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On restrictions of the Picard bundle

L. Brambila-Paz, L. Hidalgo-Solís, and J. Muciño-Raymundo

ABSTRACT. The Picard sheaves have been studied from differents aspects. Li proves the stability of the Picard bundle $\mathcal{W}$ over the moduli space $\mathcal{M}(n,d)$ of stable bundles of rank n and degree d. In general the restrictions of stable bundles need not be stable. In this paper we study the restiction $\mathcal{W}_\xi$ of the Picard bundle $\mathcal{W}$ to the subvariety $\mathcal{M}(n,\xi)$ of stable bundles with fixed determinant ξ. We give a condition to get polystability. If such condition is satisfied for rank 2 then $\mathcal{W}_\xi$ is stable and the connected component of the moduli space of stable bundles over $\mathcal{M}(2,\xi)$ with the same Hilbert polynomial as $\mathcal{W}_\xi$ containing $\mathcal{W}_\xi$ is isomorphic to the Jacobian J of the curve.

Let X be a non-singular projective curve of genus $g \geq 2$ over $\mathbf{C}$. Let $\mathcal{M}(n,d)$ be the moduli space of stable bundles of rank n and degree d over X. If $(n,d)=1$ then $\mathcal{M}(n,d)$ is a fine moduli space, i.e. there exists a universal family, $\mathcal{U}$, called a Poincaré bundle, over $X \times \mathcal{M}(n,d)$. The direct images of $\mathcal{U}$ are called Picard sheaves over $\mathcal{M}(n,d)$.

The Picard sheaves have been studied from differents aspects (see [**13**], [**4**], [**7**], [**11**], [**1**]). If $d >> 0$, the Picard sheaf, $\pi_*\mathcal{U} = \mathcal{W}$, is actually a vector bundle. For $n = 1$, the stability of $\mathcal{W}$ was proved by Kempf [**7**] for $d = 2g-1$ and by Ein and Lazarfeld [**4**] for $d \geq 2g$. For $n \geq 2$ and $d > 2gn$, Y. Li proved the stability of $\mathcal{W}$, in [**9**].

The moduli space $\mathcal{M}(n,d)$ has natural subvarieties, namely the Jacobian of the curve and the moduli space $\mathcal{M}(n,\xi)$ of stable bundles with fixed determinant ξ.

Li in [**9**], Theorem 2.5, proved that the restriction $\mathcal{W}_J$ of the Picard bundle $\mathcal{W}$ to the Jacobian is stable with respect to any theta divisor. By the results of Uhlenbeck and Yau (see [**18**]) there are Hermitian Einstein metrics h and h_1 in $\mathcal{W}$ and $\mathcal{W}_J$ respectively. If such metrics coincide then we prove (see Theorem 3.1)

The restriction $\mathcal{W}_\xi$ of the Picard bundle $\mathcal{W}$ to $\mathcal{M}(n,\xi)$ is polystable with respect to any polarization of $\mathcal{M}(n,\xi)$.

If Theorem 3.1 holds and $n=2$ then

$\mathcal{W}_\xi$ is stable.

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Moreover, let H be any ample divisor on $\mathcal{M}(2,\xi)$ and $\mathcal{M}(\mathcal{W}_\xi)$ the moduli space of H-stable bundles with same Hilbert polynomial as $\mathcal{W}_\xi$ on $\mathcal{M}(2,\xi)$ then

The connected component $\mathcal{M}(\mathcal{W}_\xi)^0$ of $\mathcal{M}(\mathcal{W}_\xi)$ containing $\mathcal{W}_\xi$ is isomorphic to the Jacobian $J(X)$.

In section 1 we shall recall the relation between Hermitian-Einstein bundles and stable bundles. We prove that given a Hermitian-Einstein vector bundle E over the product of two Kähler manifolds, $M \times N$, such that the restriction in one factor is Hermitian-Einstein then the restriction to the other factor is Hermitian-Einstein, with respect to the respective forms (see Lemma 1.2).

In section 2 we prove that there is a Hermitian-Einstein bundle over $J_0(X) \times \mathcal{M}(n,\xi)$ such that the restriction to each factor is $\mathcal{W}_J$ and $\mathcal{W}_\xi$ respectively. In section 3 we prove that $\mathcal{W}_\xi$ is polystable if certain condition in the metrics is satisfied. If such condition is satisfied for rank 2, we use Balaji and Vishwanath results in [**1**], to see that in this case $\mathcal{W}_\xi$ is stable for any polarisation and that the connected component $\mathcal{M}(\mathcal{W}_\xi)^0$ of $\mathcal{M}(\mathcal{W}_\xi)$ containing $\mathcal{W}_\xi$ is isomorphic to the Jacobian $J(X)$.

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1. Hermitian-Einstein bundles

In this section we recall the definition of Hermitian-Einstein bundles and prove two Lemmas that we will use.

Let (M,g) be a compact Kähler manifold of complex dimension m with Kähler form Φ. A Hermitian-Einstein vector bundle (E,h) with respect to g is a holomorphic vector bundle E, together with a Hermitian metric h in E such that the curvature $R(\nabla)$ of the canonical connection ∇ is of bidegree $(1,1)$, and satisfies the condition

$$R(\nabla) \wedge \Phi^{m-1} = \lambda \Phi^m \otimes \mathbf{1}_E$$

for some constant $\lambda \in \mathbf{C}$.

Let $\mathcal{E}$ be a coherent sheaf over M. The Φ-slope of $\mathcal{E}$, is by definition the quotient

$$\mu_\Phi(\mathcal{E}) := \frac{d_\Phi(\mathcal{E})}{rk(\mathcal{E})}$$

where $rk(\mathcal{E})$ is the rank and $d_\Phi(\mathcal{E})$ is the degree of $\mathcal{E}$ with respect to Φ, i.e.

$$d_\Phi(\mathcal{E}) := \frac{1}{(m-1)!}\int_M c_1(\mathcal{E}) \wedge \Phi^{m-1}.$$

A vector bundle E is called Φ-stable if for any coherent subsheaf F with $rk(F) < rk(E)$,

$$\mu_\Phi(F) < \mu_\Phi(E).$$

A bundle E is Φ-polystable if it is the direct sum of Φ-stable bundles of the same slope. It is clear that Φ-polystable and simple implies Φ-stable.

REMARK 1.1. Kobayashi in [**8**] and Lübke in [**10**] prove that every Hermitian-Einstein vector bundle (E,h) with respect to g is Φ-polystable. Moreover, Uhlenbeck and Yau in [**18**] proved that every Φ-stable bundle over (M,g) admits a Hermitian-Einstein metric with respect to g, which is unique up to multiplication by a positive scalar. When M is a compact Riemann surface, such results were obtained by Narasimhan and Seshadri in [**15**] (see also [**5**]).

In general the restriction of a Hermitian-Einstein vector bundle does not need to be Hermitian-Einstein. However, when the base is a product of two Kähler manifolds we can say something.

Let (M, g_M) and (N, g_N) be compact Kähler manifolds, of complex dimension m and n respectively. Let g be the Kähler metric on $M \times N$ given by $g = g_M + g_N$ and (E, h) be a Hermitian holomorphic vector bundle over $M \times N$. Let Φ_M, Φ_N and Φ denote the Kähler forms of (M, g_M), (N, g_N) and $(M \times N, g)$, respectively, so $\Phi = \Phi_M + \Phi_N$.

LEMMA 1.2. *Suppose that (E, h) is Hermitian-Einstein with respect to g and that for every point $x \in M$, the restriction (E_x, h_x) of (E, h) to $\{x\} \times N$ is Hermitian-Einstein with respect to g_N. Then, for every point $y \in N$, the restriction (E^y, h^y) of (E, h) to $M \times \{y\}$ is Hermitian-Einstein with respect to g_M.*

Proof. Let U and V be holomorphic coordinate domains on M and N, respectively, and fix a point $(x_0, y_0) \in U \times V$. Denote by Ω, Ω_{x_0} and Ω^{y_0} the curvature matrices of (E, h), (E_{x_0}, h_{x_0}) and (E^{y_0}, h^{y_0}) with respect to the frames S, S_{x_0} and S^{y_0}, respectively.

We know that

$$\Phi^{m+n-1} = (a\Phi_M^m \Phi_N^{n-1} + b\Phi_M^{m-1}\Phi_N^n)$$

and

$$\Phi^{m+n} = c\Phi_M^m \Phi_N^n.$$

Since U and V are coordinate domains, on the restrictions we have

$$\Omega = \Omega_{x_0} + \sum_{i=1}^{r} \beta_{U,i} \wedge \gamma_{V,i} + \Omega^{y_0},$$

where $\beta_{U,i} \in A^1(U)$, $\gamma_{V,i} \in A^1(V)$.

Thus,

$$\begin{aligned} \Omega\Phi^{m+n-1} &= (\Omega_{x_0} + \textstyle\sum_{i=1}^{r} \beta_{U,i} \wedge \gamma_{V,i} + \Omega^{y_0})(a\Phi_M^m \Phi_N^{n-1} + b\Phi_M^{m-1}\Phi_N^n) \\ &= a'\Omega_{x_o}\Phi_M^m\Phi_N^{n-1} + b'\Omega^{y_0}\Phi_M^{m-1}\Phi_N^n \end{aligned} \tag{1}$$

where a', b' are constants.

By hypothesis (E_x, h_x) and (E, h) are Hermitian-Einstein,

$$\text{i.e. } \Omega_{x_0}\Phi_N^{n-1} = \lambda'\Phi_N^n I \quad \text{and} \quad \Omega\Phi^{m+n-1} = \lambda\Phi^{m+n} I \tag{2}$$

where I is the identity matrix.

From (1) and (2) we have

$$\begin{aligned} c\lambda\Phi_M^m\Phi_N^n I &= \lambda\Phi^{m+n} I \\ &= \Omega\Phi^{m+n-1} \\ &= a'\Omega_{x_o}\Phi_M^m\Phi_N^{n-1} + b'\Omega^{y_0}\Phi_M^{m-1}\Phi_N^n \\ &= a''\Phi_M^m\Phi_N^n I + b''\Omega^{y_0}\Phi_M^{m-1}\Phi_N^n. \end{aligned} \tag{3}$$

Thus, $\Omega^{y_0}\Phi_M^{m-1} = \lambda'\Phi_M^m I$, i.e. (E^{y_0}, h^{y_0}) is Hermitian-Einstein with respect to g_M. $\square$

We shall quote the following result since we will need it later.

LEMMA 1.3. Let $f : (A, g_A) \to (B, g_B)$ be a local biholomorphism between two compact Kähler manifolds. Let g_B and g_A be the Kähler metrics of A and B, respectively, and $f^*(g_B) = g_A$. If (E, h) is Hermitian-Einstein over B with respect to Φ_B then $(f^*(E), f^*(h))$ is Hermitian-Einstein with respect to Φ_A, where Φ_B and Φ_A are the respectively Kähler forms.

Proof. The proof follows from [**8**], IV, Prop. 1.8, since it is a local biholomorphism.

2. Stability

Let A be a smooth complex projective variety of dimension n with an ample divisor H on A and E a vector bundle over A. The H-slope of E, denoted by $\mu_H(E)$, is the quotient

$$\mu_H(E) = c_1(E)\cdot[H]^{n-1}/rk(E)$$

where $rk(E)$ is the rank of E. The vector bundle E is call H-stable if for every coherent subsheaf F with $rk(F) < rk(E)$ one has

$$\mu_H(F) < \mu_H(E). \tag{4}$$

The bundle E is H-polystable if it is direct sum of H-stable bundles of the same slope.

REMARK 2.1. If H is an ample line bundle and Φ is a closed $(1,1)$-form representing the first Chern class $c_1(H)$, then we say either H-stable or Φ-stable.

Note that if A is a curve then the stability of the vector bundle is independent of the line bundle H.

In general it is not easy to see when a vector bundle is stable. Mehta and Ramanathan in [**12**] proved that if $[\omega]$ is the hyperplane class of $A \subset \mathbf{P}^N$ and X the intersection of A with a hypersurface of degree $d >> 0$, then a bundle E over A is ω-stable if and only if the restriction of E to X is $\omega_{|X}$-stable. In the case that A is a principal polarized abelian variety we have the following result.

THEOREM 2.2. Let (A, θ_A) be a principally polarized abelian variety of dimension g and E a vector bundle over A. There exists a non-singular irreducible projective curve X and an embedding $\phi : X \to A$ such that the θ_A-stability of E is implied from the stability of $\phi^*(E)$.

Proof: Welters proved in [**19**] that for any principally polarized abelian variety (A, θ_A) there is non-singular irreducible projective curve X, (not unique) such that (A, θ_A) is a Prym-Tyurin variety for the curve X of exponent $e = 3^{g-1}(g-1)!$. By Welters' Criterion (see [**19**]) there is a embedding $\phi : X \to A$ such that the algebraic 1-cycles $\phi_*(X)$ and $\frac{e}{(g-1)!}[\theta_A]^{g-1}$ are numerically equivalent.

Now let $F \subset E$ be a torsion free subsheaf of E such that E/F is also a torsion free sheaf. Since Codim $(Sing(F)) \geq 2$ and X generates A as an abelian group we may assume $\phi(X) \subset A - Sing(F)$, thus $\phi^*(F)$ is a subbundle of $\phi^*(E)$. If $\phi^*(E)$ is stable then

$$\mu(\phi^*(F)) < \mu(\phi^*(E))$$

i.e.

$$\frac{c_1(\phi^*(F))}{rk(\phi^*(F))} < \frac{c_1(\phi^*(E))}{rk(\phi^*(E))}. \tag{5}$$

Since

$$c_1(\phi^*(-)) = \phi_*(X)\cdot c_1(-) \equiv c_1(-)\cdot \frac{e}{(g-1)!}[\theta_A]^{g-1} \tag{6}$$

and $\mu_{\theta_A}(-) = c_1(-)\cdot[\theta_A]^{g-1}/rk(-)$, we have from (5) and (6) that

$$\mu_{\theta_A}(F) < \mu_{\theta_A}(E).$$

Hence, E is θ_A-stable.

REMARK 2.3. When A is the Jacobian of a non-singular curve, Theorem 2.2 was proved by Li in [**9**], Lemma 2.8. Actually, we use similar arguments for the proof.

Some projective varieties, like abelian varieties or moduli spaces, carries natural bundles different from the trivial or the induced from the tangent bundle, namely the Picard bundles. Mukai in [**13**] use the Fourier functor to study Picard sheaves over abelian varieties. However, we consider Picard sheaves in the following sense.

Let $\mathcal{M}(n,d) = \mathcal{M}$ be the moduli space of stable bundles of rank n and degree d over a non-singular algebraic curve X of genus $g \geq 2$. If $(n,d) = 1$ then there is a universal family called the Poincaré bundle. Denote by $\mathcal{U}$ a Poincaré bundle over $X \times \mathcal{M}$. Note that $\mathcal{U}$ is unique up to tensoring by a line bundle from $\mathcal{M}$.

The direct image $\pi_*(\mathcal{U}) = \mathcal{W}$ over $\mathcal{M}(n,d)$ is a sheaf that is called the *Picard sheaf*. If $d > n(2g-2)$ then the direct image $\pi_*(\mathcal{U}) = \mathcal{W}$ over $\mathcal{M}(n,d)$ is a vector bundle that is called the *Picard bundle*. By Riemann-Roch's theorem the rank of $\mathcal{W}$ is $d + n(1-g)$.

Let θ be the generalized theta divisor over $\mathcal{M}(n,d)$. Li in [**9**] proved that if $d > 2ng$ then $\mathcal{W}$ is θ-stable.

REMARK 2.4. There is a natural Kähler metric on $\mathcal{M}(n,d)$ (see [**14**]) called the *theta metric*, which we denote by g_θ. If Φ_θ denotes the Kähler form of this metric, then by [**3**] Theorem 3.27, the cohomology class of $n\Phi_\theta$ equals θ. Therefore, from the results of Uhlembeck and Yau (see [**18**]) and the θ-stability of $\mathcal{W}$, there exists a Hermitian metric k in $\mathcal{W}$, which is Hermitian-Einstein with respect to g_θ.

We shall consider the restriction of $\mathcal{W}$ to natural subvarieties of $\mathcal{M}(n,d)$.

If $\mathcal{M}(n,\xi) = \mathcal{M}_\xi$ denote the moduli space of stable bundles in $\mathcal{M}(n,d)$ with fixed determinat ξ and $J_0(X)$ the Jacobian of line bundles over X of degree 0 then we have the map

$$f : J_0(X) \times \mathcal{M}(n,\xi) \rightarrow \mathcal{M}(n,d)$$

defined as $(L,E) \mapsto L^{-1} \otimes E$.

For generic $L \in J_0(X)$ and $E \in \mathcal{M}(n,\xi)$ the restrictions of

$$f : J_0(X) \times \mathcal{M}(n,\xi) \rightarrow \mathcal{M}(n,d)$$

to $J_0(X) \times \{E\}$ and $\{L\} \times \mathcal{M}(n,\xi)$ induce embeddings. Actually, if J^n denote the group of n-torsion points in $J_0(X)$ then

$$\mathcal{M}(n,\xi) \times J_0(X) \rightarrow \mathcal{M}(n,d).$$

is a fibration with fibre J^n (see [**9**], 4.8).

Denote by θ_J the theta divisor in $J_0(X)$ and by θ_ξ the restriction of the generalized theta divisor θ in $\mathcal{M}(n,d)$ to $\mathcal{M}(n,\xi)$. As in Remark 2.4, (see also [**3**]) we have Kähler metrics $g_{n\theta_J}$ and g_{θ_ξ} in $J_0(X)$ and $\mathcal{M}(n,\xi)$, respectively, such that the associated Kähler forms are in the class of θ_J and θ_ξ, respectively.

From [**2**] we have that

$$f^*(\theta) = n\theta_J + \theta_\xi := \Theta \tag{7}$$

and

$$g_\Theta = g_{n\theta_J} + g_{\theta_\xi}.$$

Hence, since $\mathcal{W}$ is θ-stable we have

PROPOSITION 2.5. *$f^*(\mathcal{W})$ is semistable. Moreover, there exists a metric h in $f^*(\mathcal{W})$ such that $(f^*(\mathcal{W}), h)$ is Hermitian-Einstein with respect to g_Θ.*

Proof: The semistability follows from [**6**], lemma 3.2.2. To prove that is Hermitian-Einstein, we have from Remark 2.4, that there exists a metric k in $\mathcal{W}$ such that $(\mathcal{W}, k)$ is Hermitian-Einstein with respect to g_θ. Denote by h the pull-back $f^*(k)$ under the morphism $f : J_0(X) \times \mathcal{M}(n,\xi) \rightarrow \mathcal{M}(n,d)$. Hence, from Lemma 1.1 $(f^*(\mathcal{W}), h)$ is Hermitian-Einstein with respect to g_Θ since f is a local biholomorphism (see [**9**], 4.8).

3. Restrictions

From Proposition 2.5 we have a Hermitian-Einstein bundle

$$p : (f^*(\mathcal{W}), h) \rightarrow J_0(X) \times \mathcal{M}(n,\xi)$$

with respect to $g_\Theta = g_{n\theta_J} + g_{\theta_\xi}$. For generic $L \in J_0(X)$ and $E \in \mathcal{M}(n,\xi)$, denote by $\mathcal{W}_{J,E}$ and $\mathcal{W}_{\xi,L}$ the restriction of the Picard bundle $\mathcal{W}$ over $\mathcal{M}(n,d)$ to the embeddings of $J_0(X)$ and $\mathcal{M}(n,\xi)$ given by the restriction of $f : J_0(X) \times \mathcal{M}(n,\xi) \rightarrow \mathcal{M}(n,d)$. That is,

$$f^*(\mathcal{W})_{|J_0(X)\times\{E\}} = \mathcal{W}_{J,E} \quad \text{and} \quad f^*(\mathcal{W})_{|\{L\}\times\mathcal{M}_\xi} = \mathcal{W}_{\xi,L}.$$

Li in [**9**], Theorem 2.5, proved that the restriction $\mathcal{W}_{J,E}$ is stable with respect to any theta divisor in $J(X)$. In this case, the stability of $\mathcal{W}_{J,E}$ follows from the stability of the restriction of $\mathcal{W}_{J,E}$ to the curve, which is embedded in $J_0(X)$ by the Abel-Jacobi map (see [**9**]).

Since $\mathcal{W}_{J,E}$ is θ_J-stable and therefore $n\theta_J$-stable, there exists a metric h_E such that $(\mathcal{W}_{J,E}, h_E)$ is Hermitian-Einstein with respect to $g_{n\theta_J}$. Let h' be the metric on $f^*(\mathcal{W})$ such that $h'_{|J\times\{E\}} = h_E$, for a generic E.

THEOREM 3.1. *If the metrics h' and h on $f^*(\mathcal{W})$ are gauge equivalent then the restriction $\mathcal{W}_{\xi,L}$ of the Picard bundle $\mathcal{W}$ to $\mathcal{M}(n,\xi)$ is θ_ξ-polystable.*

Proof: From Proposition 2.5 $(f^*(\mathcal{W}), h)$ is Hermitian-Einstein with respect to g_Θ. By assumption the restriction of h to $J_0(X) \times \{E\}$ is h_E. Hence, for a generic E the restriction of $(f^*(\mathcal{W}), h)$ to $J_0(X) \times \{E\}$ (which is $(\mathcal{W}_{J,E}, h_E)$), is Hermitian-Einstein with respect to $g_{n\theta_J}$. Therefore, from Lemma 1.1 we have that for a generic $L \in J_0(X)$, $(\mathcal{W}_{\xi,L}, h_|) = (f^*(\mathcal{W}), h)_{|\{L\}\times\mathcal{M}_\xi}$, is Hermitian-Einstein with respect to g_{θ_ξ}. Therefore, from Remark 1.1, $\mathcal{W}_{\xi,L}$ is θ_ξ-polystable.

For rank 2, Balaji and Vishwanath in [**1**] proved that

$$H^0(\mathcal{M}_\xi, \mathrm{ad}(\mathcal{W}_\xi)) = 0 \tag{8}$$

and

$$H^1(\mathcal{M}_\xi, \mathrm{ad}(\mathcal{W}_\xi)) = g. \tag{9}$$

They use Thaddeus description of the moduli space of stable pairs (see [**17**]). The equality (8) implies that $\mathcal{W}_\xi$ is simple. Therefore, if the hypothesis of Theorem 3.1 holds then for rank 2 we have that *$\mathcal{W}_\xi$ is θ_ξ-stable.*

Denote by $\mathcal{M}(\mathcal{W}_\xi)$ the moduli space of stable vector bundles over $\mathcal{M}(2,\xi)$ that have the same Hilbert polynomial as $\mathcal{W}_\xi$. Denote by $\mathcal{M}^0(\mathcal{W}_\xi)$ the connected component of $\mathcal{M}(\mathcal{W}_\xi)$ that contains $\mathcal{W}_\xi$. By deformation theory we have that

$$\dim H^1(\mathcal{M}_\xi, \mathrm{End}(\mathcal{W}_\xi)) = \dim T\mathcal{M}^0(\mathcal{W}_\xi)_{|\mathcal{W}_\xi}, \tag{10}$$

where $T\mathcal{M}^0(\mathcal{W}_\xi)_{|\mathcal{W}_\xi}$ is the tangent space of $\mathcal{M}^0(\mathcal{W}_\xi)$ at $\mathcal{W}_\xi$.

Since $H^1(\mathcal{M}_\xi, \mathrm{End}(\mathcal{W}_\xi)) = H^1(\mathcal{M}_\xi, \mathcal{O}) \oplus H^1(\mathcal{M}_\xi, \mathrm{ad}(\mathcal{W}_\xi))$ and $H^1(\mathcal{M}_\xi, \mathcal{O}) = 0$ (see [**16**]), we have from equality (9) that $\dim H^1(\mathcal{M}_\xi, \mathrm{End}(\mathcal{W}_\xi)) = g$. Hence, from (10) we have that

$$\dim \mathcal{M}^0(\mathcal{W}_\xi) = g.$$

Let $\mathcal{U}_\xi$ be a universal bundle over $X \times \mathcal{M}_\xi$ and p_i the projection in the i-factor, for $i = 1, 2$. The natural map $\phi : J(X) \to \mathcal{M}^0(\mathcal{W}_\xi)$ defined as $L \mapsto p_{2*}(\mathcal{U}_\xi \otimes p_1^* L)$ is injective (see [**1**]). Since $J(X)$ is complete, ϕ is injective and has the same dimension, thus ϕ is an isomorphism. That is,

The connected component $\mathcal{M}(\mathcal{W}_\xi)^0$ of $\mathcal{M}(\mathcal{W}_\xi)$ containing $\mathcal{W}_\xi$ is isomorphic to the Jacobian $J(X)$. i.e.

$$\mathcal{M}^0(\mathcal{W}_\xi) \cong J(X).$$

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CIMAT, Apdo. Postal 402, C.P. 36240 Guanajuato, México
E-mail address: `lebp@fractal.cimat.mx`

Departamento de Matemáticas, UAM - Iztapalapa México, 09340, D.F.
E-mail address: `lhso@xanum.uam.mx`

Instituto de Matemáticas, Unidad Morelia, UNAM Morelia Michoacán.
E-mail address: `muciray@servidor.unam.mx`