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Spaces determined by selections ☆

Michael Hrušák *, Iván Martínez-Ruiz

Instituto de Matemáticas, UNAM, Apartado Postal 61-3, Xangari, 58089, Morelia, Michoacán, Mexico

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ABSTRACT

A function $\psi : [X]^2 \to X$ is a called a *weak selection* if $\psi(\{x, y\}) \in \{x, y\}$ for every $x, y \in X$. To each weak selection ψ, one associates a topology τ_ψ, generated by the sets $(\leftarrow, x) = \{y \neq x\colon\ \psi(x, y) = y\}$ and $(x, \rightarrow) = \{y \neq x\colon\ \psi(x, y) = x\}$. Answering a question of S. García-Ferreira and A.H. Tomita [S. García-Ferreira, A.H. Tomita, A non-normal topology generated by a two-point selection, Topology Appl. 155 (10) (2008) 1105–1110], we show that (X, τ_ψ) is completely regular for every weak selection ψ. We further investigate to what extent the existence of a continuous weak selection on a topological space determines the topology of X. In particular, we answer two questions of V. Gutev and T. Nogura [V. Gutev, T. Nogura, Selection problems for hyperspaces, in: E. Pearl (Ed.), Open Problems in Topology 2, Elsevier B.V., 2007, pp. 161–170].

1. Introduction

E. Michael initiated the study of continuous selections in 1951 with his seminal paper [7]. He considered the hyperspace 2^X of all non-empty closed subsets of X, equipped with the *Vietoris topology*, i.e. the topology on 2^X generated by sets of the form

$$\langle U; V_0, \dots, V_n\rangle = \{F \in 2^X\colon\ F \subseteq U \text{ and } F \cap V_i \neq \emptyset \text{ for any } i \leqslant n\},$$

where $U, V_0, \dots, V_n$ are open subsets of X.

A function ψ defined on $[X]^2$, the collection of all subsets of X with exactly two points, such that $\psi(\{x, y\}) \in \{x, y\}$ for every $x, y \in X$ is called a *weak selection on* X. A weak selection is *continuous* if it is continuous with respect to the Vietoris topology on $[X]^2$, treating $[X]^2$ as a subspace of 2^X.

The general question studied in Michael's article, and many subsequent articles, is: *When does a space admit a continuous weak selection?* In his paper, E. Michael has proved that every space that admits a weaker topology generated by a linear order, i.e. that the space is *weakly orderable*, also admits a continuous weak selection. The natural question whether the converse is also true, implicit in Michael's paper, was stated explicitly by van Mill and Wattel in [8]: *Is every space that admits a continuous weak selection weakly orderable?*

We have recently answered this question in the negative by constructing a separable, first countable locally compact space X which admits a continuous weak selection but is not weakly orderable [6].

In this paper we investigate to what extent the existence of a continuous weak selection on a topological space determines the topology of X. We show that for every weak selection ψ on a set X, the topology τ_ψ induced by ψ is Tychonoff,

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* Corresponding author.
E-mail addresses: michael@matmor.unam.mx (M. Hrušák), ivan@matmor.unam.mx (I. Martínez-Ruiz).

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answering a question of S. García-Ferreira and A.H. Tomita [1]. We introduce the notion of a space *determined by selections* and its weak and strong form. We study these classes of spaces. In particular, we answer two questions of V. Gutev and T. Nogura [5]. We conclude with some open problems.

All spaces considered here are at least Hausdorff. The set-theoretic and topological notation used is standard, possibly with one exception, we denote by $f''A$ the forward image of a set A via a function f.

Given a weak selection on a set X and $x, y \in X$, we write $y \to x$ (or equivalently $x \leftarrow y$) if $\psi(x, y) = x$. Some authors use the notation $x \leqslant_\psi y$ to denote $x \leftarrow y$ or $x = y$ (see [7], for example). If $A \subseteq X$ and $B \subseteq X$, we write $A \rightrightarrows B$ whenever $a \to b$ for every $a \in A$ and $b \in B$, and we say that $A \parallel B$ if $A \rightrightarrows B$ or $B \rightrightarrows A$.

It is well known that the relation $\leqslant_\psi$ is reflexive and antisymmetric but, in general, it is not transitive. However, as in the case of an order, it induces a topology on X. Indeed, for every $x \in X$, consider the following sets:

$$(\leftarrow, x)_\psi = \{z \in X\colon z \leftarrow x\},$$

$$(x, \to)_\psi = \{z \in X\colon x \leftarrow z\}.$$

We denote by τ_ψ the topology generated by sets of the form $(\leftarrow, x)_\psi$ and $(x, \to)_\psi$, $x \in X$, and call it the *topology generated* by the weak selection ψ.

Analogously, we introduce the following notation:

$$(\leftarrow, x]_\psi = \{x\} \cup (\leftarrow, x)_\psi,$$

$$[x, \to)_\psi = \{x\} \cup (x, \to)_\psi,$$

$$(x, y)_\psi = (x \to)_\psi \cap (\leftarrow, y)_\psi, \quad \text{and}$$

$$[x, y]_\psi = [x \to)_\psi \cap (\leftarrow, y]_\psi.$$

2. Topological properties of τ_ψ

Topologies generated by weak selections were studied in [4]. In particular, the following result holds:

Proposition 2.1. *([4]) Let ψ be a weak selection defined on X. Then (X, τ_ψ) is a regular space.*

In the same paper, the authors ask if (X, τ_ψ) is always normal. This question was recently answered in the negative:

Example 2.2. ([1]) There is a weak selection ψ defined on $\mathbb{P}$, the set of irrational numbers, such that $(\mathbb{P}, \tau_\psi)$ is not normal.

This example is not normal but it is Tychonoff. Motivated by this observation, the original question was reformulated in [1] as follows:

Question 2.3. Are there a set X and a weak selection ψ on X such that the space (X, τ_ψ) is not Tychonoff?

In order to answer this question in the negative, let us first analyze an immediate consequence of the existence of special triples on X with respect to a given weak selection.

Given x, y, z in X and ψ a weak selection on X, we say that the triple $\{x, y, z\}$ is a 3-*cycle* with respect to ψ if $x \to y \to z \to x$ (or $x \leftarrow y \leftarrow z \leftarrow x$).

Notice that if a set X does not admit 3-cycles with respect to a weak selection ψ, then the relation $\leqslant_\psi$ induced by ψ is transitive and the space (X, τ_ψ) is orderable.

On the other hand, every 3-cycle naturally determines a clopen partition of X, as the following proposition shows. This observation appears in [6], we present the simple proof here for the sake of completeness.

Proposition 2.4. *Let ψ be a weak selection on a set X and let $x, y, z \in X$ be such that $\{x, y, z\}$ is a 3-cycle with respect to ψ. Then there is a (canonical) partition $\mathcal{P}$ of X so that $|\mathcal{P}| \leqslant 5$, P is τ_ψ-clopen and $|\{x, y, z\} \cap P| \leqslant 1$ for every $P \in \mathcal{P}$.*

Proof. Assume that $x \to y \to z \to x$. Consider the following sets:

$$P_0 = (y, z)_\psi,$$

$$P_1 = (z, x)_\psi,$$

$$P_2 = (x, y)_\psi,$$

$$P_3 = (\leftarrow, x)_\psi \cap (\leftarrow, y)_\psi \cap (\leftarrow, z)_\psi, \quad \text{and}$$

$$P_4 = (x, \to)_\psi \cap (y, \to)_\psi \cap (z, \to)_\psi.$$

It is easy to see that $\mathcal{P} = \{P_i\colon\ i < 5\}$ is a partition of X and, clearly, P_i is open (hence clopen) for every $i < 5$. Also, $x \in P_0$, $y \in P_1$ and $z \in P_2$. $\square$

Given a space X, we denote by C_x the *quasicomponent* of x on X and by C_x^* the *component* of x:

$$C_x = \bigcap\{C \subseteq X\colon\ C \text{ is clopen and } x \in C\},$$
$$C_x^* = \bigcup\{C \subseteq X\colon\ C \text{ is connected and } x \in C\}.$$

The following result is due to Gutev and Nogura.

Lemma 2.5. *([2]) Let ψ be a weak selection on a set X. If $x \in X$ and $y, z \in C_x$, where C_x is the τ_ψ-quasicomponent of x, then $[y, z]_\psi \subseteq C_x$.*

Proof. Suppose that $y, z \in C_x$ are such that $y \leftarrow z$. If there is a $w \in [y, z]_\psi \setminus C_x$ then, since $w \notin C_x$, we can find a clopen subset $V \subseteq X$ with $x \in V$ (and so $C_x \subseteq V$) and $w \notin V$. Then the clopen set $W = V \cap (\leftarrow, w]_\psi = V \cap (\leftarrow, w)_\psi$ is such that $y \in W$ and $z \notin W$, which is a contradiction. $\square$

In a similar way, it is also proved in [2] that $[y, z]_\psi$ must be connected, and so $C_x = C_x^*$, i.e. C_x is connected.

Lemma 2.6. *Let $x \neq y \in X$ and let ψ be a weak selection on X such that $x \leftarrow y$. Then there are τ_ψ-continuous functions $f : X \to [0, 1]$ and $g : X \to [0, 1]$ such that:*

(1) $f(x) = 1$ *and* $f''[y, \to)_\psi = \{0\}$,
(2) $g(y) = 1$ *and* $g''(\leftarrow, x]_\psi = \{0\}$.

Proof. We will prove (1), the proof of (2) is completely analogous. There are two possible cases:

Case 1: There is a clopen $C \subseteq X$ such that $x \in C$ and $y \notin C$.

In this case, let $U = C \cap (\leftarrow, y)_\psi$. Notice that also $U = C \cap (\leftarrow, y]_\psi$ and so U is a clopen subset containing x. Define $f : X \to [0, 1]$ by $f(z) = 1$ if $z \in U$ and $f(z) = 0$ otherwise.

Case 2: For every $C \subseteq X$ clopen, $x \in C$ if and only if $y \in C$.

Notice first that, by Lemma 2.5, the point x determines a finite partition $\mathcal{P}$ of X, which consists of the closed connected subset C_x and two open subsets: $U_0 = \{z \in X \setminus C_x\colon\ C_x \rightrightarrows \{z\}\}$ and $U_1 = \{z \in X \setminus C_x\colon\ \{z\} \rightrightarrows C_x\}$. The idea of the proof will be to first define the desired continuous function on a particular closed subset of C_x containing x and y and to finally extend it to the whole space.

Consider the quasicomponent C_x. By Proposition 2.4, $\leqslant_\psi \restriction (C_x \times C_x)$ is a transitive relation since, as otherwise, there would be a $z \in C_x$ and $C \subseteq C_x$ clopen such that $x \in C$ and $z \notin C$, which is not possible. Therefore C_x, as a subspace of (X, τ_ψ), is a connected orderable space. In particular, C_x is normal and $[x, y]_\psi$, being a closed subset of C_x, is normal also.

Let $h : [x, y]_\psi \to [0, 1]$ be a continuous function such that $h(x) = 1$ and $h(y) = 0$.

Finally, define $f : X \to [0, 1]$ by

$$f(u) = \begin{cases} 1, & \text{if } u \in (\leftarrow, x]_\psi, \\ h(u), & \text{if } u \in [x, y]_\psi, \\ 0, & \text{if } u \in [y, \to)_\psi. \end{cases}$$

The function f is well defined because $(\leftarrow, x]_\psi \cap [x, y]_\psi = \{x\}$, $[x, y]_\psi \cap [y, \to)_\psi = \{y\}$ and $(\leftarrow, x]_\psi \cap [y, \to)_\psi = \emptyset$. Moreover, since f is continuous on each of these τ_ψ-closed sets, it is continuous on X. $\square$

Theorem 2.7. *Let ψ be a weak selection on a set X. Then (X, τ_ψ) is Tychonoff.*

Proof. Let $x \in X$ and let U be a basic neighbourhood of x. Then there are $z_0, \ldots, z_n \in X \setminus \{x\}$, for some $n \in \omega$, such that $U = \bigcap\{U_i\colon\ i \leqslant n\}$, where $U_i = (z_i, \to)_\psi$ if $z_i \leftarrow x$ and $U_i = (\leftarrow, z_i)$ otherwise. By Lemma 2.6, for every $i \leqslant n$ we can find a continuous function $f_i : X \to [0, 1]$ such that $f_i(x) = 1$ and $f_i''[X \setminus U_i] = \{0\}$. Let $f : X \to [0, 1]$ be defined by $f = \prod\{f_i\colon\ i \leqslant n\}$. Clearly, f is continuous and $f(x) = 1$. If $z \notin U$ then $z \notin U_i$ for some $i \leqslant n$ and so $f_i(z) = 0$, which implies that $f(z) = 0$. Therefore, $f''(X \setminus U) = \{0\}$. We conclude that (X, τ_ψ) is Tychonoff. $\square$

3. Topologies generated by selections

The first result that establishes a relationship between a (continuous) weak selection defined on a space and the topology this selection generates is the following:

Proposition 3.1 *([3]). Let ψ be a continuous weak selection on a Hausdorff space (X,τ). Then $\tau_\psi \subseteq \tau$.*

As mentioned above, the answer to van Mill and Wattel's question is negative, i.e. there is a space X which admits a continuous weak selection but which is not weakly orderable. One might ask, whether this question has a positive answer assuming that there is a closer relationship between the original topology on X and the topology generated by the weak selection on X. Motivated by this, we introduce the following definitions.

Definition 3.2. Let (X,τ) be a topological space. We say that:

(1) X is *weakly determined by selections* (***wDS***) if there is a weak selection ψ on X so that $\tau = \tau_\psi$.
(2) X is *determined by selections* (***DS***) if there is a continuous weak selection ψ on X so that $\tau = \tau_\psi$.
(3) X is *strongly determined by selections* (***sDS***) if X is **DS** and $\tau = \tau_\psi$ for every continuous weak selection ψ on X.

Given a weak selection ψ on a space (X,τ), it is not always true that ψ is τ_ψ-continuous, even when ψ is τ-continuous [3]. On the other hand, the next result states that if there is a coarser topology on a given set so that a weak selection defined on it is continuous, this topology must be precisely the topology determined by the weak selection itself. This answers Question 7 of Gutev and Nogura [5] in the negative.

Proposition 3.3. *Let ψ be a weak selection on a set X. Then τ_ψ is the intersection of all Hausdorff topologies τ on X such that ψ is τ-continuous.*

In particular, there exists the coarsest topology τ^* on X such that ψ is τ^*-continuous if and only if ψ is τ_ψ-continuous, and then $\tau^* = \tau_\psi$.

Proof. Since τ_ψ is contained in any Hausdorff topology on X for which the weak selection ψ is continuous, if we consider the topology:

$$\tau^* = \bigcap\{\tau\colon\ \tau \text{ is a Hausdorff topology on } X \text{ and } \psi \text{ is } \tau\text{-continuous}\},$$

we have that $\tau_\psi \subseteq \tau^*$. We only need to prove that $\tau^* \subseteq \tau_\psi$.

For $x \in X$, define the set:

$$\mathfrak{N}_x = \{U \subseteq X\colon\ x \in U \text{ and } U \text{ is } \tau_\psi\text{-open}\}.$$

For every $x \in X$, let τ_x be the topology on X generated by $\mathfrak{N}_x \cup \{\{y\}\colon\ y \in X \setminus \{x\}\}$. Let $y \in X \setminus \{x\}$ and, without loss of generality, suppose that $x \leftarrow y$. Then $U_x = (\leftarrow, y)_\psi$ and $U_y = \{y\}$ are disjoint τ_x-open neighbourhoods of x and y respectively, and so τ_x is Hausdorff. Moreover, since $\{y\} \rightrightarrows U_x$, the weak selection ψ is τ_x-continuous.

Therefore, $\tau^* \subseteq \bigcap\{\tau_x\colon\ x \in X\}$. However, this implies that $\tau^* \subseteq \tau_\psi$. □

Now we turn our attention to **DS** spaces. Any orderable space is a **DS** space: The selection min determines the order topology. The next example shows that orderability is not a necessary condition.

Denote by $\mathbb{R}_l$ the *Sorgenfrey line*, i.e. the real numbers $\mathbb{R}$ equipped with the topology τ_l given by the basis

$$\mathcal{B} = \big\{[a,b)\colon\ a,b \in \mathbb{R},\ a < b\big\},$$

and by $\mathbb{R}_l^*$ the topological space on the real line having as basis the collection:

$$\mathcal{B}^* = \big\{(a,b]\colon\ a,b \in \mathbb{R},\ a < b\big\},$$

which is, of course, homeomorphic to $\mathbb{R}_l$.

It is well known that $\mathbb{R}_l$ is a suborderable space which is not orderable.

The next result states that the topology on the Sorgenfrey line can be determined by a continuous weak selection defined on it.

Example 3.4. $\mathbb{R}_l$ is a suborderable **DS** space which is not orderable.

Proof. Let $X = \bigcup\{X_n \times \{n\}\colon\ n \in \omega\}$, where $X_n = \mathbb{R}_l$ if n is odd and $X_n = \mathbb{R}_l^*$ if n is even. Notice that $(\mathbb{R}_l, \tau_l) \cong (X,\tau)$, where τ is the topology of disjoint sum. Define $\psi : [X]^2 \to X$ as follows:

$\psi(\{(x,n),(y,m)\}) = (x,n)$ if and only if one of the following occurs:

(1) $x < y$ and $|n - m| \leqslant 1$,
(2) $x = y$, $n = 2k + 1$ for some $k \in \omega$ and $|n - m| = 1$,
(3) $m - n > 2$, or
(4) $n - m = 2$.

Let us first prove that $\tau \subseteq \tau_\psi$.

Fix $n \in \omega$ and let $x, y \in \mathbb{R}$ be such that $x < y$. Let

$$U = \big((x, n+1), \rightarrow\big)_\psi \cap \big(\leftarrow, (y, n+1)\big)_\psi \cap \big(\leftarrow, (x, n+3)\big)_\psi.$$

Then $U = (x, y] \times \{n\}$ if n is odd and $U = [x, y) \times \{n\}$ if n is even. This proves that $\tau \subseteq \tau_\psi$ and, in particular, $X_n \times \{n\}$ is τ_ψ-clopen for every $n \in \omega$.

To prove that $\tau_\psi \subseteq \tau$, it is enough to verify that ψ is τ-continuous. For this, let $(x, n), (y, m) \in X$ be such that $(x, n) \neq (y, m)$ and $\psi((x, n), (y, m)) = (x, n)$. There are three possible cases:

Case 1: $n = m$.

Let $z \in \mathbb{R}$ be such that $x < z < y$. Then $U = (x - 1, z) \times \{n\}$ and $V = (z, y + 1) \times \{n\}$ are disjoint τ-open neighbourhoods of (x, n) and (y, n) respectively such that $V \rightrightarrows U$. Therefore, ψ is continuous at $\{(x, n), (y, n)\}$.

Case 2: $|n - m| = 1$.

If $x < y$ then continuity is verified as in Case 1. If $x = y$ then n is odd and m is even. In this case, $U = (x - 1, x] \times \{n\}$ and $V = [x, x + 1) \times \{m\}$ are τ-neighbourhoods of (x, n) and (y, m) respectively, with $V \rightrightarrows U$, which implies continuity of ψ on $\{(x, n), (y, n)\}$.

Case 3: $|n - m| > 1$.

$U = X_n \times \{n\}$ and $V = X_m \times \{m\}$ are neighbourhoods of (x, n) and (y, m) with $V \rightrightarrows U$. □

It is also easy to see that suborderable spaces do not have to be **DS**.

Example 3.5. $X = (0, 1) \cup \{2\}$, as subspace of $\mathbb{R}$, is suborderable but not a **DS** space.

Proof. If ψ is a continuous weak selection on X then note that $\psi \restriction [(0, 1)]^2$ must be either the weak selection min or the weak selection max. Without loss of generality, let us suppose that $\tau \restriction [(0, 1)]^2 = \min$. If there is a point $z \in (0, 1)$ so that $\psi(z, 2) = z$ then $\psi(z, 2) = z$ for all $x \in (0, 1)$ and so $(X, \tau_\psi) \cong (0, 1]$.

On the other hand, if $(z, 2) = 2$ for some $z \in (0, 1)$ then $\{2\} \rightrightarrows (0, 1)$, which implies that $(X, \tau_\psi) \cong [0, 1)$. In any case, $(X, \tau_\psi) \neq (X, \tau)$. □

In an earlier version of this article, we asked if every **wDS** space must be weakly orderable and if every normal **wDS** space is **DS**. As pointed out by the referee, the following example answers both questions in the negative.

Example 3.6. Let $X = \{(x, 0) \in \mathbb{R}^2\colon\ x \in [-1, 1]\} \cup \{(0, \frac{1}{n}) \in \mathbb{R}^2\colon\ n \in \omega \setminus \{0\}\}$ with the subspace topology. Define $\psi : [X]^2 \to X$ by

(1) $\psi(\{(x, 0), (y, 0)\}) = (\min(x, y), 0)$,
(2) $\psi(\{(0, \frac{1}{n}), (0, \frac{1}{m})\}) = (0, \max\{n, m\})$, and
(3) $\psi(\{(x, 0), (0, \frac{1}{n})\}) = (x, 0)$ if and only if $x \leqslant 0$.

Then τ_ψ is the usual topology on X, as a subspace of $\mathbb{R}^2$, but X does not admit a continuous weak selection.

As far as **sDS** spaces are concerned, every weakly orderable **sDS** space is, in fact, orderable. On the other hand, every compact **DS** space is **sDS**. It is also true that every connected locally connected **DS** space is **sDS** (see [9]). The following question was asked in [5].

Question 3.7. Is there a non-compact **sDS** that is neither connected nor locally connected?

The following example answers this question in the affirmative.

Example 3.8. There is a **sDS** space which is neither compact nor locally compact nor connected nor locally connected.

Proof. Let $X = \bigcup\{U_n\colon n \in \omega\}$, where $U_0 = (-1, 0]$ and $U_n = (\frac{1}{n+1}, \frac{1}{n})$ for every $n > 0$, with the subspace topology.

The space X is obviously not compact or connected and it is neither locally compact or locally connected at the point 0.

Clearly X is a **DS** space (the weak selection min induces the topology on X). To prove that X is **sDS**, let ψ be a continuous weak selection on X.

For any $n, m \in \omega$, $U_n \parallel U_m$ and either $\psi \restriction [U_n]^2 = \min$ or $\psi \restriction [U_n]^2 = \max$. Notice that if $x \in X \setminus \{0\}$ and U is an open neighbourhood of x, then there are $a, b \in X$ such that $x \in (\leftarrow, b)_\psi \cap (a, \rightarrow)_\psi \subseteq U$. Therefore, we only need to prove that any basic open neighbourhood of 0 in X contains an open τ_ψ-neighbourhood of it.

Let $U = (a, b) \cap X$ be an open neighbourhood of 0 and suppose that $\psi(a, 0) = a$ (the case when $\psi(a, 0) = 0$ is completely analogous). We can also suppose that $b = \frac{1}{n}$ for some $n \in \omega$ and, by continuity of ψ, that $(U \setminus U_0) \rightrightarrows U_0$. Let $F = \{0 < k < n\colon U_k \rightrightarrows U_0\}$. If F is empty then, for every $k < n$, $\{a\} \rightrightarrows U_k$, which guarantees that $(a, \rightarrow)_\psi \cap (\leftarrow, b)_\psi \subseteq (a, b)$ and, in this case, $W = (a, \rightarrow)_\psi \cap (\leftarrow, b)_\psi$ is as desired. We can suppose then that F is non-empty.

Notice that $U_k \rightrightarrows \{0\}$ for every $k \in F$ and so, by continuity of ψ, we can find an $m > n$ such that $\bigcup\{U_k\colon k \in F\} \rightrightarrows \bigcup\{U_s\colon s > m\}$. Let $z \in U_{m+1}$ and consider the neighbourhood $W = (a, \rightarrow)_\psi \cap (\leftarrow, z)_\psi$.

As $z \in U \setminus U_0$, W is an open τ_ψ-neighbourhood of 0. If $x \in X \setminus U$ then either $x \in (0, a]$ or $x \in U_k$ for some $k < n$. In the first case, $\psi(x, a) = x$ and then $x \notin W$. Otherwise, if $x \in U_k$ for some $k \notin F$ then, since $U_0 \rightrightarrows U_k$, $\psi(x, a) = x$ and again $x \notin W$. Finally, if $x \in U_k$ for some $k \in F$ then $\psi(x, z) = z$ and then $x \notin W$. We conclude that $W \subseteq U$ and so $\tau_\psi = \tau$. □

We conclude with some open problems.

Question 3.9. Is every **DS** space weakly orderable?

Question 3.10. Is every **DS** space normal?

Question 3.11. Is there a characterization of **DS** spaces in terms of an orderability property?

Question 3.12. Is every **sDS** space orderable?

Question 3.13. Let X be a non-compact **sDS** space. Is then the set $\{x\colon X$ is locally connected at $x\}$ dense in X?

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