LIFE IN THE SACKS MODEL

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Abstract. This note contains results which *everybody knows* are true but the proofs of which are not to be found in the literature. In particular, we prove that certain cardinal invariants of the continuum are small in the Sacks model and provide a proof of a theorem of J. Baumgartner stating that $\clubsuit$ holds in the side-by-side Sacks model.

I. Introduction.

In many ways the models obtained by adding many Sacks reals to a model of CH are viewed as "the opposite" of Martin's Axiom. J. Baumgartner in [Ba] showed that, indeed, if one adds many Sacks reals to a model of CH Martin's Axiom fails totally. In particular, many cardinal invariants of the continuum are small in both the side-by-side and iterated Sacks models. It usually follows either from the fact that the Sacks forcing has the Sacks property or from the fact that it preserves P-ultrafilters (see [BaL] or [BJ]).

In this note we develop what we believe to be comprehensible approach to countable support iteration of Sacks forcing (Section II.) and then use it (Section III.) to show that some other cardinal invariants are small in the iterated Sacks model. In Section IV. we introduce the notion of (κ, λ)-semidistributivity of forcing notions and use it to prove an unpublished result of J. Baumgartner that $\clubsuit$ holds in the side-by-side Sacks model.

The set theoretic notation is mostly standard and follows [Ku]. Recall the definitions of the following $\diamondsuit$-like principles:

The $\clubsuit$ principle asserts that

$$\exists\{A_\alpha : \alpha \in Lim(\omega_1)\} \text{ such that } \forall \alpha \in Lim(\omega_1) \quad A_\alpha \subseteq \alpha,\ sup(A_\alpha) = \alpha$$

$$\text{and } \forall X \in [\omega_1]^{\omega_1} \quad \exists \alpha \in Lim(\omega_1) \text{ such that } A_\alpha \subseteq X.$$

A weakening of both $\clubsuit$ and CH, denoted by $\uparrow$, states that

$$\exists X \subseteq [\omega_1]^\omega \quad |X| = \aleph_1 \text{ such that } \forall y \in [\omega_1]^{\omega_1} \quad \exists x \in X : x \subseteq y.$$

The $\clubsuit$ principle has been used by Ostaszewski (see [Os]) to construct the famous Ostaszewski space - a countably compact non-compact S-space with closed sets

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either countable or co-countable. In the presence of CH, ♣ is equivalent to ◊. The principle ⫯ was first considered in [BGKT].

The forcing notions mentioned throughout the text are standard as are the cardinal invariants of the continuum with possibly the following exceptions:

$\mathfrak{a}_e = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \omega^\omega$ is a maximal family of eventually different functions$\}$
$\mathfrak{a}_p = \min\{|\mathcal{A}| : \mathcal{A}$ is a maximal almost disjoint family of graphs of permutations on $\omega\}$
$\mathfrak{a}_T = \min\{|\mathcal{A}| : \mathcal{A}$ is an uncountable maximal almost disjoint family of subtrees $2^{<\omega}\}$

The cardinal invariant $\mathfrak{a}_e$ was studied by A. Miller in [Mi2]; $\mathfrak{a}_p$ was considered by S. Thomas, P. Cameron, Y. Zhang and others. The cardinal invariants $\mathfrak{a}_e$ and $\mathfrak{a}_p$ are larger or equal than $non(\mathcal{M})$ (see [BrSZ]). $\mathfrak{a}_T$ was studied (without being given a name) in [Mi1] and [Ne]. It is easily seen that $\mathfrak{a}_T$ is equal to the minimal size of a partition of the Baire space ω^ω into compact sets, hence is greater or equal to $\mathfrak{d}$. The author believes that $Con(\mathfrak{d} < \mathfrak{a}_T)$ is an open problem (despite a cryptic note in [Ne]).

II. Countable support iteration of Sacks reals.

This section uses a classical treatment of iterated Sacks forcing (see [BaL]) and ideas from [SS]. Recall that the Sacks forcing $\mathbb{S}$ is the set of all perfect subtrees of $2^{<\omega}$ ordered by inclusion. A $p \subseteq 2^{<\omega}$ is a *perfect tree* provided that $\forall s \in p\ \forall n \in \omega$ $s \restriction n \in p$ and $\forall s \in p\ \exists n \in \omega\ \exists t \neq t' \in 2^n \cap p$ such that $s \subseteq t, t'$. For $p \in \mathbb{S}$ and $s \in 2^{<\omega}$ we let $p_s = \{t \in p : t \subseteq s$ or $s \subseteq t\}$. Notice that $p_s \in \mathbb{S}$ iff $s \in p$. For a perfect tree p let $[p] = \{f \in 2^\omega : \forall n \in \omega \quad f \restriction n \in p\}$.

$\mathbb{S}$ is an ω^ω-bounding proper forcing. In fact $\mathbb{S}$ satisfies Axiom A. As in [BaL] we shall use the following notation: If $p, q \in \mathbb{S}$ and $m, n \in \omega$ then we say that $(p, m) < (q, n)$ provided that $p \leq q$, $m > n$ and $\forall s \in q \cap 2^n\ \exists t \neq t' \in p \cap 2^m$ such that $s \subseteq t, t'$. The following is the standard Fusion Lemma.

Lemma II.1. *([BaL]) If $\{(p_i, n_i) : i \in \omega\}$ is such that $(p_{i+1}, n_{i+1}) < (p_i, n_i)$ for every i, then $p_\omega = \bigcap\{p_i : i \in \omega\} \in \mathbb{S}$.*

Let $\mathbb{S}_\alpha$ denote a countable support iteration of $\mathbb{S}$ of length α. We shall need a version of the Fusion Lemma also for $\mathbb{S}_\alpha$. If $p, q \in \mathbb{S}_\alpha$, $m, n \in \omega$ and $F \in [supp(q)]^{<\omega}$ we will write $(p, m) <_F (q, n)$, when $p \leq q$ and $\forall \beta \in F\ p \restriction \beta \Vdash$ "$(p(\beta), m) < (q(\beta), n)$". Abusing the notation slightly, we can state the Fusion Lemma as follows.

Lemma II.2. *([BaL]) Let $\{(p_i, n_i, F_i) : i \in \omega\}$ be such that $p_i \in \mathbb{S}_\alpha$, $n_i \in \omega$, $F_i \subseteq F_{i+1}$, $\bigcup F_i = \bigcup supp(p_i)$ and $(p_{i+1}, n_{i+1}) <_{F_i} (p_i, n_i)$ for every i. Define p so that $supp(p) = \bigcup supp(p_i)$ and $\forall \beta \in supp(p)\ p(\beta) = \bigcap\{p_i(\beta) : \beta \in supp(p_i)\}$. Then $p \in \mathbb{S}_\alpha$.*

Let $p \in \mathbb{S}_\alpha$, $F \in [supp(p)]^{<\omega}$ and $\sigma : F \longrightarrow 2^n$. Denote by $p \restriction \sigma$ the function with the same domain as p such that

$$(p \restriction \sigma)(\beta) = \begin{cases} p(\beta) & \text{if } \beta \notin F \\ p(\beta)_{\sigma(\beta)} & \text{if } \beta \in F. \end{cases}$$

The function $p \restriction \sigma$ does not necessarily have to be a condition. We will say that σ is *consistent with* p if $p \restriction \sigma \in \mathbb{S}_\alpha$ (i.e. if $\forall \beta \in F$ $(p \restriction \sigma) \restriction \beta \Vdash$ "$\sigma(\beta) \in p(\beta)$"). A condition p is said to be (F,n)-*determined* provided that $\forall \sigma : F \longrightarrow 2^n$ either σ is consistent with p or $\exists \beta \in F$ s.t. $\sigma \restriction (F \cap \beta)$ is consistent with p and $(p \restriction \sigma) \restriction \beta \Vdash$ "$\sigma(\beta) \notin p(\beta)$".

Lemma II.3. *([BaL]) Let $p \in \mathbb{S}_\alpha$, $F \in [supp(p)]^{<\omega}$, $n \in \omega$ and $\sigma : F \longrightarrow 2^n$. Then:*

1. *If $maxF < \beta < \alpha$ then $(p \restriction \sigma) \restriction \beta = (p \restriction \beta) \restriction \sigma$.*
2. *p is $(\{0\}, n)$-determined for every $n \in \omega$.*
3. *If $k \geq n$, $F \subseteq G$, $(q, m) <_G (p, k)$ and p is (F, n)-determined then so is q.*
4. *If $maxF < \beta < \alpha$ then p is (F, n)-determined iff $p \restriction \beta$ is (F, n)-determined.*
5. *There is $q \in \mathbb{S}_\alpha$, $q \leq p$ such that for some $\sigma : F \longrightarrow 2^n$ $q = q \restriction \sigma$.*
6. *If p is (F, n)-determined and $q \leq p$ then there is $\sigma : F \longrightarrow 2^n$ such that σ is consistent with p and, q and $p \restriction \sigma$ are compatible.*

Proof. See [BaL]. $\square$

A condition $p \in \mathbb{S}_\alpha$ is *continuous* iff $\forall F \in [supp(p)]^{<\omega}$ $\forall n \in \omega$ $\exists m \geq n$ $\exists G \in [supp(p)]^{<\omega}$, $F \subseteq G$ so that p is (G, m)-determined.

Lemma II.4. *([BaL]) Let $p \in \mathbb{S}_\alpha$, $n \in \omega$ and $F \in [supp(p)]^{<\omega}$. There is $(q, m) <_F (p, n)$ such that q is (F, n)-determined.*

Proof. The lemma will be proved by induction on α.
$\underline{\alpha = 1}$: This is true since every $p \in \mathbb{S}_1$ is $(\{0\}, n)$-determined for every n.
$\underline{\alpha = \beta + 1}$: Only the case when $\beta \in F$ has to be considered. There are $\mathbb{S}_\beta$-names $\dot{q}$ and $\dot{m}$ such that $p \restriction \beta \Vdash$ "$(\dot{q}, \dot{m}) < (p(\beta), n)$". By the inductive hypothesis there is a q' which is $(F \setminus \{\beta\}, n)$-determined, $(q', m') <_{F \setminus \{\beta\}} (p \restriction \beta, n)$ and q' decides $\dot{q} \cap 2^n$. For every σ consistent with q' let m_σ be such that $q' \restriction \sigma \Vdash$ "$\dot{m} = m_\sigma$". Put $q = q' {}^\frown \dot{q}$ and $m = max\{\{m'\} \cup \{m_\sigma : \sigma \text{ is consistent with } q'\}\} + 1$.
$\underline{\alpha\text{-limit}}$: Choose β such that $maxF < \beta < \alpha$. Let $q' \in \mathbb{S}_\beta$ be such that $(q', m) <_F (p \restriction \beta, n)$ and q' is (F, n)-determined. Then put

$$q(\gamma) = \begin{cases} q'(\gamma) & \text{if } \gamma < \beta \\ p(\gamma) & \text{if } \gamma \geq \beta \end{cases}$$

It is easy to see that this works. $\square$

Lemma II.5. *For every $p \in \mathbb{S}_\alpha$ there is a continuous $q \leq p$.*

Proof. Use the previous lemma to construct recursively $p_i \in \mathbb{S}_\alpha$, $n_i \in \omega$ and F_i a finite subset of α satisfying the following:

1. $p_0 = p$, $n_0 = 1$, $F_0 = \{\min(supp(p))\}$,
2. p_{i+1} is (F_i, n_i)-determined,
3. $(p_{i+1}, n_{i+1}) <_{F_i} (p_i, n_i)$,
4. $\bigcup\{F_i : i \in \omega\} = \bigcup\{supp(p_i) : i \in \omega\}$,
5. $F_i \subseteq F_{i+1}$.

Let q be the fusion of this sequence. Then q is obviously a continuous extension of p $\square$

We shall make use of the fact that every continuous condition q is fully described by the sequence $\{(F_i, n_i, \Sigma_i) : i \in \omega\}$ where F_i, n_i are as above, and $\Sigma_i = \{\sigma : F_i \longrightarrow 2^{n_i}$ such that σ is consistent with $q\}$. The important property of this representation is that (informally) each condition is forced to branch between levels n_i and n_{i+1}. Notice that if $\{(F_i, n_i, \Sigma_i) : i \in \omega\}$ is a representation of a continuous q and $f \in \omega^\omega$ is a strictly increasing function, then $\{(F_{f(i)}, n_{f(i)}, \Sigma_{f(i)}) : i \in \omega\}$ also represents the same q.

Lemma II.6. *Let $q \leq p \in \mathbb{S}_\alpha$ be continuous conditions. There are $\{(F_i^q, n_i^q, \Sigma_i^q) : i \in \omega\}$ a representation of q and $\{(F_i^p, n_i^p, \Sigma_i^p) : i \in \omega\}$ a representation of p such that*

$$\forall i \in \omega \quad F_i^q \cap supp(p) \subseteq F_i^p \text{ and } n_i^q < n_i^p < n_{i+1}^q.$$

Proof. By induction using previous remark. $\square$

Let a^* be a countable set of ordinals. Define $\mathbb{S}_{a^*}$ as a countable support iteration of Sacks forcing with domain a^*, i.e. $\mathbb{S}_{a^*}$ is isomorphic to $\mathbb{S}_\delta$ where δ is the order type of a^*. Even though, in general, it is not obvious that every condition in $\mathbb{S}_{a^*}$ can be viewed as a condition in $\mathbb{S}_{\omega_2}$ it is obviously so for continuous ones. Since the set of continuous conditions is dense in $\mathbb{S}_{\omega_2}$ and closed under fusion we can (and will) from now on assume that **all conditions mentioned are continuous**.

Lemma II.7. *Let a^* be a countable subset of $\alpha < \omega_2$. Let $p^* \in \mathbb{S}_{a^*}$, $q \in \mathbb{S}_\alpha$ such that $q \leq p^*$. Then there is a $q^* \in \mathbb{S}_{a^*}$, $q^* \leq p^*$ such that, every $r^* \in \mathbb{S}_{a^*}$ incompatible with q is incompatible with q^*.*

Proof. Let $q \leq p^*$ be given together with their representations $\{(F_i^q, n_i^q, \Sigma_i^q) : i \in \omega\}$ and $\{(F_i^{p^*}, n_i^{p^*}, \Sigma_i^{p^*}) : i \in \omega\}$. Without loss of generality we can assume that $\bigcup\{F_i^{p^*} : i \in \omega\} = a^*$ and the representations are as in Lemma II.6. Define q^* via a representation by putting for every $i \in \omega$:

$F_i^{q^*} = F_i^{p^*}$,

$n_i^{q^*} = n_i^{p^*}$ and

$\Sigma_i^{q^*} = \{\sigma \in \Sigma_i^{p^*} : \exists \tau \in \Sigma_{i+1}^q \quad \forall \beta \in F_i^{q^*} \quad \sigma(\beta) \subseteq \tau(\beta)\}$.

It is easy to see that this, indeed, defines a representation of a condition. Another way of describing the same procedure is as a fusion of $p_i = \bigcup\{p \restriction \tau : \tau \in \Sigma_{i+1}^q\}$. So $q^* \in \mathbb{S}_{a^*}$ and obviously $q^* \leq p^*$.

Let $r^* \in \mathbb{S}_{a^*}$ be compatible with q^*. Let $s^* \in \mathbb{S}_{a^*}$ be their common extension. Let $\{(F_i^q, n_i^q, \Sigma_i^q) : i \in \omega\}$ and $\{(F_i^{s^*}, n_i^{s^*}, \Sigma_i^{s^*}) : i \in \omega\}$ be representations of q and s^* such that for every $i \in \omega$ $F_i^{s^*} \subseteq F_i^q$ and $n_i^{s^*} < n_i^q < n_{i+1}^{s^*}$. As in Lemma II.6. this is very easy to provide. Define a common extension t of s^* and q by putting

$F_i^t = F_i^q$,

$n_i^t = n_i^q$ and

$\Sigma_i^t = \{\sigma \in \Sigma_i^q : \exists \tau \in \Sigma_{i+1}^{s^*} \quad \forall \beta \in F_i^{s^*} \quad \sigma(\beta) \subseteq \tau(\beta)\}$.

The condition t also has an alternative description using fusion. It should be obvious that $t \leq q, s^*$. This finishes the proof. $\square$

Note that the lemma says that $\mathbb{S}_{a^*}$ is "nearly" regularly embedded into $\mathbb{S}_{\omega_2}$. A virtually identical analysis (for a forcing notion different that the Sacks forcing) is contained in [HSZ].

III. Cardinal invariants in the Sacks model.

It is well known (see c.f. [BJ]) that iteration of any forcing having the Sacks property (i.p. the Sacks forcing itself) preserves that the ground model meager sets are cofinal. Hence $cof(\mathcal{M}) = \omega_1$ in the Sacks model. It is also known that $\mathbb{S}$ preserves P-points, hence $\mathfrak{u} = \omega_1$ in the Sacks model. As a consequence, most cardinal invariants are small in the Sacks model. There are, however, cardinal invariants the smallness of which (in the Sacks model) does not follow from the above. The aim of this section is to show that some of these cardinal invariants are also small in the Sacks model. The main tool used here is the Lemma II.7.

It is tempting to say that the following lemma is probably folklore but the same could be said for any of the results contained in this note.

Lemma III.1. *(CH) For every proper ω^ω-bounding forcing $\mathbb{P}$ of size ω_1 there is a $\mathbb{P}$-indestructible MAD family.*

Proof. Using properness of $\mathbb{P}$ (and CH) it is possible to construct a sequence $\{(p_\alpha, \tau_\alpha) : \alpha < \omega_1\}$, where $p_\alpha \in \mathbb{P}$, τ_α is a $\mathbb{P}$-name, so that if τ is a $\mathbb{P}$-name and $p \Vdash$ "$\tau \in [\omega]^\omega$" then there is an $\alpha \in \omega_1$ such that $p_\alpha \leq p$ and $p_\alpha \Vdash$ "$\tau = \tau_\alpha$". Having fixed such a sequence an almost disjoint family $\mathcal{A} = \{A_\alpha : \alpha < \omega_1\}$ will be constructed by induction.

Let $\{A_i : i \in \omega\}$ be a partition of ω into infinite sets. At stage α consider the pair (p_α, τ_α). If $p_\alpha \nVdash$ "$\forall \beta < \alpha\ |\tau_\alpha \cap A_\beta| < \omega$" then let A_α be any infinite set almost disjoint from all the A_β, $\beta < \alpha$. If $p_\alpha \Vdash$ "$\forall \beta < \alpha\ |\tau_\alpha \cap A_\beta| < \omega$" let $\{B_m : m \in \omega\}$ be an enumeration of pairwise disjoint finite modifications of $\{A_\beta : \beta < \alpha\}$. Let ρ be a name such that $p_\alpha \Vdash$ "$\rho \in \omega^\omega$ and $\forall m \in \omega\ B_m \cap \tau_\alpha \subseteq \rho(m)$". As $\mathbb{P}$ is ω^ω-bounding, there is an $f \in \omega^\omega$ and a $q \leq p_\alpha$ such that $q \Vdash$ "$\rho \leq f$". Put

$$A_\alpha = \bigcup_{m \in \omega} B_m \cap f(m).$$

To finish the proof it is sufficient to show that $\Vdash_{\mathbb{P}}$ "$\mathcal{A}$ is MAD". To that end assume the contrary. That is, there is a $\mathbb{P}$-name for a real τ and a condition $p \in \mathbb{P}$ such that $p \Vdash$ "$\forall \alpha < \omega_1 : |\tau \cap A_\alpha| < \aleph_0$". There is a β such that $p_\beta \leq p$ and $p_\beta \Vdash$ "$\tau = \tau_\beta$". Then, however, $p_\beta \Vdash$ "$\tau \subseteq A_\beta$" which is a contradiction. $\Box$

Theorem III.2. *$\mathfrak{a} = \omega_1$ in the Sacks Model.*

Proof. Let $\mathcal{A}$ be an $\mathbb{S}_{\omega_1}$-indestructible MAD family. CH holds in the ground model and even though $\mathbb{S}_{\omega_1}$ itself does not have cardinality $\aleph_1$ it has a dense subset of cardinality $\aleph_1$. Take for instance the set of all continuous conditions. So the Lemma III.1. applies. The plan is to show that $\mathcal{A}$ is in fact $\mathbb{S}_{\omega_2}$-indestructible.

To that end assume that there is a $\mathbb{S}_\alpha$-name τ for a real and a $p^* \in \mathbb{S}_\alpha$ such that $p^* \Vdash_{\mathbb{S}_\alpha}$ "$\forall A \in \mathcal{A} \quad |\tau \cap A| < \aleph_0$". Let N be a countable elementary submodel of $H(\omega_2)$ such that $p^*, \mathbb{S}_\alpha, \tau, \mathcal{A} \in N$. Let $D_n = \{p \in \mathbb{S}_\alpha : p$ decides whether $n \in \tau\}$.

Recall that all conditions involved are assumed to be continuous, hence absolute. Let $a^* = \alpha \cap N$ and let $q^* \leq p^*$ be $(N, \mathbb{S}_\alpha)$-generic such that $q^* \in \mathbb{S}_{a^*}$. Then

(1) $\forall n \in \omega\ D_n \cap N$ is predense below q^* and $D_n \cap N \subseteq \mathbb{S}_{a^*}$ and
(2) there is an $\mathbb{S}_{a^*}$-name τ' such that $q^* \Vdash_{\mathbb{S}_\alpha}$ "$\tau = \tau'$".

Since $\mathcal{A}$ is $\mathbb{S}_{\omega_1}$-indestructible it is also $\mathbb{S}_{a^*}$-indestructible. Using that and the existential completeness of forcing,

$$\exists r^* \in \mathbb{S}_{a^*} \quad r^* \leq q^* \quad \exists A \in \mathcal{A} \quad r^* \Vdash_{\mathbb{S}_{a^*}} \text{“}|A \cap \tau'| = \aleph_0\text{”}.$$

However, since $r^* \leq p^*$ and $p^* \Vdash_{\mathbb{S}_\alpha}$ "$\forall A \in \mathcal{A} \quad |\tau \cap A| < \aleph_0$",

$$\exists q \in \mathbb{S}_\alpha \quad q \leq r^* \quad \exists M \in \omega \quad q \Vdash_{\mathbb{S}_\alpha} \text{“}\tau \cap A \subseteq M\text{”},$$

which means that q is not compatible with those elements of D_n for $n > M$, $n \in A$ which force $n \in \tau$. By Lemma II.7. there is $s^* \in \mathbb{S}_{a^*}$, $s^* \leq r^*$ such that every $t^* \in \mathbb{S}_{a^*}$ incompatible with q is also incompatible with s^*. Therefore $s^* \Vdash_{\mathbb{S}_{a^*}}$ "$\tau \cap A \subseteq M$" which is contradictory to the fact that $r^* \Vdash_{\mathbb{S}_{a^*}}$ "$|A \cap \tau| = \aleph_0$". □

Next it is shown that $\mathfrak{a}_T = \omega_1$ in the Sacks model.

Lemma III.3. *(CH) There is a $\mathbb{S}_{\omega_1}$-indestructible partition of ω^ω into compact sets .*

Proof. Fix a sequence $\{(p_\alpha, \tau_\alpha) : \alpha < \omega_1\}$, where $p_\alpha \in \mathbb{S}_{\omega_1}$, τ_α is a $\mathbb{S}_{\omega_1}$-name, such that if τ is a $\mathbb{S}_{\omega_1}$-name and $p \Vdash$ "$\tau \in \omega^\omega$" then there is an $\alpha \in \omega_1$ such that $p_\alpha \leq p$ and $p_\alpha \Vdash$ "$\tau = \tau_\alpha$".

Construct a sequence $\langle T_\alpha : \alpha < \omega_1 \rangle$ of finitely branching subtrees of $\omega^{<\omega}$ by induction on α so that:

(1) $[T_\alpha] \cap \bigcup_{\beta<\alpha} [T_\beta] = \emptyset$ and
(2) $\exists q \leq p_\alpha \quad \exists \beta \leq \alpha : q \Vdash$ "$\tau_\alpha \in [T_\beta]$".

First find a $q \leq p_0$ and $g \in \omega^\omega$ such that $q \Vdash$ "$\tau_0 \leq g$" and let $T_0 = \bigcup_{n\in\omega} \{\sigma \in 2^n : \sigma \leq g \restriction n\}$. At stage α consider the pair (p_α, τ_α).

If there is a $p' \leq p_\alpha$ such that $p' \Vdash$ "$\tau_\alpha \in \bigcup_{\beta<\alpha} [T_\beta]$" let T_α be arbitrary satisfying (1). Then, of course, there is a $q \leq p'$ and a $\beta < \alpha$ such that $q \Vdash$ "$\tau_\alpha \in [T_\beta]$".

If not, find a $p' \leq p_\alpha$ and a $g \in \omega^\omega$ such that $p' \Vdash$ "$\tau_\alpha \notin \bigcup_{\beta<\alpha} [T_\beta]$ and $\tau_\alpha \leq g$". Enumerate $\alpha = \{\alpha_n : n \in \omega\}$ and construct a fusion sequence $(q_{i+1}, m_{i+1}) <_{F_i} (q_i, m_i)$ such that $q_0 \leq p'$ and for every $\sigma : F_i \longrightarrow 2^{m_i}$ consistent with p_i there is an $s_\sigma \in \omega^{<\omega}$ such that $q_i \restriction \sigma \Vdash$ "$s_\sigma \subseteq \tau_\alpha$ and $s_\sigma \notin T_{\alpha_i}$". Let q be the fusion of the sequence and let $T_\alpha = \{t \in \omega^{<\omega} : \exists i \in \omega \quad \exists \sigma : F_i \longrightarrow 2^{m_i}$ consistent with q such that $q \restriction \sigma \Vdash$ "$t \subseteq s_\sigma$"$\}$. Note that T_α is a compact tree as every $f \in [T_\alpha]$ is dominated by g. Obviously $q \Vdash$ "$\tau_\alpha \in [T_\alpha]$". □

Theorem III.4. $\mathfrak{a}_T = \omega_1$ *in the Sacks model.*

Proof. Fix a partition $\mathcal{T} = \{T_\alpha : \alpha < \omega_1\}$ as in the previous lemma (CH holds in the ground model). It will be shown that $\mathcal{T}$ is not only $\mathbb{S}_{\omega_1}$-indestructible but also $\mathbb{S}_{\omega_2}$-indestructible.

Assume that it is not the case. Then there is an $\alpha < \omega_1$, a $p \in \mathbb{S}_\alpha$, and an $\mathbb{S}_\alpha$-name $\dot{f}$ for a real such that $p \Vdash_{\mathbb{S}_\alpha}$ "$\dot{f} \notin \bigcup\{[T_\alpha] : \alpha < \omega_1\}$". Again, we can assume that p and all conditions mentioned later are continuous. Fix a countable elementary submodel N containing $\mathbb{S}_\alpha, p, \dot{f}, \mathcal{T}$ and let $a^* = N \cap \alpha$. Then $p \in \mathbb{S}_{a^*}$ and $\mathcal{T}$ is $\mathbb{S}_{a^*}$-indestructible. Let $r^* \leq p$ be $(N, \mathbb{S}_\alpha)$-generic such that $r^* \in \mathbb{S}_{a^*}$. There is a $\beta < \omega_1$ and $p^* \in \mathbb{S}_{a^*}$ such that $p^* \leq r^*$ and $p^* \Vdash_{\mathbb{S}_{a^*}}$ "$\dot{f} \in [T_\beta]$". On the other hand, there is a $q \leq p^*$ and a $\sigma \in \omega^{<\omega} \setminus [T_\beta]$ such that $q \Vdash_{\mathbb{S}_\alpha}$ "$\sigma \subseteq \dot{f}$". By Lemma II.7. there is a $q^* \in \mathbb{S}_{a^*}$, $q^* \leq p^*$, incompatible with all the elements of $\mathbb{S}_{a^*}$ which are incompatible with q.

As r^* is $(N, \mathbb{S}_\alpha)$-generic we can treat $\dot{f}$ also as a $\mathbb{S}_{a^*}$-name. Let D be the set of those $p \in \mathbb{S}_\alpha$ which decide $\dot{f} \restriction |\sigma|$. Then $D \in N$, $D \cap N \subseteq \mathbb{S}_{a^*}$ and $D \cap N$ is predense below r^*. As q is incompatible with all $s^* \in \mathbb{S}_{a^*}$ which force that $\dot{f} \restriction |\sigma| \neq \sigma$, so is q^*. That, however, means that $q^* \Vdash_{\mathbb{S}_{a^*}}$ "$\sigma \subseteq \dot{f}$" which contradicts the fact that $p^* \Vdash_{\mathbb{S}_{a^*}}$ "$\dot{f} \in [T_\beta]$". □

Next it will be shown that $\mathfrak{a}_e = \mathfrak{a}_p = \omega_1$ in the Sacks model. First it will be proved that, assuming CH, there are maximal families corresponding to the cardinal invariants indestructible by $\mathbb{S}_{\omega_1}$ and then the Lemma II.7. will be used to show that they are, in fact, $\mathbb{S}_{\omega_2}$-indestructible.

Lemma III.5. *(CH) There is an $\mathbb{S}_{\omega_1}$-indestructible maximal family of eventually different functions.*

Proof. Fix a sequence $\{(p_\alpha, \tau_\alpha) : \alpha < \omega_1\}$, where $p_\alpha \in \mathbb{S}_{\omega_1}$, τ_α is a $\mathbb{S}_{\omega_1}$-name, such that if τ is a $\mathbb{S}_{\omega_1}$-name and $p \Vdash$ "$\tau \in \omega^\omega$" then there is an $\alpha \in \omega_1$ such that $p_\alpha \leq p$ and $p_\alpha \Vdash$ "$\tau = \tau_\alpha$".

We will construct a sequence $\langle f_\alpha : \alpha < \omega_1 \rangle$, each $f_\alpha \in \omega^\omega$ by induction on α so that:

(1) f_α is eventually different from f_β for every $\beta < \alpha$ and
(2) $\exists q \leq p_\alpha \quad \exists \beta \leq \alpha : q \Vdash$ "$|\tau_\alpha \cap f_\beta| = \aleph_0$".

At stage α consider the pair (p_α, τ_α).

If there is a $q \leq p_\alpha$ and a $\beta < \alpha$ such that $q \Vdash$ "$|\tau_\alpha \cap f_\beta| = \aleph_0$", let f_α be arbitrary satisfying (1).

If it is not the case, enumerate $\alpha = \{\alpha_i : i \in \omega\}$ and construct a fusion sequence $(q_{i+1}, m_{i+1}) <_{F_i} (q_i, m_i)$, a tree $T \subseteq \omega^{<\omega}$ and an increasing sequence of integers $\langle n_i : i \in \omega \rangle$ so that

a) q_0 decides $\tau_\alpha \restriction n_0$,
b) $q_i \Vdash$ "$\tau_\alpha \cap f_{\alpha_i} \subseteq n_i \times \omega$",
c) for every $\sigma : F_i \longrightarrow 2^{m_i}$ consistent with q_i there is an $s_\sigma \in T \cap \omega^{n_i}$ such that $q_i \restriction \sigma \Vdash$ "$s_\sigma \subseteq \tau_\alpha$",
d) for every $s \in T \cap \omega^{n_i}$ there is a σ consistent with p_i such that $s = s_\sigma$ and
e) $|T \cap \omega^{n_{i+1}}| \leq n_{i+1} - n_i$.

Let q be the fusion of the sequence. Obviously, $q \Vdash$ "$\tau_\alpha \in [T]$" and also $\forall s \in T$ $\forall m \in dom(s)$ $m > n_i \Rightarrow s(m) \neq f_{\alpha_i}(m)$. Enumerate $T \cap \omega^{n_i} = \{s^i_j : j \leq J_i\}$ for every $i \in \omega$. It follows from the construction that $J_{i+1} \leq n_{i+1} - n_i$. Now let

$$f_\alpha(k) = \begin{cases} s^i_j(k) & \text{if } k = n_i + j \text{ and } j < J_i \\ min\{s(k) : s \in T \cap \omega^{k+1}\} & \text{otherwise.} \end{cases}$$

It is immediate that $q \Vdash$ "$|\tau_\alpha \cap f_\alpha| = \aleph_0$" and that f_α is eventually different from all f_β, $\beta < \alpha$. □

Theorem III.6. $\mathfrak{a}_e = \omega_1$ *in the Sacks model.*

Proof. Fix a family $\mathcal{F} = \{f_\alpha : \alpha < \omega_1\}$ as in the previous lemma (CH holds in the ground model). It will be shown that $\mathcal{F}$ is $\mathbb{S}_{\omega_2}$-indestructible.

Assume that it is not the case. Then there is an $\alpha < \omega_1$, a $p \in \mathbb{S}_\alpha$, and an $\mathbb{S}_\alpha$-name $\dot{f}$ for a real such that p forces that $\dot{f}$ is eventually different from f_α for every $\alpha < \omega_1$. Assume that p and all conditions mentioned later are continuous. Fix a countable elementary submodel N containing $\mathbb{S}_\alpha, p, \dot{f}, \mathcal{F}$ and let $a^* = N \cap \alpha$. Then $p \in \mathbb{S}_{a^*}$ and $\mathcal{F}$ is $\mathbb{S}_{a^*}$-indestructible. Let $r^* \leq p$ be $(N, \mathbb{S}_\alpha)$-generic such that $r^* \in \mathbb{S}_{a^*}$. There is a $\beta < \omega_1$ and $p^* \in \mathbb{S}_{a^*}$ such that $p^* \leq r^*$ and $p^* \Vdash_{\mathbb{S}_{a^*}}$ "$|\dot{f} \cap f_\beta| = \aleph_0$". On the other hand, there is a $q \leq p^*$ and an $n \in \omega$ such that $q \Vdash_{\mathbb{S}_\alpha}$ "$\dot{f} \cap f_\beta \subseteq n$". By Lemma II.7. there is a $q^* \in \mathbb{S}_{a^*}$, $q^* \leq p^*$, incompatible with all the elements of $\mathbb{S}_{a^*}$ which are incompatible with q.

As r^* is $(N, \mathbb{S}_\alpha)$-generic we can treat $\dot{f}$ also as a $\mathbb{S}_{a^*}$-name. Let D_m be the set of those $p \in \mathbb{S}_\alpha$ which decide $\dot{f}(m)$ for $m \geq n$. Then $D_m \in N$, $D_m \cap N \subseteq \mathbb{S}_{a^*}$ and $D_m \cap N$ is predense below r^*. As q is incompatible with all $s^* \in \mathbb{S}_{a^*}$ which force that $\dot{f}(m) = f_\beta(m)$, so is q^*. That, however, means that $q^* \Vdash_{\mathbb{S}_{a^*}}$ "$\dot{f} \cap f_\beta \subseteq n$" which contradicts the fact that $p^* \Vdash_{\mathbb{S}_{a^*}}$ "$|\dot{f} \cap f_\beta| = \aleph_0$". □

Lemma III.7. *(CH) There is an $\mathbb{S}_{\omega_1}$-indestructible maximal almost disjoint family of graphs of permutations.*

Proof. Fix a sequence $\{(p_\alpha, \tau_\alpha) : \alpha < \omega_1\}$, where $p_\alpha \in \mathbb{S}_{\omega_1}$, τ_α is a $\mathbb{S}_{\omega_1}$-name, such that if τ is a $\mathbb{S}_{\omega_1}$-name and $p \Vdash$ "$\tau \in Sym(\omega)$" then there is an $\alpha \in \omega_1$ such that $p_\alpha \leq p$ and $p_\alpha \Vdash$ "$\tau = \tau_\alpha$".

We will construct a sequence $\langle \pi_\alpha : \alpha < \omega_1 \rangle$ of permutations on ω by induction on α so that:

(1) π_α is almost disjoint from π_β for every $\beta < \alpha$ and
(2) $\exists q \leq p_\alpha \quad \exists \beta \leq \alpha : q \Vdash$ "$|\tau_\alpha \cap \pi_\beta| = \aleph_0$".

At stage α consider the pair (p_α, τ_α).

If there is a $q \leq p_\alpha$ and a $\beta < \alpha$ such that $q \Vdash$ "$|\tau_\alpha \cap \pi_\beta| = \aleph_0$" let π_α be an arbitrary permutation satisfying (1).

If it is not the case, enumerate $\alpha = \{\alpha_i : i \in \omega\}$ and construct a fusion sequence $(q_{i+1}, m_{i+1}) <_{F_i} (q_i, m_i)$, a tree $T \subseteq \omega^{<\omega}$ and an increasing sequence of integers $\langle n_i : i \in \omega \rangle$ so that

a) $q_0 \Vdash$ "$\tau_\alpha \restriction n_0 = s_0$" for some one-to-one $s_0 \in \omega^{n_0}$,
b) $q_i \Vdash$ "$\tau_\alpha \cap \pi_{\alpha_i} \subseteq n_i \times \omega$" and $q_{i+1} \Vdash$ "$rng(\tau_\alpha \restriction n_{i+1}) \supseteq n_i$",
c) for every $\sigma : F_i \longrightarrow 2^{m_i}$ consistent with q_i there is an $s_\sigma \in T \cap \omega^{n_i}$ such that $q_i \restriction \sigma \Vdash$ "$s_\sigma \subseteq \tau_\alpha$",
d) for every $s \in T \cap \omega^{n_i}$ there is a σ consistent with q_i such that $s = s_\sigma$ and
e) $|T \cap \omega^{n_{i+1}}| \leq n_{i+1} - 2n_i$.

Let q be the fusion of the sequence. Obviously, $q \Vdash$ "$\tau_\alpha \in [T]$" and also $\forall s \in T$ $\forall m \in dom(s)\ m > n_i \Rightarrow s(m) \neq \pi_{\alpha_i}(m)$. Enumerate $T \cap \omega^{n_i} = \{s^i_j : j \leq J_i\}$ for every $i \in \omega$. It follows from the construction that $J_{i+1} \leq n_{i+1} - 2n_i$. Now construct π_α by induction. Let $\pi_\alpha \restriction n_0 = s_0$. Having defined $\pi_\alpha \restriction n_i$ let $A = n_i \setminus rng(\pi_\alpha \restriction n_i)$ and define $\pi_\alpha^{-1} \restriction A$ so that $\pi_\alpha^{-1}(k) \neq \pi_{\alpha_{i'}}^{-1}(k)$, $i' \leq i$, for every $k \in A$. For every $j < J_{i+1}$ inductively find an $l < n_{i+1}$ such that l is not in the domain of the part of π_α constructed so far and also such that $s^{i+1}_j(l)$ is not in the range of the part of π_α

constructed so far. As $n_{i+1} \geq 2n_i + J_{i+1}$ there is no problem in doing so. Finally define π_α on the rest of n_{i+1} so that it is one-to-one, and so that $\pi_\alpha(k) \neq \pi_{\alpha_{i'}}(k)$ for every $k \in n_{i+1} \setminus n_i$ and for every $i' \leq i$.

Then, indeed, π_α is a permutation as $\pi_\alpha \restriction n_i$ is one-to-one and $n_i \subseteq rng(\pi_\alpha \restriction n_{i+1})$ for every $i \in \omega$. It is also true that π_α is almost disjoint from all π_β, $\beta < \alpha$ and finally $q \Vdash$ "$|\tau_\alpha \cap \pi_\alpha| = \aleph_0$". □

Theorem III.8. $\mathfrak{a}_p = \omega_1$ *in the Sacks model.*

Proof. Fix a family $\mathcal{P} = \{\pi_\alpha : \alpha < \omega_1\}$ as in the previous lemma (CH holds in the ground model). It will be shown that $\mathcal{P}$ is $\mathbb{S}_{\omega_2}$-indestructible.

Assume that it is not the case. Then there is an $\alpha < \omega_1$, a $p \in \mathbb{S}_\alpha$, and an $\mathbb{S}_\alpha$-name $\dot\pi$ for a permutation such that p forces that $\dot\pi$ is eventually different from π_α for every $\alpha < \omega_1$. Assume that p and all conditions mentioned later are continuous. Fix a countable elementary submodel N containing $\mathbb{S}_\alpha, p, \dot\pi, \mathcal{P}$ and let $a^* = N \cap \alpha$. Then $p \in \mathbb{S}_{a^*}$ and $\mathcal{P}$ is $\mathbb{S}_{a^*}$-indestructible. Let $r^* \leq p$ be $(N, \mathbb{S}_\alpha)$-generic such that $r^* \in \mathbb{S}_{a^*}$. There is a $\beta < \omega_1$ and $p^* \in \mathbb{S}_{a^*}$ such that $p^* \leq r^*$ and $p^* \Vdash_{\mathbb{S}_{a^*}}$ "$|\dot\pi \cap \pi_\beta| = \aleph_0$". On the other hand, there is a $q \leq p^*$ and an $n \in \omega$ such that $q \Vdash_{\mathbb{S}_\alpha}$ "$\dot\pi \cap \pi_\beta \subseteq n$". By Lemma II.7. there is a $q^* \in \mathbb{S}_{a^*}$, $q^* \leq p^*$, incompatible with all the elements of $\mathbb{S}_{a^*}$ which are incompatible with q.

As r^* is $(N, \mathbb{S}_\alpha)$-generic we can treat $\dot\pi$ also as a $\mathbb{S}_{a^*}$-name. Let D_m be the set of those $p \in \mathbb{S}_\alpha$ which decide $\dot\pi(m)$ for $m \geq n$. Then $D_m \in N$, $D_m \cap N \subseteq \mathbb{S}_{a^*}$ and $D_m \cap N$ is predense below r^*. As q is incompatible with all $s^* \in \mathbb{S}_{a^*}$ which force that $\dot\pi(m) = \pi_\beta(m)$, so is q^*. That, however, means that $q^* \Vdash_{\mathbb{S}_{a^*}}$ "$\dot\pi \cap \pi_\beta \subseteq n$" which contradicts the fact that $p^* \Vdash_{\mathbb{S}_{a^*}}$ "$|\dot\pi \cap \pi_\beta| = \aleph_0$". □

IV. ♣ holds in the side-by-side Sacks model

A forcing notion (complete Boolean algebra or partial order) $\mathbb{B}$ is said to be (λ, κ)-*semidistributive* if every subset of κ of size κ in a forcing extension contains a ground model subset of size λ when forcing with $\mathbb{B}$.

In what follows it will be shown that ♣ holds in the side-by-side Sacks model. We develop a slightly more general framework in hope that it has more applications.

Let $\mathbb{P}$ be an Axiom A forcing and let $\langle \leq_n : n \in \omega \rangle$ be a sequence of orderings on $\mathbb{P}$ witnessing it. Define a partial order $\mathcal{A}(\mathbb{P}) = \mathbb{P} \times \omega$ ordered by $(p, n) \leq (q, m)$ if $n > m$ and $p \leq_n q$. Properties of $\mathcal{A}(\mathbb{P})$ depend, of course, not only on $\mathbb{P}$ but also on the choice of the orderings $\leq_n$.

Given a $\mathbb{P}$-name $\dot x$ for an uncountable subset of ω_1, a condition $p \in \mathbb{P}$ and an $n \in \omega$ let

$$A_n(p, \dot x) = \{\alpha \in \omega_1 : \exists q \in \mathbb{P} \quad q \leq_n p \text{ and } q \Vdash \text{“}\alpha \in \dot x\text{”}\}.$$

A condition $p \in \mathbb{P}$ is said to be $(\dot x, n)$-*good* if $\forall q \leq_n p \quad |A_n(q, \dot x)| = \aleph_1$. A forcing notion $\mathbb{P}$ (together with an Axiom A structure) is said to be ω_1-*good* provided that for every $\mathbb{P}$-name $\dot x$ for an uncountable subset of ω_1 and for every $n \in \omega$ the set $\{p \in \mathbb{P} : p \text{ is } (\dot x, n)\text{-good}\}$ is dense in $\mathbb{P}$.

We will say that an Axiom A partial order $\mathbb{P}$ has *unique fusion* if whenever $\langle p_i : i \in \omega \rangle$ is a fusion sequence and $p, q \in \mathbb{P}$ are such that $\forall i \in \omega \ p \leq_i p_i$ and

$q \leq_i p_i$ then $p = q$. Recall also that if $\mathbb{P}$ is a forcing notion then $\mathfrak{m}(\mathbb{P})$ denotes the least number of dense subsets of $\mathbb{P}$ with no filter meeting them all.

Proposition IV.1. *Let $\mathbb{P}$ be ω_1-good. Then:*

(1) *$\mathbb{P}$ is (ω, ω_1)-semidistributive.*
(2) *If $\mathbb{P}$ has unique fusion and $\mathfrak{m}(\mathcal{A}(\mathbb{P})) > \omega_1$ (in fact, if $MA_{\aleph_1}$ holds for $\mathcal{A}(\mathbb{P})$ below every condition) then $\mathbb{P}$ is (ω_1, ω_1)-semidistributive.*

Proof. Let $\dot{x}$ be a $\mathbb{P}$-name for an uncountable subset of ω_1 and let $p \in \mathbb{P}$. Construct sequences $\langle p_i : i \in \omega \rangle$, $\langle \alpha_i : i \in \omega \rangle$ such that:
a) $\alpha_i = \alpha_j \Rightarrow i = j$,
b) $p_0 \leq p$ and $p_{i+1} \leq_i p_i$,
c) p_i is $(\dot{x}, i)$-good and
d) $p_i \Vdash$ "$\alpha_i \in \dot{x}$".
It is easy to fulfill the task given the fact that $\mathbb{P}$ is ω_1-good. Let p_ω be the fusion of the sequence $\langle p_i : i \in \omega \rangle$. Then $p_\omega \leq p$ and $p_\omega \Vdash$ "$\{\alpha_i : i \in \omega\} \subseteq \dot{x}$" witnessing the (ω, ω_1)-semidistributivity of $\mathbb{P}$.

In order to prove (2) Let $p \in \mathbb{P}$ be given and let

$$D_\alpha = \{(q, n) \in \mathcal{A}(\mathbb{P}) : q \text{ is } (\dot{x}, n)\text{-good and } q \Vdash \text{“}\beta \in \dot{x}\text{” for some } \beta \geq \alpha\}$$

and let

$$E_n = \{(q, m) \in \mathcal{A}(\mathbb{P}) : q \in \mathbb{P} \text{ and } m \geq n\}.$$

As $\mathbb{P}$ is ω_1-good the set D_α is dense in $\mathcal{A}(\mathbb{P})$ for every α. The sets E_n are obviously dense. Let G be an ultrafilter on $\mathcal{A}(\mathbb{P})$ containing $(p, 0)$ which meets all of the D_α and E_n. For each $i \in \omega$ choose $p_i \in \mathbb{P}$ and $m_i \geq i$ such that $(p_i, m_i) \in G$, $p_0 \leq p$ and $(p_{i+1}, m_{i+1}) < (p_i, m_i)$. Then the sequence $\langle p_i : i \in \omega \rangle$ is a fusion sequence in $\mathbb{P}$. Let p_ω be the fusion of the sequence. Obviously $p_\omega \in \mathbb{P}$. Let $Y = \{\alpha \in \omega_1 : p_\omega \Vdash_{\mathbb{P}}$ "$\alpha \in \dot{x}$"$\}$. All that is left to show is that Y is uncountable. If not then there is an $\alpha < \omega_1$ such that $Y \subseteq \alpha$. Let $(q, k) \in D_\alpha \cap G$. The following Claim clearly produces a contradiction, hence finishes the proof.

Claim. $p_\omega \leq q$.

In order to prove the Claim construct a sequence $\langle (q_i, k_i) \in \mathcal{A}(\mathbb{P}) : i \in \omega \rangle$ such that
a) $(q_0, k_0) = (q, k)$,
b) $(q_{i+1}, k_{i+1}) \leq (q_i, k_i)$,
c) $(q_{i+1}, k_{i+1}) \leq (p_i, m_i)$.
To accomplish the goal simply pick $(q_{i+1}, k_{i+1}) \in G$ extending both (q_i, k_i) and (p_i, m_i). The sequence $\langle q_i : i \in \omega \rangle$ is a fusion sequence. Let q_ω be the fusion of the sequence. Note that $q_\omega \leq_i p_i$ for every $i \in \omega$. As $\mathbb{P}$ has unique fusion $q_\omega = p_\omega$ and hence $p_\omega \leq q$. □

Examples. Cohen forcing $Fn(\omega, 2)$ is trivially (ω_1, ω_1)-semidistributive. Other forcing notions such as random forcing, Hechler forcing, Mathias forcing, Laver forcing and Sacks forcing are (ω, ω_1)-semidistributive and, in some models, these forcings are even (ω_1, ω_1)-semidistributive.

Here we concentrate on Sacks forcing. Recall that if $p \in \mathbb{S}$ then $t \in p$ is a *branching node* of p if $t^\frown 0, t^\frown 1 \in p$. The standard Axiom A orderings for the

Sacks forcing ($p \leq_n q$ if $p \leq q$ and the first n-many branching levels of q are contained in p) obviously have unique fusion property. For $p \in \mathbb{S}$ and $k \in \omega$ let $p \restriction k = \{t \restriction k : t \in p\}$ and if $a \subseteq p$ let $p\langle a\rangle = \{t \in p : \exists s \in a\ s \subseteq t \text{ or } t \subseteq s\}$.

To show that $\mathbb{S}$ is ω_1-good it is enough to show that whenever $\dot{x}$ is a name for an uncountable subset of ω_1, $p \in \mathbb{S}$ and $m \in \omega$ then the following holds:

Claim IV.2. *If $p \in \mathbb{S}$ is $(\dot{x}, m)$-good then there is a $q \leq_m p$ such that q is $(\dot{x}, m+1)$-good.*

Suppose the Claim fails. Construct a sequence $\langle p_n : n \in \omega\rangle \subseteq \mathbb{S}$ and for every p_n an integer k_n so that

a) $p_0 \leq_m p$, $|p_0 \cap 2^{k_0}| = 2^m$ and every $t \in 2^{k_0}$ contains m-many branching nodes,

b) $(p_{n+1}, k_{n+1}) < (p_n, k_n)$ and

c) if $a \in [p_n \cap 2^{k_n}]^{2^{m+1}}$ and $(\forall t \in p_0 \restriction k_0\ \exists t^0 \neq t^1 \in a \text{ s.t. } t \subseteq t^0 \cap t^1)$ then $|A_{m+1}(p_{n+1}\langle a\rangle, \dot{x})| < \aleph_1$.

To do this suppose that p_n, k_n have been already constructed. Enumerate all $a \subseteq p_n \cap 2^{k_n}$ relevant for c) as $\{a_i : i < I\}$. Construct $\{p_n^i : i < I+1\}$ so that

d) $p_n^0 = p_n$,

e) $p_n^{i+1} \leq p_n^i$

f) $p_n \restriction k_n \subseteq p_n^i$ and

g) $|A_{m+1}(p_n^{i+1}\langle a_i\rangle, \dot{x})| < \aleph_1$.

At step i find $\bar{p}_n^i \leq p_n^i\langle a_i\rangle$ such that $a_i \subseteq \bar{p}_n^i$ and $|A_{m+1}(\bar{p}_n^i, \dot{x})| < \aleph_1$ (Note that if this is not possible then the Claim holds as then $p_n^i\langle a_i\rangle \leq_{m+1} p$ and is $(\dot{x}, m+1)$-good). Let

$$p_n^{i+1} = \bigcup\{\bar{p}_n^i\langle t\rangle : t \in a_i\} \cup \bigcup\{p_n^i\langle t\rangle : t \in p_n \cap 2^{k_n} \setminus a_i\}$$

and finally let $p_{n+1} = p_n^I$ and let k_{n+1} be such that $(p_{n+1}, k_{n+1}) < (p_n, k_n)$.

Now let p_ω be the fusion of the sequence and let

$$A = \bigcup\{A_{m+1}(p_\omega\langle a\rangle, \dot{x}) : a \in [p_n \cap 2^{k_n}]^{2^{m+1}} \text{ for some } n \in \omega \text{ as in c)}\}$$

and note that A is countable. Choose $\gamma \in A_m(p_\omega, \dot{x}) \setminus A$. Then there is a $p' \leq_m p_\omega$ such that $p' \Vdash$ "$\gamma \in \dot{x}$". Choose n such that the $m+1$-branching subtree of p' is contained in $p' \restriction k_n$, i.e. there is an $a \in [p_\omega \cap 2^{k_n}]^{2^{m+1}}$ satisfying the condition in c) such that $p'\langle a\rangle \leq_m p_\omega$. Then, however, $\gamma \in A_{m+1}(p_\omega\langle a\rangle, \dot{x})$ which is impossible. $\square$

So we have shown that $\mathbb{S}$ is (ω, ω_1)-semidistributive. As $\mathcal{A}(\mathbb{S})$ is proper (see e.g. [CL]) by Proposition IV.1. PFA implies that $\mathbb{S}$ is (ω_1, ω_1)-semidistributive.

J. Baumgartner (in an unpublished note) showed that ♣ holds in a model obtained from a model of $V = L$ by adding many Sacks reals side-by-side. A proof of this fact is presented here. The side-by-side Sacks forcing for adding κ many Sacks reals is denoted by $\mathbb{S}^\kappa$. Let F be a finite subset of κ, let $\dot{x}$ be a $\mathbb{S}^\kappa$ name for an uncountable subset of ω_1 and let m, n be integers. A condition $p \in \mathbb{S}^\kappa$ is said to be $(\dot{x}, F, n)$-*good* if $\forall (q, m) \leq_F (p, n)\quad |A_{(F,n)}(q, \dot{x})| = \aleph_1$, where $A_{(F,n)}(p, \dot{x}) = \{\alpha < \omega_1 : \exists (q, m) <_F (p, n) \text{ such that } q \Vdash \text{"}\alpha \in \dot{x}\text{"}\}$.

Lemma IV.3. *Let $p \in \mathbb{S}^\kappa$, let $F \subseteq G$ be finite subsets of κ, let $\dot{x}$ be a $\mathbb{S}^\kappa$-name for an uncountable subset of ω_1 and let n be an integer. If p is $(\dot{x}, F, n)$-good then there are $q \in \mathbb{S}^\kappa$ and $m > n$ such that $(q, m) <_F (p, n)$ and q is $(\dot{x}, G, m)$-good.*

Proof. The proof is an easy, though technical, extension of an analogous result for $\mathbb{S}$ in IV.2. □

Lemma IV.4. *($\diamondsuit$) There is a $\clubsuit$-sequence $\langle X_\alpha : \alpha \in Lim(\omega_1)\rangle$ such that for every $p \in \mathbb{S}^{\omega_1}$ and every $\mathbb{S}^{\omega_1}$-name $\dot{x}$ for an uncountable subset of ω_1 there are $q \leq p$ and $\alpha \in Lim(\omega_1)$ such that $q \Vdash$ "$X_\alpha \subseteq \dot{x}$".*

Proof. First identify every $\mathbb{S}^{\omega_1}$-name $\dot{y}$ for a subset of ω_1 with a set $Y \subseteq \mathbb{S}^{\omega_1} \times \omega_1$ by putting a pair (p, α) into Y if and only if $p \Vdash$ "$\alpha \in \dot{y}$".

Claim. *($\diamondsuit$) There is a sequence $\langle (p_\alpha, A_\alpha, M_\alpha) : \alpha \in Lim(\omega_1)\rangle$ such that if $p \in \mathbb{S}^{\omega_1}$, $A \subseteq \mathbb{S}^{\omega_1} \times \omega_1$ and $C \subseteq [H(\omega_2)]^{\aleph_0}$ is a closed and unbounded set of elementary submodels then there is an $M \in C$ and an $\alpha < \omega_1$ such that $M \cap H(\omega_1) = M_\alpha$, $M_\alpha \cap \omega_1 = \alpha$, $p = p_\alpha \in M_\alpha$ and $A \cap M_\alpha = A_\alpha$.*

To see this fix a $\diamondsuit$-sequence $\{D_\alpha : \alpha < \omega_1\}$ (i.e. a sequence such that $D_\alpha \subseteq \alpha$ for every $\alpha < \omega_1$ and such that for every $D \subseteq \omega_1$ there are stationarily many α such that $D \cap \alpha = D_\alpha$).

First (using CH, a consequence of $\diamondsuit$) construct a sequence $\langle M_\alpha : \alpha \in C'\rangle$ (for some closed unbounded set $C' \subseteq \omega_1$) such that

a) M_α is an elementary submodel of $H(\omega_1)$,
b) $M_\alpha \subseteq M_\beta$ for $\alpha < \beta$, $M_\beta = \bigcup\{M_\alpha : \alpha < \beta\}$ for β limit in C',
c) $\{M_\alpha : \alpha \in C'\}$ is a closed unbounded subset of $[H(\omega_1)]^{\aleph_0}$ and
d) $M_\alpha \cap \omega_1 = \alpha$ for every $\alpha \in C'$.

Doing this is straightforward. For $\alpha \notin C'$ let M_α be arbitrary. Note that $\bigcup\{M_\alpha : \alpha \in C'\} = H(\omega_1)$ and that for every $C \subseteq [H(\omega_2)]^{\aleph_0}$ closed and unbounded set of elementary submodels $\{\alpha < \omega_1 : \exists M \in C$ such that $M \cap H(\omega_1) = M_\alpha\}$ is a closed unbounded subset of C'.

Fix also a bijection $\Phi : \omega_1 \longrightarrow H(\omega_1)$ such that $\Phi[\alpha] = M_\alpha$ for every $\alpha \in C'$. Now we are ready to define p_α, A_α. If $\Phi[D_\alpha] = \{p\} \times A \in \mathcal{P}(\mathbb{S}^{\omega_1}) \times \mathcal{P}(\mathbb{S}^{\omega_1} \times \omega_1)$ and $\alpha \in C'$, let $p_\alpha = p$ and let $A_\alpha = A$. Otherwise let p_α and A_α be arbitrary.

To see that the construction works let p, A, C be as required (WLOG $p, A \in M$ for every $M \in C$) and let $D = \Phi^{-1}[\{p\} \times A]$. Let $C'' = \{\alpha \in C' : \exists M \in C$ such that $M_\alpha = M \cap H(\omega_1)\}$. Note that C'' is a closed unbounded subset of ω_1. There is an $\alpha \in C''$ such that $D_\alpha = D \cap \alpha$, as $\{D_\alpha : \alpha < \omega_1\}$ is a $\diamondsuit$-sequence. This, of course, implies that $p = p_\alpha$ and $A \cap M_\alpha = A_\alpha$. As $\alpha \in C''$ also $M_\alpha \cap \omega_1 = \alpha$ and there is an $M \in C$ such that $M \cap H(\omega_1) = M_\alpha$. This finishes the proof of the claim.

Having fixed a sequence like this, construct X_α as follows:

If there is a $p \in \mathbb{S}^{\omega_1}$, $A \subseteq \mathbb{S}^{\omega_1} \times \omega_1$ a name for an uncountable subset of ω_1 and an elementary submodel M containing p and A such that $p_\alpha = p$, $M_\alpha = M \cap H(\omega_1)$ and $A_\alpha = A \cap M$ $(= A \cap M_\alpha)$ then fix a sequence $\langle \alpha_i : i \in \omega\rangle \nearrow \alpha$ and construct a sequence $\langle (q_i, n_i, F_i, \beta_i) : i \in \omega\rangle$ such that

(1) $F_i \subseteq F_{i+1}$ and $\bigcup_{i\in\omega} F_i = \alpha$
(2) $\alpha_i \leq \beta_i < \alpha$,
(3) $q_0 \leq p_\alpha$,
(4) $q_i \in \mathbb{S}^{\omega_1} \cap M$,
(5) $(q_{i+1}, n_{i+1}) <_{F_i} (q_i, n_i)$,
(6) q_i is (A, F_i, n_i)-good and
(7) $q_i \Vdash$ "$\beta_i \in A$".

Finally put $X_\alpha = \{\beta_i : i \in \omega\}$. It is easy to go through the construction using previous lemma (and the fact that M is an elementary submodel).

If the triple $(p_\alpha, A_\alpha, M_\alpha)$ does not satisfy the above requirements let X_α be an arbitrary sequence increasing to α.

In order to verify that the construction works let $p \in \mathbb{S}^{\omega_1}$ and $\dot{x}$ be as required. Let $X \subseteq \mathbb{S}^{\omega_1} \times \omega_1$ be the "nice" name corresponding to $\dot{x}$. Let C be a closed unbounded set of elementary submodels of $H(\omega_2)$ containing p and X. Then there is an $\alpha \in Lim(\omega_1)$ and an $M \in C$ such that $p = p_\alpha$, $X \cap M_\alpha = A_\alpha$ and $M \cap H(\omega_1) = M_\alpha$. Let q be the fusion of the sequence constructed at stage α. Note that even though the model in which q was constructed was probably different from M and the name for an uncountable subset of ω_1 was most likely not X, in the construction we never had to go outside $H(\omega_1)$ on which the two models agree. So $q \Vdash$ "$X_\alpha \subseteq \dot{x}$". □

Theorem IV.5 (J. Baumgartner). *If $\diamondsuit$ holds in the ground model then $\clubsuit$ holds in the side-by-side Sacks extension.*

Proof. Let $\langle X_\alpha : \alpha \in Lim(\omega_1)\rangle$ be the $\clubsuit$-sequence constructed in the previous lemma. What remains to be proved is that it is still a $\clubsuit$-sequence after forcing with $\mathbb{S}^\kappa$. To that end let $\dot{x}$ be a name for an uncountable subset of ω_1 and let p be a condition. As all antichains in $\mathbb{S}^\kappa$ are of size at most $\aleph_1$ there is a set $X \subseteq \kappa$ of cardinality $\aleph_1$ and a $\mathbb{S}^X$-name $\dot{y}$ such that $p \in \mathbb{S}^X$ and $\Vdash_{\mathbb{S}^\kappa}$ "$\dot{x} = \dot{y}$. Recall also that $\mathbb{S}^\kappa \simeq \mathbb{S}^X \times \mathbb{S}^{\kappa \setminus X}$. Now, as $\mathbb{S}^X \simeq \mathbb{S}^{\omega_1}$, by previous lemma there is an $\alpha \in Lim(\omega_1)$ and a $q \in \mathbb{S}^X$ such that $q \leq p$ and $q \Vdash_{\mathbb{S}^X}$ "$X_\alpha \subseteq \dot{y}$". In fact $q \Vdash_{\mathbb{S}^\kappa}$ "$X_\alpha \subseteq \dot{x}$". □

Corollary IV.6. *If $\diamondsuit$ holds in the ground model then $\diamondsuit_{\mathfrak{d}}$[1] holds in the side-by-side Sacks extension.*

Proof. As $\clubsuit$ and $\mathfrak{d} = \omega_1$ both hold in the side-by-side Sacks model, so does $\diamondsuit_{\mathfrak{d}}$ by Proposition I.3. of [Hr]. □

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