PERSISTENT CYCLES FOR HOLOMORPHIC FOLIATIONS HAVING A MEROMORPHIC FIRST INTEGRAL*

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The objective of this work is to extend Iliashenko's methods introduced in his paper "The origin of Limit Cycles under Perturbation of the equation $dw/dz = -R_z/R_w$, where $R(z,w)$ is a polynomial" [13]. In this seminal paper, Iliashenko introduces the notion of a persistent cycle under deformations and gives a way to compute the linear holonomy of a first order variation of a foliation in terms of an Abelian integral. We will give a general setting where these notions extend.

Let F be a codimension one holomorphic foliation with singularities in a complex manifold M and let δ be a closed loop contained in a leaf of F. If $\{F_t\}_{t\in T}$ denotes a family of foliations with $F_0 = F$, then it is possible to decide whether the loop δ may be deformed to other loops δ_t contained in leaves of F_t. This is possible since a closed loop may be detected as a fixed point of the holonomy map (or Poincaré return map) of F around δ, and the set of fixed points of the holonomy map of $\{F_t\}$ around δ will determine free homotopy classes of loops contained in leaves of F_t.

Let U be an open set in M where the foliation F may be described by a non-vanishing closed holomorphic 1-form ω. Choosing this 1-form chooses for us the following structure in $F|_U$. The integral $\int\omega$ gives rise to a multiple-valued holomorphic function on

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U whose fibers $(\int\omega)^{-1}(c)$ are the leaves of F. Choosing a covering of U, the integral $\int\omega$ will induce a family of submersions to $\mathbb{C}$ whose fibers are plaques of the foliation F. The holonomy of $F|_U$ may be computed by composing with the changes of coordinates of the submersions, which may be seen to be translations, since ω is closed. Hence, $F|_U$ has a transversely parallelisable structure. If the family of foliations $\{F_t\}$ may be described in U by $\Omega(z,t) = $ $= w(z) + \eta(z)t + \cdots$ then the first order variation F^1 is determined by $\omega + \eta t$. If δ is a closed loop contained in a leaf of F, then the linear holonoly of F^1 around δ takes the form

$$(t,s) \to (t, s + (\textstyle\int_\delta \eta) \cdot t) \qquad (0.1)$$

If M is now a compact complex manifold, we denote by $\mathcal{F}o\ell\ (M,L)$ the set of foliations of codimension 1 in M defined by maps $L \to T^*M$. In section 1 we show that $\mathcal{F}o\ell\ (M,L)$ has the structure of a projective variety and that its tangent space at the point ω, $T_\omega \mathcal{F}o\ell\ (M,L)$, may be identified with the infinitesimal deformations of F_ω. If U is an open set in M where ω may be defined by a closed holomorphic 1-form and δ is a loop contained in a leaf of F_ω in U, then the map

$$T_\omega \mathcal{F}o\ell\ (M,L) \to \mathbb{C} \qquad \eta \to \textstyle\int_\delta \eta \qquad (0.2)$$

associates to an infinitesimal deformation of F_ω the linear holonomy (0.1) of the deformation around δ. The map (0.2) depends on the choises made for defining it, but only up to multiplication by a nonzero constant. Hence δ determines in a canonical form a point in Proj $(T^*_\omega \mathcal{F}o\ell\ (M,L))$, which we have called an Iliashenko point. Note that a hyperplane in Proj $(T^*_\omega \mathcal{F}o\ell\ (M,L))$ corresponds to a line in $T_\omega \mathcal{F}o\ell\ (M,L)$, and hence to an infinitesimal direction of deformation. Also note that the transversally parallelizable structure of $F|_U$ implies that the loop δ may be deformed to nearby leaves of F_ω as closed loops, and hence the corresponding Iliashenko points give rise to a holomorphic curve, that we have called an Iliashenko curve.

The intersection of the Iliashenko curves with the hyperplanes in Proj $(T^*_\omega \mathcal{F}o\ell\ (M,L))$ determines which loops persist under the deformation specified by the hyperplane. We apply these results to foliations having a meromorphic first integral and we show that the Iliashenko curves associated to the indeterminacy locus are conics. We will extend these results in [8] and [14].

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1. THE SPACE OF CODIMENSION ONE HOLOMORPHIC FOLIATIONS.

In this section we introduce codimension one holomorphic foliations in a complex manifold, and show how to put a complex analytic structure in the set of all foliations that have isomorphic defining cotangent bundles. We exemplify through the case of projective spaces.

1.1. Codimension 1 Holomorphic Foliations.

A codimension 1 holomorphic foliation (with singularities) on the complex manifold M may be given by a family of integrable holomorphic 1-forms ω_α defined on an open cover $\{U_\alpha\}$ of M, $\omega_\alpha \wedge d\omega_\alpha = 0$ satisfying $\omega_\alpha = \xi_{\alpha\beta} \cdot \omega_\beta$, with $\xi_{\alpha\beta}$ holomorphic never vanishing functions. Viewing the 1-forms as inducing bundle maps on U_α

$$U_\alpha \times \mathbb{C} \to T^*M|_{U_\alpha} \qquad (p,t) \mapsto \omega_\alpha(p) \cdot t$$

where T^*M is the cotangent bundle of M and similarly

$$U_\beta \times \mathbb{C} \to T^*M|_{U_\beta} \qquad (p,s) \mapsto \omega_\beta(p)\cdot s$$

we obtain on $U_\alpha \cap U_\beta$ the equality

$$\omega_\beta(p)s = \xi_{\alpha\beta}(p)\omega_\beta(p)t$$

If we glue $U_\beta \times \mathbb{C}$ with $U_\alpha \times \mathbb{C}$ over $U_\alpha \cap U_\beta$ with the bundle isomorphism $(p,s) \mapsto (p,\xi_{\alpha\beta}^{-1}\cdot s) = (p,t)$, we will obtain a bundle map $\omega\colon L \to T^*M$, where L is the bundle on M formed with the cocycle $(\xi_{\alpha\beta}^{-1})$. We say that $\omega\colon L \to T^*M$ is equivalent to $\omega'\colon L' \to T^*M$ if there is a holomorphic bundle isomorphism $\rho\colon L \to L'$ such that $\omega' \circ \rho = \omega$.

DEFINITION 1.1. *A codimension 1 holomorphic foliation (with singularities)* F in the complex manifold M is an equivalence class of holomorphic bundle maps $\omega\colon L \to T^*M$ from a line bundle L to the cotangent bundle of M such that ω does not vanish identically on any connected component of M and such that in local trivializing coordinates, ω is given by integrable 1-forms ω_α (i.e. $\omega_\alpha \wedge d\omega_\alpha = 0$). The *singular set* Sing F *of* F is the analytic subspace of M defined by $\omega = 0$.

If $\omega\colon L \to T^*M$ is a holomorphic bundle map on the connected manifold M, described in trivializing connected coordinates $\{U_\alpha\}$ by the 1-forms ω_α; then if one of them satisfies the integrability condition $\omega_0 \wedge d\omega_0 = 0$, then all of them satisfy the integrability condition. Namely, if $U_0 \cap U_1 \neq \phi$, then in $U_0 \cap U_1$ we have $\omega_1 \wedge d\omega_1 = \xi_{10}\omega_0 \wedge d(\xi_{10}\omega_0) = \xi_{10}^2\omega_0 \wedge d\omega_0 = 0$, and hence ω_1 satisfies the integrability condition in U_1, by analytic continuation.

In M-Sing F we may obtain by the theorem of Frobenins ([2] p. 89) a cover $\{U_\alpha\}$ and biholomorphisms $\varphi_\alpha\colon U_\alpha \to V_\alpha$, with V_α open balls in $\mathbb{C}^n$ such that $\varphi_\alpha^*(dz_n) = \omega_\alpha$ are 1-forms describing the foliation. In such a coordinate cover, the transition of coordinates $\varphi_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1}\colon \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ are sending the hyperplanes defined by $z_n = K$ to themselves, i.e. if $\varphi = (\varphi^1,\ldots,\varphi^n)$ then the coordinate function φ^n is only a function of z_n.

Introduce in $\mathbb{C}^n = \mathbb{C}^{n-1} \times \mathbb{C}$ the Euclidian topology in the first factor and the discrete topology in the second. With this topology $\mathbb{C}^n$ becomes an uncountable complex manifold of dimension $n-1$, and the maps $\varphi_{\alpha\beta}$ are biholomorphisms of this (n-1)-dimensional manifold. We may induce in M-Sing F a new topology, called the leaf topology, by means of $(\varphi_{\alpha}, \varphi_{\alpha\beta})$ as an uncountable (n-1)-dimensional complex manifold. A connected component of M-Sing F with this new topology will be called a *leaf of* F, and M-Sing F is a disjoint union of all its leaves, M-Sing $F = \amalg L_i$.

If L is a holomorphic line bundle on the compact connected complex manifold M, then the set of holomorphic bundle maps from L to T*M form a finite dimensional $\mathbb{C}$-vector space E(L), by the Cartan-Serre theorem of finiteness of cohomology groups applied to the sheaf of sections of the bundle Hom (L,T*M) ([9] p.152). Let $\omega_0, \ldots, \omega_N$ be a $\mathbb{C}$-basis of E(L), and choose a trivialization of L on an open set U of M so that ω_i is represented in U by the 1-forms $\tilde{\omega}_i$. Let $(a_0, \ldots, a_N)$ be coordinates of E(L) with respect to the basis ω_i. If $\omega = \sum_{i=0}^{N} a_i\omega_i$, then as previously observed, ω determines a holomorphic foliation if and only if $\tilde{\omega} = \sum_{i=0}^{N} a_i\tilde{\omega}_i \not\equiv 0$ satisfies the integrability conditions in U:

$$0 = \tilde{\omega} \wedge d\tilde{\omega} = (\Sigma\, a_i\tilde{\omega}_i) \wedge (\Sigma\, a_j d\tilde{\omega}_j) = \sum_{i,j=0}^{N} a_i a_j (\tilde{\omega}_i \wedge d\tilde{\omega}_j) \qquad (1.1)$$

Hence we obtain that a finite number of quadratic equations (1.1) in E - {0} determine those maps ω: L → T*M that describe foliations.

Two maps ω, ω': L → T*M determine the same foliation if and only if there is an isomorphism ρ: L → L such that $\omega' \circ \rho = \omega$. Assuming M compact, we claim that all isomorphisms ρ: L → L consist of multiplication by a non-zero scalar. To see this, let $\{U_\alpha\}$ be an open cover of M where L is given by the cocycle $(\xi_{\alpha\beta})$, and ρ by a collection of holomorphic functions $\rho_\alpha: U_\alpha \to \mathbb{C}^*$. In $U_\alpha \cap U_\beta$ they satisfy the compatibility condition $\rho_\alpha \xi_{\alpha\beta} = \xi_{\alpha\beta}\rho_\beta$. Since all the terms are non-zero scalars we obtain $\rho_\alpha = \rho_\beta$. Hence ρ is given by a holomorphic function on M, which is a non-zero constant by the assumption that M is compact (by the maximum principle). Hence we have:

DEFINITION 1.2. The complex analytic subset *Fol* (M,L) of the projective space Proj E(L) defined by equations (1.1) will be called the *space of foliations of codimension* 1 *in* M *and with cotangent* *space* L. There is a one to one correspondence between the points in $\mathcal{Fol}\,(M,L)$ and equivalence classes of foliations defined by $\omega\colon L \to T^*M$, and the complex structure in $\mathcal{Fol}\,(M,L)$ may be seen to satisfy a universal property (see [4], [6]). Note that if the dimension of M is 2, then $\mathcal{Fol}\,(M,L) = \mathrm{Proj}\, E(L)$, since the conditions (1.1) are automatically satisfied.

1.2. Projective Spaces.

We exemplify with M the projective space $\mathbb{C}P^n$. $\mathbb{C}P^n$ may be constructed as the quotient of the space $\mathbb{C}_0^{n+1} = \mathbb{C}^{n+1} - \{0\}$ by the action of multiplication by non-zero scalars, $\lambda \cdot (z_0, \ldots, z_n) = (\lambda z_0, \ldots, \lambda z_n)$. One obtains affine coordinates of $\mathbb{C}P^n$ by considering in $\mathbb{C}^{n+1}$ the hyperplanes H_j defined by $z_j = 1$ that project as homeomorphisms $H_j \to U_j \subset \mathbb{C}P^n$ onto its image. The open covering $\{U_j\}$ of $\mathbb{C}P^n$, with coordinates $(z_{j0}, \ldots, \hat{z}_{jj}, \ldots, z_{jn}) \in \mathbb{C}^n \cong U_j$ and transition coordinates

$$\varphi_{ij}(z_{j0}, \ldots, z_{jn}) = \left(\frac{z_{j0}}{z_{ji}}, \ldots, \frac{1}{z_{ji}}, \ldots, \frac{\hat{z}_{ji}}{z_{ji}}, \ldots, \frac{z_{jn}}{z_{ji}}\right) =$$

$$= (z_{i0}, \ldots, z_{ij}, \ldots, \hat{z}_{ii}, \ldots, z_{in})$$

defined on $z_{ji} \neq 0$ onto $z_{ij} \neq 0$ give a coordinate description of $\mathbb{C}P^n$.

Given an integer e, we define a bundle H(e) on $\mathbb{C}P^n$ as $U_j \times \mathbb{C}$ in the above coordinates, and gluing cocycle $\xi_{ij}\colon (U_i \cap U_j) \times \mathbb{C} \hookrightarrow U_j \times \mathbb{C} \to (U_i \cap U_j) \times \mathbb{C} \hookrightarrow U_i \times \mathbb{C}$ defined as $\xi_{ij}(z_{j0}, \ldots, z_{jn}) = z_{ji}^{-e}$. It may be shown (see [9] p. 144) that any holomorphic line bundle on $\mathbb{C}P^n$ is isomorphic to one and only one H(e), where e is the Chern class of H(e) in $H^2(\mathbb{C}P^2, Z)$. The bundle H(-1) is called the Hopf bundle, and it may also be obtained as the subline bundle of $H(-1) \hookrightarrow \mathbb{C}P^n \times \mathbb{C}^{n+1}$ such that $H(-1)_p$ is the line in $\mathbb{C}^{n+1}$ that p represents.

PROPOSITION 1.3. 1) There is a one to one correspondence between holomorphic maps $\omega\colon H(-e) \to T^*\mathbb{C}P^n$ and polynomial 1-forms

$\omega = \omega_{e-1} + \omega_{e-2} + \cdots + \omega_0$ in one of the canonical coordinate charts, say U_0, of $\mathbb{C}P^n$, where ω_j is a homogeneous 1-form of degree j in $(z_{01}, \ldots, z_{0n})$ and the evaluation of ω_{e-1} on the radial vector field vanishes, $\omega_{e-1} \circ \left(\sum_{i=1}^{n} z_{0i} \frac{\partial}{\partial z_{0i}} \right) = 0$.

2) The set $E(H(-e))$ of maps from $H(-e)$ to $T^*\mathbb{C}P^n$ form a vector space of dimension

$$\binom{e+n-1}{e}(e-1)$$

3) For $\mathbb{C}P^2$, $\dim E(H(-e)) = \dim \mathcal{F}o\ell\,(\mathbb{C}P^2, H(-e)) + 1 = e^2 - 1$.

Proof. To simplify notation, let $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ be coordinates of $U_0 \cong \mathbb{C}^n$ and $U_1 \cong \mathbb{C}^n$ respectively, with transition coordinates

$$\varphi(x_1, \ldots, x_n) = \left(\frac{1}{x_1}, \frac{x_2}{x_1}, \ldots, \frac{x_n}{x_1} \right) = (y_1, y_2, \ldots, y_n) \qquad (1.2)$$

Let ω be a polynomial 1-form on U_1 of degree less than of equal to d, and let $\omega = \sum_{i=1}^{n} \sum_{j=0}^{d} a_{ij} dy_i$, where a_{ij} are homogeneous polynomials of degree j in $y_1, \ldots, y_n$. ω is transformed to U_0 as

$$\varphi^*(\omega) =$$

$$= \sum_{j=0}^{d} \left[a_{ij}\left(\frac{1}{x_1}, \ldots, \frac{x_n}{x_1} \right) \left(-x_1^{-2} dx_1 \right) + \sum_{i=2}^{n} a_{ij}\left(\frac{1}{x_1}, \ldots, \frac{x_n}{x_1} \right) x_1^{-2} (-x_i dx_1 + x_1 dx_i) \right]$$

$$= \sum_{j=0}^{d} \left[\left(-a_{1j}(1, \ldots, x_n) - \sum_{i=2}^{n} a_{ij}(1, \ldots, x_n) x_i \right) x_1^{-j-2} dx_1 + \right.$$

$$\left. + \sum_{i=2}^{n} a_{ij}(1, \ldots, x_n) x_1^{-j-1} dx_i \right]$$

Hence $\varphi^*(\omega)$ has a pole of order $d+2$ on $\mathbb{C}P^n - U_1$, unless the term

$$a_{1d}(1, x_2, \ldots, x_n) + \sum_{i=2}^{n} a_{id}(1, x_2, \ldots, x_n) x_i = 0 \qquad (1.3)$$

in which case it will have a pole of smaller order. Dividing (1.3) by x_1^{d+1}, (1.3) transform to the y_i-coordinates as

$$0 = \sum_{i=1}^{n} a_{id}(y_1,\dots,y_n)y_i = \left(\sum_{i=1}^{d} a_{id}dy_i\right)\cdot\left(\sum_{i=1}^{n} y_i \frac{\partial}{\partial y_i}\right) \qquad (1.4)$$

Hence we obtain that the set of 1-forms on $\mathbb{C}P^n$, holomorphic in U_1 and having a pole of order less than or equal to e on $\mathbb{C}P^n - U_1$ has a representation on U_1 as a polynomial 1-form

$\omega = \omega_{e-1} + \omega_{e-2} + \cdots + \omega_0$ and $\omega_{e-1} \circ \sum_{i=1}^{n} y_i \frac{\partial}{\partial y_i} = 0$. Any holomorphic 1-form on U_1 extends to $\mathbb{C}P^n$ as a holomorphic map $H(-e) \to$ $\to T^*\mathbb{C}P^n$ if and only if it exteds as a meromorphic 1-form with a pole on $\mathbb{C}P^n - U_1$ of order less than or equal to e. This proves part 1.

The dimension of $\{\omega_{e-1} + \cdots + \omega_0\}$ is $n \cdot \dim$ {homogeneous polynomials of degree $(e-1)$ in $(n+1)$-variables} $= n\binom{e+n-1}{e-1}$.

The condition that $\omega_{e-1} \circ \left(\sum_{i=1}^{n} z_{0i} \frac{\partial}{\partial z_{0i}}\right) = 0$ imposes as many conditions as the dimension of {homogeneous polynomials of degree e in n-variables}; hence the dimension of $E(H(-e))$ is

$$n\binom{e+n-1}{e-1} - \binom{e+n-1}{e} = \binom{e+n-1}{e}\left[\frac{ne}{n} - 1\right] = \binom{e+n-1}{e}(e-1)$$

This proves part 2. Part 3 follows since the integrability conditions are automatically satisfied. ∎

1.3. Families of Foliations.

We will end this section by defining the notion of families and deformations of holomorphic foliations.

LEMMA 1.4. Let M be a compact complex manifold, L a holomorphic line bundle on M, $E(L)$ the $\mathbb{C}$-vector space of holomorphic bundle maps $L \to T^*M$, Proj $E(L)$ the projective space of lines through 0 in $E(L)$ and $H(-1)$ the Hopf bundle on Proj $E(L)$. Then, there is a bundle map on Proj $E(L) \times M$, $\mathcal{W}: \Pi_1^*H(-1) \otimes \Pi_2^*L \to \Pi_2^*T^*M$, such that for any $\omega \in$ Proj $E(L)$ the restriction of $\mathcal{W}$ to $\omega \times M$ is the map represented by ω.

Proof. Let $\omega_0, \ldots, \omega_n$ be a basis of $E(L)$, $(a_0, \ldots, a_n)$ the coordinates of $E(L)$ associated with this basis, and $\tilde{\Pi}_i$ the projections of $E(L) \times M$ to both factors, $i = 1,2$. We may form a tautological bundle map on $E(L) \times M$

$$\tilde{W}\colon \tilde{\Pi}_2^* L \to \tilde{\Pi}_2^* T^*M \qquad \tilde{W}(a_0, \ldots, a_n; p) = \sum_{i=0}^{N} a_i \omega_i(p)\colon L_p \to T_p^*M \tag{1.5}$$

Note that $\tilde{W}(\lambda a_0, \ldots, \lambda a_n; p) = \lambda \tilde{W}(a_0, \ldots, a_n; p)$, so that $\tilde{W}$ does not descend to a bundle map on $\mathrm{Proj}\ E(L) \times M$.

Let $\Pi\colon E(L) - \{0\} = E(L)_0 \to \mathrm{Proj}\ E(L)$ be the natural projection. There is a natural section σ of the line bundle $\Pi^*H(-1)$ on $E(L)_0$, such that $\sigma(\omega) = \omega \in \Pi^*H(-1)_\omega = \mathbb{C} \cdot \omega$. This section has the property that $\sigma(\lambda\omega) = \lambda\sigma(\omega)$. Hence the bundle map on $E(L)_0 \times M$, $\frac{1}{\sigma}\tilde{W}\colon \Pi^*H(-1) \otimes \tilde{\Pi}_2^*L \to \Pi_2^*T^*M$ is invariant under multiplication by $\lambda \in \mathbb{C}^*$, so induces the map W in the statement of the Lemma. ‖

DEFINITION 1.5. Let L be a holomorphic line bundle on the compact complex manifold M, $\mathit{Fol}\ (M,L) \subset \mathrm{Proj}\ E(L)$ the space of foliations of codimension 1 in M with cotangent space L and $W\colon \Pi_1^*H(-1) \otimes \Pi_2^*L \to \Pi_2^*T^*M$ the restriction of W in Lemma 1.4 to $\mathit{Fol}\ (M,L) \times M$. *A family of holomorphic foliations in* M *(with cotangent space* L*) parametrized by the complex analytic space* S may be specified by a holomorphic map $f\colon S \to \mathit{Fol}\ (M,L)$, and the bundle map on $S \times M$

$$f^*W\colon f^*\Pi_1^*H(-1) \otimes \Pi_2^*L \to \Pi_2^*T^*M \tag{1.6}$$

defines the family of foliations. If F is a foliation represented by $\bar{\omega}_0 \in \mathit{Fol}\ (M,L)$, then a *deformation of* F *parametrized by a germ of an analytic space* $(S,0)$ may be given by a germ of a holomorphic map $(S,0) \to (\mathit{Fol}\ (M,L), \bar{\omega}_0)$.

Remark. It may be shown using Douady's thesis [4] that the space $\mathit{Fol}\ (M,L)$ is universal for an intrinsic definition of holomorphic families.

Let $f\colon S \to \mathit{Fol}\ (M,L)$ be a map specifying a family $\tilde{F}$ of foliations in $S \times M$ where S is a complex manifold, as in (1.6). The singular set of the family $\tilde{F}$, is the analytic subspace $\mathrm{Sing}\ \tilde{F}$ of $S \times M$ defined by $f^*W = 0$. We may apply the theorem of Frobenius with parameters to obtain a covering $\{U_\alpha\}$ of $S \times M\text{-}\mathrm{Sing}\ \tilde{F}$ and biholomorphisms over S, $\Phi_\alpha : U_\alpha \to S_\alpha \times V_\alpha \hookrightarrow S \times \mathbb{C}^n$, where S_α is an open set in S, V_α is open in $\mathbb{C}^n$, $\Pi_1 = \Pi_1 \circ \Phi_\alpha$ and such that the relative 1-form $\Phi_\alpha^*(dz_n)$ is a defining equation for $\tilde{F}$ in U_α. The transition of coordinates $\Phi_{\alpha\beta} = \Phi_\alpha \circ \Phi_\beta^{-1}$ are biholomorphisms over S such that $\dfrac{\partial \Phi^n_{\alpha\beta}}{\partial z_k} = 0$, $k = 1,\dots,n-1$. Putting in $S \times \mathbb{C}^{n-1} \times \mathbb{C}$ the discrete topology in the first and third factors, and the Euclidean topology in the middle one, we obtain a structure in $S \times M\text{-Sing}\,\tilde{F}$ of an uncountable (n-1)-dimensional manifold. A connected component L will be called a *leaf of* $\tilde{F}$, and any such leaf is contained in a Π_1-fibre.

2. FIRST VARIATIONS OF HOLOMORPHIC FOLIATIONS.

In this section we will interpret a tangent vector to the space of codimension one holomorphic foliations as a family of foliations parametrized by the analytic space A associated to the dual numbers $\mathbb{C} \oplus t\mathbb{C}$, $t^2 = 0$. We introduce the holonomy pseudogroup and we show how to compute its first variation. We finish the section by introducing the notion of persistent cycles under deformations.

2.1. First Variations and $T\,\mathcal{F}o\ell\,(M,L)$.

In this subsection we will give a method to construct the tangent space to $\mathcal{F}o\ell\,(M,L)$ at a point in $\mathcal{F}o\ell\,(M,L)$ given by $\omega: L \to T^*M$.

We will begin by recalling the description of the tangent space to the analytic space S at a point p. Let $\mathcal{O}_{S,p}$ be the local algebra of germs of holomorphic functions on S at p, with maximal ideal m_p, then the tangent space to S at p is $T_pS =$ $= Hom_{\mathbb{C}}(m_p/m_p^2, \mathbb{C})$ (see [15] p. 75). Let A be the analytic space consisting of one point 0 and ring of holomorphic functions isomorphic to $\mathbb{C} \oplus \mathbb{C} \cdot t$, with $t^2 = 0$.

LEMMA 2.1. There is a one to one correspondence between points in T_pS and holomorphic maps $\phi: A \to S$ such that $\phi(0) = p$.

Proof. A holomorphic map $\phi: A \to S$, $\phi(0) = p$, is completely determined by the morphism of local $\mathbb{C}$-algebras $\phi^*: \mathcal{O}_{S,p} \to \mathbb{C} \oplus \mathbb{C}t$ (i.e. $\phi^{*-1}(\mathbb{C}t) = m_p$, since the morphism is local ([11], p. 73)). We then have maps $\phi^*: m_p \to \mathbb{C}t$ and $\phi^*(m_p^2) \subset \phi^*(m_p)^2 = 0$, so that ϕ^* induces a map $\phi^*: \frac{m_p}{m_p^2} \to \mathbb{C}t$ giving rise to an element in $\mathrm{Hom}_{\mathbb{C}}(m_p/m_p^2, \mathbb{C}) = T_pS$. Conversely, given a $\mathbb{C}$-linear map $\lambda: m_p/m_p^2 \to \mathbb{C}$ construct the local algebra morphism $\lambda: \mathcal{O}_{S,p} \to \mathbb{C} \oplus \mathbb{C}t$ as $\lambda(f) = f(0) + \lambda \circ \Pi(f - f(0))$, where $\Pi: m_p \to m_p/{}_p^2$ is the quotient map. ‖

Given a complex manifold, we will examine the analytic space $A \times M$, where A is the analytic space associated to the dual numbers $\mathbb{C} \oplus t\mathbb{C}$, $t^2 = 0$. We will view it as a family of complex manifolds parametrized by A. The analytic space $A \times M$ is set theoretically the same as M, it has also the same topology, but it has a different function theory. If U is an open set of M and $\mathcal{O}_U$ denotes the ring of holomorphic functions of M defined in U, then the ring of holomorphic functions on $A \times U$ is the ring $\mathcal{O}_U \oplus t\mathcal{O}_U$, $t^2 = 0$.

If L is a holomorphic line bundle on M, defined on an open cover $\{U_\alpha\}$ by the cocycle $(\xi_{\alpha\beta})$, then Π_2^*L is a holomorphic line bundle on $A \times M$ defined in the open cover $\{A \times U_\alpha\}$ by the cocycle $(\xi_{\alpha\beta} \cdot \mathrm{Id})$: Namely, on U_α it is the bundle $U_\alpha \times (\mathbb{C} \oplus \mathbb{C}t)$ and we glue with the cocycle $\xi_{\alpha\beta} \cdot \mathrm{Id}$, where Id is the 2×2 identity matrix. Hence we have that $\Pi_2^*L = L \oplus tL$. Similarly, the bundle $\Pi_2^*T^*M$ on $A \times M$, called the *relative cotangent bundle* of $A \times M \to M$, is isomorphic to $T^*M \oplus tT^*M$. If $f + tg$ is a function on $A \times U$, and $\omega + t\eta$ is a relative 1-form, then $(f + tg) \cdot (\omega + t\eta) =$
$= f\omega + t(g\omega + f\eta)$.

A bundle map $\Phi: L \oplus tL \to T^*M \oplus tT^*M$ on $A \times M$ is a vector bundle map on M which is $\mathcal{O}_U \oplus t\mathcal{O}_U$-linear over any open set U. Since $L \oplus tL$ is locally isomorphic to $\mathcal{O}_U \oplus t\mathcal{O}_U$, we may find a cover $\{U_\alpha\}$ of M where L is described by the cocycle $(\xi_{\alpha\beta})$, and in this description Φ is given in U_α by multiplication with a relative 1-form $\omega_\alpha + t\eta_\alpha$. From the cocycle condition obtained, we observe that (ω_α) and (η_α) glue to give bundle maps $\omega, \eta: L \to T^*M$, so that $\Phi = \omega + t\eta$.

LEMMA 2.2. Let M be a compact complex manifold, L a holomorphic line bundle on M and $E(L)$ the finite dimensional vector space of bundle maps from L to T^*M. Let $\widetilde{\mathcal{Fol}}\ (M,L)$ be the analytic subspace of $E(L) - \{0\}$ formed by those maps $\omega\colon L \to T^*M$ that satisfy the integrability conditons (1.1), then:

1. There is a one to one correspondence between the tangent space $T_\omega E$ to E at ω and bundle maps $\omega + t\eta\colon L \oplus tL \to T^*_M \oplus tT^*_M$ on $M \times A$.

2. If $\omega \in \widetilde{\mathcal{Fol}}\ (M,L)$, then $\omega + t\eta$ represents a vector tangent to $\widetilde{\mathcal{Fol}}\ (M,L)$ at ω if and only if for local 1-forms (ω_α), (ω_α) describing ω and η, we have

$$\omega_\alpha \wedge d\eta_\alpha + \eta_\alpha \wedge d\omega_\alpha = 0 \tag{2.1}$$

Proof. By Lemma 2.1 there is a one to one correspondence between $T_\omega E$ and holomorphic maps $\phi\colon (A,0) \to (E(L),\omega)$. By pulling back the family $\widetilde{w}$ in Lemma 1.4, we obtain for each such ϕ a bundle map over $M \times A$: $L \oplus tL \to T^*_M \oplus tT^*_M$ which as shown above is of the form $\omega + t\eta$, for $\eta \in E(L)$. Using the expression (1.5) of $\widetilde{w}$, we see that if ϕ represents $\eta \in T_\omega E(L)$ then $\phi^*(\widetilde{w}) = \omega + t\eta$. This proves part 1.

To prove 2, let $\omega + t\eta$ represent a tangent vector to $E(L)$ at ω giving rise to a map $\phi\colon (A,0) \to (E(L),\omega)$. ϕ is tangent to $\widetilde{\mathcal{Fol}}\ (M,L)$ if and only if the pull back to A of equation (1.1) is satisfied identically; namely

$$(\omega_\alpha + t\eta_\alpha) \wedge (d\omega_\alpha + td\eta_\alpha) = \omega_\alpha \wedge d\omega_\alpha + t(\omega_\alpha \wedge d\eta_\alpha + \eta_\alpha \wedge d\omega_\alpha) \quad (\text{mod } t^2) \tag{2.2}$$

Since $\omega_\alpha \wedge d\omega_\alpha = 0$ by hypothesis, the vanishing of (2.2) is equivalent to vanishing of (2.1). ‖

PROPOSITION 2.3. With the same hypothesis as Lemma 2.2, let $\omega\colon L \to T^*M$ be a map defining a point $\bar{\omega} \in \operatorname{Proj} E(L)$, then:

1. There is a one to one correspondence between the tangent space $T_{\bar{\omega}}\operatorname{Proj} E(L)$ to $\operatorname{Proj} E(L)$ at $\bar{\omega}$ and points in $E(L)/\mathbb{C}\cdot\omega$; the correspondence is establishen by associating to a class $[\eta] \in E(L)/\mathbb{C}\cdot\omega$ the map $\omega + t\eta\colon L \oplus tL \to T^*M \oplus tT^*M$.

2. If $\bar{\omega} \in \mathcal{Fol}\ (M,L)$, then $T_{\bar{\omega}}\ \mathcal{Fol}(M,L)$ is the subspace of

$E/\mathbb{C}\cdot\omega$ formed of classes $[\eta]$ satisfying (2.1).

Proof. Let $\Pi: E(L) - \{0\} \to \mathrm{Proj}\ E$ be the map defining $\mathrm{Proj}\ E(L)$. The tangent spaces at ω and $\bar{\omega}$ are related by the exact sequence

$$0 \to \mathbb{C}\cdot\omega \to T_\omega E(L) \xrightarrow{D\Pi} T_{\bar{\omega}}\mathrm{Proj}\ E(L) \to 0$$

The proposition follows from this sequence and Lemma 2.2. ‖

DEFINITION 2.4. Let F be a holomorphic foliation of codimension 1 represented by a point $\bar{\omega} \in \mathcal{F}o\ell\,(M,L)$. The family of foliations on M parametrized by the analytic space A associated to the dual numbers $\mathbb{C}\oplus\mathbb{C}t$ constructed from the tangent vector $\bar{\eta} \in T_{\bar{\omega}}\mathcal{F}o\ell\,(M,L)$ will be called an *infinitesimal deformation of* F (keeping M and L fixed), and will be denoted by F^1 or $F^1_{\bar{\eta}}$.

We will now see how we may choose coordinate covers of M adapted to a foliation F defined by a bundle map $\omega: L \to T^*M$. Let $M' = M\text{-Sing}\ F$ and let $\{U_\alpha\}$ be a coordinate cover of M' with coordinates $\phi_\alpha: U_\alpha \to V_\alpha \hookrightarrow \mathbb{C}^n$ with coordinates $(z_{\alpha 1},\dots,z_{\alpha n})$ where V_α is an open ball in $\mathbb{C}^n$, with transition coordinates $\phi_{\alpha\beta}$: $\phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$, and assume that F is described by $dz_{\alpha n}$ in U_α. Note that any 1-form defining F in U_α is of the form $f_\alpha(z_{\alpha 1},\dots,z_{\alpha n})dz_{\alpha n}$, and if we require that this 1-form is closed, then $\omega_\alpha = f_\alpha(z_{\alpha n})dz_{\alpha n}$. Assume that (ω_α) is a family of closed 1-forms defining F in $M\text{-Sing}\ F$. If $\xi_{\alpha\beta}: U_\alpha \cap U_\beta \to \mathbb{C}^*$ is the cocycle defined by $\omega_\alpha = \xi_{\alpha\beta}\omega_\beta$, then the cocycle $(\xi_{\alpha\beta})$ depends only on the variable $z_{\beta n}$; that is, if a foliation is defined by closed 1-forms, then the cocycle $(\xi_{\alpha\beta})$ is constant along the leaves.

LEMMA 2.5. Let $\{U_\alpha\}$, ω_α and $\xi_{\alpha\beta}$ be as above with ω_α closed 1-forms, and let $F^1_{\bar{\eta}}$ be an infinitesimal deformation of F associated to $\eta: L \to T^*M$. Then

1. In the above coordinates of M', η is described by a cocycle (η_α), $\eta_\alpha = \xi_{\alpha\beta}\eta_\beta$, such that η_α is F-closed; i.e. for any leaf L of F, η restricted to L as a 1-form in L is closed.

2. Let ω_1 be a non-vanishing closed 1-formed defined in the open subset U of M defining F, then the infinitesimal deformation $F^1_{\bar{\eta}}$ is defined in U by a 1-form η_1 that is F-closed.

Proof. The integrability condition (2.1) reduces to $\omega_\alpha \wedge d\eta_\alpha = 0$

if ω_α is a closed 1-form. Choosing local coordinates so that $\omega_\alpha = f(z_n)dz_n$ we have

$$\omega_\alpha \wedge d\eta_\alpha = (fdz_n) \wedge d\left[\sum_{i=1}^{n} \eta_i dz_i\right] = f\left[\sum_{j=1}^{n-1} \frac{\partial}{\partial z_j}\left(\sum_{i=1}^{n-1} \eta_j\right) dz_j \wedge dz_i\right] \wedge dz_n$$

Hence the term inside the bracket is zero, which means that the 1-form on the leaves of $\mathcal{F}$ is $d_{\mathcal{F}}$-closed. This proves part 1. Part 2 also follows since ω_1 gives a trivialization of L restricted to U. ∎

The following Lemma is a Frobenius Theorem for infinitesimal deformations:

LEMMA 2.6. Let $\mathcal{F}$ be a holomorphic foliation in M, $M' = M\text{-Sing}\,\mathcal{F}$ and let $\{U_\alpha\}$ be an open cover of M' with biholomorphisms $\phi_\alpha: U_\alpha \to D_\alpha \times W_\alpha \subset \mathbb{C}^{n-1} \times \mathbb{C}$, where D_α are balls in $\mathbb{C}^{n-1}$ and W_α open sets in $\mathbb{C}$, $\phi_\alpha^* dz_n$ defines $\mathcal{F}$ in U_α and the transition coordinates of the submersions $\bar{\phi}_\alpha = \Pi_2 \circ \phi_\alpha: U_\alpha \to W_\alpha$ are $\bar{\phi}_{\alpha\beta}: \bar{\phi}_\beta(U_\alpha \cap U_\beta) \to \bar{\phi}_\alpha(U_\alpha \cap U_\beta)$. Then, for any infinitesimal deformation $\mathcal{F}^1$ of $\mathcal{F}$ there are holomorphic maps $\bar{\phi}_\alpha + t\bar{\psi}_\alpha: A \times U_\alpha \to A \times W_\alpha$ with transition coordinates $\bar{\phi}_{\alpha\beta} + t\bar{\psi}_{\alpha\beta}$ such that $(\bar{\phi}_\alpha + t\bar{\psi}_\alpha)^* dz_{\alpha n}$ defines $\mathcal{F}^1$ in U_α.

Proof. Let $\omega_\alpha = \phi_\alpha^* dz_{\alpha n}$ be the 1-forms defining $\mathcal{F}$, and η_α the $\mathcal{F}$-closed 1-forms describing $\mathcal{F}^1$, as in Lemma 2.5, so that $\omega_\alpha + t\eta_\alpha$ describes $\mathcal{F}^1$. Note that for every holomorphic function f_α in U_α, $\mathcal{F}^1$ may also be defined by $(1 + tf_\alpha)(\omega_\alpha + t\eta_\alpha) = \omega_\alpha + t(\eta_\alpha + f_\alpha\omega_\alpha)$. Hence any 1-form $\tilde{\eta}_\alpha$ whose restriction to every leaf of $\mathcal{F}$ in U_α coincides with η_α, also serves to define $\mathcal{F}^1$ in U_α.

Choose a section Σ_α to the foliation in U_α and define $\bar{\psi}_\alpha(p) = \int_\delta \eta_\alpha$, where δ is a path contained in a leaf of $\mathcal{F}$ going from a point in Σ_α to p. Clearly $d\bar{\psi}_\alpha$ coincides with η_α restricted to any leaf, hence $(\phi_\alpha + t\psi_\alpha)^* dz_{\alpha n} = \omega_\alpha + t\psi_\alpha^* dz_{\alpha n}$ defines $\mathcal{F}^1$ in U_α. Define $\psi_\alpha: U_\alpha \to \mathbb{C}^n$ as $\psi_\alpha = (0,\ldots,0,\bar{\psi}_\alpha)$, then $(\phi_\alpha + t\psi_\alpha): A \times U_\alpha \to A \times (D_\alpha \times W_\alpha)$ is an isomorphism of analytic spaces and $(\phi_\alpha + t\psi_\alpha)dz_n$ defines F^1 in U_α. If $\phi_{\alpha\beta} + t\psi_{\alpha\beta} = (\phi_\alpha + t\psi_\alpha)^{-1} \circ (\phi_\beta + t\psi_\beta)$ is the transition of coordinates, then

$$(\phi_{\alpha\beta} + t\psi_{\alpha\beta})^* dz_{\beta n} = \sum_{i=1}^{n} \left(\frac{\partial \phi^n_{\alpha\beta}}{\partial z_{\alpha i}} + t \frac{\partial \psi^n}{\partial z_{\alpha i}} \right) dz_{\alpha i}$$ is of the form

$(f + tg) \cdot dz_{\alpha n}$ since $(\phi_{\alpha\beta} + t\psi_{\alpha\beta})$ is preserving the foliations on $A \times \mathbb{C}^n$. Hence $\phi^n_{\alpha\beta}$ and $\psi^n_{\alpha\beta}$ are functions of $z_{\beta n}$ alone, which we denote by $\bar{\phi}^n_{\alpha\beta}$ and $\bar{\psi}^n_{\alpha\beta}$. The transition coordinates of the submersions $\bar{\phi}_\alpha + t\bar{\psi}_\alpha$ are then $\bar{\phi}^n_{\alpha\beta} + t\bar{\psi}^n_{\alpha\beta}$. This proves the lemma. ‖

2.2. First Variations of Holonomy Maps.

In this section we define the holonomy pseudogroup of a foliation, and show how to obtain from a holomorphic family of foliations, an induced family of holonomy pseudogroups. Then we will obtain from an infinitesimal deformation of a foliation an infinitesimal deformation of the holonomy pseudogroup, and we will show how to compute some of its elements using the preceeding subsection.

Let F be a holomorphic foliation of codimension 1 in M, defined by $\omega: L \to T^*M$. In $M' = M\text{-Sing}\, F$ we obtain a non-singular foliation of codimension 1. Let $\{U_\alpha\}$ be a countable open cover of M' and biholomorphisms $\phi_\alpha: U_\alpha \to V_\alpha$, with V_α open balls in $\mathbb{C}^n$, such that $\phi^*_\alpha(dz_n) = \omega_\alpha$ are 1-forms describing F, as in section 1.3. We will denote by $\bar{\phi}_\alpha: U_\alpha \to \bar{V}_\alpha \subset \mathbb{C}$ the maps obtained by applying ϕ_α and then composing with projection to the last factor, and we assume that $\bar{V}_\alpha \cap \bar{V}_\beta = \phi$ for $\alpha \neq \beta$ (obtained after suitable translations) and that $\bar{\phi}_\alpha$ has connected fibers. Denote by $\bar{\phi}_{\alpha\beta}: \bar{\phi}_\beta(U_\alpha \cap U_\beta) \to \bar{\phi}_\alpha(U_\alpha \cap U_\beta)$ the biholomorphism of open sets in $\mathbb{C}$ defined as $\bar{\phi}_\alpha \circ \bar{\phi}_\beta^{-1}$.

DEFINITION 2.7. The collection of biholomorphisms between open sets of $\mathbb{C}$ $\{\bar{\phi}_{\alpha_r\alpha_{r-1}} \circ \cdots \circ \bar{\phi}_{\alpha_1\alpha_0}\}$, with domain of definition the maximal open set where the composition makes sense, is called the *holonomy pseudogroup of the foliation* F (with respect to the submersions $\bar{\phi}_\alpha$). Many properties of the holonomy pseudogroup do not depend on the submersions $\{\bar{\phi}_\alpha: U_\alpha \to \bar{V}_\alpha \subset \mathbb{C}\}$; for a notion of equivalence of pseudogroups, see [10].

Given points p_0 and p_1 in a leaf L of F, let $p_0 \in U_0$ and $p_1 \in U_1$, and for every close path $\delta: [0,1] \to L$, $\delta(0) = p_0$, $\delta(1) = p_1$, we will show how to obtain an element of the holonomy pseudogroup, $h_\delta: (V_0, \phi_0(p_0)) \to (V_1, \phi_1(p_1))$ which tells us how the leaves of F near p_0 are distributed near p_1 following the path δ. To define h_δ, let $0 = t_0 < t_1 < \cdots < t_r = 1$ be a partition of $[0,1]$, and $U_{\alpha_0} = U_0, U_{\alpha_1}, \dots, U_{\alpha_{r-1}} = U_1$ be elements of the cover $\{U_\alpha\}$ such that $\delta[t_i, t_{i+1}] \subset U_{\alpha_i}$, $i = 0,\dots,r-1$; then $h_\delta = \bar{\phi}_{\alpha_{r-1}, \alpha_{r-2}} \circ \cdots \circ \bar{\phi}_{\alpha_1, \alpha_0}$. It is shown in [10] that h_δ does not depend on the covering or the partition used, and that ~~it~~ is also independent of the homotopy class of δ, with fixed end points. If $p_0 = p_1$ and $U_0 = U_1$, we obtain the *holonomy representation*

$$h: \Pi_1(L,p_0) \to \mathcal{B}ih\ (\mathbb{C},\phi_0(p_0))$$

of the fundamental group of L at p_0 into the germ of local biholomorphisms of $\mathbb{C}$ at $\phi_0(p_0)$. The *linear holonomy representation*

$$Dh: \Pi_1(L,p_0) \to \mathbb{C}^*$$

is obtained by taking the derivatives at $\phi_0(p_0)$ of the holonomy representation.

Let $\tilde{F}$ be a family of holomorphic foliations in M parametrized by the complex manifold S, or by the analytic space $S = A$ associated to the dual numbers $\mathbb{C} \oplus t\mathbb{C}$, and let $\{\tilde{U}_\alpha\}$ be an open cover of $S \times M\text{-Sing}\,\tilde{F}$ with biholomorphisms $\Phi_\alpha: \tilde{U}_\alpha \to S_\alpha \times V_\alpha \subset S \times \mathbb{C}^n$ over S such that the relative 1-forms $\Phi^*_\alpha(dz_n)$ define $\tilde{F}$ in U_α, as in section 1.3. Let $\bar{\Phi}_\alpha: \tilde{U}_\alpha \to S_\alpha \times \bar{V}_\alpha \subset S \times \mathbb{C}$ be the maps obtained by applying Φ_α and then composing with the projection $S \times \mathbb{C}^n \to S \times \mathbb{C}$ to the z_n-coordinate, and we assume that $\bar{V}_\alpha \cap \bar{V}_\beta = \phi$ for $\alpha \neq \beta$ and that $\bar{\Phi}_\alpha$ has connected fibers. Denote by $\bar{\Phi}_{\alpha\beta}: \bar{\Phi}_\beta(\tilde{U}_\alpha \cap \tilde{U}_\beta) \to \bar{\Phi}_\alpha(\tilde{U}_\alpha \cap \tilde{U}_\beta)$ the biholomorphism over S of open sets in $S \times \mathbb{C}$ defined as $\bar{\Phi}_\alpha \circ \bar{\Phi}_\beta^{-1}$.

DEFINITION 2.8. The collection of biholomorphisms between open sets of $S \times \mathbb{C}$ $\{\bar{\Phi}_{\alpha_r\alpha_{r-1}} \circ \cdots \circ \bar{\Phi}_{\alpha_1\alpha_0}\}$ with domain of definition the maximal open set where the composition makes sense is called the *holonomy pseudogroup of the family* $\tilde{F}$.

If $\tilde{F}$ is a deformation of $F = \tilde{F}_0$ parametrized by a germ of a complex manifold, and L is a leaf of F, then the *holonomy representation of the deformation along* L is the representation

$$h: \Pi_1(L,p_0) \to \mathcal{B}ih\ (S \times \mathbb{C},\ \Phi_0(p_0))$$

obtained for $\delta \in \Pi_2(L,p_0)$ as the composition $\Phi_\delta = \bar{\Phi}_{\alpha_{r-1}\alpha_{r-2}} \circ \cdots \circ \bar{\Phi}_{\alpha_1\alpha_0}$ associated to a partition $0 < t_0 < \quad < t_r = 1$ and elements $\tilde{U}_{\alpha_0} = \tilde{U}_0, \tilde{U}_{\alpha_1}, \ldots, \tilde{U}_{\alpha_{r-1}} = U_0$ such that $\delta[t_i, t_{i+1}] \subset \tilde{U}_{\alpha_i}$. The *linear holonomy representation of the deformation along* L

$$Dh: \Pi_1(L,p_0) \to \left\{\left(\begin{array}{c|c} I_n & \mathbb{C}^n \\ \hline & \mathbb{C}^* \end{array}\right)\right\} \subset GL(n+1,\ \mathbb{C})$$

is the representation obtained by taking the derivate at $\Phi_0(p_0)$ of the holonomy representation h; n is the dimension of the tangent space to S at 0, and I_n is the identity in T_0S.

THEOREM 2.9. Let F be a codimension 1 holomorphic foliation in M defined by ω: $L \to T^*M$, U and open set in M where F may be defined by a non-vanishing holomorphic closed 1-form ω_1 and δ a closed loop in U contained in a leaf of F, then:

1. For any infinitesimal deformation F^1 of F the linear holonomy of F^1 along δ takes the form

$$\begin{pmatrix} 1 & a(F^1) \\ 0 & 1 \end{pmatrix},\ a(F^1) = \int_\delta \eta_1 \qquad (2.3)$$

where $\omega_1 + t\eta_1$ describes F^1 in U.

2. If M is compact, the map $F^1 \to a(F^1)$ in (2.3) induces a linear map

$$\int_\delta : T_{\bar{\omega}}\mathcal{F}o\ell\ (M,L) \to \mathbb{C} \qquad (2.4)$$

This map depends only on F^1 up to multiplication by a non-zero scalar (i.e. For other elections of ω, U, ω_1, describing F, δ up to homology in a leaf, and the submersion where the holonomy is defined, we obtain a function of F^1 which is a scalar multiple of (2.4)).

Proof. Let $\{U_\alpha\}$, $\phi_\alpha: U_\alpha \to D_\alpha \times W_\alpha \hookrightarrow \mathbb{C}^{n-1} \times \mathbb{C}$ be an open cover by coordinate charts of U, as in Lemma 2.6, with $\omega_\alpha = \phi_\alpha^*(dz_{\alpha n})$ equal to ω_1 in U_α, and let $\bar\phi_\alpha: U_\alpha \to W_\alpha$ be the projection to the second factor and $\bar\phi_{\alpha\beta}(z_{\beta n}) = z_{\beta n} + c_{\beta n}$ be the transitions of coordinates. Since $\omega_\alpha = \omega_\beta$ in $U_\alpha \cap U_\beta$, the transition of coordinates for L in this cover are $\xi_{\alpha\beta} = 1$ and hence the infinitesimal deformation F^1 is described by a 1-form η in U with $\eta_\alpha = \eta \mid U_\alpha$ (see Lemma 2.6). As in the proof of Lemma 2.6, define

$$\bar\psi_\alpha(p) = \int_{p_\alpha}^{p} \eta_\alpha$$

where the integral is taken in a path from $p_\alpha \in \Sigma_\alpha$ to p in a leaf, then $\bar\phi_\alpha + t\bar\psi_\alpha$ are submersions defining the infinitesimal deformation in U_α. If $U_\alpha \cap U_\beta \neq \phi$, the transition of coordinates $(Id + c_{\alpha\beta}) + t\bar\psi_{\alpha\beta}$ satisfies for $p \in U_\alpha \cap U_\beta$

$$\bar\phi_\alpha(p) + t\int_{p_\alpha}^{p} \eta_\alpha = \left((Id + c_{\alpha\beta}) + t\bar\psi_{\alpha\beta}\right)\left(\bar\phi_\beta(p) + t\int_{p_\beta}^{p} \eta_\beta\right) =$$

$$= (\bar\phi_\beta(p) + c_{\alpha\beta}) + t\left(\int_{p_\beta}^{p} \eta_\beta + \bar\psi_{\alpha\beta}(\bar\phi_\beta(p))\right)$$

and since $\bar\phi_\alpha(p) = \bar\phi_\beta(p) + c_{\alpha\beta}$ and using that $\eta \mid U_\alpha = \eta_\alpha$ and $\eta \mid U_\beta = \eta_\beta$ we obtain

$$\bar\psi_{\alpha\beta}(\bar\phi_\beta(p)) = \int_{p_\alpha}^{p} \eta_\alpha - \int_{p_\beta}^{p} \eta_\beta = \int_{p_\alpha}^{p_\beta} \eta$$

The transition of coordinates is then

$$\left(Id + t\int_{p_\alpha}^{p_\beta} \eta\right) + c_{\alpha\beta}$$

If $U_{\alpha_0}, \ldots, U_{\alpha_{r-1}} = U_{\alpha_0}$ is a covering of δ as in Definition 2.8, the holonomy map obtained by following δ has the form

$$z \mapsto z + t\int_{\delta_z} \eta \tag{2.5}$$

where δ_z is the closed loop near δ obtained by joining points on Σ_α over z and the constant is zero since δ is a closed loop. Hence we obtain (2.3).

From the expression (2.3), it follows that the map (2.4) is

$\mathbb{C}$-linear. Since the matrix (2.3) is the linear holonomy of $\mathcal{F}^1$, it is uniquely defined up to conjugation and since

$$\begin{pmatrix}1 & b\\ 0 & c\end{pmatrix}^{-1}\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & b\\ 0 & c\end{pmatrix} = \begin{pmatrix}1 & ca\\ 0 & 1\end{pmatrix}$$

we obtain that fixing any election of ω, U, ω_1, δ and submersion, the map (2.4) obtained as a function of $\mathcal{F}^1$ is a non-vanishing constant times (2.4). ‖

DEFINITION 2.10. With the assumptions in the theorem, there is a canonically defined point $\int_\delta \in \text{Proj } T^*_\omega \mathcal{F}o\ell(M,L)$, that we will call the *Iliashenko point of* δ.

2.3. Persistent Cycles under Deformations.

If a closed loop δ in a leaf of a foliation $\mathcal{F}$ has trivial holonomy, then there is a canonical way to associate to near leaves a homotopy class $[\delta_u]$ of loops near δ. If $\tilde{\mathcal{F}}$ is a deformation of $\mathcal{F}$, then some of the homotopy classes $[\delta_u]$ persist under the deformation, while others open up. In this section we will analyse this process, as well as the associated infinitesimal concept.

Let $\mathcal{F}$ be a holomorphic foliation of codimension 1 in M, L a leaf of $\mathcal{F}$ in M-Sing $\mathcal{F}$ and δ a curve in L representing a free homotopy class in L with trivial holonomy, $h_\delta = \text{id}$. Let $\{U_0,\dots,U_r\}$ be a covering of δ by coordinate charts as in Lemma 2.6, such that there is a partition $0 = t_0 < t_1 < \cdots < t_{r-1} = 1$ with $\delta[t_i, t_{i+1}] \subset U_i$ for $i = 0,\dots,r-1$. The element of the holonomy pseudogroup $\bar{\phi}_{r-1,r-2} \circ \cdots \circ \bar{\phi}_{1,0}$ is the holonmy of δ, which by hypothesis is the identity. Choose transversals Σ_i to $\mathcal{F}$ at $\delta(t_i)$, $i = 0,\dots,r-1$, $\Sigma_{r-1} = \Sigma_0$. For any point near to $\delta(t_0)$ on Σ_0 there is a unique homotopy class of paths in U_i in a leaf going from a point in Σ_0 to Σ_1; and then in U_2 from Σ_1, to Σ_2, etc. In this way we construct homotopy classes of loops $\{\delta_u\}$ near to δ and on leaves near L.

DEFINITION 2.11. Let $\tilde{\mathcal{F}}$ be a deformation of $\mathcal{F} = \tilde{\mathcal{F}}_0$ parametrized by a germ of a complex manifold S, L a leaf of $\mathcal{F}$ and δ a free homotopy class in $\Pi_1(L)$.

Let $\Phi_\delta(s,z) = (s,h(s,z))$: $(S \times \mathbb{C},(0,0)) \to (S \times \mathbb{C},(0,0))$ be the holonomy map associated to δ in definition 2.8. Let Z be the germ of a subset of $S \times \mathbb{C}$ determined by the equation $h(s,z) - z = 0$. Choosing transversals $\tilde{\Sigma}_i$ to the foliation $\tilde{F}$ in $S \times$ M-Sing F at δ, we may as before for every point z in Z sufficiently close to $(0,0)$ associate a free homotopy class δ_z in a leaf of $\tilde{F}$. We will say that δ_z *is obtained by following* δ *in* $\tilde{F}$. Note that Z is $S \times \mathbb{C}$ or has codimension 1 in $S \times \mathbb{C}$. If $Z_1 \cap (0 \times \mathbb{C})$ has 0 as an isolated zero, we will say that δ is a *limit cycle*, and the multiplicity will be called the *multiplicity of the limit cycle*. If δ is a limit cycle, then the projection to the first factor $Z \to S$ is a finite map, and the sum of the multiplicities of the cycles over $s \in S$ equals the multiplicity of δ (see [5]).

Assume now that S has dimension 1 and that the holonomy of δ in F is the identity. Then, we may write $h(s,z) - z = sg(s,z)$, where g is a germ of a holomorphic functions. If $g(0,0) = 0$ and $s \neq g(s,z)$, we will say that δ *is persistent thorugh the 1-dimensional deformation*; and otherwise we will say that δ *is not persistent*.

Let again δ be a free homotopy class in a leaf L of F with trivial holonomy, F^1 be an infinitesimal deformation of F, and $h_\delta = Id + th'$: $(A \times \mathbb{C},0) \to (A \times \mathbb{C},0)$ the holonomy representation of F^1. We will say that δ *is persistent for the infinitesimal deformation* if 0 is an isolated zero of h'.

PROPOSITION 2.12. Let δ be a free homotopy class in a leaf L of a foliation F with trivial holonomy, $\tilde{F}$ a deformation of F parametrized by $(\mathbb{C},0)$ and F^1 the induced infinitesimal deformation, then:

1) If δ is persistent for F^1, then it is persistent for $\tilde{F}$.

2) If ω_1 is a closed non-vanishing 1-form defining F in the open subset U containing δ and if $\{\delta_u\}$ are the free homotopy classes obtained from δ in the leaves $\{L_u\}$ of F near L, then F^1 may be defined by a 1-form $\omega + t\eta$ in U and the map which associates to δ_u its Iliasenko point in Proj $T^*_\omega Fol(M,L)$ gives rise to a holomorphic map

$$u \mapsto \int_{\delta_u} \qquad (2.6)$$

proof. Using the previously used notation, the holonomy of $\tilde{F}$ around δ is $h(s,z) = z + sg(s,z)$, and the holonomy of F^1 is $h(z) = z + th'(z)$, where $h'(z) = \frac{\partial h}{\partial s}(0,z) = g(0,z)$. If δ is persistent for F^1, then 0 is an isolated zero of h', the implies $g(0,0) \neq 0$ and s does not divide g. Hence δ is persistent for F^1. This proves 1.

In (2.5) it was shown that by choosing special coordinates, the holonomy map around δ has the form $h_\delta = Id + t\int_{\delta_u} \eta$, hence the linear holonomy at u, which is $\frac{\partial h}{\partial s}(u) = \int_{\delta_u} \eta$. If $\eta_0, \ldots, \eta_N$ is a basis of $T^*_\omega Fol(M,L)$ then since the maps $\int_{\delta_u} \eta_i$ are holomorphic, we obtain that (2.6) is a holomorphic map. ‖

3. GOOD MEROMORPHIC FIRST INTEGRALS.

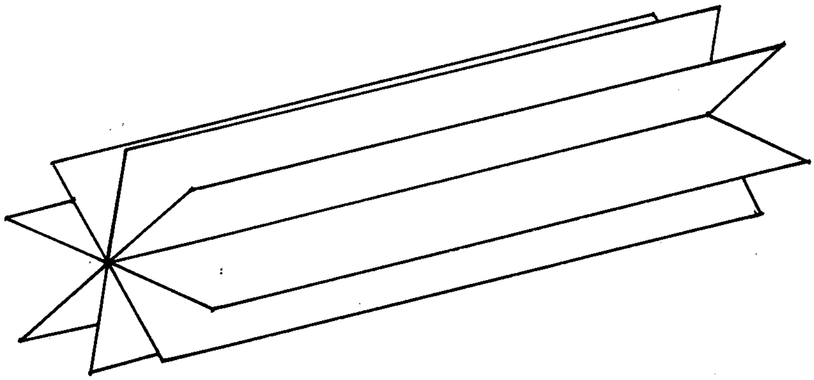
In this section we describe a class of meromorphic functions having simple singularities, and modelled in Lefschetz's pencils (see [1]). A holomorphic foliation of codimension 1 whose leaves coincide with the fibers of one of these functions will be said to have a good meromorphic first integral.

3.1. Lefschetz's Pencils.

Let $(z_0 : \cdots : z_N)$ be homogeneous coordinates of $\mathbb{C}P^N$. A hyperplane in $\mathbb{C}P^N$ is described by an equation $\sum_{i=0}^{N} a_i z_i = 0$, and the set of hyperplanes of $\mathbb{C}P^N$ is parametrized by $(\mathbb{C}P^N)^* = \{(a_0 : \cdots : a_N)\}$.

Let $\ell = \Sigma a_i z_i$ and $\ell' = \Sigma b_i z_i$ be equations defining two distinct hyperplanes in $\mathbb{C}P^N$. The linear family of hyperplanes $\{\alpha\ell + \beta\ell' \mid (\alpha : \beta) \in \mathbb{C}P^1\} \subset (\mathbb{C}P^N)^*$ consist of those hyperplanes that contain the codimension 2 linear subvariety

$$K = \{z \in \mathbb{C}P^N \quad \ell(z) = \ell'(z) = 0\}$$



This family of hyperplanes may also be described as the closure of the fibers of the rational map $f = \frac{\ell}{\ell'}$ that has its indeterminacy locus at K. Using homogeneous coordinates for the range, $f = (\ell : \ell')$ is a rational map from $\mathbb{C}P^N$ to $\mathbb{C}P^1$ giving rise to a holomorphic map outside of K, $f: \mathbb{C}P^N - K \to \mathbb{C}P^1$.

The set of linear families of hyperplanes is parametrized by the Grassmanian of projective lines in $(\mathbb{C}P^N)^*$, Grass $(\mathbb{C}P^1, (\mathbb{C}P^N)^*)$.

LEMMA 3.1. To any linear family of hyperplanes $f = (\ell : \ell') : \mathbb{C}P^N - \{\ell = \ell' = 0\} \to \mathbb{C}P^1$ we may asociate a holomorphic foliation of codimension one $df: H(-2) \to T^*\mathbb{C}P^N$ whose leaves are the f-fibers. This family of foliations in $\mathrm{Proj}\,\Gamma(\mathbb{C}P^N, H(-2)) \otimes \otimes\, T^*\mathbb{C}P^N)$ has dimension $2(N-1)$.

Proof. Given $f = (\ell : \ell')$, choose coordinates $(z_0, \ldots, z_N)$ of $\mathbb{C}^{N+1}$ such that $\ell = z_0$ and $\ell' = z_1$. In affine coordinates $(y_1, \ldots, y_N) = \left(\frac{z_1}{z_0}, \ldots, \frac{z_N}{z_0}\right)$ $f = y_1$. The 1-form $df = dy_1$ is tangent to the f-fibers and by Lemma 1.3 it has a pole of order 2 as a rational 1-form on $\mathbb{C}P^N$, so it gives rise to a holomorphic bundle map $df: H(-2) \to T^*\mathbb{C}P^N$. Since dim Grass $(\mathbb{C}P^1, (\mathbb{C}P^N)^*) = 2(N-1)$, the Lemma is proved. ‖

We will now blow up $\mathbb{C}P^N$ along K, and we will analyse the induced foliation. Let $f = (\ell : \ell') : \mathbb{C}P^N \to \mathbb{C}P^1$ be the rational map associated to a linear family of hyperplanes, with indeterminacy locus the linear subvariety $K = \{\ell = \ell' = 0\}$ of codimension 2.

Let $\widetilde{\mathbb{C}P}^N$ be the subvariety of $\mathbb{C}P^N \times \mathbb{C}P^1$ defined by the equation

$$\ell w_1 = \ell' w_0 \qquad (z_0 : \cdots : z_N) \in \mathbb{C}P^N;\ (w_0 : w_1) \in \mathbb{C}P^1$$

and let $\sigma\colon \widetilde{\mathbb{C}P}^N \to \mathbb{C}P^N$ and $\tilde{f}\colon \widetilde{\mathbb{C}P}^N \to \mathbb{C}P^1$ be the restriction to $\widetilde{\mathbb{C}P}^N$ of the projections to the factors of $\mathbb{C}P^N \times \mathbb{C}P^1$. σ is called the *blow up of* $\mathbb{C}P^N$ *along* K and $\sigma^{-1}(K) = K \times \mathbb{C}P^1$ is called the *exceptional divisor*. $\sigma\colon \widetilde{\mathbb{C}P}^N - \sigma^{-1}(K) \to \mathbb{C}P^N - K$ is a biholomorphism, $\tilde{f}$ is a $\mathbb{C}P^{N-1}$-fiber bundle and $\tilde{f} = f \circ \sigma$ (see [15] p. 98).

Any holomorphic line bundles on $\mathbb{C}P^N \times \mathbb{C}P^1$ is isomorphic to a bundle of the form $\Pi_1^*(H(n_1)) \otimes \Pi_2^*(H(n_2))$, and any holomorphic line bundle on $\widetilde{\mathbb{C}P}^N$ is isomorphic to one an only one of the restrictions of the above bundles to $\widetilde{\mathbb{C}P}^N$, that we will denote by $H(n_1, n_2)$. The exceptional divisor $K \times \mathbb{C}P^1$ is the zero set of a holomorphic section of the bundle $H(1,-1)$, which we will denote by e; it is defined uniquely up to multplication by a non-zero scalar.

LEMMA 3.2. Let $\omega\colon H(-2) \to T^*\mathbb{C}P^N$ be the holomorphic foliation associated to the linear family of hyperplanes $f = (\ell : \ell')$, having singular locus along K. Let $\sigma\colon \widetilde{\mathbb{C}P}^N \to \mathbb{C}P^N$ be the blow up of $\mathbb{C}P^N$ along K, and let $\tilde{\omega}\colon H(-2,0) = \sigma^*(H(-2)) \to T^*\widetilde{\mathbb{C}P}^N$ be the holomorphic bundle map induced from ω by means of the coderivative of σ: $\tilde{\omega} = {}^t(D\sigma)\omega$. Then $\tilde{\omega}$ vanishes of order 2 along the exceptional divisor $K \times \mathbb{C}P^1$, and $\frac{1}{e^2}\tilde{\omega}\colon H(0,-2) \to T^*\widetilde{\mathbb{C}P}^N$ is never vanishing and it describes the non-singular foliation whose leaves are the $\tilde{f} = f \circ \sigma$ fibers.

Proof. Take coordinate charts $(z_0 : \cdots : z_N)$ of $\mathbb{C}P^N$ such that $\ell = z_0$ and $\ell' = z_1$.

A typical coordinate chart of $\widetilde{\mathbb{C}P}^N \hookrightarrow \mathbb{C}P^N \times \mathbb{C}P^1$ is

$$\mathbb{C}^N \ni (z_1, \ldots, z_{N-1}, w_0) \to (z_1 w_0 : z_1 : \cdots : z_{N-1} : 1\,;\, w_0 : 1) \in \widetilde{\mathbb{C}P}^N$$

so that the blowing up receives an expression in affine charts

$$\sigma(z_1, \ldots, z_{N-1}, w_0) = (z_1 w_0, z_1, \ldots, z_{N-1})$$

the foliation in $\mathbb{C}P^N$ is defined in $\mathbb{C}^N$ as $z_1 dz_0 - z_0 dz_1$, hence

$$\sigma^*(z_1 dz_0 - z_0 dz_1) = z_1(z_1 dw_0 + w_0 dz_1) - (z_0 w_0) dz_1 = z_1^2 dw_0$$

Since the exceptional divisor is defined by $z_1 = 0$, we have that

$\sigma^*(\omega)$ vanishes of order 2 along $K \times \mathbb{C}P^1$, and hence $\frac{1}{e^2}\,{}^t(D\sigma)\omega =$ $= \omega$: $H(-2,0) \otimes H(2,-2) \to T^*\mathbb{C}P^N$ describes a non-singular foliation, whose leaves are the $\tilde{f}$-fibers. Note that $\tilde{\omega} = D\tilde{f}$: $\tilde{f}^*T^*\mathbb{C}P^1 \to T^*\mathbb{C}P^N$. ‖

Let now M be a compact complex manifold embedded in $\mathbb{C}P^N$ $i: M \hookrightarrow \mathbb{C}P^N$ and such that it is not contained in any hyperplane. Let $f = (\ell : \ell')$ be a linear family of hyperplanes such that $K = \{\ell = \ell' = 0\}$ intersects M in a subvariety $K' = K \cap M$ of codimension 2 in M. The restriction of f to M induces a rational function on M, having indeterminacy locus on K'; and the restriction of f to $M - K'$ induces a holomorphic map $M - K' \to \mathbb{C}P^1$. Let ω: $H(-2) \to T^*\mathbb{C}P^N$ be the codimension one holomorphic foliation associated to f, $\omega = df$, and L is the line bundle $i^*(H(-2))$ on M obtained by restriction, the coderivative induces on M a codimension 1-holomorphic foliation F' defined by $\omega' = (di)^*\omega$: $L \to T^*M$. The singular set of F' is K' plus the tangency points of M with the family of hyperplanes $\{\lambda\ell + \mu\ell'\}$. The leaves of F' in M-Sing F' are the intersection of the hyperplanes $\lambda\ell + \mu\ell' = 0$ with M. We will say that *the foliation* F' *is induced in* $M \hookrightarrow \mathbb{C}P^N$ *by the family of hyperplanes* $\{\lambda\ell + \mu\ell'\}$.

EXAMPLE 3.3. Let $d > 0$ be a positive integer, and let $z_0^d, z_0^{d-1}z_1^d, \ldots, z_n^d$ be a basis of monomials of degree d in $\mathbb{C}^{n+1}$, and let ρ_d be the holomorphic map

$$\rho_d = (z_0^d : \cdots : z_n^d) : \mathbb{C}P^n \to \mathbb{C}P^N \qquad N = \binom{n+d}{n} - 1$$

ρ_d is an embedding of $\mathbb{C}P^n$, called the d-uple embedding (see [11] p. 13). A hyperplane H in $\mathbb{C}P^N$ with coordinates $\{(w_{i_0,\ldots,i_n}) = (w_I) \mid |I| = i_0 + \cdots + i_n = d\}$ is given by an equation $\sum_{|I|=d} a_I w_I = 0$; hence $\rho_d(\mathbb{C}P^n) \cap M$ is the subvariety of $\mathbb{C}P^n = \rho_d(\mathbb{C}P^n)$ defined by $\Sigma a_I z^I = \Sigma a_{i_0,\ldots,i_n} z_0^{i_0},\ldots,z_n^{i_n} = 0$, and hence there is a one to one correspondance between hypersurfaces of degree d in $\mathbb{C}P^n$ and hyperplanes in $\mathbb{C}P^N$. A linear family of hyperplanes $(\ell : \ell')$ in $\mathbb{C}P^N$ determines when intersecting with $\rho_d(\mathbb{C}P^n)$ a linear family of hypersurfaces of degree d; namely the family $\{\lambda(\ell \circ \rho_d) + \mu(\ell' \circ \rho_d)\}$ and hence $f \circ \rho_d = \frac{\ell \circ \rho_d}{\ell' \circ \rho_d}$ is a rational function on $\mathbb{C}P^n$ where numerator and denumerator are homogeneous polynomials of degree d. The condition that $K' = K \cap \rho_d(\mathbb{C}P^n)$ has codimension 2 in $\rho_d(\mathbb{C}P^n)$ is equivalent to asking that $\ell \circ \rho_d$ and $\ell' \circ \rho_d$ have no common factor.

DEFINITION 3.4. A holomorphic foliation of codimension 1 in M is a *Lefschetz pencil* if it is the foliation induced in M by a linear family of hyperplanes $\{\lambda\ell + \mu\ell'\}$ in $\mathbb{C}P^N$ via an embedding $i: M \hookrightarrow \mathbb{C}P^N$ and satisfying

1) $K = \{\ell = \ell' = 0\}$ intersects M transversely.

2) Each hyperplane $\lambda\ell + \mu\ell' = 0$ intersects M transversely, except possibly at one point, where it has a non-degenerate tangency (i.e. the holomorphic map $\frac{\ell}{\ell'}: M - K \to \mathbb{C}P^1$ has only Morse-type singularities, each with a distinct value).

The smooth submanifold K is called the *basis of the pencil*.

LEFSCHETZ THEOREM ([1]). Let M be a connectd complex submanifold of $\mathbb{C}P^N$ not contained in any hyperplane of $\mathbb{C}P^N$, then

1) There exists a Zariski dense open subset in Grass $(\mathbb{C}P^1, (\mathbb{C}P^N)^*)$ such that the holomorphic foliation they induce in M is a Lefschetz pencil.

2) If H is a hyperplane in $\mathbb{C}P^N$ intersecting transversely M, then the inclusion $M \cap H \hookrightarrow M$ induces isomorphisms between the homotopy groups $\Pi_k(M \cap H) \to \Pi_k(M)$, for $k = 0, \ldots, \dim M - 2$.

3.2. Good Meromorphic First Integrals.

Based on the Lefschetz pencils, we introduce the following definition:

DEFINTION 3.6. Let $\omega: L \to T^*M$ be a holomorphic foliation of codimension 1 in M.

1) ω has a *meromorphic first integral* if there is a meromorphic function f on M and a Zarizki dense subset U of M where $f: U \to \mathbb{C}P^1$ is holomorphic and the leaves of ω in U coincide with the connected components of the fibers of $f|_U$.

2) ω has a *good meromorphic first integral* if the singular locus $\{\omega = 0\}$ of ω has codimension bigger than 1 and if there

is a meromorphic function f on M such that

a) The indeterminacy locus K of f is a submanifold of M of codimension two, and around each point p of K we may find coordinates $(z_1, \dots, z_n)$ of an open set W in M, $p = 0$, such that the foliation in $W - K \cap W$ is described by $z_2^2 d(z_1 z_2^{-1}) = = z_2 dz_1 - z_1 dz_2$.

b) The critical points of the holomorphic map $f_1: M - K \to \mathbb{C}P^1$ obtained by restricting f to $M - K$ has codimension bigger than one (i.e. $\operatorname{cod}\{Df_1 = 0\} > 1$).

Remark. If ω has a good meromorphic first integral f and if W and $(z_1, \dots, z_n)$ are as above, then f restricted to W receives an expression of the form $f(z_1, \dots, z_n) = g(z_1 z_2^{-1})$, where $g: \mathbb{C}P^1 \to \mathbb{C}P^1$ is a rational map of $\mathbb{C}P^1$. Assumption b forces g to be a non-branched map, and so g has degree one. Hence, after a change of coordinates we actually have $f = z_1 z_2^{-1}$ in W.

LEMMA 3.6. Let $\omega: L \to T^*M$ be a holomorphic foliation of codimension 1 in M with a good meromorphic first integral f. If K is the indeterminacy locus of K, and if $\lambda_1, \dots, \lambda_r$ are the critical values in $\mathbb{C}P^1$ of f on $M - K$, then the map $f_2: M - [\bigcup_i f^{-1}(\lambda_i) - K] \to \mathbb{C}P^1 - \cup\{\lambda_i\}$ obtained by restricting f has the structure of a C^∞-fibre bundle.

Proof. Let $\sigma: \tilde{M} \to M$ be the blowing up morphism of M along K and let $\tilde{\omega} = {}^t(D\sigma)\omega: \sigma^*L \to T^*\tilde{M}$ be the holorphic bundle map induced from ω by means of the coderivative of σ. Similar as in Lemma 3.2 $\tilde{\omega}$ vanishes of order two along the exceptional divisor $\sigma^{-1}(K)$, and if $\sigma^{-1}(K)$ is defined by $e = 0$, then $\frac{1}{e^2}\tilde{\omega}$ describes a foliation $\tilde{F}$ in M. The closure of the leaves of $\tilde{F}$ coincide with the fibers of the holomorphic map $\tilde{f} = f \circ \sigma: \tilde{M} \to \mathbb{C}P^1$. By generic smoothness ([11] p. 272) there are only a finite number $\lambda_1, \dots, \lambda_r$ of critical values, and $\tilde{f}$ restricted to $\tilde{M} - \tilde{f}^{-1}\{\lambda_1, \dots, \lambda_r\}$ has the structure of a C^∞-fibre bundle. The map $\tilde{f}$ restricted to $\sigma^{-1}(K)$ is also a fibre bundle, with fiber isomorphic to K. $\tilde{f}$ restricted to $\tilde{M} - (f^{-1}\{\lambda_1, \dots, \lambda_r\} \cup \sigma^{-1}(K))$ is also a C^∞-fibre bundle, and it is biholomorphic to f_2 via σ. This proves the Lemma. ‖

We will illustrate for projective spaces some of the preceeding definitions.

LEMMA 3.7. 1) The set of rational functions $\mathbb{C}P^n \to \mathbb{C}P^1$ such that the inverse image of a generic point is a hypersurface of degree d is a projective variety of dimension $2\binom{d+n}{n} - 4$ which is naturally embedded in $\mathrm{Proj}\ E(H(-2d))$, that has dimension $\binom{2d+n-1}{2d}(2d-1) - 1$.

2) For $n = 2$, the above set of rational function has dimension $d^2 + 3d - 2$, and $\mathrm{Proj}\ E(H(-2d)) = \mathcal{F}o\ell\ (\mathbb{C}P^2, H(-2d))$ has dimension $4d^2 - 2$.

Proof. The vector space of homogeneous polynomials of degree d in $n + 1$ variables has dimension $\binom{d+n}{d}$, so the Grassman manifold of 2-planes has dimension $2\left[\binom{d+n}{d} - 2\right]$. Part 1 follows then from Proposition 1.3, since $g^2 d\left(\frac{f}{g}\right) = gdf - fdg$ has degree $2d - 1$, and its terms of top degree annhilate the radial vector field. Part 2 follows from part 1. ∥

Remark. For every $d \geqslant 1$, we obtain a subvariety of $\mathcal{F}o\ell\ (\mathbb{C}P^n, H(-2d))$. We will show elsewhere ([8]) that for $n > 2$ this is an irreducible component of $\mathcal{F}o\ell\ (\mathbb{C}P^n, H(-2d))$; for $n = 2$ it has codimension $3d^2 - 3d$. There are other foliations having meromorphic first integrals, due to cancellations in branching sets, i.e. $d(f^m/g^{m'})$.

LEMMA 3.10. 1) A foliation $\mathcal{F}$ in $\mathcal{F}o\ell\ (\mathbb{C}P^2, H(-e))$ with isolated singularities of multiplicity 1 has $e^2 - 3e + 3$ singular points.

2) If $\mathcal{F}$ is a Lefschetz pencil in $\mathcal{F}o\ell\ (\mathbb{C}P^2, H(-2d))$, then the indeterminacy locus has d^2 points, and there are $3d^2 - 6d + 3$ Morse type singular points.

Proof. The number of singular points is computed as the second Chern class of $T^*\mathbb{C}P^2 \otimes H(e)$ (see [3]), which using the Euler Sequence ([11] p. 176) tensored with $H(e)$.

$$0 \to \Omega^1_{\mathbb{C}P^n} \to H(-1)^{\oplus (n+1)} \to H(0) \to 0$$

and the properties of Chern classes ([12])

$$(1+dt)[(1+c_1(\Omega^1(d))t+c_2(\Omega^1_d))t^2] = (1+(d-1)t)^3$$

$$1+(d+c_1(\Omega^1(d))t+(c_2(\Omega^1(d))+dc_1(\Omega^1(d)) = 1+3(d-1)+3(d-1)^2t^2$$

Hence $c_1(\Omega^1(d)) = 2d-3$ and $c_2(\Omega^1(d)) = d^2-3d+3$.

To prove 2, observe that the indeterminacy locus is obtained by intersecting two elements of the pencil, which by Bezout's theorem is d^2. Since the foliation of the pencil is in $\mathcal{Fol}\ (\mathbb{C}P^2, H(-2d))$, by part 1 there are $4d^2 - 6d + 3$ singular points, so there are $3d^2 - 6d + 3$ Morse type singularities. ‖

4. THE ILIASHENKO CURVES.

In this section we define the Iliashenko curves of a foliation having a good meromorphic first integral, and show that some of these curves are conics.

4.1. The Iliashenko Curves.

Let $\omega: L \to T^*M$ be a holomorphic foliation of codimension 1 in the compact manifold M, having a good meromorphic first integral $f: M \to \mathbb{C}P^1$ and indeterminacy locus K. The fibers of f restricted to $M-K$ will be denoted by $F_\lambda = f^{-1}(\lambda)$, and we will denote by $\Lambda = \{\lambda_1, \ldots, \lambda_r\}$ the set of critical values of $f|_{M-K}$.

Let $\tau: W \to \mathbb{C}P^1 - \Lambda$ be the universal covering space of $\mathbb{C}P^1 - \Lambda$ viewed as homotopy classes of paths with fixed end points, starting in λ_0. By Lemma 3.8, the restriction of f, $f_2: M - [K \cup f^{-1}(\Lambda)] \to \mathbb{C}P^1 - \Lambda$, has the structure of a C^∞-fibre bundle, and pulling it back to W, we obtain a fibre bundle that is C^∞-bundle isomorphic to $F_{\lambda_0} \times W$ if W is biholomorphic to $\mathbb{C}$ or to the unit disc (i.e. if $r \geqslant 1$. If there are no critical values we still have $f_{2*}H_1(F_\lambda, Z) = H_1(F_\lambda, Z) \times \mathbb{C}P^1)$. Hence, given $\delta \in \Pi_1(F_{\lambda_0})$ for any element $\lambda \in W$ there is a well defined free homotopy class δ_λ obtained from δ by deforming it along the path λ to a free homotopy class in $\Pi_1(F_{\tau(\lambda)})$. Composing f_2 in the range with a Moebius transformation, we may obtain $f_2(\delta_\lambda) \neq \infty$, and we may use df_2 in $M - (K \cup f_2^{-1}(\infty))$ to apply theorem 2.9 to δ_λ, and hence the Iliashenko point of δ_λ is well defined for $\lambda \in W$.

DEFINITION 4.1. Let $\mathcal{F}$ be a holomorphic foliation of codimension 1 having a good meromorphic first integral $f: M \to \mathbb{C}P^1$ and $\delta \in \Pi_1(F_{\lambda_0})$, then the map

$$I_\delta : W \to \text{Proj } T^*_{\mathcal{F}} \, \mathcal{F}o\ell \, (M,L) \tag{4.1}$$

obtained by associating to $\lambda \in W$ the Iliashenko point of δ_λ (see definition 2.10) will be called the *Iliashenko curve* of δ.

THEOREM 4.2. Let $\mathcal{F}$ be a holomorphic foliation of codimension 1 in the compact manifold M having a good meromorphic first integral $f: M \to \mathbb{C}P^1$, then:

1) The Iliashenko curve I_δ is holomorphic, and depends only on the homology class of δ in $H_1(F_{\lambda_0}, Z)$.

2) If $\delta, \delta' \in H_1(F_{\lambda_0}, Z)$, then $I_{\delta+\delta'} = I_\delta + I_{\delta'}$.

3) There is a one to one correspondance between infinitesimal directions of deformations (i.e. given by $\omega + t(a\eta)$, $a \in \mathbb{C}^*$) and hyperplanes H_η in $\text{Proj } T^*_{\mathcal{F}} \, \mathcal{F}o\ell \, (M,L)$. If $I_\delta(W) \not\subset H_\eta$ then the set of points $I_\delta(W) \cap H_\eta$ corresponds to those homology classes δ_λ which are persistent for the infinitesimal direction η of deformation.

Proof. Part 1 follows from Proposition 2.12, and the expression of the map as an integral, which is zero for commutator paths.

Part 2 follows from the additive properties of the integral (2.6).

The equations defining a hyperplane in $\text{Proj } T^*_{\mathcal{F}} \, \mathcal{F}o\ell \, (M,L)$ belong to $\text{Hom } (T^*_{\mathcal{F}} \, \mathcal{F}o\ell \, (M,L), \mathbb{C}) = T_{\mathcal{F}} \, (\mathcal{F}o\ell \, (M,L))$, and so hyperplanes correspond to points in $\text{Proj } T_{\mathcal{F}} \, \mathcal{F}o\ell \, (M,L)$. A point λ lies in $I_\delta(W) \cap H_\eta$ if and only

$$\int_{\delta_\lambda} \eta = 0$$

Hence by Theorem 2.9 δ_λ is persistent for the infinitesimal deformation $\mathcal{F}_\eta$. ‖

4.2. The Conics Associated to the Indeterminacy locus.

Let $\mathcal{F}$ be a codimension 1 holomorphic foliation having a good meromorphic first integral $f\colon M \to \mathbb{C}P^1$ with indeterminacy locus K; and let $(z_1, \ldots, z_n)$ be coordinates of an open set W in M, with $0 \in K \cap W$ where the foliation is described by the 1-form $z_2dz_1 - z_1dz_2$. Let W' be the two dimensional manifold transversal to K defined by $z_3 = \cdots = z_n = 0$ with coordinates (z_1, z_2). The foliation $\mathcal{F}'$ in W' induced from $\mathcal{F}$ consist of lines through the origin in W'. If S_r' denotes a small sphere $|z_1|^2 + |z_2|^2 = r$, then S_r' will intersect each leaf $L_{(a:b)} = \{t(a,b) \in W' \mid t \in \mathbb{C}^*\}$ in a closed loop $\delta_{(a:b)}$. Since we may take f to be $z_1z_2^{-1}$ in W', then it is clear that the family of loops $\delta_{(a:b)}$ are obtained from the other by deformation inside the leaves of $\mathcal{F}$. Since the critical points of f are far from W', we also see that there is no monodromy around the critical values for these loops, and that they may be extended to loops over the critical values. This implies that the Iliashenko curve is defined as a map with domain $\mathbb{C}P^1 - \Lambda$, since there is no monodromy around the critical values for $\delta_{(a:b)}$, and by continuity of the formula (2.6) it extends also to the points of Λ. Hence the Iliashenko curve is described by a holomorphic map

$$I_\delta\colon \mathbb{C}P^1 \to \operatorname{Proj} T^*_{\mathcal{F}}\, \mathcal{Fol}\, (M,L) \tag{4.2}$$

We will now give an expression for this map. To do this, let $\omega = z_2dz_1 - z_1dz_2$ define $\mathcal{F}$ in W and let $\eta_1, \ldots, \eta_N$ be the holomorphic 1-forms in W obtained from a basis of $T_{\mathcal{F}}\, \mathcal{Fol}\, (M,L)$ using the trivialization of L given by ω. Since

$$d\left(\frac{z_1}{az_1 + bz_2}\right) = \frac{b}{(az_1 + bz_2)^2}\,(z_2dz_1 - z_1dz_2) = \frac{b\omega}{(az_1 + bz_2)^2}$$

it follows that dividing ω by $(az_1 + bz_2)^2$ we make ω a closed 1-form. Choose coordinates so that $a = 0$, $b = 1$ (to simplify notation), and in the trivialization of L on $W_1 - \{z_2 = 0\}$ induced by the closed 1-form $z_2^{-2}\omega$, a basis of $T_{\mathcal{F}}\, \mathcal{Fol}\, (M,L)$ is $z_2^{-2}\eta_1, \ldots, z_2^{-2}\eta_N$. Expand η_k in power series

$$\eta_k = \sum_{i,j \geqslant 0} a_{ij} z_1^i z_2^j dz_1 + \sum_{i,j \geqslant 0} b_{ij} z_1^i z_2^j dz_2 \tag{4.3}$$

The form $z_2^{-2}\eta_k$ restricted to the line $t \mapsto t(a,b)$ has the expression in t

$$b^{-2}t^{-2}\left(\sum_{i,j\geq 0} (a_{ij}a^{i+1}b^j + b_{ij}a^i b^{j+1})t^{i+j}dt\right)$$

and hence the integral of $z_2^{-2}\eta_k$ along the loop $\delta_{(a:b)}$ is equal by the Residue Theorem to :

$$\int_{\delta_{(a:b)}} \eta_k = 2\Pi i b^{-2} \sum_{i+j=1} (a_{ij}a^{i+1}b^j + b_{ij}a^i b^{j+1}) =$$

$$= \frac{2\Pi i}{b^2}[a_{10}a^2 + (a_{01} + b_{10})ab + b_{01}b^2] \qquad (4.4)$$

Repeating a similar calculation for the other forms η_k, the map (4.2) has an expression

$$(a:b) \mapsto (c_1a^2 + c_1'ab + c_1''b^2 : \cdots : c_Na^2 + c_N'ab + c_N''b^2)$$

We have proved:

PROPOSITION 4.3. Let δ be a loop going once around one of the connected components of the indeterminacy locus, then either:

1) The Iliashenko curve of δ is not defined (namely, the integrals in (2.3) vanish for any $\delta_{(a:b)}$).

2) The Iliashenko curve degenerates to a point (namely, the linear maps in (2.4) for any $\delta_{(a:b)}$ are constant multiples one of the others).

3) The Iliashenko curve of δ is a smooth conic in Proj $T^*_{\mathcal{F}}$ $\mathcal{F}o\ell$ (M,L).

We may interpret the conic Q corresponding to the above loop δ as follows. Let H_η be a hyperplane in Proj $T^*_{\mathcal{F}}$ $\mathcal{F}o\ell$ (M,L) defined by the infinitesimal deformation η in $T_{\mathcal{F}}$ $\mathcal{F}o\ell$ (M,L). If $Q \cap H_\eta$ consists of two points, this is selecting for us two fibers of f, such that when deforming in the direction of η the corresponding two loops are persisting. The singularity of $\mathcal{F}$ at K is an example of a Kupka phenomena (see [8]). The Kupka phenomena is persistent in the sense that if $\{\mathcal{F}_t\}_{t\in\mathbb{C}}$ is a 1-parameter deformation of $\mathcal{F}$ there is a smooth submanifold K_s of codimension 2 near K formed of singular points of $\mathcal{F}_t$, $\mathcal{F}_t \cap W' = \mathcal{F}_t'$ is a foliation by curves in the complex surface W' with an isolated singularity at

$K_t \cap W' = \{p_t\}$ and the foliation $\mathcal{F}_t$ in a neighbourhood of K_t is locally modelled on the foliation $\mathcal{F}'_t$ in W' near p_t product a disc in $\mathbb{C}^{n-2}$. Hence we may restrict to W' and analyse the family $\mathcal{F}'_t$. Assume that $\mathcal{F}'_t$ is defined by

$$(z_2 dz_1 - z_1 dz_2) + t(\Sigma a_{ij} z_1^i z_2^j dz_1 + \Sigma b_{ij} z_1^i z_2^j dz_2) + t^2(\cdots) \qquad (4.5)$$

with first order part (4.3). The vector field

$$X_t = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + t\left\{\begin{pmatrix} -b_{00} \\ a_{00} \end{pmatrix} + \begin{pmatrix} -b_{10} & -b_{01} \\ a_{10} & a_{01} \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \cdots\right\} + t^2(\cdots)$$

is also determining $\mathcal{F}_t$. Assume that the linear part of the first order variation has distinct eigenvalues. Then by a linear change of coordinates in $W' = \mathbb{C}^2$ we may put this linear part in diagonal form, so that $a_{10} = b_{01} = 0$ and $-b_{10} \neq a_{01}$. The linear part of X_t at p_t is

$$\begin{pmatrix} 1 - b_{10}t & 0 \\ 0 & 1 + a_{01}t \end{pmatrix} + t^2(\quad)$$

and hence the eigenspaces of X_t at p_t for $t \neq 0$ small are approximating the z_1 and z_2 axis. Note that in this coordinates the vanishing of the integral in (4.4) is $ab = 0$, hence it is also detecting the two axes.

Formulating the conclusion in intrinsec terms: Assume that the deformation $\mathcal{F}_t$ is such that the linear part of the first order variation of the transversal model (4.5) of the Kupka phenomena along K_t of $\mathcal{F}_t$ has distinct eigenvalues, then the eigendirections of the linear parts of X_t tend to the directions at p specified by the solutions of (4.4).

If one further assumes that the quotient of the above eigenvalues is a non-real complex number, then by Pincaré's Linearization Theorem we may conclude that there are two separatrix manifolds passing through K_t, for $t \neq 0$, and tending t to 0 they are approaching the fibers $F_{(a:b)}$ of f, with $(a:b)$ the points of intersection $Q \cap H_\eta$.

We will now see that if the indeterminacy locus K of a good meromorphic first integral $f: M \to \mathbb{C}P^1$ has at least two irreducible components, then we have an infinite number of Iliashenko curves that are conics. This is the case if M has dimension 2, where

K consists of a finite number of points. By Lemma 3.10, the indeterminacy locus of a Lefschetz pencil in $\mathcal{Fol}\ (\mathbb{C}P^2, H(-2d))$ consists of d^2 points.

Let δ and δ' be loops around distinct connected components of K and let $\delta'' = m\delta + n\delta'$ with $m,n \in Z$. Since the monodromy action around the critical values of δ and δ' is trivial, we obtain that it acts trivially also on δ'' (as a 1-cycle in homology). In $M - (K \cup F_\infty)$ the foliation is defined by the closed holomorphic 1-form df, and we obtain as in Theorem 2.9 maps

$$I, I', I'': \mathbb{C}P^1 - \{\infty\} \to T_F\ \mathcal{Fol}\ (M,L)$$

whose projectivisation are the Iliashenko curves associated to δ, δ' and δ''. By the defining formula (2.3), we have $I'' = mI + nI'$. From (4.4) we see that I'' has a quadratic expression in a/b; hence the Iliashenko curve $Q_{m,n}$ is also a conic (if non-degenerate, as in Proposition 4.3). Since $\frac{m}{n} I + \frac{m'}{n'} I' = \frac{1}{mn'} [mn'I + m'nI']$ observe that if the conics $Q_{1,0}$ and $Q_{0,1}$ are non degenerate and distinct, then the closure of $\{Q_{m,n} \mid m,n \in Z\}$ is the image of $Q_{1,0} \times \mathbb{R}P^1$. Similar constructions holds for any 1-homology class in the subgroup of $H_1(F_0, Z)$ generated by the loops around the indeterminacy locus of f.

We finish by observing that the formalism presented here allows one to extend the results in Iliashenko's work [13] to rational functions. A polynomial function is not a good first integral since it is branching on the line at infinity. Using the techniques developed in [7], one may extend the formalism of this paper allowing the complex structure of M and L to vary. This line of approach will be continued in [14].

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