On the structure of Yang–Mills fields in compactified Minkowski space

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In this work the Yang–Mills functional with gauge group SU(2) in compactified Minkowski space is reduced to a mechanical Lagrangian which enables one to find new classes of exact dual and nonself-dual solutions. The technique used in the reduction allows a simple interpretation of meron-type solutions. The formalism is then used to produce a good simple analytic approximation to static solutions of Skyrme's field equation, relevant to low energy hadronic interactions.

I. INTRODUCTION

In recent years there has been a great deal of work on the study of critical points of the Yang–Mills functional. In particular the study of minima (instantons, anti-instantons, multi-instantons, etc.) has been motivated by the physics of electroweak and strong interactions,[1] and on the mathematical side by the study of simply connected four-manifolds using the theory of Donaldson.[2] An important simplification of these analyses is that in Euclidean space minima of the functional satisfy the self-dual equations. This allows the construction of moduli spaces of instantons, using ideas of patching individual solutions. The study of internal properties such as holonomy and the behavior of the Chern–Simons functional is also possible for these solutions. To date the possibility of a similar program for nonself-dual solutions is still to a great extent an open problem.

As a contribution to the study of nonself-dual solutions, this work analyzes the Yang–Mills system with SU(2) gauge group in compactified Minkowski space ($S^1\times S^3$ with a canonical pseudo-Riemannian metric). Our approach is based on the ideas suggested by Urakawa[3] on the use of invariant connections over cohomogeneity one manifolds, combined with the reduction to ordinary differential equations used by Parker.[4] This reduction allows the construction of explicit solutions. Moreover this approach unifies several solutions previously obtained and produces a large new class of periodic solutions. The case of nonself-dual solutions for $S^1\times S^3$ with a Riemannian metric was studied by Parker[4] and Wang.[5]

The simplest nonself-dual solution is shown to be the meron solution discovered by De Alfaro *et al.*[6–8] The present formalism allows, using the Chern–Simons formula, to generalize the meron solutions to solutions with higher positive and negative charge.

The article is organized as follows: In Sec. II we obtain the explicit expressions for invariant connections [under nontransive SU(2) action] on $\mathbb{R}\times S^3$, i.e., the invariant connections over a cohomogeneity one manifold. The third section is devoted to the reduction of the Yang–Mills functional to a Lagrangian for a simple mechanical system. In Sec. IV we provide a phase-space classification of the possible closed form bi-invariant solutions. This provides a unified treatment of previously known solutions and new solutions as well.

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The fifth section is devoted to the study of of the Chern–Simons integral and its relation to the physical charge of merons, and to the introduction of special gauge transformations on single merons which leads to objects of arbitrary fractional charge.

In Sec. VI we construct an approximate solution for the standard spherically symmetric static skyrmion based on the observation that the functional form of the bi-invariant connections developed in the previous sections is such that, when interpreted as currents, they satisfy Skyrme's field equations identically on a conformally flat Minkowski space–time, and also result in an integer barionic charge. The simple closed form of this approximate solution makes it an interesting candidate for calculations in low-energy hadronic physics. Finally, in a separate Appendix we give the basic coordinate and metric relations for compactified Minkowski space which serve to fix the notation used throughout the article, and also helps to make the presentation more self-contained.

II. INVARIANT CONNECTIONS ON $\mathbb{R}\times S^3$

In this section we find the expressions for the invariant connections over the appropriate cohomogeneity one manifold.

Let $\mathbb{R}\times S^3$ be a Lorentz manifold with a canonical pseudo-Riemannian metric of signature $(+,-,-,-)$. Let P be a trivial G bundle with base space $M=\mathbb{R}\times S^3$.

It is well known that for pseudo-Riemannian manifolds, the condition of (anti)self-duality on the Yang–Mills gauge fields requires either a noncompact gauge group or that the gauge potentials be complex. We thus take $G=\mathrm{SL}(2,\mathbb{C})$ as the structure group for P.

Consider now S^3 as an homogeneous space given by

$$S^3=\frac{\mathrm{Spin}(4)}{\mathrm{Spin}(3)}\cong\frac{\mathrm{SU}(2)\times\mathrm{SU}(2)}{(\mathrm{SU}(2)\times\mathrm{SU}(2))_D}=\mathrm{SU}(2)$$

so that $S=\mathrm{SU}(2)\times\mathrm{SU}(2)$ is the symmetry group acting transitively from the left on S^3, and the diagonal subgroup $J=(\mathrm{SU}(2)\times\mathrm{SU}(2))_D\simeq\mathrm{SU}(2)$ is the group of isotropy. On each slice $\{t\}\times S^3$ of M, we can construct an $\mathrm{SU}(2)\times\mathrm{SU}(2)$-invariant connection by making use of Wang's theorem[9] which establishes a bijective correspondence between S-invariant connections and the linear transformation $\Lambda:L(S)\rightarrow L(G)$ of Lie algebras such that

$$\text{(A)}\quad \Lambda(\mathrm{Y})=\mu_{p_0}(\mathrm{Y}),\quad \text{for}\quad \mathrm{Y}\in L(J_{x_0}),$$
$$\text{(B)}\quad \Lambda[\mathscr{A}\delta_j(\mathrm{X})]=\mathscr{A}\delta_{\mu(j)}(\Lambda(\mathrm{X})),\quad \text{for}\quad \mathrm{X}\in L(S),\quad j\in J_{x_0}, \tag{2.1}$$

where $\mu:J_{x_0}\rightarrow G$ is a homomorphism of Lie groups, and $x_0=\pi(p_0)$ is a fixed point in M.

If ω is the S-invariant connection corresponding to Λ, we have

$$\Lambda(\mathrm{X})=\omega(\hat{\mathrm{X}}_{p_0}),\quad \text{with}\quad \hat{\mathrm{X}}_{p_0}=\frac{d}{dt}(\exp t\mathrm{X}\cdot p_0)|_{t=0}\in L(S). \tag{2.2}$$

Making use of Eq. (2.1) and the fact that $L(\mathrm{SL}(2,\mathbb{C}))=L(\mathrm{SU}(2))\otimes\mathbb{C}$, it follows by a fairly straightforward algebraic computation that the map $\Lambda:L(\mathrm{SU}(2)\times\mathrm{SU}(2))\rightarrow L(\mathrm{SU}(2))\otimes\mathbb{C}$ is explicitly given by the matrix

$$u^\mu = \pm \frac{2x^\mu}{\tau}, \quad u^4 = \pm \frac{(1+x^2)}{\tau}, \quad u^5 = \pm \frac{(1-x^2)}{\tau}. \tag{A10}$$

Metric two-form for $S^1 \times S^3$. Let

$$\begin{aligned} \theta^1 &= 2[-u_2\, du_1 + u_1\, du_2 + u_4\, du_3 - u_3\, du_4], \\ \theta^2 &= -2[u_3\, du_1 + u_4\, du_2 - u_1\, du_3 - u_2\, du_4], \\ \theta^3 &= 2[-u_4\, du_1 + u_3\, du_2 - u_2\, du_3 + u_1\, du_4] \end{aligned} \tag{A11}$$

be the one-forms dual to the left-invariant vector fields given in Eq. (5.4) in the text. If is easy to show that

$$\theta^1 \otimes \theta^1 + \theta^2 \otimes \theta^2 + \theta^3 \otimes \theta^3 = 4(du_1^2 + du_2^2 + du_3^2 + du_4^2). \tag{A12}$$

Also, since u_0 and u_5 are coordinates of S^1, we can write

$$u_0 = \cos\psi, \quad u_5 = \sin\psi. \tag{A13}$$

Thus, in terms of this left-invariant basis, the natural Lorentzian metric for $S^1 \times S^3$ is

$$g = du_0^2 + du_5^2 - du_1^2 - du_2^2 - du_3^2 - du_4^2 = d\psi \otimes d\psi - \tfrac{1}{4}(\theta^1 \otimes \theta^1 + \theta^2 \otimes \theta^2 + \theta^3 \otimes \theta^3).$$

If we now set $d\psi = dt/2$, then

$$g = \tfrac{1}{4}(dt^2 - \theta^1 \otimes \theta^1 - \theta^2 \otimes \theta^2 - \theta^3 \otimes \theta^3). \tag{A14}$$

This is the metric that we use throughout the text. Note in particular that since $S^1 \times S^3$ is actually the double covering of compactified Minkowski space, the angle ψ in Eq. (A13) must be restricted to the range $0 \leq \psi \leq \pi$, and so we need that $0 \leq t \leq 2\pi$ (since $t := 2\psi$).

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