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ORDERING MAD FAMILIES A LA KATĚTOV

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Abstract. An ordering ($\leq_K$) on maximal almost disjoint (MAD) families closely related to destructibility of MAD families by forcing is introduced and studied. It is shown that the order has antichains of size $\mathfrak{c}$ and decreasing chains of length $\mathfrak{c}^+$ bellow every element. Assuming $\mathfrak{t} = \mathfrak{c}$ a MAD family equivalent to all of its restrictions is constructed. It is also shown here that the Continuum Hypothesis implies that for every ω^ω-bounding forcing $\mathbb{P}$ of size $\mathfrak{c}$ there is a Cohen-destructible, $\mathbb{P}$-indestructible MAD family. Finally, two other orderings on MAD families are suggested and an old construction of Mrówka is revisited.

§1. Introduction. In this note we dust off an old ordering on ideals (filters) introduced by M. Katětov in [16]. It will be used to classify MAD (maximal almost disjoint) families on a countable set. One reason for doing this is to develop a comprehensive structural theory of MAD families, similar to that studied for ultrafilters, and another reason is to try to better understand the general question of destructibility of MAD families by forcing, aiming to wards a solution of an old problem, sometimes attributed to J. Roitman, of whether the existence of a dominating family of size ω_1 implies the existence of a MAD family of size ω_1.

Recall that an infinite family $\mathscr{A} \subseteq [\omega]^\omega$ is an *almost disjoint* (AD) family if every two distinct elements of $\mathscr{A}$ have finite intersection and it is maximal (MAD) if it is maximal with that property. It is an old result of Sierpiński [26] that there is a MAD family of size continuum.

If $\mathscr{A}$ is a MAD family then $\mathscr{I}(\mathscr{A})$ denotes the ideal of all subsets of ω which can be almost covered by finitely many elements of $\mathscr{A}$, $\mathscr{I}^+(\mathscr{A}) = \mathscr{P}(\omega) \setminus \mathscr{I}(\mathscr{A})$ denotes the family of sets of positive measure. The following definition can be found in [16], there formulated for filters. We use the (dual) language of ideals as it is more suitable in the given context.

DEFINITION 1.1. Let $\mathscr{I}, \mathscr{J}$ be ideals on ω. Let $\mathscr{I} \leq_K \mathscr{J}$ if there is a function $f : \omega \longrightarrow \omega$ such that $f^{-1}[I] \in \mathscr{J}$ for every $I \in \mathscr{I}$. If $\mathscr{A}$ and $\mathscr{B}$ are MAD families then we write $\mathscr{A} \leq_K \mathscr{B}$ instead of $\mathscr{I}(\mathscr{A}) \leq_K \mathscr{I}(\mathscr{B})$.

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We will refer to the ordering as *Katětov ordering*. When restricted to maximal ideals (dually ultrafilters) the ordering coincides with the Rudin-Keisler order. Just to point out a subtle difference recall the definition of the Rudin-Keisler order on ideals (see e.g., [11]) $\mathscr{I} \leq_{RK} \mathscr{J}$ if there is a function $f : \omega \longrightarrow \omega$ such that $f^{-1}[I] \in \mathscr{J}$ if and only if $I \in \mathscr{I}$.

Given a forcing notion $\mathbb{P}$ a MAD family $\mathscr{A}$ is $\mathbb{P}$-*indestructible* if $\mathscr{A}$ remains MAD after forcing with $\mathbb{P}$. This is obviously equivalent to $\mathbb{P}$ not diagonalizing the dual filter $\mathscr{I}^*(\mathscr{A})$ (not adding a pseudo-intersection to $\mathscr{I}^*(\mathscr{A})$, i.e., not adding an infinite set almost contained in all elements of $\mathscr{I}^*(\mathscr{A})$). If a MAD family is not $\mathbb{P}$-indestructible we say that it is $\mathbb{P}$-*destructible*. The definitions extend to all proper ideals $\mathscr{I}$ on ω. All ideals considered in this paper are proper, contain *Fin* (the ideal of all finite subsets of ω) and are *tall*, i.e., $\mathscr{I}^*$ does not have a pseudo-intersection. Note that for an AD family $\mathscr{A}$, $\mathscr{I}(\mathscr{A})$ is tall if and only if $\mathscr{A}$ is MAD if and only if $\mathscr{I}(\mathscr{A}) \not\leq_K Fin$. The following easy proposition shows the relevance of the Katětov ordering to the question of destructibility of MAD families.

Proposition 1.2. *Let $\mathbb{P}$ be a forcing notion and let $\mathscr{I}$, $\mathscr{J}$ be ideals on ω. If $\mathscr{I} \leq_K \mathscr{J}$ and $\mathbb{P}$ diagonalizes $\mathscr{J}$ then $\mathbb{P}$ diagonalizes $\mathscr{I}$.*

Proof. Let f be a witness to $\mathscr{I} \leq_K \mathscr{J}$ and let $A \in [\omega]^\omega \cap V[G]$ be a set diagonalizing $\mathscr{J}$ (G denotes a filter $\mathbb{P}$-generic over V). Let $B = f[A]$. First note that B is infinite. To see that B diagonalizes $\mathscr{I}$ assume the contrary, i.e., there is an $I \in \mathscr{I} \cap [B]^\omega$. As $f^{-1}[I] \in \mathscr{J}$, that would, however, contradict the assumption that $[A]^\omega \cap \mathscr{J} \subseteq Fin$. ⊣

Note that the converse of the above proposition does not hold (see the remark following Proposition 2.5). In [15] it is shown that the hierarchy of MAD families is stratified using natural ideals on ω. The paper [15] contains slight but serious mistakes which are rectified in the upcoming paper [7] of Brendle and Yatabe. In these two papers it is shown that for many nicely definable forcing notions $\mathbb{P}$ such as Cohen, Sacks, Miller there are corresponding ideals $\mathscr{I}_{\mathbb{P}}$ on ω such that a tall ideal $\mathscr{I}$ is $\mathbb{P}$-destructible if and only if $\mathscr{I} \leq_K \mathscr{I}_{\mathbb{P}}$. A particular instance of this is the following theorem used later on in the text.

Theorem 1.3 ([15]). *A MAD family $\mathscr{A}$ is Cohen-indestructible if and only if $\mathscr{A} \not\leq_K$ nwd, where nwd denotes the ideal of nowhere dense subsets of the rationals.*

The set-theoretic notation used here is mostly standard and follows [17]. For the definitions of the cardinal invariants and forcing notions consult e.g., [30] and [2].

§2. Elementary properties of the Katětov order on MAD families. When looking at the Katětov order restricted to MAD families there is a particular segment which is interesting. Recall that $Fin \times Fin = \{A \subseteq \omega \times \omega : \{n \in \omega : \{m \in \omega : (n,m) \in A\} \notin Fin\} \in Fin\}$. Throughout the text, given a set $A \subseteq \omega$, e_A denotes the function increasingly enumerating A.

Proposition 2.1. *For every MAD family $\mathscr{A}$, $Fin \leq_K \mathscr{A} \leq_K Fin \times Fin$.*

Proof. $Fin \leq_K \mathscr{A}$ is trivially true. To see that $\mathscr{A} \leq_K Fin \times Fin$ choose $\{A_n : n \in \omega\} \subseteq \mathscr{A}$ so that $\bigcup\{A_n : n \in \omega\} =^* \omega$ and let $B_i = A_i \setminus \bigcup_{j<i} B_j$. Define $f : \omega \times \omega \longrightarrow \omega$ by putting $f((n,m)) = e_{B_n}(m)$. Such f is easily seen to be a witness to $\mathscr{A} \leq_K Fin \times Fin$. ⊣

Recall that a MAD family $\mathscr{A}$ *refines* a MAD family $\mathscr{B}$ ($\mathscr{A} \prec \mathscr{B}$) if $\forall A \in \mathscr{A}$ $\exists B \in \mathscr{B}$ $A \subseteq B$. Given a MAD family $\mathscr{A}$ and a set $X \in \mathscr{I}^+(\mathscr{A})$ let $\mathscr{A} \restriction X = \{A \cap X : A \in \mathscr{A} \text{ and } |A \cap X| = \aleph_0\}$. Note that $\mathscr{A} \restriction X$ is a MAD family of subsets of X. Let $\mathscr{A}$ be a MAD family and let $\langle \mathscr{B}_A : A \in \mathscr{A}\rangle$ be a collection of (not necessarily distinct) MAD families. Then let

$$\Sigma_{\mathscr{A}}\langle \mathscr{B}_A : A \in \mathscr{A}\rangle = \{e_A[B] : A \in \mathscr{A}, B \in \mathscr{B}_A\}.$$

if $\mathscr{B}_A = \mathscr{B}$ for every $A \in \mathscr{A}$ denote $\Sigma_{\mathscr{A}}\langle \mathscr{B}_A : A \in \mathscr{A}\rangle$ by $\mathscr{B} \otimes \mathscr{A}$. It is easy too see that $\Sigma_{\mathscr{A}}\langle \mathscr{B}_A : A \in \mathscr{A}\rangle$ is a MAD family. Note that:

(1) If $\mathscr{A} \prec \mathscr{B}$ then $\mathscr{A} \leq_K \mathscr{B}$,
(2) $\mathscr{A} \leq_K \mathscr{A} \restriction X$ for every $X \in \mathscr{I}^+(\mathscr{A})$,
(3) $\Sigma_{\mathscr{A}}\langle \mathscr{B}_A : A \in \mathscr{A}\rangle \prec \mathscr{A}$ and
(4) $\Sigma_{\mathscr{A}}\langle \mathscr{B}_A : A \in \mathscr{A}\rangle \leq_K \mathscr{B}_A$ for every $A \in \mathscr{A}$.

PROPOSITION 2.2. *Every collection of at most $\mathfrak{c}$-many MAD families has a common $\leq_K$-lower bound.*

PROOF. Let $\{\mathscr{A}_\alpha : \alpha < \kappa \leq \mathfrak{c}\}$ be a collection of MAD families. Let $\mathscr{A} = \{A_\alpha : \alpha < \mathfrak{c}\}$ be a MAD family of size $\mathfrak{c}$. Let $\mathscr{B}_{A_\alpha} = \mathscr{A}_\alpha$, for $\alpha < \kappa$ and let $\mathscr{B}_{A_\alpha}$ be arbitrary otherwise. Put $\mathscr{A}' = \Sigma_{\mathscr{A}}\langle \mathscr{B}_A : A \in \mathscr{A}\rangle$. It is obvious that $\mathscr{A}'$ is the required lower bound. ⊣

PROPOSITION 2.3. *For every MAD family $\mathscr{A}$ there is a MAD family $\mathscr{B}$ such that $\mathscr{B} \prec \mathscr{A}$ (i.p. $\mathscr{B} \leq_K \mathscr{A}$) and $\mathscr{A} \not\leq_K \mathscr{B}$.*

PROOF. Given $\mathscr{A}$, let $\mathscr{A}' \prec \mathscr{A}$ be a MAD family of size $\mathfrak{c}$. List all elements of $\mathscr{A}'$ as $\{A_\alpha : \alpha < \mathfrak{c}\}$ and also enumerate $\omega^\omega = \{f_\alpha : \alpha < \mathfrak{c}\}$. For every $\alpha < \mathfrak{c}$ choose $B_\alpha \in \mathscr{I}(\mathscr{A})$ a subset of $f_\alpha[A_\alpha]$ such that $|f_\alpha^{-1}[B_\alpha] \cap A_\alpha| = \aleph_0$; if $f_\alpha[A_\alpha]$ is finite let $B_\alpha = f_\alpha[A_\alpha]$ and if $f_\alpha[A_\alpha]$ is infinite, by maximality of $\mathscr{A}$ there is an $A \in \mathscr{A}$ such that $A \cap f_\alpha[A_\alpha]$ is infinite and then let $B_\alpha = A \cap f_\alpha[A_\alpha]$. Now, let $\mathscr{B}_\alpha$ be any MAD family of subsets of A_α such that $f_\alpha^{-1}[B_\alpha] \cap A_\alpha \notin \mathscr{I}(\mathscr{B}_\alpha)$. Finally, let $\mathscr{B} = \bigcup_{\alpha<\mathfrak{c}} \mathscr{B}_\alpha$. It follows from the construction that $\mathscr{B}$ is a MAD family and that $\mathscr{B} \prec \mathscr{A}$.

Now assume that f is a witness to $\mathscr{A} \leq_K \mathscr{B}$. Then f is listed as f_α for some $\alpha < \mathfrak{c}$. Then, however, $B_\alpha \in \mathscr{I}(\mathscr{A})$ and $f^{-1}[B_\alpha] \notin \mathscr{I}(\mathscr{B})$ which is a contradiction. ⊣

COROLLARY 2.4. There is a strictly decreasing chain of length $\mathfrak{c}^+$ below every MAD family $\mathscr{A}$ in the Katětov order.

PROOF. It is easy to construct the chain by induction of length $\mathfrak{c}^+$, at isolated steps using Proposition 2.3 and at limit steps using Proposition 2.2. ⊣

PROPOSITION 2.5. *There is a collection of $\mathfrak{c}$-many pairwise Katětov incomparable MAD families $\leq_K$-below every MAD family $\mathscr{A}$.*

PROOF. Let $\mathscr{B}$ be a refinement of $\mathscr{A}$ of cardinality $\mathfrak{c}$. Enumerate $\mathscr{B}$ as $\{B_\xi : \xi < \mathfrak{c}\}$. Enumerate also $\omega^\omega \times \mathfrak{c}$ as $\{(f_\xi, i_\xi) : \xi < \mathfrak{c}\}$. Inductively choose ordinals β_ξ, γ_ξ and a MAD family $\mathscr{B}_\xi$ of subsets of B_{γ_ξ} so that:

(1) $f_\xi^{-1}[B_{\beta_\xi}] \cap B_{\gamma_\xi}$ is infinite,
(2) if $\beta_\xi \notin \{\gamma_\eta : \eta < \xi\}$ let $\mathscr{B}_\xi$ be a MAD family of subsets of B_{γ_ξ} such that $f_\xi^{-1}[B_{\beta_\xi}] \cap B_{\gamma_\xi} \in \mathscr{I}^+(\mathscr{B}_\xi)$,

(3) if $\beta_\xi = \gamma_\eta$ for some $\eta < \xi$ let $\mathscr{B}_\xi$ be a MAD family of subsets of B_{γ_ξ} such that there is a $B \in \mathscr{B}_{\gamma_\eta}$ such that $f_\xi^{-1}[B] \cap B_{\gamma_\xi} \in \mathscr{I}^+(\mathscr{B}_\xi)$.

(4) $\gamma_\xi \notin \{\gamma_\eta, \beta_\eta : \eta < \xi\}$.

Having done this let, for $\alpha, \xi < \mathfrak{c}$,

$$\mathscr{C}_\xi^\alpha = \begin{cases} \mathscr{B}_\xi & \text{if } \xi = \gamma_\eta \text{ and } \alpha = i_\eta, \\ \{B_\xi\} & \text{otherwise.} \end{cases}$$

Then let

$$\mathscr{A}_\alpha = \bigcup_{\xi < \mathfrak{c}} \mathscr{C}_\xi^\alpha.$$

It is immediate from the construction that $\mathscr{A}_\alpha$ is a MAD family and is a refinement of $\mathscr{B}$. Hence $\mathscr{A}_\alpha \leq_K \mathscr{A}$, for every $\alpha < \mathfrak{c}$.

All that is left to verify is that for distinct $\alpha, \alpha' < \mathfrak{c}$, $\mathscr{A}_\alpha \not\leq_K \mathscr{A}_{\alpha'}$. Assume, that it is not true, i.e., there are distinct $\alpha, \alpha' < \mathfrak{c}$ and an $f \in \omega^\omega$ such that f witnesses that $\mathscr{A}_\alpha \leq_K \mathscr{A}_{\alpha'}$. Then, however, the pair (f, α') is listed as (f_ξ, i_ξ) and it follows from the construction that there is a $B \in \mathscr{A}_\alpha$ ($B = B_{\beta_\xi}$ or $B \in \mathscr{B}_{\gamma_\eta}$ for some $\eta < \xi$ with $\beta_\xi = \gamma_\eta$) such that $f^{-1}[B] \notin \mathscr{I}(\mathscr{B}_\xi)$ so $f^{-1}[B] \notin \mathscr{I}(\mathscr{A}_{\alpha'})$. ⊣

Call two MAD families $\mathscr{A}$, $\mathscr{B}$ *equidestructible* if for every forcing notion $\mathbb{P}$ $\mathscr{A}$ is $\mathbb{P}$-indestructible if and only if $\mathscr{B}$ is $\mathbb{P}$-indestructible. All Sacks-destructible MAD families are, in fact, destroyed by any forcing adding reals (see [7]) and as a result are equidestructible. Also, Sacks-destructible MAD families exist in ZFC, for instance the classical construction of Sierpiński produces a Sacks-destructible MAD family. Proposition 2.5 therefore shows that there are MAD families which are equidestructible, yet K-incomparable.

Two MAD families $\mathscr{A}, \mathscr{B}$ are *K-equivalent* ($\mathscr{A} \simeq_K \mathscr{B}$) if $\mathscr{A} \leq_K \mathscr{B}$ and $\mathscr{B} \leq_K \mathscr{A}$. Let $T_K(\mathscr{A}) = \{\mathscr{B} : \mathscr{B} \simeq_K \mathscr{A}\}$, $P_K(\mathscr{A}) = \{\mathscr{B} : \mathscr{B} \leq_K \mathscr{A}\}$ and $S_K(\mathscr{A}) = \{\mathscr{B} : \mathscr{B} \geq_K \mathscr{A}\}$.

LEMMA 2.6. *$\mathscr{A} \leq_K \mathscr{B}$ implies $|\mathscr{B}| \leq |\mathscr{A}|$*

PROOF. Let f witness $\mathscr{A} \leq_K \mathscr{B}$. Let for $A \in \mathscr{A}$, $F_A = \{B \in \mathscr{B} : |f^{-1}[A] \cap B| = \aleph_0\}$. Note that F_A is finite for every $A \in \mathscr{A}$ and $\mathscr{B} = \bigcup\{F_A : A \in \mathscr{A}\}$. ⊣

PROPOSITION 2.7. *Let $\mathscr{A}$ be a MAD family. Then*:

(1) $|T_K(\mathscr{A})| = |S_K(\mathscr{A})| = 2^{|\mathscr{A}|}$ *and*

(2) $|P_K(\mathscr{A})| = 2^{\mathfrak{c}}$.

PROOF. To see that $2^{|\mathscr{A}|} \leq |T_K(\mathscr{A})|$ split every $A \in \mathscr{A}$ into two infinite subsets B_A^0, B_A^1. For $f \in 2^{\mathscr{A}}$ let $\mathscr{B}_f = \{A : f(A) = 0\} \cup \{B_A^0, B_A^1 : f(A) = 1\}$. All $\mathscr{B}_f$ are K-equivalent to $\mathscr{A}$, the identity being a witness.

$|T_K(\mathscr{A})| \leq |S_K(\mathscr{A})|$ follows directly from the definition. For $|S_K(\mathscr{A})| \leq 2^{|\mathscr{A}|}$ use Lemma 2.6; If $\mathscr{B} \in S_K(\mathscr{A})$ then $|\mathscr{B}| \leq |\mathscr{A}|$, so there is an $f : \mathscr{A} \longrightarrow \mathscr{P}(\omega)$ such that $\mathscr{B} = rng(f)$. Hence $|S_K(\mathscr{A})| \leq |(\mathscr{P}(\omega))^{\mathscr{A}}| \leq 2^{|\mathscr{A}|}$.

Clause (2) follows easily from clause (1) and the fact that every MAD family has a refinement of size $\mathfrak{c}$. ⊣

It follows from Proposition 2.3 that there are no K-minimal MAD families. We do not know whether there are (can be) K-maximal MAD families. We will show next that, at least consistently, there is no K-largest MAD family. Recall that $\mathfrak{b}$

denotes the unboundedness number (see [30]), $\mathfrak{a}$ denotes the minimal cardinality of a MAD family, and that $\mathfrak{b} \leq \mathfrak{a} \leq \mathfrak{c}$.

PROPOSITION 2.8. ($\mathfrak{b} = \mathfrak{c}$) *For every MAD family $\mathscr{A}$ there is a MAD family $\mathscr{B}$ such that $\mathscr{A}$ and $\mathscr{B}$ are K-incomparable.*

PROOF. Enumerate ω^ω as $\{f_\alpha : \alpha < \mathfrak{c}\}$. We will construct $\mathscr{B}$ as a union of an increasing chain of almost disjoint families $\mathscr{B}_\alpha$, $\alpha < \mathfrak{c}$ by induction so that

(1) $\mathscr{B}_0$ is a partition of ω into infinitely many infinite sets,
(2) $|\mathscr{B}_\alpha| = |\alpha + \omega|$ for every $\alpha < \mathfrak{c}$,
(3) $(\forall \alpha < \mathfrak{c})(\exists A \in \mathscr{A})(|\{B \in \mathscr{B}_{\alpha+1} : |f_\alpha^{-1}[A] \cap B| = \aleph_0\}| = \aleph_0)$ and
(4) $(\forall \alpha < \mathfrak{c})(\exists B \in \mathscr{B}_{\alpha+1})(f_\alpha^{-1}[B] \notin \mathscr{I}(\mathscr{A}))$.

It is obvious that then $\mathscr{B} = \bigcup\{\mathscr{B}_\alpha : \alpha < \mathfrak{c}\}$ is as required.

To see that we can proceed with the construction assume that $\mathscr{B}_\beta$ have been defined for every $\beta < \alpha$. For α limit let $\mathscr{B}_\alpha = \bigcup_{\beta<\alpha} \mathscr{B}_\beta$. If $\alpha = \gamma + 1$ consider f_γ. As $\mathfrak{a} \geq \mathfrak{b}$, the family $\mathscr{B}_\gamma$ is not maximal, i.e., there is an infinite subset C of ω almost disjoint from all elements of $\mathscr{B}_\gamma$. Choose $A \in \mathscr{A}$ such that $|f_\gamma^{-1}[A] \cap C| = \aleph_0$ and partition the set $f_\gamma^{-1}[A] \cap C$ into infinitely many infinite sets $\{D_i : i \in \omega\}$.

If there is a $B \in \mathscr{B}_\gamma \cup \{D_i : i \in \omega\}$ such that $f_\gamma^{-1}[B] \notin \mathscr{I}(\mathscr{A})$, let $\mathscr{B}_\alpha = \mathscr{B}_\gamma \cup \{D_i : i \in \omega\}$. If $f_\gamma^{-1}[B] \in \mathscr{I}(\mathscr{A})$ for every $B \in \mathscr{B}_\gamma \cup \{D_i : i \in \omega\}$, let

$$\{A_i : i \in \omega\} \subseteq \mathscr{A} \setminus \{A \in \mathscr{A} : (\exists B \in \mathscr{B}_\gamma \cup \{D_i : i \in \omega\})(|f_\gamma^{-1}[B] \cap A| = \aleph_0)\}$$

This can be done as $|\mathscr{A}| = \mathfrak{c}$ and $|\{A \in \mathscr{A} : (\exists B \in \mathscr{B}_\gamma \cup \{D_i : i \in \omega\})(|f_\gamma^{-1}[B] \cap A| = \aleph_0)\}| < \mathfrak{c}$. Let $\{C_i : i \in \omega\}$ be a disjoint refinement of $\{f_\gamma[A_i] : i \in \omega\}$. Let for every $B \in \mathscr{B}_\gamma \cup \{D_i : i \in \omega\}$ and $i \in \omega$, $g_B(i) = \max(C_i \cap B) + 1$. Note that this is well-defined as $C_i \cap B$ is finite for every $i \in \omega$ and every $B \in \mathscr{B}_\gamma \cup \{D_i : i \in \omega\}$. Now, as $\mathfrak{b} = \mathfrak{c}$, there is an $f \in \omega^\omega$ which eventually dominates all g_B, $B \in \mathscr{B}_\gamma \cup \{D_i : i \in \omega\}$. Let

$$E = \bigcup_{i\in\omega}(C_i \setminus f(i)).$$

Then $E \cap B$ is finite for every $B \in \mathscr{B}_\gamma \cup \{D_i : i \in \omega\}$ and $f_\gamma^{-1}[E] \notin \mathscr{I}(\mathscr{A})$ as $f_\gamma^{-1}[E] \cap A_i$ is infinite for every $i \in \omega$. Set $\mathscr{B}_\alpha = \mathscr{B}_\gamma \cup \{D_i : i \in \omega\} \cup \{E\}$. Conditions (1)-(4) are then obviously satisfied. ⊣

§3. Special MAD families. MAD families with special properties have been studied in the literature. In this section we consider these properties in connection with the Katětov order. In our context the most important properties are those of forcing indestructibility (see [15, 7]).

Another property of MAD families was investigated in [13] in connection with topological games on Fréchet spaces. Recall that $T \subseteq \omega^{<\omega}$ is a *tree* if for every $s \in T$ and every $t \subseteq s$, $t \in T$. A tree T is *$\mathscr{I}^+(\mathscr{A})$-branching* if $succ_T(t) = \{n \in \omega : t^\frown n \in T\} \in \mathscr{I}^+(\mathscr{A})$ for every $t \in T$. Finally, $[T] = \{f \in \omega^\omega : \forall n \in \omega\ f \restriction n \in T\}$ denotes the set of branches through T.

A MAD family $\mathscr{A}$ is *+-Ramsey* if for every $\mathscr{I}^+(\mathscr{A})$-branching tree $T \subseteq \omega^{<\omega}$ there is a branch $f \in [T]$ such that $rng(f) \in \mathscr{I}^+(\mathscr{A})$. Among other things it is shown in [13] that +-Ramsey MAD families exist if $cov(\mathscr{M}) = \mathfrak{c}$.

A property closely related to Cohen-indestructible MAD families was introduced in [20]. A MAD family $\mathscr{A}$ is *tight* (in terminology of [20] and [18] $\aleph_0$-*MAD*) if $(\forall \langle I_n : n \in \omega \rangle \subseteq \mathscr{I}^+(\mathscr{A}))(\exists A \in \mathscr{I}(\mathscr{A}))(\forall n \in \omega)\ |A \cap I_n| = \aleph_0$. Next proposition provides a useful characterization of tight MAD families.

Proposition 3.1. *A MAD family $\mathscr{A}$ is tight if and only if $(\forall f : \mathbb{Q} \longrightarrow \omega)$ $(\exists A \in \mathscr{I}(\mathscr{A}))$ $(f^{-1}[A]$ is either dense or has non-empty interior$)$.*

Proof. Let $\{U_n : n \in \omega\}$ be an enumeration of a basis for the topology on $\mathbb{Q}$.

Let $\mathscr{A}$ be tight and let $f : \mathbb{Q} \longrightarrow \omega$ be given. Put $I_n = f[U_n]$ for every $n \in \omega$. If there is an $n \in \omega$ such that $I_n \in \mathscr{I}(\mathscr{A})$ then we are done as $f^{-1}[I_n] \supseteq U_n$, hence has a non-empty interior. If $I_n \notin \mathscr{I}(\mathscr{A})$ for every $n \in \omega$ then, as $\mathscr{A}$ is tight, there is an $A \in \mathscr{I}(\mathscr{A})$ such that $|A \cap I_n| = \aleph_0$ for every $n \in \omega$. This implies that $f^{-1}[A] \cap U_n \neq \emptyset$, so $f^{-1}[A]$ is dense. This finishes the proof of the left to right implication.

For the other direction let $\mathscr{A}$ be such that $(\forall f : \mathbb{Q} \longrightarrow \omega)$ $(\exists A \in \mathscr{I}(\mathscr{A}))$ $(f^{-1}[A]$ is either dense or has non-empty interior), and let $\langle I_n : n \in \omega \rangle \subseteq \mathscr{I}^+(\mathscr{A})$ be given. Without loss of generality we can assume that $I_n \cap I_m = \emptyset$ for distinct $m, n \in \omega$. Let $\{A_n : n \in \omega\}$ be a partition of $\mathbb{Q}$ into non-empty open intervals. Let $\{C^n_m : m \in \omega\}$ be a disjoint refinement of the family $\{U_k : U_k \subseteq A_n\}$ for every $n \in \omega$, and let $\{J^n_m : m \in \omega\}$ be a partition of I_n such that $J^n_m \in \mathscr{I}^+(\mathscr{A})$ for every $n, m \in \omega$. Now let $f : \mathbb{Q} \longrightarrow \omega$ be such that $f \restriction C^n_m$ is a bijection between C^n_m and J^n_m for every $m, n \in \omega$. Using the property of $\mathscr{A}$, there is an $A \in \mathscr{I}(\mathscr{A})$ such that $f^{-1}[A]$ is either dense or has non-empty interior. If $f^{-1}[A]$ is dense then $|f^{-1}[A] \cap A_n| = \aleph_0$ for every $n \in \omega$ which implies that $|A \cap I_n| = \aleph_0$ for every $n \in \omega$. To finish the proof it is enough to show that the other alternative leads to a contradiction. So, assume that $f^{-1}[A]$ has non-empty interior for some $A \in \mathscr{I}(\mathscr{A})$. Then there are $n, m \in \omega$ such that $C^n_m \subseteq f^{-1}[A]$. This, in turn, implies that $J^n_m \subseteq A$ which contradicts the assumption that $A \in \mathscr{I}(\mathscr{A})$. ⊣

Corollary 3.2. Let $\mathscr{A}$ be a tight MAD family. Then:

(1) ([20, 18]) $\mathscr{A}$ is Cohen-indestructible,
(2) $\forall \mathscr{B}$ MAD $\mathscr{A} \leq_K \mathscr{B}$ implies that $\mathscr{B}$ is tight, and
(3) $\mathscr{A}$ is $+$-Ramsey.

Proof. (1) and (2) follow directly from Theorem 1.3 and Proposition 3.1.

In order to prove (3) let $T \subseteq \omega^{<\omega}$ be an $\mathscr{I}^+(\mathscr{A})$-branching tree. Let $I^0_t = succ_T(t)$ for every $t \in T$. As $\mathscr{A}$ is tight, there is an $A_0 \in \mathscr{I}(\mathscr{A})$ intersecting each I^0_t in an infinite set. Let $B_0 = \bigcup\{A \in \mathscr{A} : |A \cap A_0| = \aleph_0\}$. Note that $B_0 \in \mathscr{I}(\mathscr{A})$. Having defined B_i, let $I^{i+1}_t = I^i_t \setminus B_i$. By tightness of $\mathscr{A}$ there is an A_{i+1} intersecting each I^{i+1}_t in an infinite set and let $B_{i+1} = \bigcup\{A \in \mathscr{A} : |A \cap A_{i+1}| = \aleph_0\}$. This procedure produces a family $\{B_i : i \in \omega\} \subseteq \mathscr{I}(\mathscr{A})$ of disjoint sets such that:
a) $\forall t \in T\ \forall i \in \omega\ |B_i \cap succ_T(t)| = \aleph_0$ and
b) $\forall B \subseteq \omega$ if B intersects each B_i in an infinite set then $B \in \mathscr{I}^+(\mathscr{A})$.
Now we are ready to construct a branch $b \in [T]$. Let $\langle m, n \rangle = 2^m(2n+1) - 1$ be the standard pairing function. Construct b by induction so that $b(\langle m, n \rangle) \in succ_T(b \restriction \langle m, n \rangle) \cap B_m$. Then $|rng(b) \cap B_m| = \aleph_0$ for every $m \in \omega$, hence by b) $rng(b) \in \mathscr{I}^+(\mathscr{A})$. ⊣

The following nice fact from [18] shows that the existence of a tight MAD family follows from the existence of a Cohen-indestructible MAD family, i.p. they exist assuming $\mathfrak{b} = \mathfrak{c}$ ([15, 18]). For the reader's convenience we provide a proof here (slightly different from the proof contained in [18]).

PROPOSITION 3.3 ([18]). *If $\mathscr{A}$ is a Cohen-indestructible MAD family then there is an $X \in \mathcal{I}^+(\mathscr{A})$ such that $\mathscr{A} \restriction X$ is a tight MAD family.*

PROOF. First note that as every element of $\mathcal{I}^+(\mathscr{A})$ can be partitioned into infinitely many elements of $\mathcal{I}^+(\mathscr{A})$, a MAD family $\mathscr{A}$ is tight if and only if for every partition $\langle I_n : n \in \omega \rangle \subseteq \mathcal{I}^+(\mathscr{A})$ there is an $A \in \mathcal{I}(\mathscr{A})$ such that $A \cap I_n$ is non-empty for every $n \in \omega$. We will work with this alternative definition.

Assume that $\mathscr{A} \restriction X$ is not tight for any $X \in I^+(\mathscr{A})$. We will show that $\mathscr{A}$ is Cohen-destructible. Construct a family $\langle X_\sigma : \sigma \in \omega^{<\omega} \rangle$ so that

(1) $X_\emptyset = \omega$,
(2) $X_\sigma \in I^+(\mathscr{A})$ for every $\sigma \in \omega^{<\omega}$,
(3) $\langle X_{\sigma^\frown n} : n \in \omega \rangle$ is a partition of X_σ for every $\sigma \in \omega^{<\omega}$,
(4) $(\forall \sigma \in \omega^{<\omega})\ (\forall A \in \mathcal{I}(\mathscr{A}))\ (\exists n \in \omega)\ A \cap X_{\sigma^\frown n} = \emptyset$ and
(5) $(\forall n \neq m \in \omega)\ (\exists \sigma \in \omega^{<\omega})\ |X_\sigma \cap \{n, m\}| = 1$.

This is very easy to do given the assumption that $\mathscr{A} \restriction X$ is not tight for any $X \in I^+(\mathscr{A})$. Note that declaring the sets X_σ an open base produces a topology on ω which is homeomorphic to the rationals $\mathbb{Q}$. By (4), every $A \in \mathcal{I}(\mathscr{A})$ is a nowhere dense set in this topology. Hence, by Theorem 1.3, $\mathscr{A}$ is Cohen-destructible. ⊣

Note that Proposition 3.3 together with Corollary 3.2 (part (3)) show that there is a +-Ramsey MAD family assuming $\mathfrak{b} = \mathfrak{c}$, which provides a partial answer to a question from [13].

Next we will show another condition sufficient for the existence of a tight MAD family. Recall the definition of the guessing principle $\diamondsuit(\mathfrak{d})$ from [23]:

For every Borel $F : 2^{<\omega_1} \to \omega^\omega$ there is a $g : \omega_1 \to \omega^\omega$ such that for every $f : \omega_1 \to 2$ the set $\{\alpha < \omega_1 : F(f \restriction \alpha) \leq^* g(\alpha)\}$ is stationary.

Recall that a function $F : 2^{<\omega_1} \to \omega^\omega$ is Borel if $F \restriction 2^\delta$ is Borel for every $\delta < \omega_1$. It was shown in [23] that $\diamondsuit(\mathfrak{d})$ holds in many models of $\mathfrak{d} = \aleph_1$ including the Sacks model.

PROPOSITION 3.4. *$\diamondsuit(\mathfrak{d})$ implies that there is a tight MAD family of size $\aleph_1$.*

PROOF. Fix for every infinite ordinal $\delta < \omega_1$ a bijection $e_\delta : \omega \longrightarrow \delta$. Using suitable coding, the function F will be defined on triples $(\langle A_\gamma : \gamma < \delta \rangle, \langle I_n : n \in \omega \rangle, B)$ such that $\{A_\gamma : \gamma < \delta\} \cup \{B\}$ is an almost disjoint family of infinite subsets of ω and $\langle I_n : n \in \omega \rangle$ is a sequence of infinite subsets of ω such that each I_n intersects infinitely many A_γ in an infinite set.[1] Let $\langle A_\gamma : \gamma < \delta \rangle, \langle I_n : n \in \omega \rangle, B$ be given. If the triple $(\langle A_\gamma : \gamma < \delta \rangle, \langle I_n : n \in \omega \rangle, B)$ does not satisfy the above requirements set $F((\langle A_\gamma : \gamma < \delta \rangle, \langle I_n : n \in \omega \rangle, B))(n) = 0$ for all $n \in \omega$, otherwise let

$$F(\langle A_\gamma : \gamma < \delta \rangle, \langle I_n : n \in \omega \rangle, B)(n) = \min\{k \in \omega : B \cap A_{e_\delta(n)} \subseteq k \text{ and } (\forall i \leq n)(|A_{e_\delta(n)} \cap I_i| = \aleph_0 \Rightarrow (k \cap A_{e_\delta(n)} \cap I_i) \setminus \bigcup_{i<n} A_{e_\delta(i)} \neq \emptyset)\}.$$

[1] The coding in this case is quite simple: for $t \in 2^\delta$ ($\delta = \omega^\alpha$ for some $\alpha > \omega$) let:

(1) $n \in B$ iff $t(n) = 1$,
(2) $n \in I_m$ iff $t(\omega \cdot (m+1) + n) = 1$ and
(3) $n \in A_\gamma$ iff $t(\omega \cdot (\omega + \gamma) + n) = 1$.

It is easy to verify that F is a Borel function. Suppose that $g : \omega_1 \to \omega^\omega$ is a $\diamondsuit$-sequence for F. Define an almost disjoint family $\{A_\delta : \delta < \omega_1\}$ by recursion as follows. Let $\{A_i : i \in \omega\}$ be a partition of ω into infinite sets. At stage $\delta \geq \omega$ let

$$A_\delta = \omega \setminus \bigcup_{n\in\omega} (A_{e_\delta(n)} \setminus g(\delta)(n))$$

if the set $\omega \setminus \bigcup_{n\in\omega}(A_{e_\delta(n)} \setminus g(\delta)(n))$ is infinite. Otherwise, pick A_δ arbitrary almost disjoint from $A_\gamma, \gamma < \delta$.

Clearly, $\mathscr{A} = \{A_\delta : \delta < \omega_1\}$ is an almost disjoint family. In order to see that it is maximal and that it is tight, let $B \subseteq \omega$ and let $\langle I_n : n \in \omega\rangle$ be such that for every $i \in \omega$ there are infinitely many $\delta < \omega_1$ such that $|A_\delta \cap I_i| = \aleph_0$. If B is almost disjoint from A_γ for all $\gamma < \omega_1$. Let $\alpha < \omega_1$ be such that $|A_\beta \cap I_i| = \aleph_0$ for infinitely many $\beta < \alpha$ and pick a $\delta \geq \alpha$ such that $g(\delta)$ eventually dominates $F(\langle A_\gamma : \gamma < \delta\rangle, B)$. Then $A_\delta = \omega \setminus \bigcup_{n\in\omega}(A_{e_\delta(n)} \setminus g(\delta)(n)) \supseteq^* \omega \setminus \bigcup_{n\in\omega}(A_{e_\delta(n)} \setminus F(\langle A_\gamma : \gamma < \delta\rangle, \langle I_n : n \in \omega\rangle, B)(n))$, in particular, $B \subseteq^* A_\delta$ and $|A_\delta \cap I_i| = \aleph_0$ for every $i \in \omega$. $\dashv$

The notion of a tight MAD family suggests the following natural weakening: Call a MAD family $\mathscr{A}$ *weakly tight* provided that $(\forall \langle I_n : n \in \omega\rangle \subseteq \mathscr{I}^+(\mathscr{A}))(\exists A \in \mathscr{I}(\mathscr{A}))(\exists^\infty n \in \omega)\ |A \cap I_n| = \aleph_0$. Recall that, given a MAD family $\mathscr{A}$, a set $X \in [\omega]^\omega$ is a *partitioner of* $\mathscr{A}$ if for every $A \in \mathscr{A}$ either $A \subseteq^* X$ or $A \cap X =^* \emptyset$. A partitioner X is *trivial* if either $X \in \mathscr{I}(\mathscr{A})$ or $\omega \setminus X \in \mathscr{I}(\mathscr{A})$. It is easy to see that the equivalence classes (mod *fin*) of partitioners form a subalgebra of $\mathscr{P}(\omega)/fin$. Denote it by $Part(\mathscr{A})$. The algebra $Part(\mathscr{A})/\mathscr{I}(\mathscr{A})$ is the *partitioner algebra* of $\mathscr{A}$. For more on partitioner algebras see e.g., [5, 9] or [8].

Proposition 3.5. *If $\mathscr{A}$ is weakly tight then $\mathscr{A} \not\leq_K \mathscr{A} \otimes \mathscr{A}$.*

Proof. First note that if $\mathscr{A}$ is weakly tight then the algebra $\mathbb{B} = Part(\mathscr{A})/\mathscr{I}(\mathscr{A})$ is finite. Obviously, a Boolean algebra $\mathbb{B}$ is finite if and only if every antichain in $\mathbb{B}$ is finite. Now, $\langle I_n : n \in \omega\rangle \subseteq \mathscr{I}^+(\mathscr{A})$ represents an infinite antichain in $\mathbb{B}$ if and only if for every $A \in \mathscr{A}$ and every $n \in \omega$ $A \subseteq^* I_n$ or $A \cap I_n =^* \emptyset$. It easily follows that if $\mathbb{B}$ has an infinite antichain then $\mathscr{A}$ is not weakly tight.

Next observe that if $\mathscr{A}$ and $\mathscr{B}$ are MAD families then $\mathscr{A} \otimes \mathscr{B}$ is not weakly tight. (In fact, $\mathscr{B}$ embeds into $Part(\mathscr{A} \otimes \mathscr{B})/\mathscr{I}(\mathscr{A} \otimes \mathscr{B})$.) This follows immediately from the definition of $\mathscr{A} \otimes \mathscr{B}$, as $\mathscr{B} \subseteq Part(\mathscr{A} \otimes \mathscr{B})$ and $B \in \mathscr{I}^+(\mathscr{A} \otimes \mathscr{B})$ for every $B \in \mathscr{B}$.

To finish the proof it is enough to note that being weakly tight is upwards closed in the Katětov order. $\dashv$

We do not know whether MAD families maximal in the Katětov order exist. However, (weakly) tight MAD families are almost maximal. The following lemma follows easily from the fact that $\mathscr{I}^+(\mathscr{A})$ is a Happy family for every MAD $\mathscr{A}$ (see e.g., [22]).

Lemma 3.6. *Let $\mathscr{A}$ be a MAD family and let $f : \omega \longrightarrow \omega$. Then there is an $X \in \mathscr{I}^+(\mathscr{A})$ such that $f \restriction X$ is constant or $f \restriction X$ is one-to-one.*

Proposition 3.7. *Let $\mathscr{A}$ be a weakly tight MAD family and let $\mathscr{B}$ be a MAD family. If $\mathscr{A} \leq_K \mathscr{B}$ then there is an $X \in \mathscr{I}^+(\mathscr{A})$ such that $\mathscr{B} \leq_K \mathscr{A} \restriction X$.*

Proof. Let $\mathscr{A}$ and $\mathscr{B}$ be as above and let f be a witness to $\mathscr{A} \leq_K \mathscr{B}$. Obviously, $f^{-1}(n) \in \mathscr{I}(\mathscr{B})$ for every $n \in \omega$. By Lemma 3.6, there is a $Y \in \mathscr{I}^+(\mathscr{B})$ such that $f \restriction Y$ is one-to-one.

CLAIM. *The set* $\mathscr{F} = \{B \in \mathscr{B} : f[B \cap Y] \in \mathscr{I}^+(\mathscr{A})\}$ *is finite.*

Aiming toward a contradiction assume that $\{B_i : i \in \omega\}$ is an infinite subset of $\mathscr{F}$. Let $I_i = f[B_i \cap Y]$ for every $i \in \omega$. As $\mathscr{A}$ is weakly tight, there is an $A \in \mathscr{I}(\mathscr{A})$ such that $A \cap I_i$ is infinite for infinitely many $i \in \omega$. This, however, implies that $f^{-1}[A] \cap B_i$ is infinite for infinitely many $i \in \omega$ which, of course, contradicts the fact that $f^{-1}[A] \in \mathscr{I}(\mathscr{B})$.

Now, let $X = f[Y \setminus \bigcup \mathscr{F}]$ and let $g = f^{-1} \restriction X$. Then, $X \in \mathscr{I}^+(\mathscr{A})$ and $g^{-1}[B] = f \restriction (Y \setminus \bigcup \mathscr{F})[B] \in \mathscr{I}(\mathscr{A})$, so $\mathscr{B} \leq_K \mathscr{A} \restriction X$. ⊣

We will say that a MAD family $\mathscr{A}$ is *K-uniform* if $\mathscr{A} \geq_K \mathscr{A} \restriction X$ (or, equivalently, $\mathscr{A} \simeq_K \mathscr{A} \restriction X$) for every $X \in \mathscr{I}^+(\mathscr{A})$.

COROLLARY 3.8. If a MAD family $\mathscr{A}$ is weakly tight and K-uniform then it is K-maximal.

PROOF. Follows directly from Proposition 3.7. ⊣

We conclude this section by showing that at least consistently K-uniform MAD families do exist. Recall that $\mathfrak{t}$ denotes the minimal cardinality of an unfilled tower in $\mathscr{P}(\omega)/\text{fin}$ and that $\mathfrak{t} \leq \mathfrak{b} \leq \mathfrak{a}$.

THEOREM 3.9. ($\mathfrak{t} = \mathfrak{c}$) *There is a K-uniform MAD family.*

In order to prove the theorem we will need two technical lemmata. Given an almost disjoint family $\mathscr{A}$ and a family $\mathscr{F} \subseteq \omega^\omega$ consisting of one-to-one functions, we say that $\mathscr{A}$ *respects* $\mathscr{F}$ if $f^{-1}[A] \in \mathscr{I}(\mathscr{A})$ for every $A \in \mathscr{A}$ and $f \in \mathscr{F}$.

LEMMA 3.10. *Given* $\mathscr{A}$, $\mathscr{F}$ *such that* $\mathscr{A}$ *respects* $\mathscr{F}$, *and* $|\mathscr{A}|, |\mathscr{F}| < \mathfrak{t}$ *and given an* $X \in \mathscr{I}^+(\mathscr{A})$, *there is an almost disjoint family* $\mathscr{B}$ *extending* $\mathscr{A}$, $|\mathscr{B}| < \mathfrak{t}$, *which respects* $\mathscr{F}$ *such that* $\mathscr{B} \cap [X]^\omega \neq \emptyset$.

PROOF. Assume that $\mathscr{A} \cap [X]^\omega = \emptyset$, as otherwise there is nothing to prove. Let $\mathscr{F}'$ be the closure of $\mathscr{F}$ under compositions of functions. Obviously, $\mathscr{A}$ also respects $\mathscr{F}'$. Enumerate $\mathscr{F}'$ as $\{f_\alpha : \alpha < \kappa\}$, where $\kappa = |\mathscr{F}| < \mathfrak{t}$. Recursively choose a $\subseteq^*$-decreasing sequence $\{T_\alpha : \alpha < \kappa\}$ of infinite subsets of X so that:

(1) $T_0 \subseteq X$ is almost disjoint from all elements of $\mathscr{A}$,
(2) For $\alpha < \kappa$, $f_\alpha^{-1}[T_\alpha] \in \mathscr{I}(\mathscr{A})$ or $f_\alpha^{-1}[T_\alpha]$ is almost disjoint from all elements of $\mathscr{A}$,
(3) For every $\beta, \gamma \leq \alpha < \kappa$ such that $f_\beta^{-1}[T_\alpha], f_\gamma^{-1}[T_\alpha] \notin \mathscr{I}(\mathscr{A})$, $(f_\beta^{-1}[T_\alpha] \cap f_\gamma^{-1}[T_\alpha]$ is finite) or $(f_\beta^{-1} \restriction T_\alpha =^* f_\gamma^{-1} \restriction T_\alpha)$.

Assume that T_β, $\beta < \alpha$ have been successfully constructed. Choose $S \in [X]^\omega$ such that $S \subseteq^* T_\beta$ for every $\beta < \alpha$. Consider $f_\alpha^{-1}[S]$; either there is an $S_0 \in [S]^\omega$ such that $f_\alpha^{-1}[S_0] \in \mathscr{I}(\mathscr{A})$, or $f_\alpha^{-1}[S]$ is almost disjoint from all elements of $\mathscr{A}$, in which case set $S = S_0$. Note that if T_α is any infinite subset of S_0 then conditions (1) and (2) are satisfied. In order to find T_α so that (3) holds enumerate all pairs (β, γ), $\beta, \gamma \leq \alpha$ as $\{(\beta_\xi, \gamma_\xi) : \xi < \lambda\}$. Note that $\lambda < \mathfrak{t}$. Construct another decreasing sequence $\{S_\xi : \xi < \lambda\}$ (S_0 has already been chosen) so that if

$$f_{\beta_\xi}^{-1}[S_\xi] \notin \mathscr{I}(\mathscr{A}) \text{ and } f_{\gamma_\xi}^{-1}[S_\xi] \notin \mathscr{I}(\mathscr{A})$$

then

$$|f_{\beta_\xi}^{-1}[S_{\xi+1}] \cap f_{\gamma_\xi}^{-1}[S_{\xi+1}]| < \aleph_0 \text{ or } f_{\gamma_\xi}^{-1} \restriction S_{\xi+1} =^* f_{\gamma_\xi}^{-1} \restriction S_{\xi+1}.$$

Now, this is easy to do as if there is no $S_{\xi+1} \in [S_\xi]^\omega$ such that $f_{\beta_\xi}^{-1}[S_{\xi+1}] \cap f_{\gamma_\xi}^{-1}[S_{\xi+1}]$ is finite then, in fact, $f_{\gamma_\xi}^{-1} \upharpoonright S_\xi =^* f_{\gamma_\xi}^{-1} \upharpoonright S_\xi$. Finally choose $T_\alpha \in [S_0]^\omega$ such that $T_\alpha \subseteq^* S_\xi$ for every $\xi < \lambda$. This finishes the construction.

Let $\{T_\alpha : \alpha < \kappa\}$ be a sequence satisfying the above requirements (1)–(3). As $\kappa < \mathfrak{t}$, there is a $T \in [\omega]^\omega$ such that $T \subseteq^* T_\alpha$ for every $\alpha < \kappa$. Set

$$\mathscr{B} = \mathscr{A} \cup \{T\} \cup \{f_\alpha^{-1}[T] : \alpha < \kappa,\ f_\alpha^{-1}[T] \notin \mathscr{I}(\mathscr{A}) \text{ and}$$
$$(\forall \beta < \alpha)\ f_\alpha^{-1}[T] \neq^* f_\beta^{-1}[T]\}.$$

Then $\mathscr{B}$ is an almost disjoint family (by (2) and (3)), $\mathscr{B} \cap [X]^\omega \neq \emptyset$ (by (1)) and $\mathscr{B}$ respects $\mathscr{F}$, as for every element $B = f_\alpha^{-1}[T] \in \mathscr{B} \setminus \mathscr{A}$ and every $f \in \mathscr{F}$ either $f^{-1}[B] \in \mathscr{I}(\mathscr{A})$ or $f^{-1}[B] = (f_\alpha \circ f)^{-1}[T] =^* f_\beta^{-1}[T] \in \mathscr{B}$ for some $\beta < \kappa$. ⊣

LEMMA 3.11. *Given $\mathscr{A}$, $\mathscr{F}$ such that $\mathscr{A}$ respects $\mathscr{F}$, $|\mathscr{A}| < \mathfrak{a}$ and $X \in \mathscr{I}^+(\mathscr{A})$ there is a one-to-one $f \in \omega^\omega$ such that $rng(f) \subseteq X$ and $\mathscr{A}$ respects $\mathscr{F} \cup \{f\}$.*

PROOF. Let f be a bijection between ω and a subset of X almost disjoint from every element of $\mathscr{A}$. ⊣

Having these two lemmata at hand, the proof of Theorem 3.9 is now straightforward.

PROOF OF THEOREM 3.9. Assume $\mathfrak{t} = \mathfrak{c}$. We construct the K-uniform MAD family $\mathscr{A}$ as the union of an increasing chain of almost disjoint families $\mathscr{A}_\alpha, \alpha < \mathfrak{c}$. In order to do this, enumerate $[\omega]^\omega$ as $\{X_\alpha : \alpha < \mathfrak{c}\}$. By recursion construct an increasing sequence $\mathscr{A}_\alpha, \alpha < \mathfrak{c}$ of almost disjoint families and an increasing sequence $\mathscr{F}_\alpha, \alpha < \mathfrak{c}$ of subsets of ω^ω consisting of one-to-one functions so that $\mathscr{A}_0$ is a partition of ω into infinitely many infinite pieces and $\mathscr{F}_0 = \emptyset$ and for every $\alpha < \mathfrak{c}$:

(1) $|\mathscr{A}_\alpha|, |\mathscr{F}_\alpha| < \mathfrak{c}$,
(2) $\mathscr{A}_\alpha$ respects $\mathscr{F}_\alpha$,
(3) there is an $A \in \mathscr{A}_{\alpha+1}$ such that $|A \cap X_\alpha| = \aleph_0$ and
(4) if $X_\alpha \in \mathscr{I}^+(\mathscr{A}_{\alpha+1})$ then there is an $f \in \mathscr{F}_{\alpha+1}$ such that $rng(f) \subseteq X_\alpha$.

To do this is easy; For α limit let $\mathscr{A}_\alpha = \bigcup\{\mathscr{A}_\beta : \beta < \alpha\}$ and $\mathscr{F}_\alpha = \bigcup\{\mathscr{F}_\beta : \beta < \alpha\}$. If $\alpha = \beta + 1$ consider $\mathscr{A}_\beta$ and $\mathscr{F}_\beta$ and use Lemma 3.10 to extend $\mathscr{A}_\beta$ to $\mathscr{A}_\alpha$ so that $\mathscr{A}_\alpha$ respects $\mathscr{F}_\beta$ and $|A \cap X_\beta| = \aleph_0$ for some $A \in \mathscr{A}_\alpha$. Then, if $X_\beta \in \mathscr{I}^+(\mathscr{A}_\alpha)$ use Lemma 3.11 to extend $\mathscr{F}_\beta$ to $\mathscr{F}_\alpha$ so that $\mathscr{A}_\alpha$ respects $\mathscr{F}_\alpha$ and there is an $f \in \mathscr{F}_\alpha$ such that $rng(f) \subseteq X_\beta$. Note that the construction never stops as $\mathfrak{c} = \mathfrak{t}$ is a regular cardinal. This finishes the construction.

To see that $\mathscr{A} = \bigcup\{\mathscr{A}_\beta : \beta < \mathfrak{c}\}$ is, indeed, K-uniform note that $\mathscr{A}$ respects $\mathscr{F} = \bigcup\{\mathscr{F}_\beta : \beta < \mathfrak{c}\}$ let X be an element of $\mathscr{I}^+(\mathscr{A})$. Then $X = X_\alpha$ for some $\alpha < \mathfrak{c}$ and $X \in \mathscr{I}^+(\mathscr{A}_\alpha)$. By (4) there is an $f \in \mathscr{F}_{\alpha+1} \subseteq \mathscr{F}$ with $rng(f) \subseteq X$. This in turn means that f witnesses that $\mathscr{A} \geq_K \mathscr{A} \upharpoonright X$. ⊣

The question of existence of Katětov maximal MAD families remains, however, wide open.

QUESTIONS 3.12. Here is a list of questions we could not answer:

(1) Is there (consistently) a K-maximal MAD family?
(2) Is there is a K-uniform MAD family in ZFC?
(3) Can the hypothesis in Proposition 3.4 be weakened to $\diamondsuit(\mathfrak{b})$?

§4. Some remarks on indestructibility of MAD families. As mentioned in the introduction one of the main motives for these considerations stems from the question of J. Roitman as to whether the existence of a dominating family of size $\aleph_1$ implies the existence of a MAD family of size $\aleph_1$. The answer hinges on the question whether (which) MAD families can be destroyed by an ω^ω-bounding forcing. A significant step in this direction has been made by C. Laflamme in [19] where he showed that *Any F_σ-filter can be diagonalized by an ω^ω-bounding forcing.* A naive approach to the problem would be to try to show that, under CH, any MAD family is below an F_σ-ideal in the Katětov order. This (unfortunately but hardly surprisingly) is not the case as essentially also pointed out in [19].

From the results of the preceding section one may get the (false) impression that tightness or (almost equivalently) Cohen-indestructibility is the strongest possible notion of maximality of AD families. Here we would like to rectify this by looking at the notion of $\mathbb{P}$-indestructibility, where $\mathbb{P}$ is an ω^ω-bounding forcing and, in particular, at the case of Solovay's *random* forcing. Recall that $\mathbb{B}(\omega)$ denotes the measure algebra $Borel(2^\omega)/Null$. The following observation is due to J. Brendle:

PROPOSITION 4.1 ((CH)). *There is a Cohen-indestructible MAD family which is $\mathbb{B}(\omega)$-destructible.*

PROOF. Denote by $\mathcal{J}$ the ideal $\{A \subseteq \omega \times \omega : (\exists n \in \omega)(\exists m \in \omega)(\forall k \geq n)\ |\{l \in \omega : (k,l) \in A\}| \leq m\}$. $\mathcal{J}$ is $\mathbb{B}(\omega)$-destructible. It follows from the fact that $\mathbb{B}(\omega)$ adds an eventually different real (see [2]). So to prove the proposition it is sufficient to show that

CLAIM ((CH)). *There is a tight MAD family $\mathcal{A} \subseteq \mathcal{J}$.*

In order to construct $\mathcal{A}$ enumerate as $\{\langle I_n^\alpha : n \in \omega\rangle : \omega \leq \alpha < \omega_1\}$ all sequences of infinite subsets of $\omega \times \omega$. Construct $\mathcal{A} \subseteq \mathcal{J}$ as $\{A_\alpha : \alpha < \omega_1\}$ recursively so that

(1) $A_i = \{i\} \times \omega$ for $i \in \omega$ and
(2) $(\forall \alpha \geq \omega)(\exists I \in \mathcal{J}(\{A_\beta : \beta \leq \alpha\}))$ such that $I_m^\alpha \subseteq I$ for some $m \in \omega$ or $I \cap I_n^\alpha \neq \emptyset$ for all $n \in \omega$.

It is easy to see that if the construction can be carried out then $\mathcal{A}$ is indeed tight. Suppose that A_β have been constructed for all $\beta < \alpha$. Enumerate $\{A_\beta : \beta < \alpha\}$ as $\{B_n : n \in \omega\}$. If there is an $n \in \omega$ such that $\bigcup_{k<n} B_k \cap I_m^\alpha \neq \emptyset$ for all (but finitely many) $m \in \omega$ or $\bigcup_{k<n} B_k \supseteq I_m^\alpha$ for some $m \in \omega$ let A_α be an arbitrary element of $\mathcal{J}$ almost disjoint from A_β, $\beta < \alpha$. If not, choose recursively $(n_i, m_i) \in \omega \times \omega$ so that

(a) $(n_i, m_i) \in I_i^\alpha \setminus \bigcup_{k\leq i} B_k$ and
(b) $n_{i+1} > n_i$ for all $i \in \omega$.

Then set $A_\alpha = \{(n_i, m_i) : i \in \omega\}$. ⊣

On the other hand, (assuming CH) no proper ω^ω-bounding forcing of size $\mathfrak{c}$ is strong enough to destroy all Cohen-destructible MAD families:

PROPOSITION 4.2 ((CH)). *Let $\mathbb{P}$ be a proper ω^ω-bounding forcing of size $\mathfrak{c}$. There is a $\mathbb{P}$-indestructible Cohen-destructible MAD family.*

PROOF. The idea is to construct a $\mathbb{P}$-indestructible MAD family $\mathcal{A} \subseteq \mathrm{nwd}$. Enumerate $\mathbb{Q}$ as $\{x_n : n \in \omega\}$. Using properness of $\mathbb{P}$ (and CH) construct a sequence $\{(p_\alpha, \tau_\alpha) : \alpha < \omega_1\}$, where $p_\alpha \in \mathbb{P}$, τ_α is a $\mathbb{P}$-name, such that if τ is a $\mathbb{P}$-name and $p \Vdash$ "$\tau \in [\mathbb{Q}]^\omega$" then there is an $\alpha \in \omega_1$ such that $p_\alpha \leq p$ and $p_\alpha \Vdash$ "$\tau = \tau_\alpha$".

Having fixed such a sequence an almost disjoint family $\mathscr{A} = \{A_\alpha : \alpha < \omega_1\} \subseteq \mathrm{nwd}$ will be constructed recursively. To begin, let $\{A_i : i \in \omega\}$ be a partition of $\mathbb{Q}$ into infinite nowhere dense sets. At stage α consider the pair (p_α, τ_α). If $p_\alpha \not\Vdash$ "$(\forall \beta < \alpha)$ $|\tau_\alpha \cap A_\beta| < \omega$" then let A_α be any infinite nowhere dense subset of $\mathbb{Q}$ almost disjoint from all A_β, $\beta < \alpha$. If $p_\alpha \Vdash$ "$(\forall \beta < \alpha)$ $|\tau_\alpha \cap A_\beta| < \omega$" let $\{B_m : m \in \omega\}$ be an enumeration of pairwise disjoint finite modifications of $\{A_\beta : \beta < \alpha\}$. Let ρ be a $\mathbb{P}$-name such that $p_\alpha \Vdash$ "$\rho \in \omega^\omega$ and $(\forall m \in \omega)$ $B_m \cap \tau_\alpha \subseteq \{x_i : i < \rho(m)\}$". As $\mathbb{P}$ is ω^ω-bounding, there is an $f \in \omega^\omega$ and a $q \leq p_\alpha$ such that $q \Vdash$ "$\rho \leq f$". Put

$$C = \bigcup_{m \in \omega} (B_m \cap \{x_i : i < f(m)\}).$$

Note that C is almost disjoint from all A_β, $\beta < \alpha$ and $q \Vdash$ "$\tau_\alpha \subseteq C$".

Find a nowhere dense set D and a $r = r_\alpha \leq q$ such that $r \Vdash$ "$|D \cap \tau_\alpha| = \aleph_0$". The fact that such a D exists can be distilled from a result of Keremedis (see Theorem 2.4.5 of [2]) where it is in effect shown that if a forcing $\mathbb{P}$ adds a set almost disjoint from all ground model nowhere dense sets then $\mathbb{P}$ adds a Cohen real. As our $\mathbb{P}$ is ω^ω-bounding it does not add a Cohen real, hence D and r as required do exist.

Set $A_\alpha = C \cap D$. Now, $r \Vdash$ "$|A_\alpha \cap \tau_\alpha| = \aleph_0$", so in particular A_α is infinite, A_α is almost disjoint from all A_β $\beta < \alpha$ as C is and A_α is nowhere dense as D is. This concludes the construction.

The almost disjoint family $\mathscr{A} = \{A_\alpha : \alpha < \omega_1\}$ is Cohen destructible as $\mathscr{A} \subseteq \mathrm{nwd}$ by Theorem 1.3 so the only thing left to verify is that $\mathscr{A}$ is $\mathbb{P}$-indestructible. Assume not, that is there is a $\mathbb{P}$-name τ for an infinite subset of $\mathbb{Q}$ and a condition $p \in \mathbb{P}$ such that $p \Vdash$ "$(\forall \alpha < \omega_1)$ $|\tau \cap A_\alpha| < \aleph_0$". There is a β such that $p_\beta \leq p$ and $p_\beta \Vdash$ "$\tau = \tau_\beta$". Then, however, $r_\beta \Vdash$ "$|\tau \cap A_\beta| = \aleph_0$" which is a contradiction. $\dashv$

So, in particular Random-indestructibility and Cohen-indestructibility are incomparable notions. As a corollary we obtain that there is a Cohen-destructible MAD family of size $\aleph_1$ in the random real model as well as the side-by-side Sacks model. It should be also noted here that the fact that the existence of an unbounded family of size $\aleph_1$ does not imply the existence of a MAD family of size $\aleph_1$ was proved by Shelah [25] and reproved by a different method by J. Brendle in [6].

§5. Two other orderings on MAD families. One could argue that while the Katětov order is a useful tool for studying the ideals generated by MAD families it identifies MAD families with very different combinatorial (or topological) properties (see e.g., Corollary 5.8). In this section we introduce two different orderings which seem to be more adequate in this respect.

Recall the definition of the Mrówka-Isbell space associated with an almost disjoint family $\mathscr{A}$.

DEFINITION 5.1. Let $\mathscr{A}$ be an AD family. Define a space $\Psi(\mathscr{A})$ as follows: The underlying set is $\omega \cup \mathscr{A}$, all elements of ω are isolated and basic neighborhoods of $A \in \mathscr{A}$ are of the form $\{A\} \cup (A \setminus F)$ for some finite set F.

It follows immediately from the definition that $\Psi(\mathscr{A})$ is a first countable, locally compact space. If $\mathscr{A}$ is infinite then $\Psi(\mathscr{A})$ is not countably compact and $\Psi(\mathscr{A})$ is pseudocompact if and only if $\mathscr{A}$ is a MAD family.

DEFINITION 5.2. Let $\mathscr{A}, \mathscr{B}$ be MAD families. We say that:
$\mathscr{A} \leq_K^* \mathscr{B}$ if there is a function $f : \omega \longrightarrow \omega$ such that for every $A \in \mathscr{A}$ there is a $B \in \mathscr{B}$ such that $f^{-1}[A] \subseteq^* B$,
$\mathscr{A} \leq_{top} \mathscr{B}$ if there is a continuous surjection $F : \Psi(\mathscr{A}) \to \Psi(\mathscr{B})$.

It is easily seen that if $\mathscr{A} \prec \mathscr{B}$ then both $\mathscr{A} \leq_K^* \mathscr{B}$ and $\mathscr{A} \leq_{top} \mathscr{B}$. It is just as easy to see that $\Sigma_{\mathscr{A}}\langle \mathscr{B}_A : A \in \mathscr{A}\rangle \leq_K^* \mathscr{B}_A$ for every $A \in \mathscr{A}$. So the equivalents of Propositions of Section 2 hold with $\leq_K$ replaced with $\leq_K^*$. In particular:

THEOREM 5.3. (1) *Every collection of at most* $\mathfrak{c}$*-many MAD families has a common* $\leq_K^*$*-lower bound,*
(2) *There is a strictly decreasing chain of length* $\mathfrak{c}^+$ *below every MAD family* $\mathscr{A}$ *in the* $\leq_K^*$*-order,*
(3) *There is a collection of* $\mathfrak{c}$*-many pairwise* $\leq_K^*$*-incomparable MAD families* $\leq_K^*$*-below every MAD family* $\mathscr{A}$.

In [24], S. Mrówka constructed a Ψ-space with a unique compactification. His result and, more importantly, the technique of "gluing" he used is the basic tool for constructions in this section. We will say that a MAD family $\mathscr{A}$ is *Mrówka* if $|\beta\Psi(\mathscr{A}) \setminus \Psi(\mathscr{A})| = 1$, in other words, if $\Psi(\mathscr{A})$ has a unique compactification, and we will say that $\mathscr{A}$ is *connected* if $\beta\Psi(\mathscr{A}) \setminus \Psi(\mathscr{A})$ is connected. Similarly, we will say that $\mathscr{A}$ is *zero-dimensional* if $\beta\Psi(\mathscr{A}) \setminus \Psi(\mathscr{A})$ is zero-dimensional (if and only if $\Psi(\mathscr{A})$ is strongly zero-dimensional). Obviously, a MAD family $\mathscr{A}$ is Mrówka if and only if it is both zero-dimensional and connected. Teresawa and Solomon independently (see [29, 27]), were the first to observe that $\Psi(\mathscr{A})$ needs not be strongly zero-dimensional.

LEMMA 5.4. *A MAD family* $\mathscr{A}$ *is connected if and only if* $\mathscr{A}$ *has no non-trivial partitioners.*

PROOF. If $\mathscr{A}$ has a non-trivial partitioner then it has a compactification with exactly two points. Hence $\beta\Psi(\mathscr{A}) \setminus \Psi(\mathscr{A})$ is disconnected. On the other hand, if $\beta\Psi(\mathscr{A}) \setminus \Psi(\mathscr{A})$ has a non-trivial clopen subset C, then there are open (in $\Psi(\mathscr{A})$) sets U, V which separate C and $(\beta\Psi(\mathscr{A}) \setminus \Psi(\mathscr{A})) \setminus C$. Note that $U \cup V$ cover $\beta\Psi(\mathscr{A}) \setminus \Psi(\mathscr{A})$ so, $F = \mathscr{A} \cap (\beta\Psi(\mathscr{A}) \setminus U \cup V)$ is finite. It is now easy to see that $P = (U \cap \omega) \setminus \bigcup F$ is a non-trivial partitioner of $\mathscr{A}$. ⊣

Let us recall Bashkirov's results (see [3]) on zero-dimensionality of MAD families. Obviously if a space X is not strongly zero-dimensional then X maps continuously onto the unit interval.

LEMMA 5.5 ([3]). *Let* $\mathscr{A}$ *be a MAD family. Then the following are equivalent:*
(1) $\Psi(\mathscr{A})$ *maps continuously onto* $[0, 1]$,
(2) *There is a surjective map* $f : \omega \to \mathbb{Q} \cap [0, 1]$ *such that* $f[A]$ *has a unique limit point for every* $A \in \mathscr{A}$.

PROOF. For the forward implication note that if $F : \Psi(\mathscr{A}) \to [0, 1]$ is a continuous surjection then $F[\omega]$ is a countable dense set. Without loss of generality $0, 1 \in F[\omega]$. There is then a homeomorphism $h : [0, 1] \to [0, 1]$ which sends $F[\omega]$ onto $\mathbb{Q} \cap [0, 1]$. $h \circ F \restriction \omega$ is the desired f.

The reverse is even easier as f extends uniquely to a continuous surjection (it is onto as $\mathscr{A}$ is maximal). ⊣

An easy consequence is:

COROLLARY 5.6. Every Sacks-indestructible MAD family is zero-dimensional.

PROOF. Assume that $\mathscr{A}$ is not zero-dimensional and let $f : \omega \to \mathbb{Q} \cap [0, 1]$ be such that $f[A]$ has a unique limit point for every $A \in \mathscr{A}$. Let $g : \mathbb{Q} \cap [0, 1] \to \omega$ be a partial inverse of f. Then $f^{-1}[A]$ has countable closure (in $\mathbb{R}$) for every $A \in \mathscr{I}(\mathscr{A})$, hence by Theorem 2.4.5 of [7] $\mathscr{A}$ is Sacks-destructible. ⊣

LEMMA 5.7. *If $\mathscr{A}$ is a MAD family such that $|\mathscr{A} \restriction P| = \mathfrak{c}$ for every non-trivial partitioner P then there is a connected MAD family $\mathscr{B}$ such that $\mathscr{I}(\mathscr{A}) = \mathscr{I}(\mathscr{B})$. In fact, there is a continuous closed two-to-one surjective map $F : \Psi(\mathscr{A}) \to \Psi(\mathscr{B})$.*

PROOF. Enumerate all non-trivial partitioners of $\mathscr{A}$ as $\{P_\alpha : \alpha < \kappa\}$ for some $\kappa \leq \mathfrak{c}$. Recursively choose (distinct) $A_\alpha, B_\alpha \in \mathscr{A}$ so that $A_\alpha \subseteq^* P_\alpha$ and $B_\alpha \cap P_\alpha =^* \emptyset$. There is no problem choosing as there are always $\mathfrak{c}$-many possibilities and less than $\mathfrak{c}$-many already chosen. Then let

$$\mathscr{B} = \{A_\alpha \cup B_\alpha : \alpha < \kappa\} \cup \mathscr{A} \setminus \{A_\alpha, B_\alpha : \alpha < \kappa\}.$$

$\mathscr{B}$ has obviously no non-trivial partitioners, so by Lemma 5.4 it is connected. ⊣

COROLLARY 5.8 (($\mathfrak{a} = \mathfrak{c}$)). For every MAD family $\mathscr{A}$ there is a connected MAD family $\mathscr{B}$ such that $\mathscr{I}(\mathscr{A}) = \mathscr{I}(\mathscr{B})$ and $\mathscr{A} \prec \mathscr{B}$. Topologically, for every MAD family $\mathscr{A}$ there is a connected MAD family $\mathscr{B}$ such that $\Psi(\mathscr{A})$ maps continuously onto $\Psi(\mathscr{B})$ by a closed two-to-one map.

PROOF. It suffices to note that, as $\mathfrak{a} = \mathfrak{c}$, $|\mathscr{A} \restriction P| = \mathfrak{c}$ for every $P \in \mathscr{I}^+(\mathscr{A})$. ⊣

COROLLARY 5.9 ([24]). There exists a connected MAD family (in ZFC).

PROOF. Let, for $f \in 2^\omega$, $A_f = \{f \restriction n : n \in \omega\}$. Extend $\{A_f : f \in 2^\omega\}$ to a MAD family $\mathscr{B}$ (of subsets of $2^{<\omega}$) and list $\mathscr{B} \setminus \{A_f : f \in 2^\omega\}$ as $\{B_f : f \in X\}$ for some $X \subseteq 2^\omega$. For $f \in 2^\omega \setminus X$ let $B_f = \emptyset$. Let $\mathscr{C} = \{A_f \cup B_f : f \in 2^\omega\}$. All that needs to be checked is that $\mathscr{C}$ has the property required in Lemma 5.4. To that end let P be a non-trivial partitioner of $\mathscr{C}$. Then

$$\mathscr{A} \restriction P = \{A_f \cup B_f : A_f \cup B_f \subseteq^* P\} = \{A_f \cup B_f : A_f \subseteq^* P\} = \{A_f \cup B_f : f \in F\}$$

where $F = \{f \in 2^\omega : (\exists n \in \omega)(\forall m > n)\ f \restriction n \in P\}$. F is an F_σ-subset of 2^ω. So it is either countable or of size $\mathfrak{c}$. However, it can not be countable as P is a non-trivial partitioner and $\mathscr{A} \restriction P$ is a MAD family. ⊣

THEOREM 5.10 ([3]). *For every MAD family $\mathscr{A}$ there is a zero-dimensional MAD family $\mathscr{B}$ and a continuous closed two-to-one map $F : \Psi(\mathscr{A}) \to \Psi(\mathscr{B})$.*

PROOF. As all spaces of size less than $\mathfrak{c}$ are strongly zero-dimensional we can assume that $|\mathscr{A}| = \mathfrak{c}$. Enumerate $\mathbb{Q}^\omega$ as $\{f_\alpha : \alpha < \kappa\}$ and recursively (for $\alpha < \mathfrak{c}$) choose A_α, B_α distinct elements of $\mathscr{A}$ so that if f satisfies (2) of Lemma 5.5 then the limit points of $f[A_\alpha]$ and $f[B_\alpha]$ are distinct. Then let

$$\mathscr{B} = \{A_\alpha \cup B_\alpha : \alpha < \mathfrak{c}\} \cup \mathscr{A} \setminus \{A_\alpha, B_\alpha : \alpha < \kappa\}.$$

$\mathscr{B}$ is obviously a MAD family and it is zero-dimensional by Lemma 5.5. ⊣

COROLLARY 5.11. (1) [24] There is a Mrówka family (in ZFC).
(2) [4] ($\mathfrak{a} = \mathfrak{c}$) For every MAD family $\mathscr{A}$ there is a Mrówka MAD family $\mathscr{B}$ and a continuous closed finite-to-one surjection $F : \Psi(\mathscr{A}) \to \Psi(\mathscr{B})$.

Proof. (1) follows directly from Corollary 5.9 and Theorem 5.10 and (2) just as easily from Corollary 5.8 and Theorem 5.10. Note that a continuous strongly zero-dimensional image of $\Psi(\mathscr{A})$ for $\mathscr{A}$ connected has to be Mrówka. ⊣

Proposition 5.12. *For every connected MAD family $\mathscr{A}$ there is a MAD family $\mathscr{B}$ such that $\mathscr{A} \prec \mathscr{B}$, $\mathscr{I}(\mathscr{A}) = \mathscr{I}(\mathscr{B})$ and $\mathscr{B} \not\leq^*_K \mathscr{A}$.*

Proof. Let $\mathscr{A}$ be a connected MAD family of size $\mathfrak{c}$. We will construct $\mathscr{B}$ so that $\mathscr{A}$ refines $\mathscr{B}$, in fact, every element of $\mathscr{B}$ will be a union of at most two elements of $\mathscr{A}$.

Note that if f is a witness to $\mathscr{B} \leq^*_K \mathscr{A}$ then f witnesses that $\mathscr{A} \leq^*_K \mathscr{A}$. Enumerate all witnesses to $\mathscr{A} \leq^* \mathscr{A}$ as $\{f_\alpha : \alpha < \kappa\}$, where $\kappa \leq \mathfrak{c}$. For each $\alpha < \kappa$ and $A, B \in \mathscr{A}$ let

$$\Phi_\alpha(A) = B \quad \text{iff} \quad f_\alpha^{-1}[A] \subseteq^* B.$$

Note that Φ_α is finite-to-one for every $\alpha < \kappa$, as $f_\alpha[A]$ has to be a partitioner of $\mathscr{A}$ for every $A \in \mathscr{A}$ and as $\mathscr{A}$ has no non-trivial partitioners.

Now, we are ready for the construction of $\mathscr{B}$. Recursively choose B_α, C_α distinct elements of $\mathscr{A}$ so that $\Phi_\alpha(B_\alpha) \neq \Phi_\alpha(C_\alpha)$ for every $\alpha < \kappa$. Then set

$$\mathscr{B} = \{B_\alpha \cup C_\alpha : \alpha < \kappa\} \cup \mathscr{A} \setminus \{B_\alpha, C_\alpha : \alpha < \kappa\}.$$

Obviously, $\mathscr{I}(\mathscr{A}) = \mathscr{I}(\mathscr{B})$, so all that remains to be seen is that $\mathscr{B} \not\leq^*_K \mathscr{A}$. Assume not, i.e., there is an $f \in \omega^\omega$ such that for every $B \in \mathscr{B}$ there is an $A \in \mathscr{A}$ such that $f^{-1}[B] \subseteq^* A$. Then $f = f_\alpha$ for some $\alpha < \kappa$, however,

$$f^{-1}[B_\alpha \cup C_\alpha] \cap \Phi_\alpha(B_\alpha) \neq^* \emptyset, \text{ yet } f^{-1}[B_\alpha \cup C_\alpha] \not\subseteq^* \Phi_\alpha(B_\alpha)$$

which is a contradiction. ⊣

Now we have all the tools necessary to show that $\leq_K$ and $\leq^*_K$ are indeed different (pre)orderings. Assuming $\mathfrak{a} = \mathfrak{c}$, one can actually show much more. In particular, an analogue for the quest for a maximal MAD family turns out to be futile for the $\leq^*_K$-order.

Corollary 5.13. (1) There are $\mathscr{A}$ and $\mathscr{B}$ MAD such that $\mathscr{I}(\mathscr{A}) = \mathscr{I}(\mathscr{B})$ and $\mathscr{B} \not\leq^*_K \mathscr{A}$.

(2) ($\mathfrak{a} = \mathfrak{c}$) For every MAD family $\mathscr{A}$ there is a MAD family $\mathscr{B}$ such that $\mathscr{A} \prec \mathscr{B}$ (i. p. $\mathscr{A} \leq^*_K \mathscr{B}$) but $\mathscr{B} \not\leq^*_K \mathscr{A}$.

Proof. (1) follows directly from Corollary 5.9 and Proposition 5.12 and (2) from Corollary 5.8 and Proposition 5.12. ⊣

So, in particular, there are neither maximal nor minimal MAD families in the $\leq^*_K$-order.

Now we turn our attention shortly to the $\leq_{top}$-order. First we give its combinatorial reformulation.

Proposition 5.14. *$\mathscr{B} \leq_{top} \mathscr{A}$ if and only if there is a surjection $f : \omega \to \omega$ such that $f^{-1}(n) \in Part(\mathscr{B})$ for all $n \in \omega$ and $f^{-1}[A] \in Part(\mathscr{B})$ for all $A \in \mathscr{A}$.*

Proof. For the forward implication fix a continuous surjection $F : \Psi(\mathscr{B}) \to \Psi(\mathscr{A})$. Note that the set $P = \{n \in \omega : F(n) \in \mathscr{A}\}$ is a partitioner of $\mathscr{B}$ and $F[P]$ is finite. Note also that $F[\omega \setminus P] = \omega$. Set

$$f(n) = \begin{cases} F(n) & \text{if } n \in \omega \setminus P, \\ 0 & \text{if } n \in P. \end{cases}$$

For the reverse implication note that f extends uniquely to a continuous surjection $F : \Psi(\mathscr{B}) \to \Psi(\mathscr{A})$ defined by putting

$$F(A) = \begin{cases} n & \text{if } \{k \in A : f(k) \neq n\} \text{ is finite,} \\ B & \text{if } f[A] \text{ is infinite and } f[A] \subseteq^* B. \end{cases} \qquad \dashv$$

This easy observation has the following curious consequence:

PROPOSITION 5.15. *If $\mathscr{A}$ is connected then $\mathscr{A} \leq_{top} \mathscr{B}$ implies that $\mathscr{B} \leq_K \mathscr{A}$.*

PROOF. Let $f : \omega \to \omega$ be a surjection such that $f^{-1}(n) \in Part(\mathscr{A})$ for all $n \in \omega$ and $f^{-1}[B] \in Part(\mathscr{A})$ for all $B \in \mathscr{B}$. Note that as all partitioners are trivial $f^{-1}[B] \in \mathscr{I}(\mathscr{A})$ for all $B \in \mathscr{B}$, which in turn shows that f witnesses that $\mathscr{B} \leq_K \mathscr{A}$. $\dashv$

Note that Theorem 5.10 and Corollary 5.11 imply that: *For every MAD family $\mathscr{A}$ there is a zero-dimensional MAD family $\mathscr{B}$ such that $\mathscr{A} \leq_{top} \mathscr{B}$ and (assuming $\mathfrak{a} = \mathfrak{c}$), for every MAD family $\mathscr{A}$ there is a Mrówka MAD family $\mathscr{B}$ such that $\mathscr{A} \leq_{top} \mathscr{B}$.* For more on continuous functions between Ψ-spaces consult [12] and [21].

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