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PAIR-SPLITTING, PAIR-REAPING AND CARDINAL INVARIANTS OF F_σ-IDEALS

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Abstract. We investigate the pair-splitting number $\mathfrak{s}_{pair}$ which is a variation of splitting number, pair-reaping number $\mathfrak{r}_{pair}$ which is a variation of reaping number and cardinal invariants of ideals on ω. We also study cardinal invariants of F_σ ideals and their upper bounds and lower bounds. As an application, we answer a question of S. Solecki by showing that the ideal of finitely chromatic graphs is not locally Katětov-minimal among ideals not satisfying Fatou's lemma.

Introduction. The splitting number $\mathfrak{s}$ and the reaping number $\mathfrak{r}$ are cardinal invariants which play important role when we study $\mathscr{P}(\omega)/fin$.

For $X, Y \in [\omega]^\omega$ we say X *splits* Y if both $X \cap Y$ and $Y \setminus X$ are infinite. We call $\mathscr{S} \subset [\omega]^\omega$ a *splitting family* if for each $Y \in [\omega]^\omega$, there exists $X \in \mathscr{S}$ such that X splits Y. The *splitting number* $\mathfrak{s}$ is the least size of a splitting family.

We call $\mathscr{R}$ a *reaping family* if for each $X \in [\omega]^\omega$, there exists $Y \in \mathscr{R}$ such that Y is not split by X, that is, $X \cap Y$ is finite or $Y \setminus X$ is finite. The *reaping number* $\mathfrak{r}$ is the least size of a reaping family.

We shall study variations of splitting number and of reaping number and study cardinal invariants of ideals of ω.

The pair-reaping number $\mathfrak{r}_{pair}$ and the pair-splitting number $\mathfrak{s}_{pair}$ are introduced in two different contexts with the same definition independently.

One is motivated by the investigation of the dual-reaping number $\mathfrak{r}_d$ and the dual-splitting number $\mathfrak{s}_d$ which are reaping number and splitting number for the structure of all infinite partitions of ω ordered by "almost coarser" $\big((\omega)^\omega, \leq^*\big)$ respectively.

We call $A \subset [\omega]^2$ *unbounded* if for $k \in \omega$, there exists $a \in A$ such that $a \cap k = \emptyset$. For $X \in [\omega]^\omega$ and unbounded $A \subset [\omega]^2$, X *pair-splits* A if there exist infinitely many $a \in A$ such that $a \cap X \neq \emptyset$ and $a \setminus X \neq \emptyset$. We call $\mathscr{S} \subset [\omega]^\omega$ a *pair-splitting family* if for each unbounded $A \subset [\omega]^2$, there exists $X \in \mathscr{S}$ such that X pair-splits A. The *pair-splitting number* $\mathfrak{s}_{pair}$ is the least size of a pair-splitting family.

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We call $\mathscr{R} \subset \mathscr{P}([\omega]^2)$ a *pair-reaping family* if for each $A \in \mathscr{R}$, A is unbounded and for $X \in [\omega]^\omega$, there exists $A \in \mathscr{R}$ such that X does not pair-split A, that is, for all but finitely many $a \in A$, $a \cap X = \emptyset$ or $a \subset X$. The *pair-reaping number* $\mathfrak{r}_{pair}$ is the least size of a pair-reaping family.

In [13] it is proved that there is the following relationship between $\mathfrak{r}_{pair}$, $\mathfrak{s}_{pair}$ and other cardinal invariants.

Proposition 0.1. (1) $\mathfrak{s}_{pair} \leq \mathsf{non}(\mathscr{M}), \mathsf{non}(\mathscr{N})$.
(2) $\mathfrak{r}_{pair} \geq \mathsf{cov}(\mathscr{M}), \mathsf{cov}(\mathscr{N})$.
(3) $\mathfrak{s}_{pair} \geq \mathfrak{s}$.
(4) $\mathfrak{r}_{pair} \leq \mathfrak{r}, \mathfrak{s}_d$.

It is not known whether $\mathfrak{r}_d \leq \mathfrak{s}_{pair}$ or not.

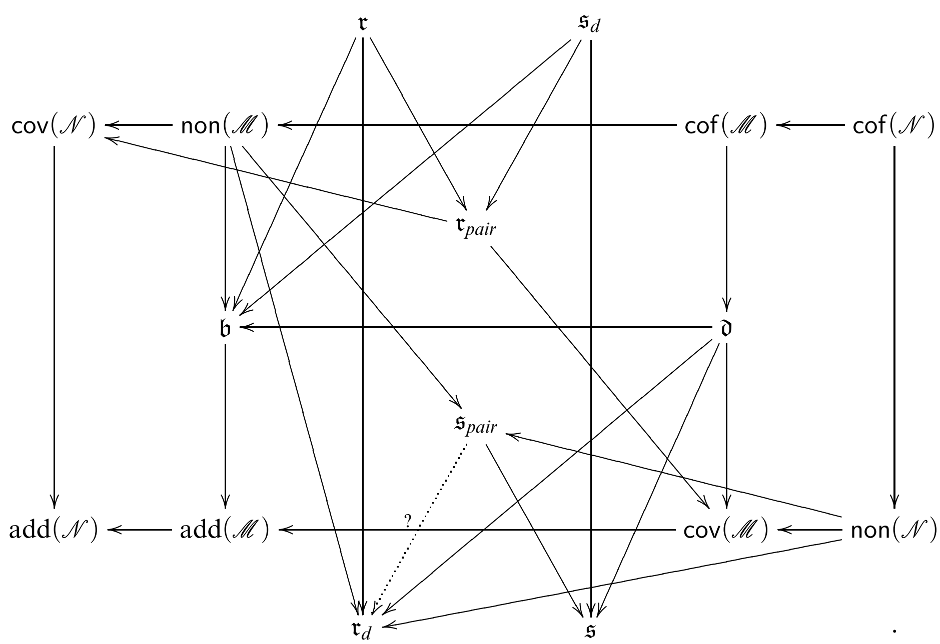
For $\mathscr{G} \subset \omega^\omega$, we call $\mathscr{G}$ a *dominating family* if for each $f \in \omega^\omega$, there exists $g \in \mathscr{G}$ such that for all but finitely many $n \in \omega$, $f(n) \leq g(n)$, denoted by $f \leq^* g$. The *dominating number* $\mathfrak{d}$ is the least size of a dominating family.

For $\mathscr{G} \subset \omega^\omega$, we call $\mathscr{G}$ an *unbounded family* if for each $f \in \omega^\omega$, there exists $g \in \mathscr{G}$ such that $g \not\leq^* f$, that is, there exist infinitely many $n \in \omega$ such that $g(n) > f(n)$. The *unbounded number* $\mathfrak{b}$ is the least size of an unbounded family.

$\mathfrak{s} \leq \mathfrak{d}$ and $\mathfrak{r} \geq \mathfrak{b}$ hold (see in [3]). Kamo proved the following statement in [13]:

Theorem 0.2. $\mathfrak{r}_d \leq \mathfrak{d}$ *and* $\mathfrak{s}_d \geq \mathfrak{b}$.

So we have the following diagram:



An arrow $\kappa \to \lambda$ denotes the inequality $\kappa \geq \lambda$.

In [13] by using finite support iteration of Hechler forcing, the following consistency results are proved.

THEOREM 0.3. *It is consistent that* $\mathfrak{s}_{pair} < add(\mathscr{M})$. *Dually it is consistent that* $\mathfrak{r}_{pair} > \mathsf{cof}(\mathscr{M})$.

$\mathfrak{r}_{pair}$ is a lower bound of $\mathfrak{r}$ and $\mathfrak{s}_d$, and $\mathfrak{s}_{pair}$ is an upper bound of $\mathfrak{s}$ (and maybe of $\mathfrak{r}_d$). So it is natural to ask the question whether $\mathfrak{s}_{pair} \leq \mathfrak{d}$ or not and whether $\mathfrak{r}_{pair} \geq \mathfrak{b}$ or not. In [14] the consistency of $\mathfrak{s}_{pair} > \mathfrak{d}$ and of $\mathfrak{r}_{pair} < \mathfrak{b}$ are shown and an upper bound of $\mathfrak{s}_{pair}$ and a lower bound of $\mathfrak{r}_{pair}$ are given.

The other source of motivation stems from the study of Borel ideals on ω.

For a set X, we call $\mathscr{I} \subset \mathscr{P}(X)$ an *ideal on X* if $\mathscr{I}$ satisfies the following:

(1) for $A, B \in \mathscr{I}$, $A \cup B \in \mathscr{I}$,
(2) for $A, B \subset X$, $A \subset B$ and $B \in \mathscr{I}$ implies $A \in \mathscr{I}$ and
(3) $X \notin \mathscr{I}$.

In this paper we assume that all ideals on X contain all finite subsets of X. We say an ideal $\mathscr{I}$ on ω is *tall* if for each $X \in [\omega]^\omega$ there exists $I \in \mathscr{I}$ such that $I \cap X$ is infinite.

If $\mathscr{I}$ is an ideal on ω and $Y \in [\omega]^\omega$, we denote by $\mathscr{I} \restriction Y$ the ideal $\{I \cap Y : I \in \mathscr{I}\}$ on Y.

The topology of $\mathscr{P}(\omega)$ is induced by identifying each subset of ω with its characteristic function, where 2^ω is equipped with the product topology of the discrete topology of $2 = \{0, 1\}$. We call $\mathscr{I}$ a Borel ideal on ω if $\mathscr{I}$ is an ideal on ω and $\mathscr{I}$ is Borel in this topology.

Let $\mathscr{I}$ be a tall ideal on ω. Then the *uniformity number* of $\mathscr{I}$, denoted by $\mathsf{non}^*(\mathscr{I})$ and the *covering number* of $\mathscr{I}$, denoted by $\mathsf{cov}^*(\mathscr{I})$ are given by

$$\mathsf{non}^*(\mathscr{I}) = \min\{|\mathscr{A}| : \mathscr{A} \subset [\omega]^\omega \wedge (\forall I \in \mathscr{I})(\exists A \in \mathscr{A})(|A \cap I| < \aleph_0)\},$$
$$\mathsf{cov}^*(\mathscr{I}) = \min\{|\mathscr{A}| : \mathscr{A} \subset \mathscr{I} \wedge (\forall X \in [\omega]^\omega)(\exists A \in \mathscr{A})(|X \cap A| = \aleph_0)\}.$$

The (pre)orderings on the family of ideals are crucial in describing some properties of ideals on ω. For example, Cohen-destructibility of an ideal $\mathscr{I}$ on ω is equivalent to the statement $\mathscr{I}$ is smaller than the nowhere dense ideal in the Katětov order ([8, 6]).

Suppose $\mathscr{I}$ and $\mathscr{J}$ are ideals on ω. Then $\mathscr{I} \leq_K \mathscr{J}$ if there exists a function $f : \omega \to \omega$ such that for each $I \in \mathscr{I}$, $f^{-1}[I] \in \mathscr{J}$. We call this ordering *Katětov order*. If the function f witnessing to $\mathscr{I} \leq_K \mathscr{J}$ is finite-to-one, then we call this ordering *Katětov–Blass order*, denoted by $\mathscr{I} \leq_{KB} \mathscr{J}$.

When we investigate the Katětov(-Blass) order, the uniformity number of ideals and the covering number of ideals are significant.

PROPOSITION 0.4. [5]

(1) *If* $\mathscr{I} \leq_K \mathscr{J}$, *then* $\mathsf{cov}^*(\mathscr{I}) \geq \mathsf{cov}^*(\mathscr{J})$.
(2) *If* $\mathscr{I} \leq_{KB} \mathscr{J}$, *then* $\mathsf{non}^*(\mathscr{I}) \leq \mathsf{non}^*(\mathscr{J})$.

In (2), we cannot replace $\leq_{KB}$ to $\leq_K$. Let $\mathscr{I}$ be an ideal with $\mathsf{non}^*(\mathscr{I}) > \omega$. Define $\mathscr{J} = \{X \subset \omega \times \omega : \exists I \in \mathscr{I}(X \subset I \times \omega)\}$. Then the projection witness $\mathscr{I} \leq_K \mathscr{J}$. However $\{\{n\} \times \omega : n \in \omega\}$ witness to $\mathsf{non}^*(\mathscr{J}) = \omega$. So $\mathsf{non}^*(\mathscr{I}) > \mathsf{non}^*(\mathscr{J})$.

In the study, the Katětov order between the finite chromatic ideal on $[\omega]^2$, denoted by $\mathscr{G}_{FC}$, which is an F_σ-ideal, and other Borel ideals is investigated. The pair-reaping number and the pair-splitting number are introduced as other descriptions of the uniformity number of $\mathscr{G}_{FC}$ and the covering number of $\mathscr{G}_{FC}$.

The encounter of these two different studies produces more general results.

In the present paper we shall investigate the relationship between $\mathfrak{r}_{pair}$, $\mathfrak{s}_{pair}$, cardinal invariants of ideals on ω and other classical cardinal invariants.

In Section 1 we shall show $\mathfrak{r}_{pair} = \mathfrak{r}_n$ for $n \geq 3$ and $\mathfrak{s}_{pair} = \mathfrak{s}_n$ for $n \geq 3$. In Section 2 we shall investigate the relation between $\mathfrak{s}_{pair}$, $\mathfrak{r}_{pair}$ and cardinal invariants of the ideal of finitely chromatic graphs. In Section 3 we shall show the consistency of $\mathsf{non}^*(\mathscr{I}) < \mathfrak{b}$ for F_σ-ideals on ω. In Section 4 we shall answer a question by Solecki from [15].

§1. *n*-splitting number and *n*-reaping number. In this section we shall show $\mathfrak{s}_{pair} = \mathfrak{s}_n$ and $\mathfrak{r}_{pair} = \mathfrak{r}_n$ for $n \geq 2$.

We call $A \subset [\omega]^n$ unbounded if for $k \in \omega$ there exists $a \in A$ such that $a \cap k = \emptyset$.

For $X \in [\omega]^\omega$ and unbounded $A \subset [\omega]^n$, X n-splits A if there exist infinitely many $a \in A$ such that $a \cap X \neq \emptyset$ and $a \setminus X \neq \emptyset$. We call $\mathcal{S} \subset [\omega]^\omega$ an n-splitting family if for each unbounded $A \subset [\omega]^n$ there exists $X \in \mathcal{S}$ such that X n-splits A. The n-splitting number $\mathfrak{s}_n$ is the least size of an n-splitting family.

We call $\mathscr{R} \subset \mathscr{P}([\omega]^n)$ an n-reaping family if for each $A \in \mathscr{R}$, A is unbounded and for $X \in [\omega]^\omega$, there exists $A \in \mathscr{R}$ such that X does not n-split A, that is, for all but finitely many $a \in A$, $a \cap X = \emptyset$ or $a \subset X$. The n-reaping number $\mathfrak{r}_n$ is the least size of an n-reaping family. So $\mathfrak{s}_{pair} = \mathfrak{s}_2$ and $\mathfrak{r}_{pair} = \mathfrak{r}_2$.

The following relations hold between $\mathfrak{s}_n$ for $n \geq 2$ and between $\mathfrak{r}_n$ for $n \geq 2$.

Proposition 1.1. (1) $\mathfrak{s}_{pair} = \mathfrak{s}_2 \geq \mathfrak{s}_3 \geq \cdots \geq \mathfrak{s}_n \geq \dots$ *and* $\mathfrak{s}_n \geq \mathfrak{s}$ *for* $n \geq 2$.
(2) $\mathfrak{r}_{pair} = \mathfrak{r}_2 \leq \mathfrak{r}_3 \leq \cdots \leq \mathfrak{r}_n \leq \dots$ *and* $\mathfrak{r} \geq \mathfrak{r}_n$ *for* $n \geq 2$.

Proof. Fix $n \geq 2$. Let $\mathcal{S}$ be an n-splitting family of cardinality $\mathfrak{s}_n$. For an unbounded $A \subset [\omega]^{n+1}$, let $A^* \subset [\omega]^n$ be the collection of the initial n-many elements of an element of A. Then there exists $X \in \mathcal{S}$ which n-splits A^*. So there exist infinitely many $a \in A^*$ such that $a \cap X \neq \emptyset$ and $a \setminus X \neq \emptyset$. Since for each $a \in A^*$, there exists $a^* \in A$ such that $a \subset a^*$, there exist infinitely many $a^* \in A$ such that $a^* \cap X \neq \emptyset$ and $a^* \setminus X \neq \emptyset$. So $\mathcal{S}$ is an $(n+1)$-splitting family. Hence $\mathfrak{s}_{n+1} \leq \mathfrak{s}_n$.

We shall show $\mathfrak{s}_n \geq \mathfrak{s}$. Let $\mathcal{S}$ be an n-splitting family of cardinality $\mathfrak{s}_n$. For $Y \in [\omega]^\omega$, fix an infinite subset A_Y of $[Y]^n$ whose elements are pairwise disjoint. Then A_Y is unbounded. Pick $X \in \mathcal{S}$ which n-splits A_Y. So there exist infinitely many $a \in A_Y$ such that $a \cap X \neq \emptyset$ and $a \setminus X \neq \emptyset$. Hence $|X \cap Y| = |Y \setminus X| = \omega$. So X splits Y. Therefore $\mathcal{S}$ is a splitting family. So $\mathfrak{s}_n \geq \mathfrak{s}$.

We shall show $\mathfrak{r}_n \leq \mathfrak{r}_{n+1}$. Let $\mathscr{R}$ be an $(n+1)$-reaping family of cardinality $\mathfrak{r}_{n+1}$. Put $\mathscr{R}^*$ the set of the initial n-many elements of an element of $\mathscr{R}$. Given $X \in [\omega]^\omega$, pick $A \in \mathscr{R}$ such that for all but finitely many $a \in A$, $a \cap X = \emptyset$ or $a \subset X$. Put A^* the set of initial segments of size n of elements of A. Then for all but finite many $a^* \in A^*$, $a^* \cap X = \emptyset$ or $a^* \subset X$. So $\mathscr{R}^*$ is an n-reaping family of cardinality $\mathfrak{r}_{n+1}$. Hence $\mathfrak{r}_n \leq \mathfrak{r}_{n+1}$.

We shall prove $\mathfrak{r} \geq \mathfrak{r}_n$. Let $\mathscr{R}$ be a reaping family of cardinality $\mathfrak{r}$. For each $Y \in \mathscr{R}$, fix an infinite subset A_Y of $[Y]^n$ whose elements are pairwise disjoint. $\mathscr{R}^*$ is the collection of A_Y with $Y \in \mathscr{R}$.

For $X \in [\omega]^\omega$, pick $Y \in \mathscr{R}$ such that $Y \setminus X$ is finite or $X \cap Y$ is finite. Then for all but finitely many $a \in A_Y$, $a \subset X$ or for all but finitely many $a \in A_Y$, a does not meet X. So $\mathscr{R}^*$ is an n-reaping family of cardinality $\mathfrak{r}$. Therefore $\mathfrak{r}_n \leq \mathfrak{r}$. ⊣

Proposition 1.1 was proved as early as $\mathfrak{r}_n$ and $\mathfrak{s}_n$ were defined. However, it was not known whether $\mathfrak{s}_{pair} = \mathfrak{s}_n$ for $n \geq 3$ or not.

Between $\mathfrak{r}_{pair}$ and $\mathfrak{r}_n$, we can prove the following statement.

PROPOSITION 1.2. $\mathfrak{r}_{pair} = \mathfrak{r}_n$ *for* $n \geq 3$.

PROOF. We shall prove that $\mathfrak{r}_{pair} \geq \mathfrak{r}_4$. Let $\mathscr{R}$ be a pair-reaping family of cardinality $\mathfrak{r}_{pair}$. Without loss of generality we can assume $\mathscr{R}$ is closed under finite changes, i.e., if $C \in \mathscr{R}$ and $|D \mathbin{\triangle} C| < \aleph_0$ then $D \in \mathscr{R}$; and A is pairwise disjoint for each $A \in \mathscr{R}$. Let e_A be a bijection from A to ω. Put

$$\mathscr{R}^* = \{C : \exists A, B \in \mathscr{R} \left(C = \{\bigcup e_A^{-1}[b] : b \in B\}\right)\}.$$

We shall prove such $\mathscr{R}^*$ is a 4-reaping family. Let $X \in [\omega]^\omega$. Then we can find $A \in \mathscr{R}$ such that for all $a \in A$, $a \cap X = \emptyset$ or $a \subset X$. Then define $Y_{A,X} \subset \omega$ so that

$$n \in Y_{A,X} \text{ if } \begin{cases} e_A^{-1}(n) \subset X & \text{if } \exists^\infty m \in \omega \left(e_A^{-1}(m) \subset X\right), \\ e_A^{-1}(n) \cap X = \emptyset & \text{otherwise} \end{cases}$$

Pick $B \in \mathscr{R}$ such that for all $b \in B$, $b \cap Y_{A,X} = \emptyset$ or $b \subset Y_{A,X}$. Let $C_{A,B} = \{\bigcup e_A^{-1}[b] : b \in B\} \in \mathscr{R}^*$. Let $b \in B$. Since for $a \in A$, $a \cap X = \emptyset$ or $a \subset X$, $e_A^{-1}(i) \cap X = \emptyset$ or $e_A^{-1}(i) \subset X$ for $i \in b$. Since $b \cap Y_{A,X} = \emptyset$ or $b \subset Y_{A,X}$ for $b \in B$, $\bigcup e_A^{-1}[b] \cap X = \emptyset$ or $\bigcup e_A^{-1}[b] \subset X$. So X does not 4-split $C_{A,B}$. Since $|\mathscr{R}^*| = \mathfrak{r}_{pair}$, $\mathfrak{r}_4 \leq \mathfrak{r}_{pair}$. By Proposition 1.2 $\mathfrak{r}_{pair} = \mathfrak{r}_3 = \mathfrak{r}_4$.

Similarly we can prove $\mathfrak{r}_{pair} = \mathfrak{r}_{2n}$ for $n \geq 2$. ⊣

David Asperó conjectured that $\mathfrak{s}_{pair} = \mathfrak{s}_3$. Shizuo Kamo gave the proof. The proofs for the splitting numbers are not dual to the proofs for the reaping numbers. It might simplify in terms of Galois–Tukey connections as in [16]. However it might be difficult. In [11] and [12], Mildenberger introduced another variation of reaping numbers $\mathfrak{r}_n$ and $\mathfrak{r}_n = \mathfrak{r}_m (= \mathfrak{r})$ holds for $n, m \in \omega$ but it is proved that there are no nice Galois–Tukey connections between Mildenberger's reaping numbers.

THEOREM 1.3. *(Kamo)* $\mathfrak{s}_{pair} = \mathfrak{s}_n$ *for* $n \geq 3$.

PROOF. We shall prove $\mathfrak{s}_{pair} = \mathfrak{s}_4$. Let ZFC$^-$ be a large enough fragment of ZFC. Suppose $\mathfrak{s}_4, \mathfrak{s}_3 < \mathfrak{s}_{pair}$ holds. Let M_0 be a model of ZFC$^-$ such that the cardinality is $\mathfrak{s}_3$ and $M_0 \cap [\omega]^\omega$ is a 3-splitting family and 4-splitting family.

Pick an infinite subset A of $[\omega]^2$ which is not 2-split by $M_0 \cap [\omega]^\omega$. Without loss of generality we can assume this A is pairwise disjoint.

Let M_1 be a model of ZFC$^-$ of cardinality $\mathfrak{s}_3$ which contains A and all elements of M_0. Pick B in M_1 such that B is an infinite subset of $[A]^2$ and B is not 2-split by any elements in $M_1 \cap [A]^\omega$. We can also assume this B is pairwise disjoint.

Let $C = \{a \cup b : \{a, b\} \in B\}$. Since $M_0 \cap [\omega]^\omega$ is a 4-splitting family, there exists $X \in M_0 \cap [\omega]^\omega$ such that X 4-splits C. Since A is not 2-split by X, there exist infinitely many $a \in A$ such that $a \subset X$ or $X \cap a = \emptyset$. So there exist infinitely many $\{a, b\} \in B$ such that $a \subset X$ and b does not meet X. Put $Y = \{a \in A : a \subset X\}$. Then $Y \in M_1$ and Y 2-splits B. However, this is a contradiction to the fact B is not split by any infinite subset of A in M_1.

Similarly we can prove that $\mathfrak{s}_{pair} = \mathfrak{s}_{2n}$ for $n \geq 2$. Therefore $\mathfrak{s}_{pair} = \mathfrak{s}_n$ for $n \in \omega$. $\dashv$

§**2. The ideal of finitely chromatic graphs.** In this section we shall investigate the relation between the finite chromatic ideal, pair-splitting number and pair-reaping number.

The *finite chromatic ideal* on $[\omega]^2$ is defined by

$$\mathscr{G}_{FC} = \{A \subset [\omega]^2 \colon \chi(\omega, A) < \aleph_0\}$$

where $\chi(\omega, A) = \min\{k \in \omega \colon (\exists f \in k^\omega)(\forall a \in A)(|f[a]| = 2)\}$.

Theorem 2.1. *The following conditions hold.*

(1) $\mathfrak{s}_{pair} = \mathsf{cov}^*(\mathscr{G}_{FC})$,
(2) $\mathsf{non}^*(\mathscr{G}_{FC})$ *is the minimal cardinality of a family* $\mathscr{A} \subseteq [[\omega]^2]^\omega$ *such that for any finite partition* $\mathscr{P}$ *of* ω *there is an element* A *of* $\mathscr{A}$ *such that for every* $r \in A$ *there is* $P \in \mathscr{P}$ *such that* $r \subseteq P$ *and*
(3) $\mathfrak{r}_{pair} \leq \mathsf{non}^*(\mathscr{G}_{FC})$.

Proof. First we shall prove $\mathfrak{s}_{pair} \leq \mathsf{cov}^*(\mathscr{G}_{FC})$. Let $\mathscr{A}$ be a subset of $\mathscr{G}_{FC}$ such that $|\mathscr{A}| = \mathsf{cov}^*(\mathscr{G}_{FC})$ and

$$(\forall X \subset [\omega]^2)(\exists A \in \mathscr{A})\,(|X| = \aleph_0 \to |A \cap X| = \aleph_0)\,. \tag{1}$$

Claim 2.2. *If* $A \in \mathscr{G}_{FC}$, *then there exist* $n \in \omega$ *and* $A_i \subset A$ *for* $i < n$ *such that* $A = \bigcup_{i<n} A_i$ *and* $\chi(A_i) = 2$ *for* $i < n$.

Proof of Claim. Suppose $A \in \mathscr{G}_{FC}$, $k \in \omega$ and $f \colon \omega \to k$ such that for all $a \in A$ $|f[a]| = 2$. For $i, j < k$ with $i < j$, put $A_{i,j} = \{a \in A \colon f[a] = \{i, j\}\}$. Then $\chi(\omega, A_{i,j}) = 2$ and $A = \bigcup\limits_{i,j<k, i<j} A_{i,j}$. $\dashv$

By this claim, we can assume $\chi(\omega, A) = 2$ for $A \in \mathscr{A}$. For each $A \in \mathscr{A}$, fix $f \colon \omega \to 2$ so that f witnesses $\chi(\omega, A) = 2$. Put $A_0 = f^{-1}(0) \cap \bigcup A$ and $\mathscr{A}_0 = \{A_0 \colon A \in \mathscr{A}\}$.

Then $\mathscr{A}_0$ is a pair-splitting family. Let $B \subset [\omega]^2$ be infinite. Since $\mathscr{A}$ satisfies (1), there is $A \in \mathscr{A}$ such that $|A \cap B| = \aleph_0$. So there exist infinitely many $b \in B$ such that $b \in A$. So there exist infinitely many $b \in B$ such that $b \cap A_0 \neq \emptyset$ and $b \setminus A_0 \neq \emptyset$. Therefore $\mathfrak{s}_{pair} \leq \mathsf{cov}^*(\mathscr{G}_{FC})$.

We shall prove $\mathfrak{s}_{pair} \geq \mathsf{cov}^*(\mathscr{G}_{FC})$. Let $\mathscr{S} \subset [\omega]^\omega$ be a pair-splitting family. For each $S \in \mathscr{S}$, put $A_S = \{a \in [\omega]^2 \colon a \cap S \neq \emptyset \wedge a \cap \omega \setminus S \neq \emptyset\}$ and $\mathscr{A}(\mathscr{S}) = \{A_S \colon S \in \mathscr{S}\}$.

Then $\mathscr{A}(\mathscr{S})$ satisfies that for each infinite $X \in [\omega]^2$, there exists an $A_S \in \mathscr{A}(\mathscr{S})$ such that $|X \cap A_S| = \aleph_0$. Let $X \subset [\omega]^2$ be infinite. Since $\mathscr{A}$ is a pair-splitting family, there exists an $S \in \mathscr{S}$ such that S pair-splits X. So there exist infinitely many $a \in X$ such that $a \cap S \neq \emptyset$ and $a \setminus S \neq \emptyset$. Hence $|X \cap A_S| = \aleph_0$. Therefore $\mathsf{cov}^*(\mathscr{G}_{FC}) \leq \mathfrak{s}_{pair}$.

In order to prove (2), note that if P is a finite partition of ω then $G_P = \{\{n, m\} \colon (\exists a \neq b \in P)(n \in a \wedge m \in b)\} \in \mathscr{G}_{FC}$, and moreover, $\{G_P \colon P$ is a finite partition of $\omega\}$ is a base of $\mathscr{G}_{FC}$. Then, if $\mathscr{A}$ is a family as in (2) then $\mathscr{A}$ itself witnesses $\mathsf{non}^*(\mathscr{G}_{FC})$; and if $\mathscr{B}$ is a witness of $\mathsf{non}^*(\mathscr{G}_{FC})$ then defining $\mathscr{A}$

as the family of finite changes of elements of $\mathscr{B}$ we are done. (3) follows directly from (2). $\dashv$

It can be easily seen that $\mathscr{G}_{FC}$ is an F_σ-ideal. In particular, $\mathfrak{s}_{pair}$ is equal to the covering number of an F_σ-ideal and $\mathfrak{r}_{pair}$ is bounded by the uniformity number of an F_σ-ideal.

Concerning to the covering number of F_σ-ideals and $\mathfrak{b}$, we can construct a proper forcing notion which destroys tallness of an F_σ-ideal and preserves the unbounded number.

THEOREM 2.3. [9] *For each F_σ-ideal $\mathscr{I}$, there exists a proper forcing notion $\mathbb{P}_{\mathscr{I}}$ which is ω^ω-bounding and adds a new element X in the extension such that $|X \cap I| < \aleph_0$ for $I \in \mathscr{I} \cap V$.*

By using ω_2-stage countable support iteration of $\mathbb{P}_{\mathscr{I}}$, we can show the following statement.

COROLLARY 2.4. *Suppose $\mathscr{I}$ is an F_σ-ideal on ω. Then it is consistent that* $\mathsf{cov}^*(\mathscr{I}) > \mathfrak{d}$.

COROLLARY 2.5. *It is consistent that* $\mathfrak{s}_{pair} = \mathsf{cov}^*(\mathscr{G}_{FC}) > \mathfrak{d}$.

§3. The uniformities of F_σ-ideals.

The *eventually different ideal* is defined by

$$\mathscr{ED} = \{A \subset \omega \times \omega \colon (\exists m, n \in \omega)(\forall k > n)\,(|\{l \colon \langle k, l\rangle \in A\}| \leq m)\}.$$

Define $\mathscr{ED}_{fin} = \mathscr{ED} \restriction \triangle$, where $\triangle = \{\langle m, n\rangle \colon n \leq m\}$.

On the $\mathsf{cov}^*(\mathscr{ED})$ we have the following result.

LEMMA 3.1. $\mathsf{cov}^*(\mathscr{ED}) = \mathsf{non}(\mathscr{M})$.

PROOF. We will use the following lemma, due to Bartoszyński and Miller.

LEMMA 3.2 ([2], Lemma 2.4.8). *For any cardinal κ the following are equivalent*:

(a) $\kappa < \mathsf{non}(\mathscr{M})$,
(b) $(\forall F \in [\omega^\omega]^\kappa)(\exists g \in \omega^\omega)(\exists X \in [\omega]^\omega)(\forall f \in F)(\forall^\infty n \in X)(f(n) \neq g(n))$ *and*
(c) $(\forall F \in [\mathscr{C}]^\kappa)(\exists g \in \omega^\omega)(\forall S \in F)(\forall^\infty n)(g(n) \notin S(n))$

Let $\mathscr{F}$ be a subset of ω^ω of minimal cardinality such that

$$(\forall g \in \omega^\omega)(\forall X \in [\omega]^\omega)(\exists f \in \mathscr{F})(\exists^\infty n \in X)(f(n) = g(n))$$

(We are identifying every function $f \in \omega^\omega$ with its graph $\{(n, f(n)) \colon n < \omega\}$.) Define $\mathscr{A} = \mathscr{F} \cup \{\{n\} \times \omega \colon n < \omega\}$. Obviously $\mathscr{A} \subseteq \mathscr{ED}$. We claim that $\mathscr{A}$ is a covering family. Let X be an infinite subset of $\omega \times \omega$. If there exists $n < \omega$ such that $X_n = X \cap (\{n\} \times \omega)$ is infinite, then X_n is an infinite subset of an element of $\mathscr{A}$. If the set $A = \{n < \omega \colon X_n \neq \emptyset\}$ is infinite then there exists $f \in \mathscr{F}$ such that $f(n) = \min(X_n)$ for infinitely many $n \in A$. Hence, $f \cap X$ is infinite.

On the other hand, let $\mathscr{A}$ be a subset of $\mathscr{ED}$ with $|\mathscr{A}| < \mathsf{non}(\mathscr{M})$. For every $A \in \mathscr{A}$, let $n_A < \omega$ such that $|A_k| \leq n_A$ for all $k \geq n_A$, and define a slalom S_A by

$$S_A(n) = \begin{cases} \emptyset & \text{if } n < n_A \\ A_n & \text{if } n \geq n_A \end{cases}$$

Note that $|\{S_A : A \in \mathscr{A}\}| \leq |\mathscr{A}|$, and by the lemma above, there exists $g \in \omega^\omega$ such that for every $A \in \mathscr{A}$, $g(n) \notin S_A(n)$, for almost all $n < \omega$. Hence, $g \cap A$ is finite for all $A \in \mathscr{A}$, and so, $\mathscr{A}$ is not a covering family. $\dashv$

THEOREM 3.3. *If $\mathscr{I}$ is a Borel ideal on ω, then* $\mathsf{non}^*(\mathscr{I}) = \omega$ *or* $\mathscr{ED}_{fin} \leq_{KB} \mathscr{I}$. *So* $\mathsf{non}^*(\mathscr{I}) = \omega$ *or* $\mathsf{non}^*(\mathscr{ED}_{fin}) \leq \mathsf{non}^*(\mathscr{I})$.

PROOF. For a Borel ideal $\mathscr{I}$, let us consider the following two-player game: In stage k, Player I chooses a finite subset F_k of ω and then, Player II chooses a natural number $n_k \notin F_k$.

I	$F_0 \in [\omega]^{<\omega}$		$F_1 \in [\omega]^{<\omega}$		$\dots$	
II		$n_0 \notin F_0$		$n_1 \notin F_1$		$\dots$

Player I wins if $\{n_i : i \in \omega\} \in \mathscr{I}$ and Player II wins $\{n_i : i \in \omega\} \in \mathscr{I}^+$.

CLAIM 3.4. *If* Player I *has a winning strategy then* $\mathscr{ED}_{fin} \leq_{KB} \mathscr{I}$.

PROOF OF CLAIM. If Player I has a winning strategy then there is a cofinite-branching tree $T \subset \omega^{<\omega}$ such that every $t \in T$ is an increasing sequence and $rng(f) \in \mathscr{I}$ for all $f \in [T]$. Choose $g : \omega \to \omega$ a strictly increasing function such that if $n \in \omega$ and $t \in T$ with $rng(t) \subset g(n)$ then $[g(n+1), \infty) \subseteq Succ_T(t)$. Then every selector of $\{[g(n), g(n+1)) : n \in \omega\}$ is the range of a branch of T. Therefore every selector of $\{[g(n), g(n+1)) : n \in \omega\}$ is in $\mathscr{I}$.

Choose $f : \omega \to \triangle$ an injection so that for each $n \in \omega$, there exists $k \in \omega$ such that $f\,[[g(n), g(n+1))] \subset \{\langle k, l\rangle : l \leq k\}$.

We shall show this f witnesses $\mathscr{ED}_{fin} \leq_{KB} \mathscr{I}$. Let $I \in \mathscr{ED}_{fin}$ and $m \in \omega$ be such that for all but finitely many k, $|\{\langle k, l\rangle : l \leq k \wedge \langle k, l\rangle \in I\}| \leq m$. So $f^{-1}[I]$ is a union of m-many selectors of $\{[g(n), g(n+1)) : n \in \omega\}$. Since every selector of $\{[g(n), g(n+1)) : n \in \omega\}$ is in $\mathscr{I}$, $f^{-1}[I] \in \mathscr{I}$ i.e., $\mathscr{ED}_{fin} \leq_{KB} \mathscr{I}$. $\dashv$

CLAIM 3.5. *If* Player II *has a winning strategy, then* $\mathsf{non}^*(\mathscr{I}) = \omega$.

PROOF OF CLAIM. Player II has a winning strategy if and only if there exists an infinitely-branching tree $T \subset \omega^{<\omega}$ such that $rng(f) \in \mathscr{I}^+$ for all $f \in [T]$.

We shall show $\{succ_T(t) : t \in T\}$ is a witness of $\mathsf{non}^*(\mathscr{I})$. Assume to the contrary that there exists $I \in \mathscr{I}$ such that $|I \cap succ_T(t)| = \omega$ for all $t \in T$. Then there exists $b \in [T]$ such that $rng(b) \subset I \in \mathscr{I}$. This is a contradiction. Therefore $\mathsf{non}^*(\mathscr{I}) = \omega$. $\dashv$

By Borel determinacy this game is determined i.e., either Player I or Player II has a winning strategy. So $\mathscr{ED}_{fin} \leq_{KB} \mathscr{I}$ or $\mathsf{non}^*(\mathscr{I}) = \omega$. $\dashv$

Concerning to the cardinal invariants of $\mathscr{ED}_{fin}$, we have proved the following.

PROPOSITION 3.6. *The following relations hold*:

(1) $\mathsf{non}^*(\mathscr{ED}_{fin}) \leq \mathfrak{r}$,
(2) $\mathsf{cov}(\mathscr{M}) = \min\{\mathfrak{d}, \mathsf{non}^*(\mathscr{ED}_{fin})\}$ *and*
(3) $\mathsf{non}(\mathscr{M}) = \max\{\mathfrak{b}, \mathsf{cov}^*(\mathscr{ED}_{fin})\}$.

PROOF. For any $A \subseteq \Delta$ we will denote by $A_n = \{m \leq n : \langle n, m\rangle \in A\}$. Let us prove (1). We will say that a family $\mathscr{R}$ of infinite subsets of ω is *hereditarily reaping* if for every $X \in \mathscr{R}$ and every infinite subset Y of X there is R in $\mathscr{R}$ such that $R \subseteq Y$ or $R \subseteq X \setminus Y$.

LEMMA 3.7. $\mathfrak{r} = \min\{|\mathscr{R}|\colon \mathscr{R}$ *is hereditarily reaping*$\}$

PROOF. It will be enough to prove that there is a hereditarily reaping family with cardinality $\mathfrak{r}$. Let $\mathscr{Q}$ be a reaping family with cardinality $\mathfrak{r}$. Define $\mathscr{Q}_n$ by recursion on $n < \omega$. Let $\mathscr{Q}_0 = \mathscr{Q}$. Given $\mathscr{Q}_n$ and $A \in \mathscr{Q}_n$, let $\mathscr{Q}_{n+1} \restriction A$ be a reaping family on A with cardinality $\mathfrak{r}$. Put $\mathscr{Q}_{n+1} = \bigcup_{A \in \mathscr{Q}_n} \mathscr{Q}_{n+1} \restriction A$. So, $\mathscr{R} = \bigcup_{n<\omega} \mathscr{Q}_n$ is a hereditarily reaping family. ⊣

Let $\mathscr{R}$ be a hereditarily reaping family, and for every $R \in \mathscr{R}$ and $n < \omega$ define $X_{R,n} = \{(m, n)\colon m \geq n \wedge m \in R\}$. We will see that $\mathscr{A} = \{X_{R,n}\colon R \in \mathscr{R} \wedge n < \omega\}$ witnesses $\mathsf{non}^*(\mathscr{ED}_{fin})$. Let I be in $\mathscr{ED}_{fin}$, and choose $\{f_i\colon i \leq n\}$ functions such that $I \subseteq \bigcup_{i \leq n} f_i$. Define $A_j = \{k\colon (\exists i \leq n)(f_i(k) = j)\}$, for $j \leq n$. If A_j is finite for some $j \leq n$, then $I \cap X_{R,j}$ is finite for every $R \in \mathscr{R}$. So we can assume A_j is infinite for $j \leq n$. Let R_0 be in $\mathscr{R}$ such that $R_0 \cap A_0 = \emptyset$ or $R_0 \subseteq A_0$. In general, for $1 \leq j \leq n$ we can find $R_j \in \mathscr{R}$ such that $R_j \cap (R_{j-1} \cap A_j) = \emptyset$ or $R_j \subseteq R_{j-1} \cap A_j$. If the first case is true for a $j \leq n$ we are done, because for j minimal, we have that $X_{R_j,j} \cap I = \emptyset$. Suppose that $R_j \subseteq R_{j-1} \cap A_j$ for all $j \leq n$. Then, for any $k \in R_n$, $I \cap (\{k\} \times \omega) = n + 1$, and so, $X_{R_n,n+1} \cap I = \emptyset$.

In order to prove (2) we will need the following lemma, due to Bartoszyński and Miller.

LEMMA 3.8 ([2], Lemma 2.4.2). *For any cardinal* κ *the following conditions are equivalent*:

(i) $\kappa < \mathsf{cov}(\mathscr{M})$ *and*

(ii) $(\forall F \in [\omega^\omega]^\kappa)(\forall G \in [[\omega]^\omega]^\kappa)(\exists g \in \omega^\omega)(\forall f \in F)(\forall X \in G)(\exists^\infty n \in X)(f(n) = g(n))$. ⊣

Let $\mathscr{X}$ be a subset of $[\Delta]^{\aleph_0}$ with $|\mathscr{X}| < \mathsf{cov}(\mathscr{M})$. For every $X \in \mathscr{X}$ define $G_X = \{n < \omega\colon X \cap (\{n\} \times \omega) \neq \emptyset\}$ and let $f_X \in \omega^\omega$ be a function such that $f_X(n) \in X \cap (\{n\} \times \omega)$. By the previous lemma, there is a function $g \in \omega^\omega$ such that $f_X(n) = g(n)$ for infinitely many elements n of G_X, for all $X \in \mathscr{X}$. Then, $\Delta \cap g$ is an element of $\mathscr{ED}_{fin}$ having an infinite intersection with every element of $\mathscr{X}$, proving $|\mathscr{X}| < \mathsf{non}^*(\mathscr{ED}_{fin})$. So $\mathsf{cov}(\mathscr{M}) \leq \mathsf{non}^*(\mathscr{ED}_{fin})$. In addition, it is a well known fact that $\mathsf{cov}(\mathscr{M}) \leq \mathfrak{d}$. Therefore $\mathsf{cov}(\mathscr{M}) \leq \min\{\mathfrak{d}, \mathsf{non}^*(\mathscr{ED}_{fin})\}$.

We shall show $\min\{\mathfrak{d}, \mathsf{non}^*(\mathscr{ED}_{fin})\} \leq \mathsf{cov}(\mathscr{M})$. Let κ be a cardinal lower than $\mathfrak{d}$ and $\mathsf{non}^*(\mathscr{ED}_{fin})$. We will prove and use the following lemma.

LEMMA 3.9. *Let* κ *be an infinite cardinal. The following conditions are equivalent.*

(a) $\kappa < \mathsf{non}^*(\mathscr{ED}_{fin})$ *and*

(b) *for every bounded family* $\mathscr{F}$ *of* κ *functions in* ω^ω *and every family* $\mathscr{A}$ *of* κ *infinite subsets of* ω *there exists a function* $g \in \omega^\omega$ *such that for all* $f \in \mathscr{F}$ *and* $A \in \mathscr{A}$, $f(n) = g(n)$ *for infinitely many* $n \in A$.

PROOF. Suppose that κ satisfies (b) and let $\mathscr{B}$ be a family of κ infinite subsets of Δ. For every $B \in \mathscr{B}$, let $X_B = \{n\colon B_n \neq \emptyset\}$ and $f_B\colon \omega \to \omega$ such that $(n, f_B(n)) \in B$ if $n \in X_B$, and $f_B(n) = 0$ if not. The families $\mathscr{F} = \{f_B\colon B \in \mathscr{B}\}$ and $\mathscr{A} = \{X_B\colon B \in \mathscr{B}\}$ have cardinality κ, and so, there exists a function $g \in \omega^\omega$ such that for all $B \in \mathscr{B}$ there are infinitely many $n \in X_B$ such that $g(n) = f_B(n)$, showing that g has an infinite intersection with B.

On the other hand assume that $\kappa < \mathsf{non}^*(\mathscr{ED}_{fin})$, $\mathscr{F} \subseteq \omega^\omega$ and $\mathscr{A} \subseteq [\omega]^\omega$ have cardinality κ, and $\mathscr{F}$ is bounded by an increasing function $h \in \omega^\omega$. We will identify every $f \in \mathscr{F}$ with a subset of an $\mathcal{ED}_{fin}$-positive subset Δ' of Δ, as follows: Define $X = h[\omega]$, $\Delta' = \prod_{n\in X} n$, $A' = h[A]$ if $A \in \mathscr{A}$, and for $f \in \mathscr{F}$, define $f'\colon X \to \omega$ by $f'(n) = f(h^{-1}(n))$. So, $\mathscr{F}' = \{f'\colon f \in \mathscr{F}\}$ is a family of infinite subsets of Δ'. Let $\mathscr{B} = \{f' \restriction A'\colon f \in \mathscr{F} \wedge A \in \mathscr{A}\}$. Since $|\mathscr{B}| = \kappa$, there exists $I \in \mathcal{ED}_{fin}$ such that $I \cap B$ is infinite for all $B \in \mathscr{B}$. Let $\{g_i : i \leq N\}$ be a set of functions in ω^ω such that $I \subseteq \bigcup_{i\leq N} g_i$. Define $B_{f,A} = \{n \in A'\colon f'(n) = g_i(n)\}$, for some $i \leq N$ such that $|(f' \restriction A') \cap g_i| = \aleph_0$, and define $\mathscr{C} = \{B_{f,A}\colon f \in \mathscr{F} \wedge A \in \mathscr{A}\}$. By Proposition 3.6 (1) $|\mathscr{C}| \leq \kappa < \mathfrak{r}$, and so, there exists $Y \in [\omega]^\omega$ such that $|Y \cap B_{f,A}| = \omega = |B_{f,A} \setminus Y|$. We can find a partition $\{Y_0, Y_1\}$ of Y such that $|Y_0 \cap B_{f,A}| = \aleph_0 = |Y_1 \cap B_{f,A}|$, for all $f \in \mathscr{F}$ and for all $A \in \mathscr{A}$, and inductively, we can find a partition $\{Y_0, Y_1, \dots, Y_n\}$ of Y such that for every $i \leq n$, $|B_{f,A} \cap Y_i| = \aleph_0$. Now, we define $g(n) = g_i(n)$ if $n \in Y_i$ and $g(n) = 0$ if $n \notin Y$. Given f and A, if $i \leq n$ is such that $B_{f,A} = \{n \in A'\colon f'(n) = g_i(n)\}$ then $f'(n) = g(n)$ for infinitely many $n \in Y_i \cap A'$, and so, $f(n) = g(h(n))$ for infinitely many $n \in h^{-1}[Y_i] \cap A$. $\dashv$

Let us prove that $\kappa < \mathsf{cov}(\mathscr{M})$ when $\kappa < \min\{\mathfrak{d}, \mathsf{non}^*(\mathcal{ED}_{fin})\}$, by using Lemma 3.8. Let F and G be families such that $F \in [\omega^\omega]^\kappa$ and $G \in [[\omega]^\omega]^\kappa$.

Claim 3.10. *There exists $h \in \omega^\omega$ such that for all $X \in G$ and for all $f \in F$, $f(n) < h(n)$ for infinitely many $n \in X$.*

Proof of the Claim. For all $f \in F$, $X \in G$, let e_X be the enumeration of X and let $h_{f,X} \in \omega^\omega$ be such that $h_{f,X}(n) \geq f(e_X(i))$ for all $i \leq n$. Since $\kappa < \mathfrak{d}$, there is a function h which is not dominated by $\{h_{f,X}\colon X \in G \wedge f \in F\}$. This h does the work. $\dashv$

Now, for every $f \in F$ define $f' \in \omega^\omega$ such that $f'(n) = f(n)$ if $f(n) < h(n)$ and $f'(n) = 0$ otherwise; for every $f \in F$ and for every $X \in G$ define $C_{f,X} = \{n \in X\colon f(n) < h(n)\}$, $\mathscr{A} = \{C_{f,X}\colon f \in F \wedge X \in G\}$ and $\mathscr{F} = \{f'\colon f \in F\}$. $\mathscr{F}$ is bounded and so, by Lemma 3.9, there is $g \in \omega^\omega$ such that for all $f \in \mathscr{F}$ and for all $A \in \mathscr{A}$, $g(n) = f'(n)$ for infinitely many $n \in A$ and in consequence, $g(n) = f(n)$ for infinitely many $n \in C_{f,X} \subset X$ for every $X \in G$. Therefore $\kappa < \mathsf{cov}(\mathscr{M})$ by Lemma 3.9.

We shall prove (3). It is well known that $\mathfrak{b} \leq \mathsf{non}(\mathscr{M})$ and note that $\mathcal{ED} \leq_K \mathcal{ED}_{fin}$ and so, $\mathsf{cov}^*(\mathcal{ED}_{fin}) \leq \mathsf{cov}^*(\mathcal{ED}) = \mathsf{non}(\mathscr{M})$. So $\max\{\mathfrak{b}, \mathsf{cov}^*(\mathcal{ED}_{fin})\} \leq \mathsf{non}(\mathscr{M})$.

To show $\max\{\mathfrak{b}, \mathsf{cov}^*(\mathcal{ED}_{fin})\} \geq \mathsf{non}(\mathscr{M})$, we are going to use the following lemma.

Lemma 3.11 ([2], Theorem 2.4.7). $\mathsf{non}(\mathscr{M})$ *is the size of the smallest family $\mathscr{F} \subseteq \omega^\omega$ such that for every $g \in \omega^\omega$ there is an element f of $\mathscr{F}$ such that $f(n) = g(n)$ for infinitely many $n \in \omega$.* $\dashv$

Let κ be a cardinal greater than $\mathsf{cov}^*(\mathcal{ED}_{fin})$ and greater than $\mathfrak{b}$. Let $\mathscr{G} = \{f_\alpha\colon \alpha < \kappa\}$ be an unbounded family of functions in ω^ω, and let G_α a witness of $\mathsf{cov}^*(\mathcal{ED}_{fin})$ in $\Delta_\alpha = \{\langle n, m\rangle\colon m \leq f_\alpha(n)\}$, for all $\alpha < \kappa$. Without loss of generality we can assume that every element of I of G_α is the graph of a function in ω^ω. We will prove that $\mathscr{F} = \bigcup_{\alpha<\kappa} G_\alpha$ is such that for every $g \in \omega^\omega$ there is $f \in \mathscr{F}$ such that $f(n) = g(n)$ for infinitely many $n \in \omega$. Given $g \in \omega^\omega$, let $\alpha < \kappa$ be such that $f_\alpha \not\leq^* g$. Then, $g \cap \Delta_\alpha$ is infinite and so, there is $I \in G_\alpha$ such that $I \cap (g \cap \Delta_\alpha)$ is infinite. Since I is the graph of a function in $\mathscr{F}$, we are done. $\dashv$

By Proposition 3.6, it is consistent that $\mathsf{non}^*(\mathcal{ED}_{fin}) < \mathfrak{b}$. For example if the ground model satisfies Martin axiom, then the random forcing corresponding to the product space 2^{ω_1} forces $\mathsf{non}^*(\mathcal{ED}_{fin}) = \mathsf{cov}(\mathcal{M}) = \omega_1 < \mathfrak{b} = \mathfrak{c}$. However, we cannot use this argument to show the consistency of $\mathsf{non}^*(\mathcal{I}) < \mathfrak{b}$ for every F_σ-ideal $\mathcal{I}$ because $\mathsf{cov}(\mathcal{N}) \leq \mathsf{non}^*(\mathcal{G}_{FC})$ and the random forcing corresponding to the product space 2^{ω_1} forces $\mathsf{cov}(\mathcal{N}) = \mathfrak{c}$ whenever the ground model satisfies Martin axiom.

However, F_σ-ideals on ω have the following good property.

Theorem 3.12. [10] *$\mathcal{I}$ is an F_σ-ideal on ω if and only if $\mathcal{I} = Fin(\varphi)$ for some lower semi-continuous submeasure φ, where $Fin(\varphi) = \{A \subset \omega : \varphi(A) < \infty\}$. Here $\varphi : \mathcal{P}(\omega) \to [0, \infty]$ is a lower semi-continuous submeasure if*

(1) $\varphi(\emptyset) = 0$,
(2) *whenever $X, Y \subset \omega$ and $X \subset Y$, $\varphi(X \cup Y) \leq \varphi(X) + \varphi(Y)$,*
(3) *$\varphi(\{n\}) < \infty$ for $n \in \omega$ and*
(4) *$\varphi(A) = \lim_{n\to\infty} \varphi(A \cap n)$ for every $A \subset \omega$.*

To show the consistency of $\mathsf{non}^*(\mathcal{I}) < \mathfrak{b}$, we shall use the Laver forcing $\mathbb{L}$. $\mathbb{L}$ is defined by $T \in \mathbb{L}$ if $T \subset \omega^{<\omega}$ is a tree and for $s \in T$ with $stem(T) \subset s$, $|succ_T(s)| = \aleph_0$. $\mathbb{L}$ is ordered by inclusion. Then $\mathbb{L}$ adds an unbounded real.

Proposition 3.13. *Let G be a $\mathbb{L}$-generic over V and $f_G = \bigcup\{stem(T) : T \in G\}$. Then $f_G \in \omega^\omega$ and f_G dominates for all $g \in \omega^\omega \cap V$.*

Therefore, if $\mathbb{L}_{\omega_2}$ is an ω_2-stage countable support iteration of Laver forcing, then $V^{\mathbb{L}_{\omega_2}} \models \mathfrak{b} = \mathfrak{c}$.

By Proposition 3.13 it is enough to show that $\mathbb{L}_{\omega_2}$ preserves $\mathsf{non}^*(\mathcal{I})$ for each F_σ-ideal $\mathcal{I}$ on ω. We shall use the Laver property.

Definition 4. [4] A forcing notion $\mathbb{P}$ have the Laver property if for every $H : \omega \to \omega \in V$

$$\Vdash \left(\forall f \in \prod_{n\in\omega} H(n) \cap V[\dot{G}]\right) (\exists A : \omega \to \omega^{<\omega} \in V)$$
$$(\forall n \in \omega) \left(f(n) \in A(n) \wedge |A(n)| \leq 2^n\right).$$

The Laver property has the following good property.

Theorem 4.1. [4] *The Laver property is preserved under countable support iteration of proper forcing notions.*

Theorem 4.2. [2, *p.* 353] *The Laver forcing $\mathbb{L}$ has the Laver property.*

So $\mathbb{L}_{\omega_2}$ has the Laver property.

Theorem 4.3. *If $\mathcal{I}$ is an F_σ-ideal on ω, then it is consistent that* $\mathsf{non}^*(\mathcal{I}) < \mathfrak{b}$.

Proof. Let $\mathcal{I}$ be an F_σ-ideal and let φ be a lower semi-continuous submeasure such that $\mathcal{I} = Fin(\varphi)$.

If a forcing notion $\mathbb{P}$ has the Laver property, then $\mathbb{P}$ has the following good property:

Lemma 4.4. *If $\mathbb{P}$ has the Laver property, then*

$$\Vdash_{\mathbb{P}} \text{“}(\forall X \in \mathcal{I} \cap V[\dot{G}])(\exists A \in [\omega]^\omega \cap V)\,(|X \cap A| < \aleph_0)\text{”}.$$

PROOF OF LEMMA. Let $p \in \mathbb{P}$ and let $\dot{X}$ be a $\mathbb{P}$-name such that $\Vdash_{\mathbb{P}}$ "$\dot{X} \in \mathscr{I}$". Without loss of generality we can assume that there exists $n \in \omega$ such that $p \Vdash_{\mathbb{P}}$ "$\varphi(\dot{X}) < n$".

CLAIM 4.5. *Let $\varphi\colon \mathscr{P}(\omega) \to [0,\infty]$ be a lower semi-continuous submeasure such that $Fin(\varphi) = \mathscr{I}$ for some F_σ-ideal on ω. For each $k \in \omega$ and $l \in \omega$, there exists $m \in \omega$ such that $\varphi([l,m]) > k$.*

PROOF OF CLAIM. Since $[l,\infty) \notin \mathscr{I}$, $\varphi([l,\infty)) = \infty$. Because φ has the lower semi-continuous, there exists $m > l$ such that $\varphi([l,m]) > k$. $\dashv$

Let $\Pi = \langle I_j : j \in \omega\rangle$ be an interval partition of ω such that $\varphi(I_j) > 2^j \cdot n$. By the Laver property, there exist $q \le p$ and $A\colon \omega \to \bigcup_{j\in\omega} \mathscr{P}(2^{I_j}) \in V$ such that for $j \in \omega$, $A(j) \subset 2^{I_j}$ and $|A(j)| \le 2^j$ and $q \Vdash_{\mathbb{P}}$ "$\forall j \in \omega \left(\dot{X} \restriction I_j \in A(j)\right)$". Without loss of generality we can assume $\varphi(J) \le n$ for $J \in A(j)$ and for $j \in \omega$. By the finite subadditivity of φ, $\varphi(\bigcup A(j)) \le \sum_{J\in A(j)} \varphi(J) \le 2^j \cdot n$. So $I_j \setminus A_j \ne \emptyset$ for $j \in \omega$. Choose $y_j \in I_j \setminus \bigcup A(j)$ for $j \in \omega$. Put $Y = \{y_j : j \in \omega\}$. Then $q \Vdash_{\mathbb{P}}$ "$\dot{X} \cap Y = \emptyset$". Therefore $\Vdash_{\mathbb{P}}$ "$\forall X \in \mathscr{I}\, \exists Y \in [\omega]^\omega \cap V\, (|X \cap Y| < \aleph_0)$". $\dashv$

So if the ground model satisfies CH, then $V^{\mathbb{L}_{\omega_2}} \models [\omega]^\omega \cap V$ witnesses $\mathsf{non}^*(\mathscr{I})$. Therefore it is consistent $\mathsf{non}^*(\mathscr{I}) < \mathfrak{b}$. $\dashv$

In [7] Masaru Kada introduced a cardinal invariant associated with the Laver property.

We call a function from ω to $[\omega]^{<\omega}$ a slalom. Let $\mathcal{S}$ be the collection of slaloms such that $\forall \phi \in \mathcal{S} \forall n \in \omega(|\phi(n)| \le 2^n)$. $\mathfrak{l}$ is the smallest cardinal κ such that for every $h \in \omega^\omega$ there is a set $\Phi \subset \mathcal{S}$ with cardinality κ such that, for every $f \in \omega$ with $f(n) < h(n)$ for all $n < \omega$, there is $\phi \in \Phi$ such that for all but finitely many $n \in \omega$, we have $f(n) \in \phi(n)$.

Pawlikowski showed that the dual notion to the definition of $\mathfrak{l}$ characterizes $\mathsf{trans\text{-}add}(\mathscr{N})$, transitive additivity of the null ideal (see [2, p.91]). That is, $\mathsf{trans\text{-}add}(\mathscr{N})$ is the smallest size of $\le^*$-bounded family $F \subset \omega^\omega$ such that for every $\phi \in \mathcal{S}$ there is $f \in F$ such that for infinitely many $n \in \omega$, $f(n) \notin \phi(n)$.

As the proof of Theorem 4.3 we can prove the following statement.

COROLLARY 4.6. *If $\mathscr{I}$ is an F_σ-ideal, then*

(1) $\mathsf{non}^*(\mathscr{I}) \le \mathfrak{l}$ *and*
(2) $\mathsf{cov}^*(\mathscr{I}) \ge \mathsf{trans\text{-}add}(\mathscr{N})$.

PROOF OF COROLLARY. 1. Let $\mathscr{I}$ be an F_σ-ideal on ω and let φ be a lower semi-continuous submeasure such that $Fin(\varphi) = \mathscr{I}$. Choose $\Pi = \langle I_j : j \in \omega\rangle$ an interval partition of ω such that $\varphi(I_j) > 2^j \cdot j$. Choose Φ a family of functions from ω to $\bigcup_{j\in\omega} \mathscr{P}(2^{I_j})$ such that

i. $|\Phi| \le \mathfrak{l}$,
ii. for each $j \in \omega$ and $\phi \in \Phi$, $\phi(j) \in 2^{I_j}$ and $|\phi(j)| \le 2^j$ and
iii. for each $X \in [\omega]^\omega$, there exists $\phi \in \Phi$ such that for all but finitely many $j \in \omega$, $X \cap I_j \in \phi(j)$,

Without loss of generality we can assume that for each $\phi \in \Phi$ and each $j \in \omega$, $J \in \phi(j)$ implies $\varphi(J) \le j$. For each $j \in \omega$ and $\phi \in \Phi$, $\varphi(\bigcup \phi(j)) \le \sum_{J\in\phi(j)} \varphi(J) \le 2^j \cdot j$. So for each $j \in J$, $I_j \setminus \bigcup \phi(j) \ne \emptyset$.

For each $\phi \in \Phi$, choose $X_\phi \in [\omega]^\omega$ such that $X_\phi \cap I_j \setminus \bigcup \phi(j) \neq \emptyset$. Put $\mathscr{A} = \{X_\phi \colon \phi \in \Phi\}$. We shall show for each $I \in \mathscr{I}$, there exists $X \in \mathscr{A}$ such that $|A \cap I| < \aleph_0$.

Let $I \in \mathscr{I}$ and let $n \in \omega$ such that $\varphi(I) < n$. Choose $m \in \omega$ and $\phi \in \Phi$ so that for $j \geq m$, $I \cap I_j \in \phi(j)$. Then for $j \geq \max n, m$ $X_\phi \cap I_j \cap I = \emptyset$. So $|X_\phi \cap I| < \aleph_0$. Hence $\mathsf{non}^*(\mathscr{I}) \leq \mathfrak{l}$.

2. Let $\mathscr{I}$ be an F_σ-ideal. Let $\mathscr{A} \subset \mathscr{I}$ such that $|\mathscr{A}| < \mathsf{trans\text{-}add}(\mathscr{N})$. Let $\Pi = \langle I_j \colon j \in \omega \rangle$ be an interval partition of ω such that $\varphi(I_j) > 2^j \cdot j$.

Since $|\mathscr{A}| < \mathsf{trans\text{-}add}(\mathscr{N})$, there exists $\phi \colon \omega \to \bigcup_{j \in \omega} \mathscr{P}(2^{I_j})$ such that

i. for each $j \in \omega$, $\phi(j) \subset \mathscr{P}(2^{I_j})$,
ii. for each $j \in \omega$, $|\phi(j)| \leq 2^j$ and
iii. for each $I \in \mathscr{A}$ for all but finitely many $j \in \omega$, $I \cap I_j \in \phi(j)$.

Without loss of generality we can assume that for each $j \in \omega$ and $J \in \phi(j)$, $\varphi(J) < j$. By the finite subadditivity of φ, $\varphi(\bigcup \phi(j)) \leq \sum_{J \in \phi(j)} \varphi(J) \leq 2^j \cdot j$ for each $j \in \omega$. So $I_j \setminus \bigcup \phi(j) \neq \emptyset$ for $j \in \omega$.

Choose $X_\phi \in [\omega]^\omega$ such that $X_\phi \cap I_j \setminus \bigcup \phi(j)$ for $j \in \omega$. For each $I \in \mathscr{A}$, there exists $m \in \omega$ such that $j \geq m$ implies $I \cap I_j \in \phi(j)$. Then $j \geq m$ implies $I \cap I_j \cap X_\phi = \emptyset$. So $|I \cap X_\phi| < \aleph_0$. Therefore $\mathsf{trans\text{-}add}(\mathscr{N}) \leq \mathsf{cov}^*(\mathscr{I})$. $\dashv$

$$
\begin{tikzcd}
\mathsf{cov}(\mathscr{N}) \arrow[dddd] & \mathsf{cov}^*(\mathscr{ED}_{fin}) \arrow[l] \arrow[d] & \mathsf{non}(\mathscr{M}) \arrow[l] \arrow[dd] & \mathsf{cof}(\mathscr{M}) \arrow[l] \arrow[dd] & & \mathsf{cof}(\mathscr{N}) \arrow[ll] \arrow[ddl] \arrow[dddd] \\
 & \mathsf{cov}^*(\mathscr{I}) \arrow[d] & & & & \\
 & \mathsf{trans\text{-}add}(\mathscr{N}) \arrow[ddl] & \mathfrak{b} \arrow[dd] & \mathfrak{d} \arrow[l] \arrow[dd] & \mathfrak{l} \arrow[d] & \\
 & & & & \mathsf{non}^*(\mathscr{I}) \arrow[d] & \\
\mathsf{add}(\mathscr{N}) & & \mathsf{add}(\mathscr{M}) \arrow[ll] & \mathsf{cov}(\mathscr{M}) \arrow[l] & \mathsf{non}^*(\mathscr{ED}_{fin}) \arrow[l] & \mathsf{non}(\mathscr{N}) \arrow[l]
\end{tikzcd},
$$

where $\mathscr{I}$ is an F_σ-ideal on ω and $\mathsf{non}^*(\mathscr{I}) \neq \omega$.

COROLLARY 4.7. (1) *It is consistent* $\mathfrak{r}_{pair} < \mathfrak{b}$.
(2) $\mathfrak{r}_{pair} \leq \mathfrak{l}$ *and* $\mathfrak{s}_{pair} \geq \mathsf{trans\text{-}add}(\mathscr{N})$.

QUESTION 4.8. (1) $\mathfrak{r}_d \leq \mathfrak{s}_{pair}$?
(2) *If* $\mathscr{I}$ *is a Borel ideal, then* $\mathsf{non}^*(\mathscr{I}) \leq \mathsf{cof}(\mathscr{N})$?

§5. Fatou's lemma and a question of Solecki. In this section we answer a question of S. Solecki related to the Katětov order by using cardinal invariants of Borel ideals.

For a sequence of $(a_n)_{n \in \omega}$ of real numbers and an ideal $\mathscr{I}$ on ω, $\lim_{\mathscr{I}} \inf a_n = \sup\{r \in \mathbb{R} \colon \{n \in \omega \colon a_n < r\} \in \mathscr{I}\}$.

Let $(X, \mathscr{B}, \mu)$ be a σ-finite measure space with μ defined on σ-algebra $\mathscr{B}$. Let $f_n \colon X \to [0, \infty]$ be a sequence of μ-measurable functions and let $\mathscr{I}$ be an ideal on ω. We say that Fatou's lemma holds on $\langle f_n : n \in \omega \rangle$ with respect to $\mathscr{I}$ if

$$\underline{\int} \liminf_{\mathscr{I}} f_n \, d\mu \leq \liminf_{\mathscr{I}} \int f_n \, d\mu$$

where $\underline{\int}$ is the lower integral, i.e., if $g \geq 0$, then

$$\underline{\int} g \, d\mu = \sup \left\{ \int f \, d\mu \colon f \leq g \text{ and } f \text{ is } \mu\text{-measurable} \right\}.$$

Let $\mathscr{I}$ be an ideal on ω. We say that Fatou's lemma holds for $\mathscr{I}$ if Fatou's lemma holds with respect to $\mathscr{I}$ for any sequence $\langle f_n : n \in \omega \rangle$ of measurable functions from X to $[0, \infty)$ on any σ-finite measure space.

The ideal $\mathcal{S}$ is a critical (locally minimal in the Katětov order) among the ideals which satisfy Fatou's lemma. Let $\Omega = \{U \in \text{Clopen}(2^\omega) \colon \mu(U) = \frac{1}{2}\}$. $\mathcal{S}$ is an ideal on Ω generated by the set $\{I_x : x \in 2^\omega\}$ where $I_x = \{U \in \Omega \colon x \in U\}$.

THEOREM 5.1. [15] *Let $\mathscr{I}$ be a Borel ideal on ω.*

$\mathscr{I}$ does not satisfy Fatou's lemma if and only if there exists $X \in \mathscr{I}^+$ such that $\mathcal{S} \leq_K \mathscr{I} \restriction X$.

Concerning this theorem, Solecki asked the following question.

QUESTION 5.2. [15] *Can $\mathcal{S}$ be replaced by $\mathcal{G}_{FC}$?*

When we think about question related to the Katětov order, cardinal invariants of ideals are significant.

THEOREM 5.3. $\mathsf{cov}^*(\mathcal{S}) = \mathsf{non}(\mathcal{N})$.

To prove this theorem, we will use the following lemmas.

LEMMA 5.4. [5] *For any $\{U_n : n \in \omega\} \subset \Omega$,*

$$\mu(\{x \in 2^\omega \colon \exists^\infty n \, (x \in U_n)\}) \geq \frac{1}{2}.$$

PROOF OF LEMMA. Assume to the contrary that there exists $\{U_n : n \in \omega\} \in [\Omega]^\omega$ with $\mu(\{x \in 2^\omega \colon \exists^\infty n \in \omega (x \in U_n)\}) < \frac{1}{2}$. Then there exists a compact set $K \subset 2^\omega$ such that $\mu(K) > \frac{1}{2}$ and K is disjoint with $\{x \in 2^\omega \colon \exists^\infty n \in \omega (x \in U_n)\}$. Let $\delta = \mu(K) - \frac{1}{2} > 0$. Then $\mu(K \cap U_n) \geq \frac{1}{2}$ for each $n \in \omega$.

For each $k \in \omega$, define $A_k \subset K$ by

$$A_k = \{x \in K \colon |\{n \in \omega \colon x \in U_n\}| = k\}.$$

Then $\mu(K) = \sum_{k \in \omega} \mu(A_k)$. So there exists $m \in \omega$ such that $\sum_{k \geq m} \mu(A_k) < \frac{\delta}{2}$. For each $n < m$, choose a compact subset C_n of A_n so that $\mu(A_n \setminus C_n) \leq \frac{\delta}{2m}$.

Put $C = \bigcup_{n<m} C_n$. Then $\mu(\bigcup_{n<m} A_n \setminus C) \leq \frac{\delta}{2}$. Since

$$\mu(K \setminus C) = \sum_{n \geq m} \mu(A_n) + \mu(\bigcup_{n<m} A_n \setminus C) < \frac{\delta}{2} + \frac{\delta}{2} = \delta,$$

$\mu(C \cap U_n) \geq \mu(C) + \frac{1}{2} - 1 > 0$ for $n \in \omega$. However, $\sum_{n \in \omega} \mu(C \cap U_n) \leq m \cdot \mu(C) < \infty$ by $C_n \subset A_n$ for $n < m$. This is a contradiction. Therefore $\mu(\{x \in 2^\omega \colon \exists^\infty n (x \in U_n)\}) \geq \frac{1}{2}$. ⊣

LEMMA 5.5. *Given $X \subset 2^\omega$.*

(1) *If $\mu^*(X) < \frac{1}{2}$, then $\{I_x : x \in X\}$ does not witness to $\mathsf{cov}^*(\mathcal{S})$.*

(2) *If* $\{I_x : x \in X\}$ *does not witness to* $\mathsf{cov}^*(\mathcal{S})$, *then* $\mu^*(X) \leq \frac{1}{2}$.

Proof of Lemma. (1) Assume $\mu^*(X) < \frac{1}{2}$. By the definition of the outer measure, there exists a compact subset K of 2^ω such that $\mu(K) = \frac{1}{2}$ and $K \cap X = \emptyset$.

Let $\{U_n : n \in \omega\}$ be a strictly decreasing sequence of open sets such that $K = \bigcap_{n\in\omega} U_n$. Choose $V_n \in \Omega$ such that $V_n \notin \{V_i : i < n\}$ and $V_n \subset U_n$. Let $Y = \{V_n : n \in \omega\}$.

Since $K \cap X = \emptyset$, for each $x \in X$, there exists $n \in \omega$ such that $x \notin U_n$. So $|Y \cap I_x| < \omega$ for every $x \in X$.

(2) Suppose $\{I_x : x \in X\}$ does not witness to $\mathsf{cov}^*(\mathcal{S})$. Choose

$$Y = \{U_n : n \in \omega\} \in [\Omega]^\omega$$

such that $|I_x \cap Y| < \omega$. By Lemma 5.4,

$$\mu(\{x \in 2^\omega \,|I_x \cap Y| = \omega\}) = \frac{1}{2}.$$

So

$$\mu^*(X) \leq \mu(\{x \in 2^\omega : \ |I_x \cap Y| < \omega\}) \leq \frac{1}{2}. \qquad \dashv$$

Proof of Theorem 5.3. Firstly we shall show $\mathsf{cov}^*(\mathcal{S}) \leq \mathsf{non}(\mathcal{N})$.

Let X be a non-null set with $\mu^*(X) > 0$.

Claim 5.6. *There exists* $Y \subset 2^\omega$ *such that* $|Y| = |X|$ *and* $\mu^*(Y) = 1$.

Then $\{I_x : x \in Y\}$ is a witness to $\mathsf{cov}^*(\mathcal{S})$ by Lemma 5.5.

Next we shall show $\mathsf{cov}^*(\mathcal{S}) \geq \mathsf{non}(\mathcal{N})$. Let $\kappa < \mathsf{non}(\mathcal{N})$ and let $X \subset 2^\omega$ with $|X| = \kappa$. Then $\mu^*(X) = 0$. By Lemma 5.5, $\{I_x : x \in X\}$ does not witness to $\mathsf{cov}^*(\mathcal{S})$. So $\kappa < \mathsf{cov}^*(\mathcal{S})$. Therefore $\mathsf{non}(\mathcal{N}) \leq \mathsf{cov}^*(\mathcal{S})$. $\dashv$

Corollary 5.7. $\mathcal{G}_{FC} \geq_K \mathcal{S}$ *but* $\mathcal{G}_{FC} \not\leq_K \mathcal{S}$.

Proof. $\mathcal{G}_{FC} \geq_K \mathcal{S}$ is proved in [15]. We shall only show $\mathcal{G}_{FC} \not\leq_K \mathcal{S}$.

In the Cohen model, $\mathsf{cov}^*(\mathcal{G}_{FC}) = \mathfrak{s}_{pair} < \mathsf{cov}^*(\mathcal{S}) = \mathsf{non}(\mathcal{N})$ since $\mathfrak{s}_{pair} \leq \mathsf{non}(\mathcal{M})$ [13]. By Proposition 0.4, $\mathcal{G}_{FC} \not\leq_K \mathcal{S}$ in the Cohen model. By absoluteness of the Katětov order on Borel ideals, $ZFC \vdash \mathcal{G}_{FC} \not\leq_K \mathcal{S}$. $\dashv$

We need to find a Borel ideal $\mathcal{I}$ such that $\mathcal{I} \geq_K \mathcal{S}$ but for every $X \in \mathcal{I}^+$, $\mathcal{I} \restriction X \not\geq_K \mathcal{G}_{FC}$.

nwd denotes the ideal of nowhere dense subsets of $\mathbb{Q}$.

By the Sierpiński's characterization of $\mathbb{Q}$ we have the following.

Theorem 5.8. [1] $\mathsf{nwd} \simeq_K \mathsf{nwd} \restriction X$ *for every* $X \in \mathsf{nwd}^+$.

Given a forcing notion $\mathbb{P}$, we say an ideal $\mathcal{I}$ on ω is $\mathbb{P}$-indestructible if $\mathbb{P}$ does not add an infinite subset of ω which is almost disjoint from every element of $\mathcal{I}$. We say an ideal $\mathcal{I}$ is $\mathbb{P}$-destructible if $\mathcal{I}$ is not $\mathbb{P}$-indestructible. The ideal nwd is important when we think which ideals on ω are Cohen-destructible.

Theorem 5.9. [8, 6] $\mathcal{I}$ *is Cohen-destructible if and only if* $\mathcal{I} \leq_K \mathsf{nwd}$.

Using this theorem, we can decide the Katětov order between $\mathcal{G}_{FC}$ and nwd and between $\mathcal{S}$ and nwd

Theorem 5.10. (1) $\mathcal{S} \leq_K \mathsf{nwd}$.

(2) $\mathcal{G}_{FC} \not\leq_K \mathsf{nwd}$.

PROOF. First notice that the Katětov order among Borel ideal is written by a Σ^1_2-formula with reals as parameters which code Borel ideals. So it is absolute by Shoenfield absoluteness.

Work in a model V. Let $\mathbb{C}_\kappa$ be a κ-stage finite support iteration of Cohen forcing. In the model $V^{\mathbb{C}_{\mathfrak{c}^+}}$, $\mathsf{cov}^*(\mathcal{S}) = \mathsf{non}(\mathcal{N}) \geq \mathsf{cov}(\mathcal{M})$. So $\mathcal{S}$ is $\mathbb{C}_{\mathfrak{c}^+}$-destructible. By homogeneity of Cohen forcing, it destroys $\mathcal{S}$ in V. By Theorem 5.9, $V \models \mathcal{S} \leq_K \mathsf{nwd}$.

However, adding $\mathfrak{c}^+$-many Cohen reals implies that

$$\mathsf{cov}^*(\mathcal{G}_{FC}) = \mathfrak{s}_{pair} \leq \mathsf{non}(\mathcal{M}) = \mathfrak{c},$$

while $\mathsf{cov}^*(\mathsf{nwd}) \geq \mathfrak{c}^+$. So $V^{\mathbb{C}_{\mathfrak{c}^+}} \models \mathcal{G}_{FC} \not\leq_K \mathsf{nwd}$. By Shoenfield absoluteness, $V \models \mathcal{G}_{FC} \not\leq_K \mathsf{nwd}$. ⊣

By Theorem 5.8 and 5.10, $\mathcal{S}$ can not be replaced by $\mathcal{G}_{FC}$ in Theorem 5.1. So the answer of Question 5.2 is in the negative.

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