CLASSIFICATION OF GAUGE-RELATED INVARIANT CONNECTIONS

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Connection 1-forms on principal fiber bundles with arbitrary structure groups are considered, and a characterization of gauge-equivalent connections in terms of their associated holonomy groups is given. These results are then applied to invariant connections in the case where the symmetry group acts transitively on fibers, and both local and global conditions are derived which lead to an algebraic procedure for classifying orbits in the moduli space of these connections. As an application of the developed techniques, explicit solutions for SU(2) × SU(2)-symmetric connections over $S^2 \times S^2$, with SU(2) structure group, are derived and classified into non-gauge-related families, and multi-instanton solutions are identified.

1. Introduction

The principle of minimal coupling in physical theories translates, in the language of fiber bundles, into the requirement of invariance of the action density under the group $GA(P)$ of gauge transformations (i.e. of base-preserving automorphisms of a principal fiber bundle P). Given a connection 1-form ω on P, an important subgroup of $GA(P)$ is its *internal symmetry group* $I_\omega(P) = \{f \in GA(P) | f^*\omega = \omega\}$, because *it* is the generator of internal conservation laws. In studying this group Fischer [1] has shown that, if P is a connected manifold, then $I_\omega(P)$ is a finite-dimensional Lie group that acts smoothly, freely, and properly on P, and it is isomorphic to the centralizer of the holonomy group of ω.

In the present work we analyse the more general case where a gauge transformation is not necessarily an element of the internal symmetry group of the connection; i.e. we consider those automorphisms f such that $f^*\omega_1 = \omega_2$, with ω_1, ω_2 two given connection 1-forms, and arrive at results which generalize some of those obtained by Fischer and for which the latter appear as corollaries. We study in addition the space of connections which are invariant under a group S of symmetry transformations which project to a given action on the base manifold, and establish both global and local conditions for two gauge-related connections to be S-invariant. We are therefore concerned with the finite-dimensional moduli space of symmetric connections $\mathscr{M} = \{\mathscr{C}(P)/GA(P) | s * \omega = \omega\}$. There are many important results which arise from the under-

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standing of this space, and from the non-trivial relationship between local trivializations and symmetry-transformations in gauge theories (see e.g. Jackiw [2] for a review and bibliography on some previous work on the subject).

An algebraic procedure for constructing symmetric connections in the fiber bundle formalism was developed by Wang [3], and is also described in Kobayashi and Nomizu [4]. Wang's treatment assumes that there is an action of the symmetry group on the bundle which projects to a given action of the group on the base space. He also assumes that this action is transitive, so that the base space is homogeneous. However, the general problem of determining all the inequivalent lifts of the symmetry group action on the base space to an action on the principal fiber bundle is a difficult one and no general results are known [5]. In some specific cases, such as when S is compact and semisimple and the structure group G is solvable and connected, it has been shown that an essentially unique lifting exists [6]. Also, when the orbit structure is regular enough, and G is compact, Harnad et al. [7] have obtained a classification procedure for inequivalent lifts of the action of the symmetry group. Another case when such a classification can be obtained [8] is when S acts transitively on the base manifold M, so that the latter may be identified with the homogeneous space $M = S/J$, with $J = J_{x_0}$ the isotropy subgroup of S at a chosen point $x_0 = \pi(p_0)$. This last situation is the one considered in the present work.

A differential version of the solution to the problem of finding the most general gauge fields invariant under an infinitesimal symmetry transformation was presented in an independent fashion by Forgács and Manton [9], several years after the work of Wang appeared in the mathematical literature. The mathematical statement of this condition is contained in the following relation given by these authors:

$$\mathscr{L}_{\xi m} A_\mu = D_\mu W_m, \tag{1.1}$$

where ξ_m are the Killing vectors associated to the symmetry group; A_μ are the Lie algebra-valued gauge fields of the problem under consideration; and W_m are auxiliary gauge fields introduced with the explicit purpose of making the symmetry condition (1.1) covariant under the choice of gauge, and are required to satisfy the consistency condition

$$\mathscr{L}_{\xi m} W_n - \mathscr{L}_{\xi n} W_m - [W_m, W_n] - f_{mnp} W_p = 0, \tag{1.2}$$

with f_{mnp} the structure constants of the symmetry group.

We shall here give a geometric interpretation of these auxiliary fields as the differential of the function which lifts the symmetry action on the base space to a left action on the bundle, and show that, the differential formalism of Forgács and Manton is strictly equivalent to the one given by Wang only when certain global conditions hold, (as would be the case, for instance, when S is simply connected). In the case of multiply connected symmetry groups, there are restrictions to this equivalence in the sense that (1.1) and (1.2) do not integrate to an action of the symmetry group on the bundle.

With the intent of making it as self-contained as possible we have organized the paper as follows: in Sec. 2 we consider a principal fiber bundle P and discuss the problem of lifting a given symmetry action on the base manifold to a fiber transitive action on P. We derive local expressions for these actions and give a classification (along the lines of the work of Harnad et al.) in terms of conjugate classes of homomorphisms from the isotropy group to the characteristic group of the bundle. This result is contained in Proposition 2.1, and an explicit local formula for the lifting is given in (2.9). We also investigate in this section the relation between the fiber bundle formulation of symmetric connections and the one used more often in the physics literature, based on a differential approach which generalizes the condition for invariance of a field to gauge fields. We give necessary and sufficient conditions for the integrability of Eqs. (2.24) and (2.26) in Proposition 2.5, in terms of linear integrals over a finite set of generators of the homotopy group at the identity of the symmetry group S. In the case where S is simply connected, these conditions are always satisfied. To conclude the section we give the geometric interpretation of the scalar field Ψ_X of Forgács and Manton in terms of the matrix Λ associated to the symmetry connection according to Wang's theorem, deriving a relation between the gauge field and the covariant derivative of Ψ_X (see also [9]).

In Sec. 3 we prove that two connections are gauge-equivalent iff their holonomy functions at some point p_0 of P are conjugate (see Proposition 3.2), using this result we obtain as corollaries some results of Fisher (see [1]) describing the gauge transformations f such that $f^*\omega = \omega$ for some fixed connection ω (see Corollaries 3.3 and 3.4). In Sec. 4 we apply the above results to the study of gauge-equivalence of S-invariant connections. In Proposition 4.7, we give necessary and sufficient conditions for the local gauge-equivalence of two S-invariant connections. We show that if ω_1 and ω_2 are S-invariant connections, and $\Lambda_1, \Lambda_2 : L(S) \to L(G)$ are the associated linear transformations of ω_1 and ω_2, respectively, then these two connections are locally gauge equivalent in some G-invariant neighborhood of $p_0 \in P$ iff $\Lambda_2 = u^{-1}(\Lambda_1 + v_*|_e)u$ for $v : \mathscr{W} \to C_G(\mathrm{Hol}^\circ_{p_0}(\omega_1))$, where $\mathscr{W}$ is a certain open neighborhood of e in S, and $\mathrm{Hol}^\circ_{p_0}(\omega_1)$ is the group generated by the ω_1-holonomy of closed loops in M at $\pi(p_0)$ which are homotopic to the trivial loop. We also derive the restrictions which u and v must satisfy, and generalize the above conditions to the global case. For this situation it turns out that two S-invariant connections ω_1 and ω_2 are gauge equivalent iff $\Lambda_2 = u^{-1}(\Lambda_1 + v_*|_e)u$ where $v : S \to C_G(\mathrm{Hol}^\circ_{p_0}(\omega))$ and u must satisfy conditions similar to those for the local case. Moreover, here the restrictions on u and v can be expressed in a nicer way (see Proposition 4.11) and, if the connections are generic ($C_G(\mathrm{Hol}_p(\omega)) = Z(G)$), the restriction on v is that it must be a homomorphism.

As an application of our formalism, we end in Sec. 5 with an explicit derivation of $\mathrm{SU}(2) \times \mathrm{SU}(2)$-invariant connections for principal fiber bundles with base space $S^2 \times S^2$ and gauge group SU(2). The solutions correspond to a total of five non-gauge-equivalent families with 3 free parameters and characterized by a second Chern number given by $2rs$, with r, s integers. It is also noted that of these five families of solutions only the ones corresponding to the canonical connection allow for self-duality when $r = s$, and thus lead to multi-instantons with second Chern number equal to $2r^2$.

In general our notation follows the one used by Kobayashi and Nomizu [4]. However, for the sake of clarity, we list the most important symbols used in the paper:

Symbols

All the objects are smooth (C^∞).
M—a differentiable paracompact base manifold
$\pi : P \to M$ or $P(M, G)$—a principal fiber bundle with structure group G.
$\pi^{-1}(x)$—the fiber over $x \in M$ in $P(M, G)$.
S—a Lie group (the group of symmetry which acts on P).
J or J_{x_0}—a Lie subgroup of S (the isotropy group).
$M = S/J$—the manifold M as an homogeneous space.
$\sigma_\alpha : U_\alpha \to P$—a system of local sections for $P(M, G)$.
$\varphi^{(\beta,\alpha)} : U_\beta \cap U_\alpha \to G$—a function which describes how the action of S lifts to P.
$\mu : J \to G$—an homomorphism of Lie groups.
$L(G)$, $L(S)$, $L(J)$—the Lie algebras associated to the groups G, S, J.
$a\delta_j(X) = jXj^{-1}$, for $j \in S$, $X \in L(S)$—the adjoint action.
$\Lambda : L(S) \to L(G)$—a linear map.
Λ_j^α—the components of Λ.
$\mathscr{L}_Y$—the Lie derivative with respect to Y.
ω—the S-invariant connection in $P(M, G)$ associated to Λ.
A_j^α—the components of the gauge potential.
Ω or Ω^ω—the curvature associated to ω.
D_Y—the covariant derivative.
$W : M \to L(L(S), L(G))$—a C^∞ map of M to the linear functions of $L(S)$ in $L(G)$.
$W_x : L(S) \to L(G)$—the associated map at $x \in M$.
W_x^α—the components of W.
$C(P, G)$—the space of equivariant maps $\tau : P \to G$.
$f : P \to P$—a gauge transformation (given by $f(p) = p\tau(p)$, τ as above).
$C(x, y)$—the collection of paths in M from x to y ($x, y \in M$).
$H^\omega(x, y) : C(x, y) \to \mathrm{Hom}_G(\pi^{-1}(x), \pi^{-1}(y))$—the holonomy maps of the connection ω.
$h^\omega(x, y)(\alpha) : \pi^{-1}(x) \to \pi^{-1}(y)$—the holonomy map of the connection ω associated to the path α in M with end points x and y.
$h_p^\omega(\gamma)$—the holonomy map of the connection ω associated to the loop (γ), where the basis point is $p \in P$.
$\mathrm{Hol}_{p_0}(\omega)$—the holonomy group of ω based at $p_0 \in P$.
$\mathrm{Hol}_{p_0}^0(\omega)$—the restricted holonomy group of ω based at $p_0 \in P$.
$C_G(\mathrm{Hol}_{p_0}(\omega))$—the centralizer group of the holonomy group of ω in G.
$L(\mathrm{Hol}_{p_0}(\omega))$—the Lie algebra of the holonomy group of ω based at $p_0 \in P$.

2. Lifting of Actions

Let $\pi : P \to M$ be a principal fiber bundle with gauge group G. We shall say that a Lie group S acts on P by bundle automorphisms if for $s \in S$ and $p \in P$, we have $s(pg) = (sp)g$. That is, the action of S commutes with the right action of G. The transformation on M induced by this S-automorphism is defined by $s\pi(p) = \pi(sp)$. Note that the action on the base manifold is well defined, since for $p' = pg$ and $\pi(p) = x$, we have $\pi(sp') = \pi(s(pg)) = \pi((sp)g) = \pi(sp)$.

As we pointed out in the introduction, determining all the inequivalent actions of S on P which induce a given action on M is, in general, an open topological problem. Some special cases have been dealt with in the literature [5–8], and in particular when S acts fiber-transitively on P, it is known that the problem reduces to a classification of group homomorphisms. The analysis in this paper will be restricted to such homogeneous base spaces, with $M = S/J_{x_0}$ and J_{x_0} the isotropy subgroup of S at $x_0 = \pi(p_0)$ ($J_{x_0} = \{s \in S | sx_0 = x_0\}$). We shall also assume that M is paracompact in order to ensure the existence of a connection. For a discussion of a generalization of the above-mentioned problem to inhomogeneous spaces we refer the reader to the work of Harnad, Shnider and Vinet [7].

Specifically, our purpose in this section is to obtain a local expression for the action of S on P. This expression will be one of the basic tools that we shall resort to in the remainder of our work. It will allow us, in addition, to clarify the relation between the Forgács-Manton differential formalism and Wang's theorem, in the case where the base space M is homogeneous.

Let $\sigma_\alpha : U_\alpha \to P$, $\alpha \in I$, be a system of local sections. For $x \in U_\alpha \cap U_\beta$ we have $\sigma_\alpha(x) = \sigma_\beta(x) g_{\beta\alpha}(x)$, with $g_{\beta\alpha}(x) \in G$. Then, if $s \in S$, $x \in U_\alpha$, $sx \in U_\beta$, the local action of S on P is given by

$$s\sigma_\alpha(x) = \sigma_\beta(sx)\varphi_x^{(\beta,\alpha)}(s), \tag{2.1}$$

where $\varphi_x^{(\beta,\alpha)}(s) \in G$ describes how the action of S on U_α has been lifted to the fibers.

It is easy to verify that for $s, t \in S$, $x \in U_\alpha$, $sx \in U_\beta$ and $tsx \in U_\gamma$

$$\varphi_x^{(\gamma,\alpha)}(ts) = \varphi_{sx}^{(\gamma,\beta)}(t)\varphi_x^{(\beta,\alpha)}(s). \tag{2.2}$$

Also, if $x \in U_\alpha \cap U_{\alpha'}$ and $sx \in U_\beta \cap U_{\beta'}$

$$\varphi_x^{(\beta,\alpha)}(s) = g_{\beta\beta'}(sx)\varphi_x^{(\beta',\alpha')}(s)g_{\alpha\alpha'}(x)^{-1}. \tag{2.3}$$

In particular, if $x \in U_\alpha$, $sx \in U_\alpha$ and $tsx \in U_\alpha$, Eq. (2.2) becomes

$$\varphi_x^\alpha(ts) = \varphi_{sx}^\alpha(t)\varphi_x^\alpha(s), \tag{2.4}$$

where $\varphi_x^\alpha(s)$ has been used to denote $\varphi_x^{(\alpha,\alpha)}(s)$ with $x \in U_\alpha$ and $sx \in U_\alpha$.

Similarly, if x and $sx \in U_\alpha \cap U_{\alpha'}$, Eq. (2.3) becomes

$$\varphi_x^\alpha(s) = g_{\alpha\alpha'}(sx)\varphi_x^{\alpha'}(s)g_{\alpha\alpha'}(x)^{-1}. \tag{2.5}$$

Note that if $j \in J_{x_0}$ and $\sigma_\alpha(x_0) = p_0$, then $j\sigma_\alpha(x_0) = \sigma_\alpha(x_0)\mu_{p_0}(j)$, with $\mu_{p_0}(j) \equiv \varphi_{x_0}^\alpha(j) \in G$. It clearly follows from (2.4) that

$$\mu_{p_0}(j_1 j_2) = \mu_{p_0}(j_1)\mu_{p_0}(j_2), \qquad j_1, j_2 \in J_x, \tag{2.6}$$

so $\mu_{p_0}: J_{x_0} \to G$ is a homomorphism of Lie groups. Note also that

$$\mu_{p_0 g}(j) = g^{-1}\mu_{p_0}(j)g. \tag{2.7}$$

Assume now that S acts transitively on M and consider the space $S \times_{\mu_{p_0}} G$ given by the product $S \times G$ modulo the equivalence relations $(s, g) \sim (sj, \mu(j)^{-1}g)$, for $s \in S, g \in G$ and $j \in J_{x_0}$, with $\mu: J \to G$ a homomorphism of Lie groups.

Let $\hat{\pi}: S \times G \to M$ be the transformation given by $\hat{\pi}(s, g) = sx_0$, we then have that $\hat{\pi}(sj, \mu(j)^{-1}g) = sjx_0 = sx_0 = \hat{\pi}(s, g)$ and, thus, $\hat{\pi}$ induces the transformation $\pi: S \times_\mu G \to M$.

We shall show that the space $S \times_\mu G \xrightarrow{\pi} M$ is a principal fiber bundle with gauge group G. To this end, let $(\overline{s, g})$ denote the equivalence class of (s, g), and define the action of G on $S \times_\mu G$ by $(\overline{s, g})g_1 = (\overline{s, gg_1})$. This action is well defined since it preserves the equivalence relation. On the other hand we know that $S \xrightarrow{\pi} M(=S/J_{x_0})$ is a principal fiber bundle, and we consider a system of local sections $\tau_\alpha: U_\alpha \to S$ in this bundle. We can then define $\bar{\tau}_\alpha: U_\alpha \to S \times_\mu G$ by $\bar{\tau}_\alpha(x) = (\overline{\tau_\alpha(x), e})$. Observe that if $(\overline{s, g})$ is such that $\pi(\overline{s, g}) = \pi(\bar{\tau}_\alpha(x)) = x$, then there is a local trivialization $T_\alpha: \pi^{-1}(U_\alpha) \to U_\alpha \times G$ given by $T_\alpha(\bar{\tau}_\alpha(x), g) = T_\alpha(\overline{\tau_\alpha(x), g}) = (\pi(\overline{\tau_\alpha(x), g}), g) = (x, g)$. So $S \times_\mu G$ has indeed a bundle structure.

Reciprocally, assume that P is a principal fiber bundle over M with gauge group G. Then the map $\hat{\Phi}: S \times G \to P$ given by $\hat{\Phi}(s, g) = sp_0 g$ is such that $\hat{\Phi}(sj, \mu_{p_0}(j)^{-1}g) = sjp_0\mu_{p_0}(j)^{-1}g = sp_0 g = \hat{\Phi}(s, g)$. Consequently $\hat{\Phi}$ induces a transformation $\Phi: S \times_{\hat{\mu}} G \to P$, which is a homeomorphism.

The above establishes the following result [8]:

Proposition 2.1. *There is a one-to-one correspondence between equivalence classes of principal fiber bundles with gauge group G over $M = S/J_{x_0}$, admitting an S-action which projects on a given action of S on M, and the conjugate classes of homomorphisms $\mu_x: J_{x_0} \to G$.*

Given the above constructed sections $\bar{\tau}_\alpha: M \to S \times_\mu G$, we shall now derive the corresponding functions $\varphi_x^\alpha(s)$. Let $s \in S$ with $sx \in U_\alpha$ and denote by $\tilde{\varphi}_x^\alpha(s) \in J_{x_0}$ the transformations corresponding to the section $\tau_\alpha: U_\alpha \to S$. We then have $s\bar{\tau}_\alpha(x) = (\overline{s\tau_\alpha(x), e}) = (\overline{\tau_\alpha(sx)\tilde{\varphi}_x^\alpha(s), e}) = (\overline{\tau_\alpha(sx), \mu_{p_0}(\tilde{\varphi}_x^\alpha(s))}) = (\overline{\tau_\alpha(sx), e})\mu_{p_0}(\tilde{\varphi}_x^\alpha(s)) = \bar{\tau}_\alpha(sx)\mu_{p_0}(\tilde{\varphi}_x^\alpha(s))$. Consequently

$$\varphi_x^\alpha(s) = \mu_{p_0}(\tilde{\varphi}_x^\alpha(s)). \tag{2.8}$$

In order to calculate $\tilde{\varphi}_x^\alpha(s)$, note that if $\eta: S \to S/J_{x_0}$ is the natural projection given by $\eta(y) = \bar{y} = yJ_{x_0}$, then $\eta\tau_\alpha = \mathrm{id}_{S/J_{x_0}}$ and $\eta(\tau_\alpha(sx)^{-1}s\tau_\alpha(x)) = (sx)^{-1}sxJ_{x_0} = eJ_{x_0}$, which in turn implies $\tau_\alpha(sx)^{-1}s\tau_\alpha(x) \in J_{x_0}$. Thus, since $s\tau_\alpha(x) = \tau_\alpha(sx)(\tau_\alpha(sx)^{-1}s\tau_\alpha(x)$ it clearly follows that $\tilde{\varphi}_x^\alpha(s) = (\tau_\alpha(sx)^{-1}s\tau_\alpha(x))$. Substituting this result in (2.8) we finally have

$$\varphi_x^\alpha(s) = \mu_{p_0}(\tau_\alpha(sx)^{-1}s\tau_\alpha(x)). \tag{2.9}$$

This last expression proves the following:

Corollary 2.2. *Assume that the Lie group S acts transitively on M. If P is a principal fiber bundle over M with gauge group G, and S is an automorphism on P which projects on the given action of S on M, then there exist sections $\bar{\tau}_\alpha : U_\alpha \to U_\alpha \times_\mu G \approx P$ such that the corresponding functions $\varphi_x^\alpha(s)$ are equal to $\mu_{p_0}(\tau_\alpha(sx)^{-1} s\tau_\alpha(x))$, with $\tau_\alpha : U_\alpha \to S$ local sections of the bundle $S \xrightarrow{\eta} M$ and $\mu_{p_0} : J_{x_0} \to G$ the transformation function corresponding to the action of J_{x_0} on P.*

Another consequence of Proposition 2.1 is:

Corollary 2.3. (see [7]) *The bundle $P \xrightarrow{\pi} M$ is trivial iff $\mu : J_{x_0} \to G$ extends to a smooth morphism $\Gamma : S \to G$ such that $\Gamma(sj) = \Gamma(s)\mu(j)$, $s \in S$, $j \in J_{x_0}$.*

Proof. Let $\sigma : M \to P$ be a global section and let $\varphi_x(s)$ be the function associated to the action of S on P then, if $j \in J_{x_0}$, $\varphi_{x_0}(sj) = \varphi_{jx_0}(s)\varphi_{x_0}(j) = \varphi_{x_0}(s)\varphi_{x_0}(j)$. Therefore $\varphi_{x_0}(s) = \Gamma(s)$ satisfies the condition of the corollary. Reciprocally, assume that such a Γ exists. Define $\sigma(x) = (\tau_\alpha(x), \Gamma(\tau_\alpha(x))^{-1}) \in S \times_\mu G$ for $x \in U_\alpha$. If $x \in U_\alpha \cap U_\beta$, we have $\tau_\beta(x) = \tau_\alpha(x) j_{\alpha\beta}(x)$ where $j_{\alpha\beta}(x) \in J_{x_0}$. Hence $(\tau_\beta(x), \Gamma(\tau_\beta(x))^{-1}) = (\tau_\alpha(x) j_{\alpha\beta}(x), \mu_{p_0}(j_{\alpha\beta}(x))^{-1}\Gamma(\tau_\alpha(x))^{-1}) = (\tau_\alpha(x), \Gamma(\tau_\alpha(x))^{-1})$. Consequently $\sigma(x)$ is independent of the neighborhood U_α, which implies in turn that σ is a global section. □

We shall describe next an algebraic procedure for constructing gauge fields from S-invariant connection 1-forms (i.e. for $s^*\omega = \omega$) on principal fiber bundles with arbitrary gauge groups, such that they posses the symmetry of the underlying base manifold for the case when S acts transitively on the latter.

We begin by defining local invariance of a connection:

Definition 2.4. Let $N \subset M$ be an open set with $x_0 \in N$, and ω a connection 1-form defined on $\pi^{-1}(N)$. We shall say that ω is *locally S-invariant at* x_0 if for all $s \in S$ with $sx_0 \in N$ there exists a connected neighborhood $V_{(s)}$ of x_0 contained in $N \cap s^{-1}N$ and such that

$$s^*\omega|_{V_{(s)}} = \omega|_{V_{(s)}}.$$

Remark. Let $\mathscr{W} = \{s \in S | sx_0 \in N\}$. Clearly $\mathscr{W} x_0 = N$. According to the above definition, for each $s \in \mathscr{W}$ we have a connected open subset $V_{(s)}$ of N. One can indeed show that there exists a neighborhood K of $e \in S$ such that $N \cap s^{-1}N$ is connected for all $s \in K$ if N is small enough, and that in this case $V_{(s)} = N \cap s^{-1}N$. An outline of a proof of this fact is a follows:

Since M is paracompact, it is therefore metrizable and we may choose an open ball $N(x_0) \subset M$ with normal coordinates with respect to x_0, such that its boundary is orthogonal to the geodesics emmanating from x_0. If K is a sufficiently small neighborhood of the identity in S, then the open sets $s^{-1}N(x_0)$ can be made very close to $N(x_0)$ and, in particular, their boundaries $\partial(s^{-1}N(x_0))$ will be transverse to the geodesic rays emmanating from x_0 and will intersect each one only once. (Note that by the Transversality Theorem (cf. e.g. Hirsh [10]) the condition of transversality is open in the parameter s, so the construction of K is always possible.) It can then be shown that, given $q_1, q_2 \in N(x_0) \cap s^{-1}N(x_0)$, the union of geodesic arches from q_1 to x_0 and from x_0 to q_2 is contained in $N(x_0) \cap s^{-1}N(x_0)$. Hence, this intersection is convex for all $s \in K$.

Recall now Wang's theorem (see [3] and [4]) which states that there is a bijective correspondence between S-invariant connections and linear transformations $\Lambda : L(S) \to L(G)$ of Lie algebras which satisfy the following conditions:

$$\begin{aligned} &(A) \qquad \Lambda(Y) = \mu_{p_0 *}(Y), \qquad \text{for } Y \in L(J_{x_0}) \text{ (the Lie algebra of } J_{x_0}) \\ &(B) \quad \Lambda(a\delta_j(X)) = a\delta_{\mu_{p_0}(j)}(\Lambda(X)), \qquad \text{for } X \in L(S),\ j \in J_{x_0}. \end{aligned} \tag{2.10}$$

If ω is the S-invariant 1-form connection corresponding to Λ, we have

$$\Lambda(X) = \omega_{p_0}(\hat{X}_{p_0}), \qquad \text{with } \hat{X}_{p_0} = \frac{d}{dt}(\exp tX \cdot p_0)|_{t=0}, \ X \in L(S). \tag{2.11}$$

Denote now by $\tilde{X}$ the vector field on M defined by $\tilde{X}_x = \frac{d}{dt}(\exp tX \cdot x)|_{t=0}$. For $x \in U_\alpha$, and $sx \in U_\alpha$, the relation between $\tilde{X}_x$ and $\hat{X}_p$ is

$$\begin{aligned} \hat{X}_{\sigma_\alpha(x)} &= \frac{d}{dt}[\exp tX \cdot \sigma_\alpha(x)]|_{t=0} = \frac{d}{dt}[\sigma_\alpha(\exp tX \cdot x)\varphi^\alpha_x(\exp tX)]|_{t=0} \\ &= (\sigma_\alpha)_*(\tilde{X}_x) + [W^\alpha_x(X)]^*_{\sigma_\alpha(x)} \end{aligned} \tag{2.12}$$

where $W^\alpha_x \equiv (\varphi^\alpha_x)_*$, so that $x \mapsto W^\alpha_x$ gives rise to a function from M to $\mathscr{L}(L(S), L(G))$, and $[W^\alpha_x(X)]^*_{\sigma_\alpha(x)}$ is the fundamental field associated with $W^\alpha_x(X) \in L(G)$. Consequently

$$(\sigma^*_\alpha \omega)_x(\tilde{X}_x) = \omega_{\sigma_\alpha(x)}(\hat{X}_{\sigma_\alpha(x)}) - W^\alpha_x(X). \tag{2.13}$$

In addition, using the S-invariance of ω, we have

$$\omega_{\sigma_\alpha(x_0)}(\hat{X}_{\sigma_\alpha(x_0)}) = (s^*\omega)_{\sigma_\alpha(x_0)}(\hat{X}_{\sigma_\alpha(x_0)}) = \omega_{s\sigma_\alpha(x_0)}(s_* \hat{X}_{\sigma_\alpha(x_0)}). \tag{2.14}$$

But

$$s_*(\hat{X})_{\sigma_\alpha(x_0)} = \frac{d}{dt}[s \cdot \exp tX \cdot s^{-1} \cdot s\sigma_\alpha(x_0)]|_{t=0} = (\widehat{a\delta_s X})_{s\sigma_\alpha(x_0)}, \tag{2.15}$$

so

$$\begin{aligned} \omega_{\sigma_\alpha(x_0)}(\hat{X}_{\sigma_\alpha(x_0)}) &= \omega_{s\sigma_\alpha(x_0)}(\widehat{a\delta_s X}_{s\sigma_\alpha(x_0)}) = \omega_{\sigma_\alpha(sx_0)\varphi^\alpha_{x_0}(s)}(\widehat{a\delta_s X}_{\sigma_\alpha(sx_0)\varphi^\alpha_{x_0}(s)}) \\ &= a\delta_{\varphi^\alpha_{x_0}(s)^{-1}} \omega_{\sigma_\alpha(sx_0)}(\widehat{a\delta_s X}_{\sigma_\alpha(sx_0)}), \end{aligned}$$

i.e.

$$\omega_{\sigma_\alpha(sx_0)}(\hat{X}_{\sigma_\alpha(sx_0)}) = a\delta_{\varphi^\alpha_{x_0}(s)} \omega_{\sigma_\alpha(x_0)}(\widehat{a\delta_{s^{-1}} X}_{\sigma_\alpha(x_0)}). \tag{2.16}$$

Since S acts transitively on M we can set $x = sx_0$ and substituting (2.16) into (2.13) yields

$$A_\alpha(\tilde{X}_x) \equiv (\sigma_\alpha^*\omega)_x(\tilde{X}_x) = a\delta_{\varphi^\alpha_{x_0}(s)}\Lambda(a\delta_{s^{-1}}X) - W_x^\alpha(X). \tag{2.17}$$

Equation (2.17) is the general algebraic expression for our S-invariant gauge fields, and is in fact the integrated form of (1.1). That this is indeed so can be shown as follows:

Take $\sigma_\alpha : U_\alpha \to P$ to be a local section, $x \in U_\alpha$, and a fixed $s \in S$ such that $sx \in U_\alpha$. We can then find a neighborhood V_α of x such that $sV_\alpha \subset U_\alpha$, and so that for $s : \pi^{-1}(V_\alpha) \to \pi^{-1}(sV_\alpha)$, both $\pi^{-1}(V_\alpha)$ and $\pi^{-1}(sV_\alpha)$ are contained in $\pi^{-1}(U_\alpha)$.

Making use of (2.1) we have $s\sigma_\alpha(\gamma(t)) = \sigma_\alpha(s\gamma(t))\varphi^\alpha_{\gamma(t)}(s)$, from where it easily follows

$$(s_*\sigma_{\alpha*}\xi_x)_{s\sigma_\alpha(x)} = (\varphi_x^\alpha(s)^{-1}\xi_x[\varphi_x^\alpha(s)])^*_{s\sigma_\alpha(x)} + R_{\varphi^\alpha_x(s)*}\sigma_{\alpha*}s_*\xi_x, \tag{2.18}$$

where ξ is a vector field on M tangent to the curve $\gamma(t)$, and $\xi[\varphi_x^\alpha(s)]$ denotes the directional derivative of $\varphi_x^\alpha(s)$ along ξ. Applying now ω to both sides yields

$$(s^*\omega)(\sigma_{\alpha*}\xi)_{\sigma_\alpha(x)} = \varphi_x^\alpha(s)^{-1}\xi[\varphi_x^\alpha(s)] + a\delta_{\varphi^\alpha_x(s)^{-1}}\omega(\sigma_{\alpha*}s_*\xi)_{\sigma_\alpha(sx)}. \tag{2.19}$$

Furthermore, taking $A_\alpha(\xi_x) = (\sigma_\alpha^*\omega)(\xi_x)$ and recalling that $s^*\omega = \omega$, (2.19) results in

$$A_\alpha(s_*\xi_x) = \varphi_x^\alpha(s)A_\alpha(\xi_x)\varphi_x^\alpha(s)^{-1} - \xi[\varphi_x^\alpha(s)]\varphi_x^\alpha(s)^{-1}. \tag{2.20}$$

Consequently,

$$\begin{aligned}(\mathscr{L}_{\tilde{Y}}A_\alpha)(\xi_x) &= \frac{d}{dt}(A_\alpha((\exp tY)_*\xi_x))|_{t=0}\\ &= \frac{d}{dt}(\varphi_x^\alpha(s)A_\alpha(\xi_x)\varphi_x^\alpha(s)^{-1})|_{t=0} - \frac{d}{dt}(\xi_x[\varphi_x^\alpha(\exp tY)])|_{t=0}\\ &= -\xi_x[W_x(Y)] - [A_\alpha(\xi_x), W_x(Y)]\\ &\equiv D_{\xi_x}W_x^\alpha(Y),\end{aligned} \tag{2.21}$$

for $\xi \in T(M)$, $Y \in L(S)$ and all α. This expression is indeed the same as (1.1), and is the differential form on the base manifold of the S-invariance requirement of the connection 1-forms on the bundle.

Consider now the differential version of (2.4). For this purpose write $ts = t\exp(\varepsilon Y) = \exp(\varepsilon tYt^{-1})\cdot t$, so (2.4) reads $\varphi_x^\alpha(\exp(\varepsilon tYt^{-1})\cdot t) = \varphi^\alpha_{\exp(\varepsilon Y)\cdot x}(t)\varphi_x^\alpha(\exp(\varepsilon Y))$. Applying (2.4) once more to the left side of this last equation, gives

$$\varphi_{tx}^\alpha(\exp(\varepsilon tYt^{-1}))\varphi_x^\alpha(t) = \varphi^\alpha_{\exp(\varepsilon Y)\cdot x}(t)\varphi_x^\alpha(\exp(\varepsilon Y)). \tag{2.22}$$

Differentiating (2.22) with respect to ε and evaluating at $\varepsilon = 0$, results in

$$W_{tx}^\alpha(a\delta_t Y)\varphi_x^\alpha(t) = \varphi_x^\alpha(t)W_x^\alpha(Y) + \mathscr{L}_{\tilde{Y}}\varphi_x^\alpha(t). \tag{2.23}$$

Now write $t = \exp(\eta X)$ and differentiate (2.23) with respect to η and evaluate at $\eta = 0$. For the left-hand side we get

$$\frac{d}{d\eta}[W^\alpha_{tx}(a\delta_t Y)\varphi^\alpha_x(t)]|_{\eta=0} = \frac{d}{d\eta}[W^\alpha_{\exp(\eta X)\cdot x}(Y)]|_{\eta=0} + \frac{d}{d\eta}W^\alpha_x(Y + \eta[X, Y])|_{\eta=0}$$

$$+ W^\alpha_x(Y)W^\alpha_x(X)$$

$$= \mathscr{L}_{\tilde{X}}W^\alpha_x(Y) + W^\alpha_x([X, Y]) + W^\alpha_x(Y)W^\alpha_x(X).$$

Similarly for the right-hand side of (2.23) we have

$$\frac{d}{d\eta}[\varphi^\alpha_x(t)W^\alpha_x(Y) + \mathscr{L}_{\tilde{Y}}\varphi^\alpha_x(t)]|_{\eta=0} = W^\alpha_x(X)W^\alpha_x(Y) + \mathscr{L}_{\tilde{Y}}W^\alpha_x(X).$$

Thus, combining results, we arrive at

$$\mathscr{L}_{\tilde{X}}W^\alpha_x(Y) - \mathscr{L}_{\tilde{Y}}W^\alpha_x(X) + W^\alpha_x([X, Y]) - [W^\alpha_x(X), W^\alpha_x(Y)] = 0. \tag{2.24}$$

Therefore (2.24) is a consequence of the algebraic relation (2.4). We shall now observe that (2.24) is also closely related to (2.21). In fact,

$$-(\mathscr{L}_{[X,Y]}A^\alpha)(\xi) = (\mathscr{L}_{[\tilde{X},\tilde{Y}]}A^\alpha)(\xi)$$

$$= ((\mathscr{L}_{\tilde{X}}\mathscr{L}_{\tilde{Y}} - \mathscr{L}_{\tilde{Y}}\mathscr{L}_{\tilde{X}})A^\alpha)(\xi)$$

$$= \mathscr{L}_{\tilde{X}}D_\xi W^\alpha_x(Y) - \mathscr{L}_{\tilde{Y}}D_\xi W^\alpha_x(X).$$

Using now (2.22) and the Jacobi identity we have

$$-D_\xi W^\alpha_x([X, Y]) = D_\xi(\mathscr{L}_{\tilde{X}}W^\alpha_x(Y) - \mathscr{L}_{\tilde{Y}}W^\alpha_x(X) - [W^\alpha(X), W^\alpha(Y)]),$$

which is satisfied identically if (2.24) holds.

The difference in sign between some of the terms in (2.21) and (2.24), and the corresponding expressions found in [9], are due to the fact that in our discussion we consider S as a left action on M, so for $\tilde{X}_x \equiv \frac{d}{dt}(\exp tX \cdot x)|_{t=0}$ (left action) we have $[\tilde{X}_x, \tilde{Y}_x] = -\widetilde{[X_x, Y_x]}$. Had we considered a right action of S on M the results would have been identical.

Note that (2.24) is a consistency equation for the linear functions W^α_x. An additional local expression in terms of the transition functions which may be useful for explicit calculations, can be derived from an infinitesimal version of (2.5). We thus have

$$W_x^\alpha(X) = \frac{d}{dt}[g_{\alpha\alpha'}(\exp tX\cdot x)\varphi_x^{\alpha'}(\exp tX)g_{\alpha\alpha'}(x)^{-1}]|_{t=0}$$

$$= (\mathscr{L}_{\tilde{X}}g_{\alpha\alpha'}(x))g_{\alpha\alpha'}(x)^{-1} + g_{\alpha\alpha'}(x)W_x^{\alpha'}(X)g_{\alpha\alpha'}(x)^{-1}, \tag{2.25}$$

or

$$W_x^{\alpha'}(X) = -g_{\alpha\alpha'}(x)^{-1}(\mathscr{L}_{\tilde{X}}g_{\alpha\alpha'}(x)) + g_{\alpha\alpha'}(x)^{-1}W_x^\alpha(X)g_{\alpha\alpha'}(x). \tag{2.26}$$

Note furthermore that, given a principal fiber bundle $\pi: P \to M$ with structure group G, such that the $\varphi_x^\alpha(s)$ are the transformation functions corresponding to an action of S on P which induce a given action of S on M, Eqs. (2.24) and (2.26) are necessary conditions for the linear transformations W_x^α to arise from such an action of S on P.

To establish sufficiency, we need to examine the integrability of Eqs. (2.24) and (2.26). We will say that the *system* of equations (2.24), (2.26) *is integrable* if there is an action of S on P inducing the given action of S on M, such that if φ_x^α are the corresponding functions with respect to σ_α, then $(\varphi_x^\alpha)_* = W_x^\alpha$.

As the following simple example shows, the system (2.24), (2.26) is not always integrable: Take $S = S_0 \times J$, with S_0, J Lie groups, $M = S_0 \times J/e \times J = S_0$ and $P = S_0 \times G$. Then $L(S) = L(S_0) \oplus L(J)$. In the general situation there are Lie algebra-homomorphisms $W: L(J) \to L(G)$ such that there is no homomorphism of Lie groups $\varphi: J \to G$ with $\varphi_* = W$. Take one such W and define for our example $\overline{W}: L(S) \to L(G)$ with $\overline{W}|_{L(J)} = W$ and $\overline{W}_{L(S_0)} = 0$. Then define $W_x = \overline{W}$ for all $x \in M = S_0$. Clearly Eq. (2.24) holds for W_x but it is not integrable because W is not integrable ((2.26) is empty since P is trivial and we take only one section).

In order to have integrability for the system (2.24), (2.26), some additional global conditions are required. Note first that if $W: M \to \mathscr{L}(L(S), L(G))$ we can define a 1-form on S with values in $L(G)$ by putting $\underline{W}(\mathring{X}|_s) = W_{\eta(s)}(X)$ with $\mathring{X}|_s = \frac{d}{dt}\exp tX\cdot s|_{t=0}$. Recall now that if γ is a closed curve starting at e and $\tilde{\gamma}: [0,1] \to P$ is a ω-horizontal lift of γ (for ω a connection on P), then $\tilde{\gamma}(0) = \sigma_{\alpha_0}(e)$ and $\tilde{\gamma}(1) = \sigma_{\alpha_0}(e)h(\gamma)$, where $h(\gamma)$ is the holonomy of γ at $\sigma_{\alpha_0}(e)$. If γ_1, γ_2 are closed curves at e and $\gamma_1\gamma_2$ their composition, then $h(\gamma_1\gamma_2) = h(\gamma_1)h(\gamma_2)$ (see Sec. 3). We then have

Proposition 2.5. *Equations* (2.24) *and* (2.26) *are integrable if and only if there are closed curves $\gamma_1, \ldots, \gamma_s$ starting at $e \in S$ with their images generating $\pi_1(S,e)$, the homotopy group at e, such that for each $\gamma \in \{\gamma_1, \ldots, \gamma_s\}$ there are $0 = t_0 < t_1 < \cdots < t_{r+1} = 1$ with $\gamma(t_i) \in \eta^{-1}(U_{\alpha_i}) \cap \eta^{-1}(U_{\alpha_{i-1}})$, $\gamma[t_i, t_{i+1}] \subset \eta^{-1}(U_{\alpha_i})$ ($\eta: S \to M$ the projection map) and*

$$e = g_{\alpha_0\alpha_r}(\eta\gamma(t_{r+1}))\mathscr{P}\left(\exp\int_{tr}^1 \underline{W}^{\alpha_r}(\gamma'(t))\,dt\right)\cdots g_{\alpha_1\alpha_0}(\eta\gamma(t_1))\mathscr{P}\left(\exp\int_{t_0}^{t_1}\underline{W}_s^{\alpha_0}(\gamma'(t))\,dt\right). \tag{2.27}$$

Proof. Assume the system (2.24), (2.26) is integrable. Then there is an action of S on P lifting the given action on M. Moreover if $\varphi_x^\alpha(s)$ are the functions given by $s\sigma_\alpha(x) = \sigma_\alpha(x)\varphi_x^\alpha(s)$ for $x, sx \in U_\alpha$, then $(\varphi_x^\alpha)_* = W_x^\alpha$. Consider now the G-principal fiber bundle $\hat{P}$ over S with sections $\tilde{\sigma}_\alpha : V_\alpha = \eta^{-1}(U_\alpha) \to \hat{P}$ and transition functions $\underline{g}_{\alpha\beta}(s)$ with $g_{\alpha\beta}(\eta(s)) = \underline{g}_{\alpha\beta}(s)$ for $s \in \eta^{-1}(U_\alpha) \cap \eta^{-1}(U_\beta)$. We have $\Phi : \hat{P} \to P$, a map of G-fiber bundles such that the following diagram is conmutative:

$$\begin{array}{ccc} \hat{P} & \xrightarrow{\ \Phi\ } & P \\ \hat{\pi}\downarrow & & \downarrow\pi \\ S & \xrightarrow{\ \eta\ } & M \end{array} \tag{2.28}$$

Let $\psi_x^{\alpha,\beta}(s) = \varphi_{\eta(x)}^{(\alpha,\beta)}(s)$ for $x \in V_\alpha$ and $sx \in V_\beta$. Clearly these functions satisfy conditions (2.2) and (2.3) for the transition functions $\underline{g}_{\alpha\beta}$. Consequently this gives an action of S on $\hat{P}$ lifting the left action of S on S given by the multiplication in S. By (2.2) we have $\psi_x^{(\gamma,\alpha)}(ts) = \psi_{sx}^{(\gamma,\beta)}(t)\psi_x^{(\beta,\alpha)}(s)$ with $x \in V_\alpha, sx \in V_\beta, tsx \in V_\gamma$. Then taking $x = e \in V_{\alpha_0}, s \in V_\beta$, $ts \in V_\beta$, $\gamma = \beta$, we have

$$\psi_s^{(\beta)}(t) = h_\beta(ts)h_\beta(s)^{-1} \qquad \text{with } h_\beta(s) = \psi_e^{(\beta,\alpha_0)}(s),\ s \in V_\beta . \tag{2.29}$$

If $s \in V_\alpha \cap V_\beta$

$$\begin{aligned} h_\beta(s) = \psi_e^{(\beta,\alpha_0)}(s) &= \underline{g}_{\beta\alpha}(s)\psi_e^{(\alpha,\alpha_0)}(s) \\ &= \underline{g}_{\beta\alpha}(s)h_\alpha(s). \end{aligned} \tag{2.30}$$

Thus, if we put $\tau_\alpha(s) = \tilde{\sigma}_\alpha(s)h_\alpha(s)$, then for $s \in V_\alpha \cap V_\beta$

$$\tau_\beta(s) = \tilde{\sigma}_\beta(s)h_\beta(s) = \tilde{\sigma}_\alpha(s)\underline{g}_{\alpha\beta}(s)h_\beta(s) = \tilde{\sigma}_\alpha(s)h_\alpha(s) = \tau_\alpha(s).$$

Consequently the τ_α defines a global section τ in $\hat{P}$. On the other hand if X_i, $i = 1, \ldots, s$ is a basis for $L(S)$, the $\mathring{X}_i|_s = \dfrac{d}{dt}\exp tX_i s|_{t=0}$ form a basis for $T(S)$. We may therefore set

$$A^\alpha(\mathring{X}_i|_s) = -W_{\eta(s)}^\alpha(X_i) = -\psi_{s*}^\alpha(X_i) = -dh_\alpha(\mathring{X}_i|_s)(h_\alpha(s)^{-1}) = h_\alpha(s)\mathring{X}_i|_s((h_\alpha^{-1})(s)),$$

so $A^\alpha(\mathring{X}_i|_s)$ is a pure gauge.

On the section $\tau_\alpha(s) = \tilde{\sigma}_\alpha(s)h_\alpha(s)$ we have:

$$\begin{aligned} (\tau_\alpha^*\omega)(\mathring{X}_i|_s) &= h_\alpha^{-1}(s)\mathring{X}_i|_s(h_\alpha(s)) + h_\alpha^{-1}(s)[h_\alpha(s)\mathring{X}_i|_s(h_\alpha^{-1}(s))]h_\alpha(s) \\ &= h_\alpha^{-1}(s)\mathring{X}_i|_s(h_\alpha(s)) + (\mathring{X}_i|_s(h_\alpha^{-1}(s))h_\alpha(s) = 0. \end{aligned}$$

Therefore ω is the flat connection on $\hat{P}$. Now for each closed loop γ the integrals in the right-hand side of (2.27) give the holonomy $h(\gamma)$ which in the present case is e, proving our result.

Conversely, suppose that the conditions of our proposition hold. We define $\underline{W}_s^\alpha(X_i) = W_{\eta(s)}^\alpha(X_i)$ for $X_i \in L(S)$ and $s \in V_\alpha$. As before, we may define $A^\alpha(\mathring{X}_i|_s) = -\underline{W}_s^\alpha(X_i)$ for $X_i \in L(S)$, giving a connection ω_0 on $\hat{P}$. If Ω is the curvature of this connection we have

$$\begin{aligned}\sigma_\alpha^*\Omega(\mathring{X}_i, \mathring{X}_j) &= \mathring{X}_i(A^\alpha(\mathring{X}_j)) - \mathring{X}_j(A^\alpha(\mathring{X}_i)) - A^\alpha([\mathring{X}_i, \mathring{X}_j]) + [A^\alpha(\mathring{X}_i), A^\alpha(\mathring{X}_j)] \\ &= \mathring{X}_i(A^\alpha(\mathring{X}_j)) - \mathring{X}_j(A^\alpha(\mathring{X}_i)) + A^\alpha([X_i, X_j]^\circ) + [A^\alpha(\mathring{X}_i), A^\alpha(\mathring{X}_j)] \\ &= -\tilde{X}_i(W_x^\alpha(X_j)) + \tilde{X}_j(W_x^\alpha(X_i)) - W_x^\alpha([X_i, X_j]) + [W_x^\alpha(X_i), W_x^\alpha(X_j)] = 0.\end{aligned}$$

Then, because the connection ω_0 is locally flat, the holonomy of every loop homotopic to the identity is trivial, and consequently the holonomy h defines a function $h: \pi_1(S, e) \to G$ by $h(\bar{\gamma}) = h(\gamma)$, where $\bar{\gamma}$ denotes the image of γ in $\pi_1(S, e)$. Now, since by hypothesis the $\bar{\gamma}_1, \ldots, \bar{\gamma}_s$ are generators of $\pi_1(S, e)$, h is a homomorphism of groups and the right-hand side of (2.27) is just $h(\gamma_i) = h(\bar{\gamma}_i) = e$, so h is trivial. Therefore the holonomy of the connection ω_0 is trivial and thus ω_0 is the flat connection, so there is a global section $\tau: S \to \hat{P}$ such that $\tau^*\omega_0 = 0$. But then if $\tau|_{U_\alpha} = \sigma_\alpha h_\alpha$

$$0 = (\tau^*\omega_0)(\mathring{X}_i|_s) = h_\alpha^{-1}\, dh_\alpha(\mathring{X}_i|_s) + h_\alpha^{-1} A^\alpha(\mathring{X}_i|_s) h_\alpha,$$

therefore

$$\underline{W}_s^\alpha(\mathring{X}_i|_s) = -A^\alpha(\mathring{X}_i|_s) = dh_\alpha(\mathring{X}_i|_s) h_\alpha^{-1}.$$

Let us now set

$$\psi_s^{(\alpha,\beta)}(t) = h^{(\beta)}(ts)(h^\alpha(s))^{-1} \qquad \text{for } s \in V_\alpha,\ ts \in V_\beta.$$

For $s \in V_\alpha$ and $ts \in V_\alpha$, $\psi_s^{(\alpha)}(t) = h^\alpha(ts)h^\alpha(s)^{-1}$ and $(\psi_s^{(\alpha)})_*(X_i) = (\mathring{X}_i h^\alpha)_s (h^\alpha(s))^{-1} = \underline{W}_s^\alpha(X_i)$. It is easy to check that the functions $\psi_s^{(\alpha,\beta)}$ satisfy conditions (2.22) and (2.23).

We next make use of the bundle homomorphism (2.28) in order to derive the integrability of (2.24) and (2.26) for $\pi: P \to M$. For this purpose, we need to use the following results relating properties of the functions $\underline{W}_y^\alpha$ and ψ_y^α:

Lemma 2.6. *If $X \in L(S)$, then*

$$(\psi_y^\alpha)_*(\mathring{X}_i|_s) = (\underline{W}_{sy}(X_i))_{\psi_y^\alpha(s)}, \tag{2.31}$$

for $y \in V_\alpha$ and s such that $sy \in V_\alpha$.

Proof.

$$(\psi_y^\alpha)_*(\mathring{X}_i|_s) = \frac{d}{dt}\psi_y^\alpha(\exp tX_i \cdot s)|_{t=0} = \frac{d}{dt}\psi_{sy}^\alpha(\exp tX_i)\psi_y^\alpha(s)|_{t=0}$$

$$= (\underline{W}_{sy}^\alpha(X_i))_{\psi_y^\alpha(s)}. \qquad \square$$

Lemma 2.7. *If $y \in V_\alpha$, $sy \in V_\beta$ and $j \in J$ then*

$$\psi_{yj}^{(\alpha,\beta)} = \psi_y^{(\alpha,\beta)}, \tag{2.32}$$

and in particular $\psi_{yj}^\alpha = \psi_y^\alpha$.

Proof. Consider first the case $\beta = \alpha$, then by Lemma 2.6,

$$(\psi_{yj}^\alpha)_*(\mathring{X}_i|_s) = \underline{W}_{syj}(X_i)_{\psi_y^\alpha(s)} = (\psi_y^\alpha)_*(\mathring{X}_i|_s).$$

Thus $\psi_{yj}^\alpha = \psi_y^\alpha$ (here $V_\alpha y^{-1}$ is connected). Now take $y \in V_\alpha$ and $s \in S$ with $sy \in V_\beta$. For S connected, there are elements $s_1, \ldots, s_r \in S$ such that $s_1 y \in V_\alpha \cap V_{\alpha_1}$, $s_2 s_1 y \in V_{\alpha_1} \cap V_{\alpha_2}$, $s_3 s_2 s_1 y \in V_{\alpha_2} \cap V_{\alpha_3}, \ldots, s_{r-1} \ldots s_1 y \in V_{\alpha_{r-2}} \cap V_{\alpha_{r-1}}$, $s_r s_{r-1} \ldots s_1 y \in V_{\alpha_{r-1}} \cap V_\beta$.

We will prove our statement by induction on r. For $r = 2$, $s = s_2 s_1$, $y \in V_\alpha$, $s_1 y \in V_\alpha \cap V_{\alpha_1}$ and $s_2 s_1 y \in V_{\alpha_1} \cap V_\beta$, we have by (2.2) that

$$\psi_y^{(\alpha,\beta)}(s_2 s_1) = \psi_{s_1 y}^{(\alpha,\alpha_1)}(s_2)\psi_y^{(\alpha_1,\beta)}(s_2). \tag{2.33}$$

Furthermore from (2.3) we also have that $\psi_{s_1 y}^{(\alpha,\alpha_1)}(s_2) = \underline{g}_{\alpha\alpha_1}(s_2 s_1 yj)\psi_y^\alpha(s_2)$, so

$$\psi_{s_1 yj}^{(\alpha,\alpha_1)}(s_2) = \underline{g}_{\alpha\alpha_1}(s_2 s_1 yj)\psi_{yj}^\alpha(s_2)$$

$$= \underline{g}_{\alpha\alpha_1}(s_2 s_1 y)\psi_y^\alpha(s_2) = \psi_{s_1 y}^{(\alpha,\alpha_1)}(s_2).$$

In a similar way $\psi_{yj}^{(\alpha_1,\beta)}(s_1) = \psi_y^{(\alpha_1,\beta)}(s_1)$.

Consequently, substituting in (2.33), we get $\psi_{yj}^{(\alpha,\beta)}(s) = \psi_y^{(\alpha,\beta)}(s)$. The induction step is proved in the same fashion. $\square$

We can now complete the proof of proposition 2.5:

For $x \in U_\alpha$ define $\varphi_x^{(\alpha,\beta)}(s) = \psi_y^{(\alpha,\beta)}(s)$ with $\eta(y) = x$ and $sx \in U_\beta$ (here $y \in V_\alpha$, $sy \in V_\beta$). It is easy to verify (2.2) and (2.3) for these $\varphi_x^{(\alpha,\beta)}(s)$ which, therefore, determine an action of S on P such that for $x \in M$ and y in S, with $\eta(y) = x$, we have $(\varphi_x^\alpha)_* = (\psi_y^\alpha)_* = \underline{W}_y^\alpha = W_x^\alpha$.

To end this section we make use of (2.17) and (2.21) to derive an expression for $\sigma_\alpha^*\Omega$ in terms of the linear transformation Λ associated to ω, and of the functions $\varphi_{x_0}^\alpha$ related to the lifting of the action of the symmetry group on M. We get

$$(\sigma_\alpha^*\Omega)(\tilde{X}, \tilde{Y}) = \tilde{X}[A^\alpha(\tilde{Y})] - \tilde{Y}[A^\alpha(\tilde{X})] - A^\alpha([\tilde{X}, \tilde{Y}]) + [A^\alpha(\tilde{X}), A^\alpha(\tilde{Y})]$$

$$= (\mathscr{L}_{\tilde{X}} A^\alpha)(\tilde{Y}) - \tilde{Y}[A^\alpha(\tilde{X})] - [A^\alpha(\tilde{Y}), A^\alpha(\tilde{X})]$$

$$= D_{\tilde{Y}}[W^\alpha(X) + A^\alpha(\tilde{X})].$$

But according to (2.17) we have $W^\alpha(X) + A^\alpha(\tilde{X}) = a\delta_{\varphi^\alpha_{x_0}(s)}\Lambda(a\delta_{s^{-1}}X)$. Consequently,

$$(\sigma_\alpha^*\Omega)(\tilde{X}, \tilde{Y}) = D_{\tilde{Y}}(a\delta_{\varphi^\alpha_{x_0}(s)}\Lambda(a\delta_{s^{-1}}X)). \tag{2.34}$$

This result provides a criterion for S-invariance equivalent to (2.21). In addition, when (2.24) and (2.26) are integrable Eq. (2.34) gives a clear relation between Wang's algebraic formalism and the differential version of Forgács and Manton; furthermore, it allows us to give a geometrical interpretation of the scalar field Ψ_X introduced by these last authors, since $\Psi_X = a\delta_{\varphi^\alpha_{x_0}(s)}\Lambda(a\delta_{s^{-1}}X)$. Several direct and interesting physical interpretations have been found for this scalar field, such as in the "spin-from-isospin" phenomenon (cf. [2] and references therein), and the dimensional reduction of the Yang-Mills action to a gauge theory with Higgs fields related to the Ψ_X (cf. [9]).

3. Gauge-Equivalent Connections

In this section we give a characterization of gauge-equivalent connections in terms of their associated holonomy groups. We shall then apply these results to S-invariant connections in the next section, which will give us a classification of gauge fields.

Let $P(M, G)$ be a principal fiber bundle with structure group G and projection operator $\pi: P \to M$. Denote by $C(P, G)$, the space of all maps $\tau: P \to G$ which satisfy $\tau(pg) = g^{-1}\tau(p)g$ for all $g \in G$, $p \in P$. This space is isomorphic to the space of sections of the associated bundle $P \times_G G \to M$ with standard fiber G. A diffeomorphism $f: P \to P$ which satisfies $f(pg) = f(p)g$ for all $p \in P$, $g \in G$, is called a *fiber bundle automorphism.* Note that such an automorphism induces a diffeomorphism on M, $\bar{f}: M \to M$, given by $\bar{f}(\pi(p)) = \pi(f(p))$. We define a *gauge transformation* to be an automorphism $f: P \to P$ such that $\bar{f} = 1_M$. If we now let $GA(P)$ denote the group of gauge transformations on P, then it is easy to show (see Bleecker [11]) that there is a natural isomorphism between $GA(P)$ and $C(P, G)$ given by $f(p) = p\tau(p)$, with $\tau(p) \in C(P, G)$ as above.

We now introduce some more notation: Given two points x, $y \in M$, let $C(x, y)$ denote the collection of paths in M from x to y. For two paths $\alpha \in C(x, y)$, $\beta \in C(y, z)$, we have the composition of paths $C(x, y) \times C(y, z) \to C(x, z)$, given by $(\alpha, \beta) \to \beta \circ \alpha$.

A map $f: \pi^{-1}(x) \to \pi^{-1}(y)$ is called a *G-morphism* iff $f(pg) = f(p)g$ for all $p \in \pi^{-1}(x)$ and all $g \in G$. We shall denote by $\mathrm{Hom}_G(\pi^{-1}(x), \pi^{-1}(y))$ the set of G-morphisms from $\pi^{-1}(x)$ to $\pi^{-1}(y)$.

Given now a connection ω in $P(M, G)$, and two points $x, y \in M$, we can define a function $H^{\omega}(x, y): C(x, y) \to \mathrm{Hom}_G(\pi^{-1}(x), \pi^{-1}(y))$ as follows: let $\alpha(t) \in C(x, y)$ and denote by $\hat{\alpha}(t)$ the horizontal lift relative to ω of α which passes through $p \in \pi^{-1}(x)$. Then $H^{\omega}(x, y)(p) =$ end-point of $\hat{\alpha} = q \in \pi^{-1}(y)$.

Note that since $\hat{\alpha} \circ R_g = R_g \circ \hat{\alpha}$ (because a horizontal curve is mapped onto a horizontal curve by R_g) we have that $R_g\hat{\alpha}(0) = pg$ is the starting point of $R_g\hat{\alpha}$. Consequently $H^{\omega}(x, y)(pg) = (R_g \circ \hat{\alpha})(1) = \hat{\alpha}(1)g = qg = H^{\omega}(x, y)(p)g$, and $H^{\omega}(x, y)$ is indeed a G-morphism. Also, given $x, y, z \in M$, $\alpha \in C(x, y)$, $\beta \in C(y, z)$, we have

$$H^{\omega}(y, z)(\beta) \circ H^{\omega}(x, y)(\alpha) = H^{\omega}(x, z)(\beta \circ \alpha). \tag{3.2}$$

Thus taking fixed elements $r_x \in \pi^{-1}(x)$ for each $x \in M$, so that $H^{\omega}(x, y)(\alpha)(r_x) = r_y g$ for some $g \in G$, and setting $g = h^{\omega}(x, y)(\alpha)$ we get

$$\begin{aligned} H^{\omega}(x, z)(\beta \circ \alpha)(r_x) &= H^{\omega}(y, z)(\beta) \circ H^{\omega}(x, y)(\alpha)(r_x) \\ &= H^{\omega}(y, z)(\beta)(r_y h^{\omega}(x, y)(\alpha)) \\ &= r_z h^{\omega}(y, z)(\beta) h^{\omega}(x, y)(\alpha). \end{aligned}$$

That is

$$h^{\omega}(x, z)(\beta \circ \alpha) = h^{\omega}(y, z)(\beta) h^{\omega}(x, y)(\alpha); \tag{3.3}$$

from where it follows that, if $x = z$, $h^{\omega} = h^{\omega}(x, x)$ is a homomorphism from $C(x, x)$ into G.

To see how the $h^{\omega}(x, y)$ depend on the chosen $\{r_x\}_{x \in M}$, consider another set of points $s_x \in \pi^{-1}(x)$, $x \in M$, with $s_x = r_x a_x$ for some $a_x \in G$, and let $h^{\omega}_{(s)}(x, y)$ symbolize the elements taken with respect to $\{s_x\}_{x \in M}$. Then

$$H^{\omega}(x, y)(\alpha)(s_x) = s_y h^{\omega}_{(s)}(x, y)(\alpha).$$

Furthermore, since $H^{\omega}(x, y)$ is a G-morphism we also have

$$H^{\omega}(x, y)(\alpha)(r_x)a_x = H^{\omega}(x, y)(\alpha)(r_x a_x) = r_y a_y h^{\omega}_{(s)}(x, y)(\alpha).$$

Hence

$$H^{\omega}(x, y)(\alpha)(r_x) = r_y a_y h^{\omega}_{(s)}(x, y)(\alpha) a_x^{-1}$$

and

$$h^{\omega}(x, y)(\alpha) = a_y h^{\omega}_{(s)}(x, y)(\alpha) a_{\alpha}^{-1}. \tag{3.4}$$

Note that for a closed path in particular, writing $q = r_x$, $q' = qa$, and denoting $h^{\omega}_{(r)}(x, x)$ and $h^{\omega}_{(s)}(x, x)$ by $h^{\omega}_q : C(x, x) \to G$ and $h^{\omega}_{q'} : C(x, x) \to G$, respectively, the above expression yields

$$h^{\omega}_{q'} = a^{-1} h^{\omega}_q a. \tag{3.4a}$$

This is the well-known result which states that the holonomy groups with reference points q and $q' = qa$ are conjugate in G.

We are now ready to prove the following

Proposition 3.1. *Let ω_1, ω_2 be two connections on P and suppose that there exists $u \in G$ such that $h^{\omega_2}_p = u h^{\omega_1}_p u^{-1}$; that is, for all $\gamma \in C(x_0) \equiv C(x_0, x_0)$, the loop space at $x_0 = \pi(p)$, we have*

$$h^{\omega_2}_p(\gamma) = u h^{\omega_1}_p(\gamma) u^{-1}. \tag{3.5}$$

Then, for any α, $\beta \in C(x_0, x)$ it is true that

$$H^{\omega_2}(x_0, x)(\alpha) L_u H^{\omega_1}(x, x_0)(\alpha^{-1}) = H^{\omega_2}(x_0, x)(\beta) L_u H^{\omega_1}(x, x_0)(\beta^{-1}), \tag{3.6}$$

where the action $L_u : \pi^{-1}(x_0) \to \pi^{-1}(x_0)$ above is defined by $L_u(pa) = pua$.

Proof. Take the fixed set of points $\{r_x\}_{x \in M}$ such that $r_{x_0} = p$. Since $\beta^{-1}\alpha \in C(x_0)$ we have, from (3.5),

$$h^{\omega_2}_p(\beta^{-1}\alpha) = u h^{\omega_1}_p(\beta^{-1}\alpha) u^{-1},$$

i.e.

$$h^{\omega_2}(x, x_0)(\beta^{-1}) h^{\omega_2}(x_0, x)(\alpha) u = u h^{\omega_1}(x, x_0)(\beta^{-1}) h^{\omega_1}(x_0, x)(\alpha).$$

This result in turn implies that

$$h^{\omega_2}(x_0, x)(\beta) u h^{\omega_1}(x, x_0)(\beta^{-1}) = h^{\omega_2}(x_0, x)(\alpha) u h^{\omega_1}(x, x_0)(\alpha^{-1}). \tag{3.7}$$

Observing now that both sides of Eq. (3.6) are formed by a composition of G-transformations, it is sufficient to show that this equation holds at any point $r_x = q$, say. But then we get precisely (3.7) multiplied on the left by q. □

Thus if (3.5) is satisfied, we can define a map $f : P \to P$ by $f(q) = H^{\omega_2}(x_0, x)(\alpha) L_u H^{\omega_1}(x, x_0)(\alpha^{-1})(q)$, where α is any path from $x_0 = \pi(p)$ to $x = \pi(q)$. By Proposition 3.1 f does not depend on the chosen path. It is easy to verify from this definition that $f(qg) = f(q)g$ for all $g \in G$. It also follows that $\tau(q) = h^{\omega_2}(x_0, x)(\alpha) u h^{\omega_1}(x, x_0)(\alpha^{-1})$, and that $\tau(qg) = g^{-1}\tau(q)g$, so f is a gauge transformation.

To conclude this section we make use of the above results to derive the requirements for two connections to be related by a gauge transformation.

Note first that if $\hat{\alpha}$ is a ω_1-horizontal lift of α, through p, then by construction $f\hat{\alpha}$ is a ω_2-horizontal lift through pu. Thus, if $(\omega_1)_q(X) = 0$ then $(\omega_2)_{f(q)}(f_* X) = 0$, and therefore $\mathrm{Ker}(f^*\omega_2)_q \supseteq \mathrm{Ker}(\omega_1)_q$. On the other hand, we must have $\dim \mathrm{Ker}(\omega_1)_q = \dim \mathrm{Ker}(f^*\omega_2)_q$ since this is just the dimension of the horizontal subspace of the tangent space of the bundle. Thus, for any $q \in P$ we have $\mathrm{Ker}(\omega_1)_q = \mathrm{Ker}(f^*\omega_2)_q$, which means that

$$f^*\omega_2 = \omega_2 f_* = \omega_1 . \tag{3.8}$$

This proves half of the following

Proposition 3.2. *Let ω_1, ω_2 be two connections on a principal fiber bundle $P(M, G)$. Then, a gauge transformation f exists with the property $f^*\omega_2 = \omega_1$ if and only if at some point $p \in P$ we have*

$$h_p^{\omega_2} = u h_p^{\omega_1} u^{-1} \tag{3.9}$$

with $u \in G$ such that $f(p) = pu$. For a fixed p, and f such that $f^*\omega_2 = \omega_1$ and $u(p) = u$, f is unique.

Proof. The paragraph leading to (3.8) proves the result one way. It remains to be proved that if $f^*\omega_2 = \omega_1$ for some gauge transformation f then (3.9) holds. That this is so is seen as follows. Let $x_0 = \pi(p)$ and let $\hat{\alpha}$ be a ω_1-horizontal lift of $\alpha \in C(x_0)$, through p. Since $\omega_2 f_* = \omega_1$, $f \circ \hat{\alpha}$ is a ω_2-horizontal lift of α through $f(p) = pu$. But then $R_{u^{-1}} \circ f \circ \hat{\alpha}$ is a ω_2-horizontal lift of α through p, and therefore

$$\begin{aligned} p h_p^{\omega_2}(\alpha) &= (R_{u^{-1}} \circ f \circ \hat{\alpha})(1) = f(\hat{\alpha}(1))u^{-1} \\ &= f(p h_p^{\omega_1}(\alpha))u^{-1} = p u h_p^{\omega_1}(\alpha) u^{-1}, \end{aligned}$$

from which (3.9) follows. The uniqueness of f is proved in a similar way. □

Two interesting results given by Fischer [1] can now be obtained as corollaries:

Corollary 3.3. *Let p be a fixed point in P, $f \in GA(P)$, and suppose that $f^*\omega = \omega$. Then there exists $u = u(p) \in C_G(\mathrm{Hol}_p(\omega))$ with $f(p) = pu$. Conversely, for every $u \in C_G(\mathrm{Hol}_p(\omega))$ there exists a unique gauge transformation $f : P \to P$ such that $f^*\omega = \omega$ and $f(p) = pu$. (Here, $C_G(\mathrm{Hol}_p(\omega))$ denotes the centralizer in G of the holonomy group of ω with reference point p.)*

Proof. Assume that $f^*\omega = \omega$, and define $u \in G$ by $f(p) = pu$. By Proposition 3.2 it then follows that $h_p^{\omega} = u h_p^{\omega} u^{-1}$. Thus for any $\alpha \in C(\pi(p))$ we have $h_p^{\omega}(\alpha) = u h_p^{\omega}(\alpha) u^{-1}$, which implies $u \in C_G(\mathrm{Hol}_p(\omega))$.

Conversely, given $u \in C_G(\mathrm{Hol}_p(\omega))$ we have $h_p^{\omega} = u h_p^{\omega} u^{-1}$ and by Proposition 3.2 there exists a unique gauge transformation f satisfying $f^*\omega = \omega$ and $f(p) = pu$. □

Corollary 3.4. *If $f \in GA(P)$ and τ is the associated function in $C(P, G)$ the following conditions are equivalent:*

i) $f^*\omega = \omega$;

ii) *τ is constant on each ω-horizontal curve in P;*

iii) *τ is constant on the holonomy sub-bundle $P(p_0)$ of P.*

Proof. If $f^*\omega = \omega$ and $\hat{c}(t)$ is a ω-horizontal curve in P starting say at p_1 and ending at p_2, then $R_{\tau(p_1)} \circ \hat{c}(t)$ and $f(\hat{c}(t))$ are both ω-horizontal lifts of $\pi(\hat{c}(t))$, starting at the same point $p_1\tau(p_1)$ in P. Consequently for each $t \in [0, 1]$ we must have $f(\hat{c}(t)) = \hat{c}(t)\tau(p_1)$, from where it follows that $\tau(\hat{c}(t)) = \tau(p_1)$, i.e. τ is constant on $\hat{c}(t)$.

Conversely, if τ is constant on each ω-horizontal curve in P, then $f(\hat{c}(t)) = \hat{c}(t)\tau(p_1)$ for $t \in [0, 1]$, and $f(\hat{c}(t)) = R_{\tau(p_1)} \circ \hat{c}(t)$ which is also a ω-horizontal curve. Therefore $f^*\omega = \omega$. We have thus completed the proof of i)$\Leftrightarrow$ii).

Now ii)$\Rightarrow$iii) is obvious, and if iii) holds and $\hat{c}(t)$ is a horizontal curve in P, then $R_g \circ \hat{c}(t)$ is in $P(p_0)$ for some $g \in G$. Thus $\tau(R_g \circ \hat{c}(t)) = g^{-1}\tau(\hat{c}(t))g$ is constant, so $\tau(\hat{c}(t))$ must be constant. Hence iii)$\Leftrightarrow$ii). □

4. Gauge Equivalence of S-Invariant Connections

Section 2 considered the lifting of a symmetry action to an action on principal fiber bundles with arbitrary characteristic groups and homogeneous base spaces, and an algebraic procedure was given for constructing gauge fields from them, possessing the symmetry of the underlying base manifold was given. Section 3 discussed gauge transformations on connection 1-forms, and characterized their gauge equivalence in terms of their holonomy groups.

We shall here examine the following problem: given two connections, both required to be S-invariant, what are the conditions for them to be related by a gauge transformation? The answer to this question will provide us with a means of classifying symmetric gauge fields into classes modulo gauge transformations, and will be here given in both the local and global domains, in terms of their associated linear transformations (cf. Sec. 2). We start with some definitions.

Definition 4.1. Let $U \subset M$ be an open subset of our base manifold and ω_1, ω_2 two connection 1-forms in P. We then say that "ω_2 is gauge-equivalent to ω_1 on U" iff there exists a gauge transformation

$$f \in GA(\pi^{-1}(U)) \qquad \text{such that} \qquad f^*\omega_1|_{\pi^{-1}(U)} = \omega_2|_{\pi^{-1}(U)}.$$

Let now Λ_1 and Λ_2 be the linear transformations associated to two S-invariant connections ω_1 and ω_2 respectively. Recall that if $X \in L(S)$ (the Lie algebra of S) then $\Lambda_i(X) = [\omega_i(\hat{X})]_{p_0}$ where $\hat{X}_p = \frac{d}{dt}(\exp tX \cdot p)|_{t=0}$. In order to show that $f^*\omega_1 = \omega_2$ for $f \in GA(\pi^{-1}(U))$ it is sufficient to show that i) $f^*\omega_1$ is locally S-invariant at some $x_0 \in U$, and ii) $(f^*\omega_1)_{p_0} = (\omega_2)_{p_0}$. In fact for ii) it is enough to show that $(f^*\omega_1)(\hat{X})_{p_0} = \omega_2(\hat{X})_{p_0} = \Lambda_2(X)$ for $X \in L(S)$.

Writing $f(p) = p\tau(p)$ one obtains [11]

$$(f^*\omega_1)_{p_0} = \tau(p_0)^{-1}\omega_1\tau(p_0) + \tau(p_0)^{-1}\tau_*, \tag{4.1}$$

from where we must have

$$\tau(p_0)^{-1}\Lambda_1(X)\tau(p_0) + \tau(p_0)^{-1}\tau_*(\hat{X}) = \Lambda_2(X). \tag{4.2}$$

In order to find the desired relation between Λ_1 and Λ_2 it is then sufficient to find those gauge transformations $f: P \to P$ which send an S-invariant connection ω_1 into another S-invariant connection ω_2. We shall first study this situation locally:

Recall that $\mathrm{Hol}^\circ_{p_0}(\omega)$ is the subgroup of $\mathrm{Hol}_{p_0}(\omega)$ generated by loops at x_0 which are homotopic to the identity. (Here, and in what follows, $x_0 = \pi(p_0)$.) Then, given any $x \in M$, $x = \pi(p)$, there exists a neighborhood $U_0 \subset M$ of x such that $\mathrm{Hol}^\circ_p(\omega) = \mathrm{Hol}_p(\omega)(\pi^{-1}(U_0)) = \mathrm{Hol}_p(\omega)(\pi^{-1}(V))$ for any simply connected neighborhood V of x contained in U_0. In what follows we shall take neighborhoods V of x_0 such that $\mathrm{Hol}^\circ_p(\omega) = \mathrm{Hol}_{p_0}(\omega)(\pi^{-1}(V))$.

The conditions for the local S-invariance of $f^*\omega$ (cf. Definition 2.4) will be given in terms of some functions $\lambda_s(q)$ which we define by

$$\lambda_s(q) = \tau(sq)\tau(q)^{-1} \tag{4.3}$$

for all $s \in S$, $q \in \pi^{-1}(N)$, such that $sq \in \pi^{-1}(N)$.

These functions satisfy several properties which we give below. Given a neighborhood U' of s and a neighborhood N' of q, with $U' \cdot N' \subset \pi^{-1}(N)$, then the function $(s', q') \to \lambda_{s'}(q')$ is C^∞. Also given $s, t \in \mathscr{W}$ then $\lambda_{ts}(q)\tau(q) = \tau((ts)q) = \tau(t(sq)) = \lambda_t(sq)\tau(sq) = \lambda_t(sq)\lambda_s(q)\tau(q)$, i.e.

$$\lambda_{ts}(q) = \lambda_t(sq)\lambda_s(q), \tag{4.4}$$

and therefore

$$\lambda_e(q) = e. \tag{4.5}$$

In addition, since $\tau(pg) = g^{-1}\tau(p)g$, with $g \in G$, it follows that

$$\lambda_s(qg) = g^{-1}\lambda_s(q)g.$$

More properties for λ will be obtained from the following lemmas:

Lemma 4.2. *If $s \in \mathscr{W}$ then $\lambda_s(p_0) \in C_G(\mathrm{Hol}^\circ_{p_0}(w))$.*

Proof. $V_{(s)}$ is a connected neighborhood of x_0 with $sV_{(s)} \subset N$. Given $q \in \pi^{-1}(V_{(s)})$ we have

$$(s^{-1}fsf^{-1})(q) = s^{-1}fs(q\tau(q)^{-1}) = s^{-1}f(s(q)\tau(q)^{-1})$$
$$= s^{-1}s(q)\tau(sq)\tau(q)^{-1} = q\tau(sq)\tau(q)^{-1}.$$

Note that $\pi(s^{-1}fsf^{-1}(q)) = \pi(q)$. Thus the induced diffeomorphism on M is the identity transformation so $s^{-1}fsf^{-1}$ is a gauge transformation from $\pi^{-1}(V_{(s)})$ to $\pi^{-1}(V_{(s)})$ whose associated function in $C(\pi^{-1}(V_{(s)}), G)$ is precisely $\lambda_s(q) = \tau(sq)\tau(q)^{-1}$.

Now, taking $X \in T_q(\pi^{-1}(V_{(s)}))$ and using the S-invariance of ω,

$$\begin{aligned}(s^{-1}fsf^{-1})^*\omega(X) &= (f^*)^{-1}(s^{-1}fs)^*\omega(X) = (f^*)^{-1}\omega(s_*^{-1}f_*s_*X)\\ &= (f^*)^{-1}\omega(f_*s_*X) = (f^*)^{-1}s^*f^*\omega(X)\\ &= (f^*)^{-1}f^*\omega(X) = \omega(X),\end{aligned}$$

and by Corollary 3.3 this means that

$$\lambda_s(p_0) \in C_G(\mathrm{Hol}_{p_0}(\omega)(\pi^{-1}(V))) = C_G(\mathrm{Hol}^\circ_{p_0}(\omega)). \qquad \square$$

Lemma 4.3. *Let $s \in S$ and $q \in \pi^{-1}(N)$ such that $sq \in \pi^{-1}(N)$. Then*

$$\lambda_s(q) \in C_G(\mathrm{Hol}^\circ_q(\omega)).$$

Proof. We have $q \in \pi^{-1}(N)$, then $q = rp$ with $r \in S$ and $\pi(p) = x_0$. As in 4.3 one can prove that $\lambda_r(p) \in C_G(\mathrm{Hol}^\circ_p(\omega))$. Here $sq \in \pi^{-1}(N)$, so $sr \in \mathscr{W}$. Thus $\lambda_{sr}(p) \in C_G(\mathrm{Hol}^\circ_p(\omega))$. On the other hand one has $\lambda_{sr}(p) = \lambda_s(rp)\lambda_r(p)$, and consequently $\lambda_s(q) = \lambda_s(rp) \in C_G(\mathrm{Hol}^\circ_p(\omega)) = C_G(\mathrm{Hol}^\circ_{rp}(\omega)) = C_G(\mathrm{Hol}^\circ_q(\omega))$. $\square$

Note that if $J \subset S$ denotes the isotropy group which fixes x_0 (given the action of S on M) then, for $j \in J$ we have

$$\begin{aligned}\lambda_j(p_0) &= \tau(jp_0)\tau(p_0)^{-1} = \tau(p_0\mu(j))\tau(p_0)^{-1}\\ &= \mu(j)^{-1}\tau(p_0)\mu(j)\tau(p_0)^{-1},\end{aligned}$$

where μ was defined in Sec. 2. This means that $\lambda_j(p_0)$ for $j \in J$ is not arbitrary, but must satisfy

$$\lambda_j(p_0) = \mu(j)^{-1}u\mu(j)u^{-1} \in C_G(\mathrm{Hol}^\circ_{p_0}(\omega)), \qquad \text{with } u = \tau(p_0). \tag{4.6}$$

Also, for $s \in \mathscr{W}$ and $j \in J$, by Eq. (4.4) we have

$$\lambda_{sj}(p_0) = \lambda_s(jp_0)\lambda_j(p_0) = \mu(j)^{-1}\lambda_s(p_0)\mu(j)\lambda_j(p_0). \tag{4.7}$$

We have thus seen that if $f^*\omega$ is required to be S-invariant then λ_s satisfies Eqs. (4.6), (4.7). Reciprocally, suppose we have defined a function $v : \mathscr{W} \to C_G(\mathrm{Hol}^\circ_{p_0}(\omega))$ satisfying Eqs. (4.6, 4.7) for a certain $u \in G$. We will now show that in this case we can construct $\tau \in C(\pi^{-1}(N), G)$ with the properties

$$\tau(p_0) = u, \qquad \tau(sp_0)\tau(p_0)^{-1} = v_s \equiv v(s).$$

Indeed, let $q \in \pi^{-1}(N)$ and write $q = sp_0 g$, with $s \in \mathscr{W}$, $g \in G$. We define $\tau(q) := g^{-1} v_s u g$. To see that it is well defined, suppose that $q = tp_0 h$ with $t \in \mathscr{W}$, $g \in G$. Then $t = sj$, $h = \mu(j)^{-1} g$ and

$$\tau(tp_0 h) = \tau(sjp_0\mu(j)^{-1}g) = g^{-1}\mu(j)v_{sj}u\mu(j)^{-1}g$$

$$= g^{-1}\mu(j)\mu(j)^{-1}v_s\mu(j)v_j u\mu(j)^{-1}g$$

$$= g^{-1}\mu(j)\mu(j)^{-1}v_s\mu(j)\mu(j)^{-1}u\mu(j)u^{-1}u\mu(j)^{-1}g$$

$$= g^{-1}v_s ug = \tau(sp_0 g).$$

Now note that, if $q = sp_0 g \in \pi^{-1}(N)$ and $r \in S$ is such that $rq = rsp_0 g \in \pi^{-1}(N)$, then

$$\lambda_r(q) = \tau(rq)\tau(q)^{-1} = g^{-1}v_{rs}ugg^{-1}u^{-1}v_s^{-1}g = g^{-1}v_{rs}v_s^{-1}g, \tag{4.8}$$

so that v also determines λ_r. The properties mentioned above for τ follow trivially. We now want to find further necessary conditions for τ which, together with the previous ones, are also sufficient conditions for the existence of a transformation with associated τ leaving the connection S-invariant. To this effect, recall that, by Corollary 3.4, the functions $\lambda_s(q)$ with $q \in \pi^{-1}(V_{(s)})$ are constant in q over ω-horizontal curves in $\pi^{-1}(V_{(s)})$. This property on $\lambda_s(q)$ can be translated into a condition on v as follows: Recall that if $Q_s = \pi^{-1}(V_{(s)})$ then the holonomy subbundle $Q_s(p_0)$ of p_0 in Q_s is given by all $p \in Q_s$ which can be joined to p_0 by an ω-horizontal curve in Q_s. Its structure group is $\mathrm{Hol}_{p_0}(\omega)(\pi^{-1}(V_{(s)})) = \mathrm{Hol}^\circ_{p_0}(\omega)$. Let $V_{(s)}$ be a sufficiently small neighborhood of $x_0 = \pi(p_0)$ satisfying i) and ii) of Definition 2.4, such that a section $\sigma : V_{(s)} \to Q_s(p_0)$ exists. Given $x \in V_{(s)}$ and $s \in S$ with $sx \in V_{(s)}$ we can write $s\sigma(X) = \sigma(sx)\varphi_x(s)$, with $\varphi_x(s) \in G$ and these functions satisfy

$$\varphi_x(st) = \varphi_{tx}(s)\varphi_x(t) \tag{4.9}$$

whenever tx, sx, $stx \in V_{(s)}$. Also, using the constancy of λ_s we have, for $t \in S$ with $tx_0 \in V_{(s)}$ and $stx_0 \in N$,

$$\lambda_{st}(\sigma(x_0)) = \lambda_s(t\sigma(x_0))\lambda_t(\sigma(x_0))$$

$$= \lambda_s(\sigma(tx_0)\varphi_{x_0}(t))\lambda_t(\sigma(x_0))$$

$$= \varphi_{x_0}(t)^{-1}\lambda_s(\sigma(tx_0))\varphi_{x_0}(t)\lambda_t(\sigma(x_0))$$

$$= \varphi_{x_0}(t)^{-1}\lambda_s(\sigma(x_0))\varphi_{x_0}(t)\lambda_t(\sigma(x_0)),$$

so that $v(s) = \lambda_s(\sigma(x_0))$ (c.f. Eq. (4.8)) satisfies the following conditions

$$v(s) \in C_G(\mathrm{Hol}^\circ_{p_0}(\omega)) \tag{4.10a}$$

$$v(st) = \varphi_{x_0}(t)^{-1}v(s)\varphi_{x_0}(t)v(t), \qquad \text{for } s \in \mathscr{W} \text{ and } tx_0 \in V_{(s)}, st \in \mathscr{W} \tag{4.10b}$$

$$v(j) = \mu(j)^{-1}u\mu(j)u^{-1}, \qquad \text{for } j \in J. \tag{4.10c}$$

Before we arrive at the result we are after, we have a lemma and a straightforward proposition.

Lemma 4.4.
i) $\varphi_x(t) \in N_G(\mathrm{Hol}^\circ_{p_0}(\omega))$
ii) *For* $s \in \mathscr{W}$ *and* $t \in S$ *with* $tx_0 \in V_{(s)}$, *we have*

$$\varphi_{tx_0}(s) = h(s,t)\varphi_{x_0}(s), \qquad \textit{with } h(s,t) \in \mathrm{Hol}^\circ_{p_0}(\omega). \tag{4.11}$$

Proof. Since $\sigma(sx_0) \in Q(p_0)$, we have $\mathrm{Hol}^\circ_{\sigma(sx_0)}(\omega) = \mathrm{Hol}^\circ_{\sigma(x_0)}(\omega) = \mathrm{Hol}^\circ_{s\sigma(x_0)}(\omega) = \mathrm{Hol}^\circ_{\sigma(sx_0)\varphi_{x_0}(s)}(\omega) = \varphi_{x_0}(s)^{-1}\,\mathrm{Hol}^\circ_{\sigma(sx_0)}(\omega)\varphi_{x_0}(s)$; i.e. $\varphi_{x_0}(s) \in N_G(\mathrm{Hol}^\circ_{\sigma(x_0)}(\omega))$, which proves the first part. For the second part we may assume that there exists a section $\sigma : N \to Q(q)$, the holonomy subbundle of $\pi^{-1}(N)$ containing q. Now,

$$s\sigma(tx_0) = \sigma(stx_0)\varphi_{tx_0}(s) \in Q(s\sigma(x_0))$$

$$\sigma(stx_0)\varphi_{x_0}(s) \in Q(\sigma(x_0)\varphi_{x_0}(s)) = Q(s\sigma(x_0)).$$

But $\sigma(stx_0)\varphi_{x_0}(s)$ and $\sigma(stx_0)\varphi_{tx_0}(s)$ both project onto stx_0. Therefore there exists $h'(s,t) \in \mathrm{Hol}^\circ_{sp_0}(\omega) = \mathrm{Hol}^\circ_{p_0}(\omega)$ such that $\varphi_{tx_0}(s) = \varphi_{x_0}(s)h'(s,t)$. Using the fact that $\varphi_x(s) \in N_G(\mathrm{Hol}^\circ_{p_0}(\omega))$ we finally get

$$\varphi_{tx_0}(s) = \varphi_{x_0}(s)h'(s,t)\varphi_{x_0}(s)^{-1}\varphi_{x_0}(s) = h(s,t)\varphi_{x_0}(s),$$

with $h(s,t) \in \mathrm{Hol}^\circ_{p_0}(\omega)$ so defined. □

When the section σ is defined over all of N, conditions (4.10) may be restated in terms of functions which are nearly group homomorphisms, as follows:

Proposition 4.5. *Let*

$$\psi(s) = \varphi_{x_0}(s)v(s), \tag{4.12}$$

then $v(s) \in C_G(\mathrm{Hol}^\circ_{p_0}(\omega))$ *satisfies* (4.10b) *iff*

$$\psi(st) = h(s,t)\psi(s)\psi(t) \qquad (s \in \mathscr{W} \textit{ and } tx_0 \in V_{(s)}), \tag{4.13}$$

where $h(s,t)$ are the functions defined in Lemma 4.4.
N.B. *For $j \in J$ we clearly have*

$$\psi(j) = u\mu(j)u^{-1}. \tag{4.14}$$

Proof. It follows by straightforward calculation, and we omit it. □

We can combine our results so far relating to the existence of a gauge transformation which leaves ω S-invariant in the following

Proposition 4.6. *Let ω be an S-invariant connection, and N an open neighborhood of x_0. Take $\mathscr{W} = \{s \in S \mid sx_0 \in N\}$. Then a gauge transformation $f \in GA(\pi^{-1}(N))$ exists, with $f^*\omega$ locally S-invariant at x_0, iff*

i) *For each $s \in \mathscr{W}$ there exists a connected neighborhood $V_{(s)}$ of x_0, with $V_{(s)} \subset N \cap s^{-1}N$, and a local section $\sigma : V_{(s)} \to P$ with $\sigma(V_{(s)}) \subset Q_s(p_0)$ (the holonomy subbundle of p_0 in $Q_s = \pi^{-1}(V_{(s)})$).*
ii) *The associated function $\tau \in C(\pi^{-1}(N), G)$ is determined by a function $v : \mathscr{W} \to C_G(\mathrm{Hol}^\circ_{p_0}(\omega))$ satisfying Eqs.* (4.10b, c).

N.B. *The conditions* (4.10b, c) *on v can be translated into conditions* (4.13), (4.14) *on $\psi(s) = \varphi_{x_0}(s)v(s)$, when the local section σ is defined over all of N.*

Proof. The proposition has been proved above one way by construction, that is, assuming that $f^*\omega$ is locally S-invariant and for each $s \in \mathscr{W}$ taking $V_{(s)}$ so that it fulfills conditions i) and ii) of Definition 2.4. Reciprocally, suppose conditions i) and ii) of the proposition are satisfied, then v determines both $\tau \in C(\pi^{-1}(N), G)$ and $\lambda_s(q)$ for $q \in \pi^{-1}(N)$, as shown by Eq. (4.8) and the paragraph preceding it. Thus, to show that $f(p) := p\tau(p)$ is such that $f^*\omega$ is locally S-invariant at x_0 it is sufficient to show that $\lambda_s(q)$ so defined is constant for all q in $Q_s(p_0)$. To this end, note first that, for $s, t \in \mathscr{W}$ with $tx_0 \in V_{(s)}$

$$\begin{aligned}\lambda_s(\sigma(tx_0)) &= \lambda_s(t\sigma(x_0)\varphi_{x_0}(t)^{-1}) = \varphi_{x_0}(t)\lambda_s(t\sigma(x_0))\varphi_{x_0}(t)^{-1} \\ &= \varphi_{x_0}(t)\lambda_{st}(\sigma(x_0))\lambda_t(\sigma(x_0))^{-1}\varphi_{x_0}(t)^{-1} \\ &= \varphi_{x_0}(t)v(st)v(t)^{-1}\varphi_{x_0}(t)^{-1} = v(s) = \lambda_s(\sigma(x_0)),\end{aligned}$$

so that $\lambda_s(\sigma(x))$ is constant in x over $V_{(s)}$. Now, if $q \in Q_s(p_0)$ then it is of the form $q = \sigma(x)h$, with $x \in V_{(s)}$ and $h \in \mathrm{Hol}^\circ_{p_0}(\omega)$, and using the fact that $\lambda_s(\sigma(x)) \in C_G(\mathrm{Hol}^\circ_{p_0}(\omega))$ we have

$$\lambda_s(q) = \lambda_s(\sigma(x)h) = h^{-1}\lambda_s(\sigma(x))h = \lambda_s(\sigma(x)) = \lambda_s(\sigma(x_0))$$

which gives the required result. □

We may now answer, in the local domain, the question posed at the beginning of this section:

Proposition 4.7. *Let ω_1 and ω_2 be two S-invariant connections, and let Λ_1 and Λ_2 be their associated linear transformations, respectively. Then an open set $V_{(s)} \subset M$ exists containing x_0 and such that ω_1 and ω_2 are gauge equivalent over $\pi^{-1}(V_{(s)})$ if and only if there exists $u \in G$ with the following properties:*

i) $\mu(j)^{-1}u\mu(j)u^{-1} \in C_G(\mathrm{Hol}^\circ_{p_0}(\omega))$ *for all* $j \in J$.

ii) *There exists a local section* $\sigma : V_{(s)} \to Q_s(p_0)$ *for* $Q_s = \pi^{-1}(V_{(s)})$.

iii) *There exists a function* $v : \mathscr{W} \to C_G(\mathrm{Hol}^\circ_{p_0}(\omega))$ *satisfying Eqs.* (4.10b, c).

iv) $\Lambda_2 = u^{-1}(\Lambda_1 + v_*|_e)u$

N.B. *The conditions* (4.10b, c) *on* v *in* iii) *can be translated into conditions* (4.13), (4.14) *on* $\psi(s) = \varphi_{x_0}(s)v(s)$.

Proof. We have seen above that conditions i) to iv) are necessary. We now prove sufficiency. Suppose then that conditions i) to iv) are satisfied, and let $\tau \in C(\pi^{-1}(N), G)$ be the transformation associated to v as in the paragraph preceding Eq. (4.8). Given $q \in \pi^{-1}(V_{(s)})$ we know that if $f(q) = q\tau(q)$ then $f^*\omega_1|_{\pi^{-1}(V_{(s)})}$ is S-invariant. Using Eq. (4.1) we have, for $X \in L(S)$,

$$(f^*\omega_1)_{p_0}(\hat{X}) = \tau(p_0)^{-1}(\omega_i)_{p_0}(\hat{X})\tau(p_0) + \tau(p_0)^{-1}\tau_*(\hat{X}).$$

But $\tau(s\sigma(x_0)) = \lambda_s(\sigma(x_0))u = v(s)u$, so $\tau_*(\hat{X}) = \frac{d}{dt}v(\exp tX)|_{t=0}u = v_*(X)u$. Therefore

$$(f^*\omega_1)_{p_0}(\hat{X}) = u^{-1}\Lambda_1(X)u + u^{-1}v_*(X)u = \Lambda_2(X) = (\omega_2)_{p_0}(\hat{X}),$$

where we have used iv). Now S-invariance over $V_{(s)}$ gives $(f^*\omega_1)|_{\pi^{-1}(V_{(s)})} = \omega_2|_{\pi^{-1}(V_{(s)})}$, as required. □

Examples. As an application of our previous results, we consider here two examples, containing well-known results, which serve to illustrate the content of Proposition 4.7.

a) Let w_1, and w_2 be two flat connection 1-forms which are required to be S-invariant. We want to prove that these are locally gauge-equivalent.

We begin by recalling [Chap II, Theorem 9.1 of Kobayashi-Nomizu] that a connection on a principal fiber bundle is locally flat in $P|_V$ if and only if the curvature form vanishes identically in that region. Consequently

$$\Omega(\hat{X}, \hat{Y}) = d\omega(\hat{X}, \hat{Y}) + [\omega(\hat{X}), \omega(\hat{Y})]$$

$$= \hat{X}[\omega(\hat{Y})] - \hat{Y}[\omega(\hat{X})] - \omega([\hat{X}, \hat{Y}]) + [\omega(\hat{X}), \omega(\hat{Y})] = 0.$$

Moreover, since for S-invariant connections $\mathscr{L}_{\hat{X}}\omega = \mathscr{L}_{\hat{Y}}\omega = 0$, it readily follows that

$$\omega_{p_0}([\hat{X}, \hat{Y}]) + [\omega_{p_0}(\hat{X}), \omega_{p_0}(\hat{Y})] = 0,$$

and since $X \to \hat{X}$ is an algebra anti-homomorphism,

$$\Lambda([X, Y]) = [\Lambda(X), \Lambda(Y)]. \tag{4.15}$$

Thus, for locally flat connections the linear transformations Λ_1 and Λ_2 associated to ω_1 and ω_2, respectively, are homomorphisms of Lie algebras from $L(S)$ to $L(G)$.

Let us then consider the local sections $\sigma_i : V_{(s)} \to Q_{\omega_i}(p_0)$ on the holonomy subbundles of $Q_s = \pi^{-1}(V_{(s)})$.

Since the ω_i $(i = 1, 2)$ are locally flat, we have that $\mathrm{Hol}_{p_0}(\omega_i) = \mathrm{Hol}_{p_0}(\omega_i)|_{\pi^{-1}(N)} = \{e\}$, and the $h(s,t)$ of Lemma (4.4) are equal to the identity. It then follows from (4.11) and (4.9) that $\varphi_{x_0}(s)$ is a local homomorphism. Furthermore since

$$\hat{X}_{\sigma_i(x_0)} = \frac{d}{dt}(\exp tX)\sigma_i(x_0)|_{t=0} = \frac{d}{dt}[\sigma_i((\exp tX)\cdot x_0)(\varphi_i)_{x_0}(\exp tX)]_{t=0}$$

$$= (\sigma_i)_*(\tilde{X}) + [(\varphi_i)_{x_0 *}(X)]^*_{\sigma_i(x_0)}, \tag{4.16}$$

and $(\sigma_i)_*(\tilde{X})$ are horizontal lifts relative to ω_i, we get

$$\omega_i(\hat{X})_{\sigma_i(x_0)} = \Lambda_i(X) = (\varphi_i)_{x_0 *}(X). \tag{4.17}$$

Setting now $\Lambda_2 = u^{-1}(\psi_*)u$, and recalling that $\Lambda_2|_J = \mu_*$, we get $\psi_*|_J = u\mu_* u^{-1}$. Therefore $\psi(j) = u\mu(j)u^{-1}$, and thus (4.14) is satisfied. In addition, since Λ_2 is a Lie algebra homomorphism we must have

$$[\Lambda_2(X), \Lambda_2(Y)] = \Lambda_2([X, Y]),$$

i.e.

$$u^{-1}[\psi_*(X), \psi_*(Y)]u = u^{-1}\psi_*([X, Y])u.$$

Consequently ψ is a local homomorphism of Lie groups and thus it also satisfies (4.13). Finally, since $v(s) = ((\varphi_1)_{x_0}(s))^{-1}\psi(s)$ we have $v_* = -(\varphi_1)_{x_0 *} + u\Lambda_2 u^{-1}$. Hence $\Lambda_1 + v_* = u\Lambda_2 u^{-1}$, and so iv) of Proposition 4.7 is verified too.

b) Let ω be an S-invariant connection 1-form, and let $u \in C_G(\mathrm{Hol}^\circ_{p_0}(\omega))$. We shall prove here that there exists a neighborhood $V_{(s)}$ of x_0 and a local gauge transformation $f : \pi^{-1}(V_{(s)}) \to \pi^{-1}(V_{(s)})$ such that $f(p_0) = p_0 u$ and $f^*\omega = \omega$.

Indeed, since $u \in C_G(\mathrm{Hol}^\circ_{p_0}(\omega))$ and $\mu(j) \in N_G(\mathrm{Hol}^\circ_{p_0}(\omega))$ it is easy to show that $\mu(j)^{-1}u\mu(j)u^{-1} \in C_G(\mathrm{Hol}^\circ_{p_0}(\omega))$. Let now $V_{(s)}$ be a simply connected neighborhood of x_0 with $\mathrm{Hol}_{p_0}(\omega)(\pi^{-1}(V_{(s)})) = \mathrm{Hol}^\circ_{p_0}(\omega)$, and such that there exists a section $\sigma : V_{(s)} \to Q_s(p_0)$ on the holonomy subbundle of $Q_s = \pi^{-1}(V_{(s)})$. Operating on (4.16) with the connection 1-form, we get $\Lambda(X) = \omega_{\sigma(x_0)}(\hat{X}) = \omega(\sigma_* \tilde{X}_{x_0}) + \varphi_{x_0 *}(X)$, with $X \in L(S)$.

Note that since $\sigma(V_{(s)})$ is in the holonomy subbundle $Q_s(p_0)$, its structural group is $\mathrm{Hol}^\circ_{p_0}(\omega)$, so that $A(\tilde{X})_{x_0} \equiv \omega(\sigma_* \tilde{X}_{x_0}) \in L(\mathrm{Hol}^\circ_{p_0}(\omega))$.

If we now take $\psi(s) = u\varphi_{x_0}(s)u^{-1}$ for $s \in \mathscr{W} = \{s \in S | sx_0 = N\}$ then, since $u \in C_G(\mathrm{Hol}^\circ_{p_0}(\omega))$, it clearly follows from (4.11) that (4.13) is satisfied.

Furthermore since $\tau(sp_0) = v_s u$, we have that $\tau_*(\hat{X})_{p_0} = v_*(X)u$. Also since $v_s = \varphi_{x_0}(s)^{-1}\psi(s) = \varphi_{x_0}(s)^{-1}u\varphi_{x_0}(s)u^{-1}$. Consequently $\tau_*(\hat{X})_{p_0} = -\varphi_{x_0*}(X)u + u\varphi_{x_0*}(X)$, and substituting this expression into (4.1) yields

$$(f^*\omega)_{\sigma(x_0)}(\hat{X}) = u^{-1}(A(\tilde{X}_{x_0}) + \varphi_{x_0*}(X) - \varphi_{x_0*}(X) + u\varphi_{x_0*}(X)u^{-1})u$$

$$= u^{-1}A(\tilde{X}_{x_0})u + \varphi_{x_0*}(X).$$

But $u \in C_G(\mathrm{Hol}^\circ_{p_0}(\omega))$ and $A(\tilde{X})_{x_0} \in L(\mathrm{Hol}^\circ_{p_0}(\omega))$, so $u^{-1}A(\tilde{X})_{x_0}u = A(\tilde{X})_{x_0}$. Hence

$$(f^*\omega)_{\sigma(x_0)}(\hat{X}) = A(\tilde{X})_{x_0} + \varphi_{x_0*}(X) = \Lambda(X) = \omega_{\sigma(x_0)}(\hat{X}),$$

i.e.

$$f^*\omega|_{V_{(s)}} = \omega|_{V_{(s)}}.$$

The global domain

We will now show how the local results obtained so far can be extended to all of M. Thus, for example, if in Definition 2.4 we take $N = V_{(s)} = M$ then Lemma 4.2 will apply globally.

In order to derive a proposition analogous to 4.7 we need to consider an open cover $\{U_i\}_{i \in I}$ of M such that for each $i \in I$ we have a local section $\sigma_i : U_i \to P(p_0)$, where $P(p_0)$ is, as before, the holonomy subbundle of P at p_0.

Note that, if $y \in U_i \cap U_j$, then $\sigma_i(y) = \sigma_j(y)h_{ji}(y)$, where $h_{ij}(y) \in \mathrm{Hol}_{p_0}(\omega)$. Also for $x \in U_i$ and $s \in S$, there is a $j \in I$ such that $sx \in U_j$, and we can therefore write $s\sigma_i(x) = \sigma_j(sx)\varphi_x^{(j,i)}(s)$ with $\varphi_x^{(j,i)}(s) \in G$.

It is straightforward to show from the above that if $x \in U_i \cap U_j$ and $sx \in U_l \cap U_m$, then

$$\varphi_x^{(m,j)}(s) = h_{ml}(sx)\varphi_x^{(l,i)}(s)h_{ji}(x)^{-1}. \tag{4.18}$$

In addition if $x \in U_i$, $sx \in U_j$ and $tsx \in U_m$, then in analogy to (4.9) we get

$$\varphi_x^{(m,i)}(ts) = \varphi_{sx}^{(m,j)}(t)\varphi_x^{(j,i)}(s). \tag{4.19}$$

Moreover, by an argument similar to the one used to prove Lemma 4.4 we can show that the elements $\varphi_x^{(i,j)}(s)$ are contained in $N_G(\mathrm{Hol}_{p_0}(\omega))$. Therefore since $h_{ml}(sx)$ and $h_{ji}(x)^{-1} \in \mathrm{Hol}_{p_0}(\omega)$, the structure group of $P(p_0)$, it follows from (4.18) that $\varphi_x^{(m,j)}(s)$ and $\varphi_x^{(l,i)}(s)$ belong to the same equivalence class, modulo $\mathrm{Hol}_{p_0}(\omega)$. Consequently there is a function $\tilde{\varphi}_x(s) : M \times S \to \bar{N} = N_G(\mathrm{Hol}_{p_0}(\omega))/\mathrm{Hol}_{p_0}(\omega)$ which satisfies the relation

$$\tilde{\varphi}_x(ts) = \tilde{\varphi}_{sx}(t)\tilde{\varphi}_x(s). \tag{4.20}$$

Proposition 4.8. *For $s \in S$ and $x, y \in M$ one has $\tilde{\varphi}_x(s) = \tilde{\varphi}_y(s)$. Hence there is a group morphism $\tilde{\varphi} : S \to \overline{N}$ defined by $\tilde{\varphi}(s) = \tilde{\varphi}_y(s)$ for all $y \in M$.*

Proof. Let $s \in S$ be fixed and let $x \in U_i \subset M$. We will show first that there is a neighborhood N of x such that if $y \in N$, then $\tilde{\varphi}_y(s) = \tilde{\varphi}_x(s)$.

Indeed since there exists a neighborhood N of x such that $N \subset I_i$ and $sN \subset U_j$, we can proceed as in the proof of Lemma 4.4: For $y \in N$ and $g \in G$, we have $\sigma_j(sy)g \in P(p_0 g)$. In particular, for $x = y$ this implies that $\sigma_j(sx)\varphi_x^{(j,i)}(s) \in P(p_0 \varphi_x^{(j,i)}(s))$. But we also have that $\sigma_j(sx)\varphi_x^{(j,i)}(s) = s\sigma_i(x) \in P(sp_0)$. Hence $P(p_0 \varphi_x^{(j,i)}(s)) = P(sp_0)$.

Furthermore, since $s\sigma_i(y) = \sigma_j(sy)\varphi_y^{(j,i)}(x) \in P(sp_0)$, it follows from the above that both $s\sigma_i(y)$ and $\sigma_j(sy)\varphi_x^{(j,i)}(s)$ are in $P(sp_0)$ and they project onto the same point sy in M. Thus $\varphi_x^{(j,i)}(s) = \varphi_y^{(j,i)}(s)h$, with $h \in \mathrm{Hol}_{p_0}(\omega)$. That is, $\tilde{\varphi}_x(s) = \tilde{\varphi}_y(s)$.

The remainder of the proof follows from noting that if $L = \{y \in M \mid \tilde{\varphi}_y(s) = \tilde{\varphi}_x(s)\}$ then, by the continuity of $\tilde{\varphi}_y(s)$ the limit points of L are contained in L, so L is closed. It is clear that L is also open, because every point of L belongs to an open neighborhood contained in L. Finally, since M is connected we must have that $L = M$, and in consequence $\tilde{\varphi}_y(s) = \tilde{\varphi}_x(s)$ for all $y \in M$. □

Making use of the above proposition, we can readily translate the remainder of our local results to their corresponding global versions. In fact all we need is to replace $\varphi_x(s)$ by $\tilde{\varphi}(s)$ in the proofs wherever these morphisms occur and show that such an action of $\overline{N}$ is well defined. That this last assertion is indeed correct can be seen directly from the following argument:

Let $\bar{n} \in \overline{N}$ with $n \in N_G(\mathrm{Hol}_{p_0}(\omega))$. If $a \in C_G(\mathrm{Hol}_{p_0}(\omega))$, then clearly $nan^{-1} \in C_G(\mathrm{Hol}_{p_0}(\omega))$. But $na = (nan^{-1})n$, so $n \in N_G(C_G(\mathrm{Hol}_{p_0}(\omega))$. Observing finally that $\tilde{\varphi}(s)$ will appear only as an adjoint action on group elements of $C_G(\mathrm{Hol}_{p_0}(\omega))$ it is evident that such an action is in effect well defined.

We can now proceed to a global restatement of Propositions 4.6, 4.5 and 4.7. Thus if $f \in GA(P)$ and $\tau \in C(P, G)$ is its associated function, we will have, in analogy to the local case, that τ is completely determined by $v : S \to G$ if $\tau(p) = g^{-1}v(s)\tau(p_0)^{-1}g$, with $p = sp_0 g$.

The following is the global version of Proposition 4.6:

Proposition 4.9. *Let ω be an S-invariant connection 1-form, then the connection 1-form $f^*\omega$ will also be S-invariant iff f is determined by a map $v : S \to C_G(\mathrm{Hol}_{p_0}(\omega))$ which obeys the conditions*

$$v(st) = \tilde{\varphi}(t)^{-1}v(s)\tilde{\varphi}(t)v(t), \tag{4.21}$$

$$v(j) = \mu(j)^{-1}u\mu(j)u^{-1} \quad \textit{with} \quad f(p_0) = p_0 u,\, u \in G. \tag{4.22}$$

In order to give the global version of Proposition 4.5, let $\varphi_0 : S \to N_G(\mathrm{Hol}_{p_0}(\omega))$ be such that for each $s \in S$ the quantity $\varphi_0(s)$ is in the coset class of $\tilde{\varphi}(s)$. Since $\tilde{\varphi}(s)$ is a morphism, it follows that

$$\varphi_0(st) = h(s, t)\varphi_0(s)\varphi_0(t). \tag{4.23}$$

We then have

Proposition 4.10. *The map $v : S \to C_G(\mathrm{Hol}_{p_0}(\omega))$ satisfies* (4.21) *and* (4.22) *iff $\psi(s) = \varphi_0(s)v(s)$ satisfies the equations*

$$\psi(st) = h(s,t)\psi(s)\psi(t) \quad \text{for} \quad s,t \in S, \tag{4.24}$$

and

$$\psi(j) = u\mu(j)u^{-1}, \tag{4.25}$$

where $h(s,t)$ is the function which appears in (4.23).

Finally the global version of Proposition 4.7 is

Proposition 4.11. *Let ω_1 and ω_2 be two S-invariant connection 1-forms and Λ_1, Λ_2 their respective associated linear transformations, then ω_1 and ω_2 are gauge-equivalent iff there is a $u \in G$ such that*

i) $\mu(j)^{-1}u\mu(j)u^{-1} \in C_G(\mathrm{Hol}_{p_0}(\omega))$ *for all* $j \in J$,
ii) *there exists a map* $v : S \to C_G(\mathrm{Hol}_{p_0}(\omega))$ *such that* $v(st) = \tilde{\varphi}(t)^{-1}v(s)\tilde{\varphi}(t)v(t)$,
iii) $v(j) = \mu(j)^{-1}u\mu(j)u^{-1}$ *for all* $j \in J$,
iv) $\Lambda_2 = u^{-1}(\Lambda_1 + v_*|_e)u$.

Generic connections

We shall here carry our global results over to the case of generic connections. Recall that a connection 1-form ω is called generic if $C_G(\mathrm{Hol}_{p_0}(\omega)) = Z(G)$, where $Z(G)$ denotes the center of G.

In the case where P is a connected and compact manifold which admits at least one generic connection, it is known (cf. [12]) that the space of generic connections forms an open and dense subset of the space $\mathscr{C}(P)$ of all connections.

Consider then two S-invariant generic connections ω_1, ω_2, with $C_G(\mathrm{Hol}_{p_0}(\omega_i)) = Z(G)$ (the center of G), $i = 1, 2$. Since $v(s) \in Z(G)$ in this case, the condition ii) of Proposition 4.11 reduces to $v(st) = v(s)v(t)$. We thus have:

Proposition 4.12. *Let ω_1 and ω_2 be two generic S-invariant connections with associated linear transformations Λ_1 and Λ_2, respectively. Then ω_1 and ω_2 are gauge equivalent iff there exists $u \in G$ with*

i) $\mu(j)^{-1}u\mu(j)u^{-1} \in Z(G)$ *for all* $j \in J$.
ii) *There is a group homomorphism* $v : S \to Z(G)$ *such that, for* $j \in J$,

$$v(j) = \mu(j)^{-1}u\mu(j)u^{-1}.$$

iii) $\Lambda_2 = u^{-1}(\Lambda_1 + v_*|_e)u$.

Note that in this case v_* is a Lie algebra homomorphism from $L(S)$ onto an abelian subalgebra of $L(G)$, so if S or G are simple, then $v = e$ and the necessary and sufficient conditions for gauge equivalence of ω_1 and ω_2 reduce to $\Lambda_2 = u^{-1}\Lambda_1 u$, with $u \in C_G(\mu(J))$.

5. SU(2) × SU(2)-Invariant Connections over $S^2 \times S^2$

As an application of our formalism we shall derive explicit expressions for the gauge potentials corresponding to SU(2) × SU(2)-invariant connections of a principal fiber bundle with base space $S^2 \times S^2$, and gauge group SU(2). In addition, we shall provide a classification into non-gauge-related families.

The expressions obtained below correspond to five non-gauge-equivalent families with second Chern number $C_2(P) = \pm 2rs$, where r, s are integers. Furthermore, since the symmetry group SU(2) × SU(2) is assumed to act transitively on the fibers, we shall treat the base space as homogeneous with $S^2 \times S^2 = \mathrm{SU}(2) \times \mathrm{SU}(2)/\mathrm{U}(1) \times \mathrm{U}(1)$.

Our construction is as follows: Given two integers r, s, let $P_{r,s}$ be a principal fiber bundle with base space SU(2) × SU(2)/U(1) × U(1) and structure group SU(2). We choose $J = \mathrm{U}(1) \times \mathrm{U}(1)$ as the isotropy group, and the morphism $\mu : \mathrm{U}(1) \times \mathrm{U}(1) \to \mathrm{SU}(2)$ given by $\mu(p, q) = p^r q^s$, where $p^r q^s$ denotes the usual product of powers in the subgroup $\mathrm{U}(1) \subset \mathrm{SU}(2)$. Note that if r and s are different from zero, then μ does not extend to a smooth morphism from SU(2) × SU(2) to SU(2), and by Corollary 2.3 above the bundles $P_{r,s}$ are non-trivial. Furthermore, since every abelian subgroup of SU(2) is isomorphic to either U(1) or the trivial group, every morphism from J to SU(2) is, up to conjugation, as given by μ above.

Now, in order to construct S-invariant connections we must first calculate μ_*. To this end, let $[\tau_i, \tau_j] = \varepsilon_{ijk}\tau_k$ be the Lie algebra of the symmetry group SU(2). A basis for the Lie algebra of SU(2) × SU(2) is then given by $X_i = \tau_i \oplus 0$, $Y_i = 0 \oplus \tau_i$ for $i = 1, 2, 3$.

As Lie algebra of J we choose the subspace generated by X_1 and Y_1. It then follows that $J = \{\exp(\alpha X_1 + \beta Y_1) | \alpha, \beta \in \mathbb{R}\}$, and $\mu_* : L(\mathrm{U}(1)) \oplus L(\mathrm{U}(1)) \to L(\mathrm{SU}(2))$ is given by

$$\mu_*(X_1) = rX_1, \qquad \mu_*(Y_1) = sX_1. \tag{5.1}$$

We now apply Wang's theorem (cf. Eq. (2.10)) to solve for the linear transformations $\Lambda : L(\mathrm{SU}(2) \times \mathrm{SU}(2)) \to L(\mathrm{SU}(2))$. In particular, noting that $a\delta_J(m) = m$, where m is the subspace spanned by $\{X_2, X_3, Y_2, Y_3\}$, by Theorem 11.7 of Kobayashi and Nomizu [4] we have

$$\Lambda(Z) = \mu_*(Z) \quad \text{for} \quad Z \in L(J), \qquad \Lambda(Z) = \Lambda_m(Z) \quad \text{for} \quad Z \in m, \tag{5.2}$$

and

$$\Omega_{\sigma_\alpha(x_0)}(\hat{Z}_1, \hat{Z}_2) = [\Lambda_m(Z_1), \Lambda_m(Z_2)] - \Lambda_m([Z_1, Z_2]_m) - \mu_*([Z_1, Z_2]_j), \quad \text{for} \quad Z_1, Z_2 \in m \tag{5.3}$$

where $[Z_1, Z_2]_m$ denotes the m-component of $[Z_1, Z_2] \in L(\mathrm{SU}(2) \times \mathrm{SU}(2))$.

Consider first the canonical construction in $P_{r,s}$, i.e. the S-invariant connection defined by $\Lambda_m = 0$. Using the notation $\Lambda(X_i) = \Lambda_i^j X_j$, and $\Lambda(Y_i) = \Lambda_{i+3}^j X_j$, it follows immediately from (5.1) and (5.2) that

$$\Lambda_1^1 = r, \quad \Lambda_4^1 = s, \quad \Lambda_1^2 = \Lambda_1^3 = \Lambda_4^2 = \Lambda_4^3 = 0, \quad \Lambda_2^k = \Lambda_3^k = \Lambda_5^k = \Lambda_6^k = 0. \quad (5.4)$$

(Note that in this case (2.10B) is satisfied identically.)

To arrive at additional families of solutions we need to consider the general case $\Lambda_m \neq 0$, for which we have to solve Eq. (2.10B). A fairly straightforward calculation results in

$$\begin{aligned} \Lambda_l^k \exp[(\alpha(t)r + \beta(t)s)\varepsilon_{1kj}]X_j &= \exp(\alpha(t)\varepsilon_{1lk})\Lambda_k^j X_j, \\ \Lambda_{l+3}^k \exp[(\alpha(t)r + \beta(t)s)\varepsilon_{1kj}]X_j &= \exp(\beta(t)\varepsilon_{1lk})\Lambda_{k+3}^j X_j. \end{aligned} \quad (5.5)$$

Setting $\alpha(t) = \alpha\, dt$, $\beta(t) = \beta\, dt$ and expanding to first order in the infinitesimals, gives

$$\begin{aligned} (\alpha r + \beta s)\varepsilon_{1kj}\Lambda_l^k &= \alpha\varepsilon_{1lk}\Lambda_k^j, \\ (\alpha r + \beta s)\varepsilon_{1kj}\Lambda_{l+3}^k &= \beta\varepsilon_{1lk}\Lambda_{k+3}^j. \end{aligned} \quad (5.6)$$

These conditions in turn imply

$$\begin{gathered} \Lambda_1^1 = r, \quad \Lambda_4^1 = s, \quad \Lambda_1^2 = \Lambda_1^3 = \Lambda_4^2 = \Lambda_4^3 = \Lambda_2^1 = \Lambda_3^1 = \Lambda_5^1 = \Lambda_6^1 = 0, \\ (\alpha r + \beta s)\Lambda_2^2 = \alpha\Lambda_3^3, \quad -(\alpha r + \beta s)\Lambda_2^3 = \alpha\Lambda_3^2, \quad -(\alpha r + \beta s)\Lambda_5^3 = \beta\Lambda_6^2, \\ (\alpha r + \beta s)\Lambda_5^2 = \beta\Lambda_6^3, \quad (\alpha r + \beta s)\Lambda_3^3 = \alpha\Lambda_2^2, \quad (\alpha r + \beta s)\Lambda_3^2 = -\alpha\Lambda_2^3, \\ (\alpha r + \beta s)\Lambda_6^3 = \beta\Lambda_5^2, \quad (\alpha r + \beta s)\Lambda_6^2 = -\beta\Lambda_5^3. \end{gathered} \quad (5.7)$$

It is easy to verify that the admissible solutions to (6.7) are: **1)** $\alpha = -\beta, r - s = 1$; **2)** $\alpha = -\beta, s - r = 1$; **3)** $\alpha = \beta, r + s = 1$; **4)** $\alpha = \beta, r + s = -1$. In all these cases we have $Z(\mathrm{SU}(2)) = e$. Furthermore, since the Lie algebra of $\mathrm{Hol}_{p_0}\omega$ is generated by (cf. Theorem 11.8 of Kobayashi-Nomizu) $m_0 + [\Lambda(Z_l), m_0] + [\Lambda(Z_l), [\Lambda(Z_l), m_0]] + \cdots$, where m_0 is the subspace of $L(\mathrm{SU}(2))$ spanned by $\{[\Lambda(Z_l), \Lambda(Z_m)] - \Lambda([Z_l, Z_m]) | Z_l, Z_m \in L(\mathrm{SU}(2) \times \mathrm{SU}(2))\}$, it also follows that $C_G(\mathrm{Hol}_{p_0}\omega) = e$. Thus, the connections are generic and, since SU(2) is simple, we have from Proposition 4.12 that any two gauge-equivalent connections must be related by $\Lambda' = u^{-1}\Lambda u$, with $u \in C_G(\mu(J))$.

More specifically, since $u = \exp(2\gamma(t)X_1) = \cos\gamma(t)I + 2\sin\gamma(t)X_1$, it can be verified that gauge-equivalence implies

$$\begin{aligned} \Lambda_k'^1 = \Lambda_k^1, \qquad \Lambda_k'^2 &= (\cos 2\gamma)\Lambda_k^2 + (\sin 2\gamma)\Lambda_k^3, \\ \Lambda_k'^3 &= (\cos 2\gamma)\Lambda_k^3 - (\sin 2\gamma)\Lambda_k^2. \end{aligned} \quad (5.8)$$

We now consider in more detail the four cases mentioned above in order to derive explicit $SU(2) \times SU(2)$-invariant solutions.

1) Case $\alpha = -\beta, r - s = 1$.
In this case (5.7) becomes

$$\Lambda_1^1 = r, \quad \Lambda_4^1 = r - 1, \quad \Lambda_2^2 = \Lambda_3^3 = \rho, \quad \Lambda_3^2 = -\Lambda_2^3 = \sigma, \\ \Lambda_6^2 = \Lambda_5^3 = \kappa, \quad \Lambda_5^2 = -\Lambda_6^3 = \tau. \tag{5.9}$$

Now making use of (5.8) we can eliminate any one of the four parameters σ, ρ, τ, or κ. For instance, if $\sigma \neq 0$ we can set $\rho = 0$ by choosing $\tan\gamma = \frac{\sigma}{\rho}\left(-1 \pm \sqrt{1 + \left(\frac{\rho}{\tau}\right)^2}\right)$. In matrix notation we obtain

$$\Lambda = \begin{bmatrix} r & 0 & 0 \\ 0 & 0 & -\sigma \\ 0 & \sigma & 0 \\ r-1 & 0 & 0 \\ 0 & \tau & \kappa \\ 0 & \kappa & -\tau \end{bmatrix}. \tag{5.10}$$

Furthermore, it is straightforward to show that the three other solutions, obtained by gauging any one of the other parameters in (5.9), can be transformed into (5.10) by an additional gauge transformation. Consequently, case (1) leads to only one family (up to gauge) of three free parameter solutions for a given principal fiber bundle $P(r, s)$.

2) Case $\alpha = -\beta, s - r = 1$.
Here (5.7) yields

$$\Lambda_1^1 = r, \quad \Lambda_4^1 = 1 + r, \quad -\Lambda_2^2 = \Lambda_3^3 = \rho, \quad \Lambda_3^2 = \Lambda_2^3 = \sigma, \\ -\Lambda_6^2 = \Lambda_5^3 = \kappa, \quad \Lambda_5^2 = \Lambda_6^3 = \tau. \tag{5.11}$$

By a procedure entirely analogous to the one followed above, we find

$$\Lambda = \begin{bmatrix} r & 0 & 0 \\ 0 & -\rho & 0 \\ 0 & 0 & \rho \\ 1+r & 0 & 0 \\ 0 & \tau & \kappa \\ 0 & -\kappa & \tau \end{bmatrix}. \tag{5.12}$$

This constitutes another family of three free parameter solutions, and all other ones for this case are gauge-related to (5.12).

3) Case $\alpha = \beta, r + s = 1$.
Here (5.7) implies

$$\Lambda_1^1 = r, \quad \Lambda_4^1 = 1 - r, \quad \Lambda_2^2 = \Lambda_3^3 = \rho, \quad \Lambda_3^2 = -\Lambda_2^3 = \sigma,$$
$$\Lambda_6^2 = -\Lambda_5^3 = \kappa, \quad \Lambda_5^2 = \Lambda_6^3 = \tau. \tag{5.13}$$

Using (5.8) once more we get

$$\Lambda = \begin{bmatrix} r & 0 & 0 \\ 0 & \rho & -\sigma \\ 0 & \sigma & \rho \\ 1-r & 0 & 0 \\ 0 & \tau & 0 \\ 0 & 0 & \tau \end{bmatrix}, \tag{5.14}$$

which, up to gauge, is the only admissible solution in this case.

4) Case $\alpha = \beta, r + s = -1$.
For this situation we have $\Lambda_1^1 = r, \Lambda_4^1 = -1 - r, \Lambda_2^2 = -\Lambda_3^3 = \rho, \Lambda_2^3 = \Lambda_3^2 = \sigma, \Lambda_5^3 = \Lambda_6^2 = \kappa, \Lambda_5^2 = -\Lambda_6^3 = \tau$, and by an analogous procedure to the one followed before we find again that, up to gauge,

$$\Lambda = \begin{bmatrix} r & 0 & 0 \\ 0 & \rho & \sigma \\ 0 & \sigma & -\rho \\ -1-r & 0 & 0 \\ 0 & \tau & 0 \\ 0 & 0 & -\tau \end{bmatrix}. \tag{5.15}$$

To derive the gauge potentials corresponding to each of the cases above we have to resort to (2.17), for which purpose we require to know the lifting to $P(r, s)$ of the action of $SU(2) \times SU(2)$ on $S^2 \times S^2$. This lifting is determining by the functions $\varphi_x^\alpha(s)$ which we have already calculated elsewhere [13] and are given by

$$\begin{aligned}\varphi_x^\alpha(s) = {} & \left[\frac{1}{(a_1 + c_1 y_1 + d_1 y_2)^2 + (b_1 + d_1 y_1 - c_1 y_2)^2}\right]^{r/2} \\ & \cdot \left[\frac{1}{(a_2 + c_2 y_3 + d_2 y_4)^2 + (b_2 + d_2 y_3 - c_2 y_4)^2}\right]^{s/2} \cdot \\ & \cdot [(a_1 + c_1 y_1 + d_1 y_2) + 2\tau_1 (b_1 + d_1 y_1 - c_1 y_2)]^n \\ & \cdot [(a_2 + c_2 y_3 + d_2 y_4) + 2\tau_1 (b_2 + d_2 y_3 - c_2 y_4)]^s,\end{aligned} \tag{5.16}$$

where $s = (a_1 + b_1 i + c_1 j + d_1 k, a_2 + b_2 i + c_2 j + d_2 k)$ is the unit quaternion corresponding to $s \in SU(2) \times SU(2)$, and $x = (y_1, y_2; y_3, y_4)$ are local coordinates for $S^2 \times S^2$.

From (5.16) it follows that

$$W_x^\alpha(X_1) = (\varphi_x^\alpha)_*(\tau_1 \oplus 0) = r\tau_1, \qquad W_x^\alpha(X_2) = -ry_2\tau_1, \qquad W_x^\alpha(X_3) = ry_1\tau_1,$$
$$W_x^\alpha(Y_1) = s\tau_1, \qquad W_x^\alpha(Y_2) = -sy_4\tau_1, \qquad W_x^\alpha(Y_3) = sy_3\tau_1. \tag{5.17}$$

Furthermore, using left-invariant vector fields instead of right-invariant ones for the evaluation of (2.17), it can be shown that (see [14] for details)

$$A_\alpha(\tilde{X}_i^L)_x = \Lambda(X_i) - W_x^\alpha(X_i), \qquad i = 2, 3$$
$$A_\alpha(\tilde{Y}_i^L)_x = \Lambda(Y_i) - W_x^\alpha(Y_i), \qquad i = 2, 3 \tag{5.18}$$

with

$$(\tilde{X}_2^L)_x = -\frac{1}{2}(1 + y_1^2 + y_2^2)\frac{\partial}{\partial y_1}, \qquad (\tilde{X}_3^L)_x = -\frac{1}{2}(1 + y_1^2 + y_2^2)\frac{\partial}{\partial y_2},$$
$$(\tilde{Y}_2^L)_x = -\frac{1}{2}(1 + y_3^2 + y_4^2)\frac{\partial}{\partial y_3}, \qquad (\tilde{Y}_3^L)_x = -\frac{1}{2}(1 + y_3^2 + y_4^2)\frac{\partial}{\partial y_4}, \tag{5.19}$$

a local orthonormal basis in $\mathbb{R}^2 \times \mathbb{R}^2 \subset S^2 \times S^2$.

Consequently, combining the results in (5.4), (5.10), (5.12), and (5.14–18), we obtain, relative to (5.19),

$$A_\alpha = \begin{bmatrix} ry_2 & 0 & 0 \\ -ry_1 & 0 & 0 \\ sy_4 & 0 & 0 \\ -sy_3 & 0 & 0 \end{bmatrix}, \tag{5.20}$$

for the canonical connection in $P(r, s)$, and similarly

$$A_\alpha = \begin{bmatrix} ry_2 & 0 & -\sigma \\ -ry_1 & \sigma & 0 \\ sy_4 & \tau & \kappa \\ -sy_3 & \kappa & -\tau \end{bmatrix}, \qquad A_\alpha = \begin{bmatrix} ry_2 & -\rho & 0 \\ -ry_1 & 0 & \rho \\ sy_4 & \tau & \kappa \\ -sy_3 & -\kappa & \tau \end{bmatrix},$$
$$A_\alpha = \begin{bmatrix} ry_2 & \rho & -\sigma \\ -ry_1 & \sigma & \rho \\ sy_4 & \tau & 0 \\ -sy_3 & 0 & \tau \end{bmatrix}, \qquad A_\alpha = \begin{bmatrix} ry_2 & \rho & \sigma \\ -ry_1 & \sigma & -\rho \\ sy_4 & \tau & 0 \\ -sy_3 & 0 & -\tau \end{bmatrix}, \tag{5.21}$$

respectively, for cases 1) to 4) discussed above.

Calculating the gauge fields which arise from (5.20) and (5.21), it is easy to verify that our principal fiber bundles are characterized by a second Chern number given by $\pm 2rs$, and that it is only the canonical connection which allows for self-duality and thus for multi-instanton or multi-anti-instanton solutions when $r = s$ and $C_2(P_r) = \pm 2r^2$. These SU(2) × SU(2)-invariant multi-instantons form a subset of the space of solutions predicted by the Soberón-Chavez [14] classification of stable complex bundles of rank 2 over $S^2 \times S^2$, and the correspondence between stable complex bundles and self-duality established by Donaldson [15].

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