); the asymptotic values are $\mathcal{AV}(F) = R \cup P$ as in Remark 4.

We note that P is empty: otherwise one pair $(\mathcal{L}_{i,c}, \partial/\partial F_i)$, $i \in \{1, \ldots, n\}$, has a pole; then by Remark 1(1), F would be of degree greater than or equal to 2, contrary to hypothesis (f).

As a result, $\mathcal{AV}(F) = R$, and it is empty or a hypersurface (see Remark 4 and [Jel93]).

If $R = \emptyset$ then F is bijective and we can conclude that $\{\partial/\partial F_1, \ldots, \partial/\partial F_n\}$ are complete.

If $R \neq \emptyset$ then let us use a slight modification of the original idea in the Newman–Białynicki-Birula–Rosenlicht Theorem (see [BB-R62] or more recently [Gr99, Section 3.B]).

We note that $F : \mathbb{C}^n_z \to \mathbb{C}^n_w - R$ is a local biholomorphism of degree 1 (since $P = \emptyset$). Therefore,

$$H_1(\mathbb{C}^n_w - R, \mathbb{Z}) = \mathbb{Z}^{\oplus \nu},$$

where ν is the number of irreducible components of R; for the computation of this homology (see [Dim92, p. 103]). That contradicts $H_1(\mathbb{C}^n_z, \mathbb{Z}) = 0$. Thus R is empty, and assertion (b) holds. ∎

Corollary 7. *If one $(\mathcal{L}_{i,c}, \partial/\partial F_i)$ has a pole, then F^{-1} does not exist.*

Acknowledgments. The first and second authors are partially supported by the Spanish MICINN projects MTM2010-15481, MTM2011-26674-C02-02. The third author is partially supported by Conacyt, México.

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REFERENCES

[A77] S. S. Abhyankar, *Lectures on Expansion Techniques in Algebraic Geometry*, Lectures on Math. Phys. 57, Tata Institute of Fundamental Research, Bombay, 1977.

[BB-R62] A. Białynicki-Birula and M. Rosenlicht, *Injective morphisms of real algebraic varieties*, Proc. Amer. Math. Soc. 13 (1962), 200–203.

[Bro83] S. A. Broughton, *On the topology of polynomial hypersurfaces*, in: Proc. Sympos. Pure Math. 40, Amer. Math. Soc., 1983, 167–178.

[Bus03] A. Bustinduy, *Zeroes of complete polynomial vector fields*, Proc. Amer. Math. Soc. 131 (2003), 3767–3775.

[Cam97] L. A. Campbell, *Jacobian pairs and Hamiltonian flows*, J. Pure Appl. Algebra 115 (1997), 15–26.

[Dim92] A. Dimca, *Singularities and Topology of Hypersurfaces*, Universitext, Springer, New York, 1992.

[Dru91] L. M. Drużkowski, *A geometric approach to the Jacobian Conjecture in* $\mathbb{C}^2$, Ann. Polon. Math. 55 (1991), 95–101.

[Dun08] L. Dũng Tráng, *Simple rational polynomials and the Jacobian Conjecture*, Publ. RIMS Kyoto Univ. 44 (2008), 641–659.

[Eb07] W. Ebeling, *Functions of Several Complex Variables and their Singularities*, Grad. Stud. Math. 83, Amer. Math. Soc., Providence, RI, 2007.

[Ess00] A. van den Essen, *Polynomial Automorphisms and the Jacobian Conjecture*, Progr. Math. 190, Birkhäuser, Basel, 2000.

[For95] F. Forstnerič, *Limits of complete holomorphic vector fields*, Math. Res. Lett. 2 (1995), 401–414.

[FG02] K. Fritzsche and H. Grauert, *From Holomorphic Functions to Complex Manifolds*, Springer, New York, 2002.

[Gr99] M. Gromov, *Endomorphisms of symbolic algebraic varieties*, J. Eur. Math. Soc. 1 (1999), 109–197.

[Jel93] Z. Jelonek, *The set of points at which a polynomial map is not proper*, Ann. Polon. Math. 58 (1993), 259–266.

[Jel99] Z. Jelonek, *Testing sets for properness of polynomial mappings*, Math. Ann. 315 (1999), 1–35.

[Kr89] H. Kraft, *Algebraic automorphisms of affine space*, in: Topological Methods in Algebraic Transformation Groups, H. Kraft et al. (eds.), Progr. Math. 80, Birkhäuser, Boston, 1989, 81–105.

[LM00] J. L. López and J. Muciño-Raymundo, *On the problem of deciding whether a holomorphic vector field is complete*, in: Complex Analysis and Related Topics (Cuernavaca, 1996), Oper. Theory Adv. Appl. 114, Birkhäuser, Basel, 2000, 171–195.

[Me92] G. H. Meisters, *Inverting polynomial maps of n-space by solving differential equations*, in: Delay and Differential Equations (Ames, IA, 1991), World Sci., River Edge, NJ, 1992, 107–166.

[MO87] G. H. Meisters and C. Olech, *A poly-flow formulation of the Jacobian conjecture*, Bull. Polish Acad. Sci. Math. 35 (1987), 725–731.

[Muc02] J. Muciño-Raymundo, *Complex structures adapted to smooth vector fields*, Math. Ann. 322 (2002), 229–265.

[NS83] P. Nousiainen and M. E. Sweedler, *Automorphisms of polynomial power series rings*, J. Pure Appl. Algebra 29 (1983), 93–97.

[Per98] R. Peretz, *The geometry of the asymptotics of polynomial maps*, Israel J. Math. 105 (1998), 1–59.

[Reb04] J. Rebelo, *Meromorphic vector fields and elliptic fibrations*, Michigan Math. J. 52 (2004), 33–59.

[Str84] K. Strebel, *Quadratic Differentials*, Ergeb. Math. Grenzgeb. (3) 5, Springer, Berlin, 1984.

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Received 5 November 2013;
revised 30 December 2014 (6065)