CARDINAL INVARIANTS RELATED TO SEQUENTIAL SEPARABILITY

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Abstract. Cardinal invariants related to sequential separability of generalized Cantor cubes 2^κ, introduced by M. Matveev, are studied here. In particular, it is shown that the following assertions are relatively consistent with ZFC:

(1) 2^{ω_1} is sequentially separable, yet there is a countable dense subset of 2^{ω_1} containing no non-trivial convergent subsequence,
(2) 2^{ω_1} is not sequentially separable, yet it is sequentially compact.

The work contained in this paper is devoted to studying combinatorial properties of independent families and their relationship with sequential separability of generalized Cantor cubes. Connections with Q-sets and hence the existence of separable non-metrizable Moore spaces is also mentioned.

A topological space X is *sequentially separable* if there is a countable $D \subseteq X$ such that for every $x \in X$ there is a sequence $\{x_n : n \in \omega\} \subseteq D$ converging to x; such a $D \subseteq X$ will be called *sequentially dense* in X. A space X is *strongly sequentially separable* if it is separable and every countable dense subset of X is sequentially dense. Here we consider sequential separability of 2^κ equipped with the product topology.

Recall that a set A is a *pseudo-intersection* of a family $\mathcal{F} \subseteq [\omega]^\omega$ if $A \subseteq^* F$ for every $F \in \mathcal{F}$ and, $\mathcal{F}$ is *centered* if every non-empty finite subfamily of $\mathcal{F}$ has an infinite intersection. A family $\mathcal{S} \subseteq [\omega]^\omega$ is *splitting* if $\forall A \in [\omega]^\omega\ \exists S \in \mathcal{S}$ $|A \cap S| = |A \setminus S| = \omega$. A family $\mathcal{I} \subseteq [\omega]^\omega$ is *independent* provided that for every nonempty disjoint $\mathcal{F}_1, \mathcal{F}_2 \in [\mathcal{I}]^{<\omega}$ $\bigcap \mathcal{F}_1 \setminus \bigcup \mathcal{F}_2 \neq \emptyset$. $\mathcal{I} \subseteq [\omega]^\omega$ is *independent splitting* if it is both independent and splitting. The following cardinal invariants are standard.

$\mathfrak{p} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq [\omega]^\omega$ a centered system which has no infinite pseudo-intersection$\}$

$\mathfrak{s} = \min\{|\mathcal{S}| : \mathcal{S}$ is a splitting family$\}$

F. Tall in [Ta] showed that 2^κ is strongly sequentially separable for every $\kappa < \mathfrak{p}$. M. V. Matveev in [Ma] defined the following cardinal invariants (using different notation)

$\mathfrak{p}_1 = \min\{\kappa : 2^\kappa$ is not strongly sequentially separable$\}$,

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$\mathfrak{ss} = \min\{\kappa : 2^\kappa \text{ is not sequentially separable}\}$ and

$\mathfrak{is} = \min\{\kappa : \exists D \subseteq 2^\kappa \text{ with no non-trivial convergent sequences}\}$

and observed that

$\mathfrak{p}_1 = \min\{|\mathcal{I}| : \mathcal{I} \text{ is an independent family without a pseudo-intersection}\}$ and

$\mathfrak{is} = \min\{|\mathcal{I}| : \mathcal{I} \text{ is an independent splitting family}\}$.

He also noted that $\mathfrak{p} \leq \mathfrak{p}_1 \leq \mathfrak{ss} \leq \mathfrak{c}$ and $\mathfrak{p}_1 \leq \mathfrak{is} \leq \mathfrak{c}$. He asked which of the inequalities are consistently strict and what is the relationship between $\mathfrak{ss}$ and $\mathfrak{is}$. The main aim of this note is to provide answers to these questions.

We assume familiarity with the method of forcing and basic theory of cardinal invariants of the continuum. For reference consult e.g. [BJ], [Bl], [vD] or [Va]. Set theoretic notation is standard and follows [Ku].

I. ZFC results

The proof of the following proposition can be extracted from a construction contained in [DN].

Proposition I.1 (Dow-Nyikos). $\mathfrak{p}_1 = \mathfrak{p}$.

Proof. Using Matveev's observation it is sufficient to find an independent family of size $\mathfrak{p}$ without a pseudo-intersection. To do this fix an independent family $\mathcal{I} = \{I_\alpha : \alpha < \mathfrak{p}\}$ of size $\mathfrak{p}$ and a centered system $\mathcal{F} = \{F_\alpha : \alpha < \mathfrak{p}\}$ without a pseudo-intersection. Let $\Delta = \{(n, m) \in \omega \times \omega : m \leq n\}$ and let

$$J_\alpha = (F_\alpha \times I_\alpha) \cap \Delta.$$

It is easy to see that that $\{J_\alpha : \alpha < \mathfrak{p}\}$ is an independent family of subsets of Δ without infinite pseudo-intersection as if $Y \subseteq \Delta$ were a pseudo-intersection for $\{J_\alpha : \alpha < \mathfrak{p}\}$ then $X = proj(Y) = \{n : \exists m\ (n, m) \in Y\}$ would be a pseudo-intersection of $\{F_\alpha : \alpha < \mathfrak{p}\}$ which is absurd. $\Box$

We will use the following characterization of $\mathfrak{ss}$. We attribute it to folklore as we do not know how to credit it appropriately. It definitely owes to work of Tall, Przymuszynski, Fleissner and others. For $A \subseteq \omega$ let $A^0 = A$ and $A^1 = \omega \setminus A$.

Proposition I.2 (Folklore). *The following are equivalent:*

(1) $\kappa < \mathfrak{ss}$
(2) *There is an independent family $\mathcal{I}$ of size κ such that $\{I^{f(I)} : I \in \mathcal{I}\}$ has an infinite pseudo-intersection for every $f \in 2^{\mathcal{I}}$,*
(3) *There is a Q-set of size κ,*
(4) *There is a separable normal Moore space with closed discrete subset of size κ,*
(5) *There is a normal Ψ-space of size κ.*

For proof of the non-trivial implications see e.g. [GKL]. We will need the following easy fact in the next section.

Proposition I.3. *If $\kappa < \mathfrak{ss}$ then $2^\kappa = 2^\omega$.*

Proof. As $\kappa < \mathfrak{ss}$ there is countable D sequentially dense subset of 2^κ. Now, for every $f \in 2^\kappa$ fix a subset D_f of D which converges to f. This obviously defines a one-to-one map from 2^κ into $\mathcal{P}(D)$. $\square$

Corollary I.4. $2^\omega < 2^{\omega_1}$ *implies* $\mathfrak{ss} = \omega_1$.

It is not quite obvious that the cardinal invariant $\mathfrak{is}$ is well defined. In fact, the question of existence of an independent splitting family appeared on A. Miller's problem list, where he attributes it to K. Kunen. As it turns out the question was answered a long time ago by P. Simon. We include a simple construction of an independent splitting family here.

Proposition I.5. (P. Simon). *There is an independent splitting family in ZFC.*

Proof. Let $\{U_n : n \in \omega\}$ be an enumeration of a basis for the topology of the rationals $\mathbb{Q}$. Let $\{D_n : n \in \omega\}$ be a disjoint refinement of the family $\{U_n : n \in \omega\}$ and let, for each $n \in \omega$, $\{I^n_\alpha : \alpha < \mathfrak{c}\}$ be an independent family of subsets of D_n. Let for $\alpha < \mathfrak{c}$

$$J'_\alpha = \bigcup_{n\in\omega} I^n_\alpha.$$

It is easy to see that the family $\{J'_\alpha : \alpha < \mathfrak{c}\}$ is independent family of subsets of $\mathbb{Q}$ and, moreover, all combinations of its elements are dense subsets of $\mathbb{Q}$.

Enumerate all infinite nowhere dense subsets of $\mathbb{Q}$ as $\{K_\alpha : \alpha < \mathfrak{c}\}$ and split each K_α into two infinite subsets M_α and N_α. Finally let

$$J_\alpha = (J'_\alpha \setminus M_\alpha) \cup N_\alpha$$

It is obvious that $\{J_\alpha : \alpha < \mathfrak{c}\}$ is still independent as we have made only nowhere dense changes to dense sets and it is splitting as every infinite subset of $\mathbb{Q}$ contains K_α for some $\alpha < \mathfrak{c}$. $\square$

While $\mathfrak{s}$ is a natural lower bound on the minimal size of an independent splitting family, as of now, there seems to be a definite lack of upper bounds. In an unpublished note [Ny], P. Nyikos proved that assuming the existence of a scale ($\mathfrak{b} = \mathfrak{d}$), the dominating number $\mathfrak{d}$ is an upper bound. A natural question arises as to whether the assumption $\mathfrak{b} = \mathfrak{d}$ is necessary. However, we do not even know, whether $\mathfrak{s} < \mathfrak{is}$ is relatively consistent with ZFC.

Next we will point out some of the obstacles one would face trying to prove the above consistency result. According to [KW] call a family $\mathcal{S} \subseteq [\omega]^\omega$ $\aleph_0$-*splitting* if for every sequence $\{A_i : i \in \omega\}$ of infinite subsets of ω there is a $S \in \mathcal{S}$ such that $|S \cap A_i| = |A_i \setminus S| = \aleph_0$ for every $i \in \omega$, and denote by $\aleph_0 - \mathfrak{s}$ the minimal cardinality of an $\aleph_0$-splitting family[1]. Note that $\aleph_0 - \mathfrak{s}$ is the uniformity ($non(\mathcal{J})$) for the σ-ideal $\mathcal{J}$ on $\mathcal{P}(\omega)$ generated by the sets $I_A = \{B \subseteq \omega : B \subset A \text{ or } B \cap A = \emptyset\}$, where A is an infinite co-infinite subset of ω. Recall that a set $X \subseteq \mathcal{P}(\omega)$ is a *generalized $\mathcal{J}$-Luzin set* if X is uncountable and for every $A \in \mathcal{J}$ $|X \cap A| < |X|$.

[1]The consistency of $\mathfrak{s} < \aleph_0 - \mathfrak{s}$ is another open problem (see [KW] and [Br2])

Proposition I.5. *If there is a generalized $\mathcal{J}$-Luzin set of size $\mathfrak{p}$ then $\mathfrak{is} = \mathfrak{s}$.*

Proof. We will show that given a $\mathcal{J}$-Luzin set X of size $\mathfrak{p}$ one can find $Y \subseteq X$ of the same cardinality which forms an independent family. It should be obvious then that Y is indeed an independent splitting family (a subset of a generalized $\mathcal{J}$-Luzin set of the same cardinality is itself a generalized $\mathcal{J}$-Luzin set). Fix a generalized $\mathcal{J}$-Luzin set X of size $\mathfrak{p}$ and construct a family $\{I_\alpha : \alpha < \mathfrak{p}\}$ by induction as follows: Let I_0 be any infinite co-infinite element of X. At stage $\alpha < \mathfrak{p}$ let $\{f_i : i \in \omega\}$ be a (sequentially) dense subset of 2^α and let A_i be an infinite pseudo-intersection of the family $\{I_\beta^{f_i(\beta)} : \beta < \alpha\}$ for every $i \in \omega$. Such an A_i exists, as $\alpha < \mathfrak{p} = \mathfrak{p}_1$. Now, pick $I_\alpha \in X$ such that $|I_\alpha \cap A_i| = |A_i \setminus I_\alpha| = \aleph_0$ for every $i \in \omega$.

It is easy to verify that the family thus constructed is indeed independent. $\square$

Corollary I.6. *If there is a Luzin or Sierpiński set then $\mathfrak{is} = \omega_1$.*

Proof. It suffices to note that the ideal $\mathcal{J}$ defined above is σ-generated closed sets of Haar measure zero. $\square$

II. Consistency results

Here we will include the main results of this paper. Let us first start with some easy observations.

Theorem II.1. *Con*$(\mathfrak{ss} = \mathfrak{is} = \omega_1 < \mathfrak{c})$.

Proof. Let $V \models CH$ and let G be $\mathbb{C}_{\omega_2}$-generic over V, where $\mathbb{C}_{\omega_2}$ denotes the standard c.c.c. poset for adding $\aleph_2$-many Cohen reals. Let $\mathcal{I} = \langle C_\alpha : \alpha < \omega_1\rangle$ be the generic sequence of Cohen reals added by $G \cap \mathbb{C}_{\omega_1}$. The fact that $\mathcal{I}$ is an independent splitting family follows easily from the fact that a Cohen real splits every infinite subset from the ground model and the fact that the Cohen reals added are mutually generic. So $\mathfrak{is} = \omega_1$.

To see that $\mathfrak{ss} = \omega_1$ let $\dot{\mathcal{I}}$ be a name for an independent family $\{I_\alpha : \alpha < \omega_1\}$. Then there is an $\alpha < \omega_2$ such that $\mathcal{I} \in V[G \cap \mathbb{C}_\alpha]$. Let $g : \omega_1 \longrightarrow 2$ be the generic function added by the next $\aleph_1$-many Cohen reals. Assume towards contradiction that there is an $A \in V[G]$ which is an infinite pseudo-intersection of $\{I_\alpha^{g(\alpha)} : \alpha < \omega_1\}$. There is $\xi \in \omega_2$ such that

$$A \in V[G \cap (\mathbb{C}_{\alpha+\xi} \times \mathbb{C}_{\omega_2 \setminus (\alpha+\omega_1)})]$$

and hence genericity implies that there is some integer k such that $A \not\subseteq^* I_{\alpha+\xi+k}^{g(\alpha+\xi+k)}$. $\square$

It is clear that random reals would do just as well, so a similar proof gives a consistency of $\mathfrak{is} < \mathfrak{b}$ (just start with a model of $MA \& 2^{\aleph_0} > \aleph_1$ and add $\aleph_2$-many random reals).

Theorem II.2. *Con*($\mathfrak{ss} < \mathfrak{is}$).

Proof. Let $V \models$"$\mathfrak{s} = \omega_2$ and $2^{\omega_1} > \omega_2 = \mathfrak{c}$". Then $\mathfrak{is} = \omega_2$ and by Corollary I.3 $\mathfrak{ss} = \omega_1$. Such models were constructed a long time ago (see e.g. [vD], Theorem 5.4). $\square$

Consistency of $\mathfrak{p}_1 < \mathfrak{ss}$ follows directly from results in [D] as pointed out by J. Brendle in [Br]. We will produce another model for this inequality where moreover $\mathfrak{is} < \mathfrak{ss}$ and a model where $\mathfrak{ss} < add(\mathcal{N}) = \mathfrak{s} = \mathfrak{is}$.

Before doing so, recall the definition of a standard forcing for adding a pseudo-intersection to a given filter base $\mathcal{F}$. The forcing, denoted by $\mathbb{M}(\mathcal{F})$ consists of pairs (s, F), where s is a finite subset of ω and $F \in [\mathcal{F}]^{<\omega}$, ordered by $(s, F) \leq (t, G)$ if t is an initial segment of s ($s \setminus t \cap \max(t) + 1 = \emptyset$) and $s \setminus t \subseteq \bigcap G$. It is easy to see that $\mathbb{M}(\mathcal{F})$ is c.c.c. (in fact σ-centered) and that forcing with $\mathbb{M}(\mathcal{F})$ indeed produces a pseudo-intersection to the filter base $\mathcal{F}$.

As shown in Proposition II.1 the standard forcing for adding $\aleph_1$-many Cohen reals adds an independent splitting family. Our strategy for proving *Con*($\mathfrak{is} < \mathfrak{ss}$) is to first add two such families and then diagonalize all "paths" through one of them producing a witness to $\mathfrak{ss} > \omega_1$ and then show that the other one remains splitting in the extension, witnessing $\mathfrak{is} = \omega_1$.

Theorem II.3. *Con*($\mathfrak{is} < \mathfrak{ss}$).

Proof. Let $V \models GCH$. Let $\mathbb{P} = \mathbb{C}_{\omega_1} * \mathbb{P}_{\omega_2}$, where (working in V[H], where H is $\mathbb{C}_{\omega_1}$-generic over V)

$$\mathbb{P}_{\omega_2} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}} : \alpha < \omega_2 \rangle$$

is a finite support iteration such that $\mathbb{P}_0 = \mathbb{C}_{\omega_1}$ and $\Vdash_{\mathbb{P}_\alpha}$"$\dot{\mathbb{Q}} : \alpha = \mathbb{M}(\{\dot{A}_\beta^{\dot{f}(\beta)} : \beta < \omega_1\})$". Here $\dot{A}_\beta$ denotes the βth Cohen real added by $\mathbb{P}_0$ and $\dot{f}$ is a $\mathbb{P}_\alpha$-name for a function from ω_1 to 2. By a standard bookkeeping argument one can ensure that (somewhat loosely speaking) for every $\mathbb{P}_{\omega_2}$-name $\dot{f}$ for a function from ω_1 to 2 there is an $\alpha < \omega_2$ such that $\Vdash_{\mathbb{P}_\alpha}$"$\dot{\mathbb{Q}}_\alpha = \mathbb{M}(\{\dot{A}_\beta^{\dot{f}(\beta)} : \beta < \omega_1\})$".

The forcing $\mathbb{P}$ satisfies the countable chain condition and has (a dense set of) size ω_2 so, in the extension $\mathfrak{c} = 2^{\omega_1} = \omega_2$. Let G be $\mathbb{P}_{\omega_2}$-generic over $V[H]$. It follows immediately from the construction, that the family $\{A_\beta : \beta < \omega_1\}$ is independent and that the family $\{A_\beta^{f(\beta)} : \beta < \omega_1\}$ has an infinite pseudo-intersection for every $f : \omega_1 \longrightarrow 2$, hence by Proposition I.2:

Claim II.3.1. $\mathfrak{ss} = \omega_2$ *in* $V[H][G]$.

The rest of the proof is devoted to showing that the independent splitting family added by H remains splitting in the extension. This will be done by showing that, in $V[H]$, the Boolean algebra generated by $\mathbb{P}_{\omega_2}$ is *semi-Cohen* i.e. has a closed unbounded set of regularly embedded countable subalgebras. Note that this implies that every real added by G is Cohen over $V[H]$ and hence, indeed, preserves that the independent family added by H remains splitting. Note that this is actually quite subtle, as if we denote by K the filter $G \cap \mathbb{P}_0$ ($\mathbb{C}_{\omega_1}$-generic over $V[H]$) then there are many reals in $V[H][G]$ which are NOT Cohen over $V[H][K]$.

So in order to finish the proof it is sufficient to prove (working in $V[H]$) the following

Claim II.3.2. *Let $M \prec H(\omega_3)$ be (in $V[H]$) an elementary submodel containing $\mathbb{P}_{\omega_2}$ and let $\beta < \omega_2$. If $V[H][K \cap (\mathbb{P}_0 \cap M)] \models$ "$D \subseteq M \cap (\mathbb{P}_\beta/\mathbb{P}_0)$ is a maximal antichain in $M \cap (\mathbb{P}_\beta/\mathbb{P}_0)$", then $V[H][K] \models$ "D is a maximal antichain in $\mathbb{P}_\beta/\mathbb{P}_0$".*

First note that the above Claim actually makes sense as $\mathbb{P}_0 \cap M$ is regularly embedded in $\mathbb{P}_0$. Having fixed an elementary submodel M as above will prove the claim by induction on β. Let $\delta = \omega_1 \cap M$.

The Case $\beta = 0$ is vacuous.

Isolated step $(\beta + 1)$. Assume that the Lemma holds for $\beta < \omega_2$. Working in $V[H][K \cap \mathbb{C}_\delta] = V[H][K \cap (\mathbb{P}_0 \cap M)]$ let $D \subseteq M \cap \mathbb{P}_\beta/\mathbb{P}_0$ be a maximal antichain and let $p \in \mathbb{P}_0/\mathbb{C}_\delta$ be such that

$$p \Vdash \text{“}\langle q, (s, \{\dot{A}_\xi^{\dot{f}_\beta(\xi)} : \xi \in \Lambda_0 \cup \Lambda_1\})\rangle \in \mathbb{P}_{\beta+1}\text{” and } p \Vdash \text{“}\dot{f}_\beta \restriction \Lambda_0 \cup \Lambda_1 = F\text{”}$$

where $\Lambda_0 \subseteq \delta$ and $\Lambda_1 \cap \delta = \emptyset$ ($\Lambda_1 \cap M = \emptyset$) and $F : \Lambda_0 \cup \Lambda_1 \longrightarrow 2$.

Without loss of generality we can assume that $\max(s) = \max(dom(p(\gamma)))$ for every $\gamma \in \Lambda_1$. (If necessary, find $n > dom(p(\gamma))$ for $\gamma \in \Lambda_1$ and extend p to a $\bar{p}$ so that, for $m \leq n$ and $\gamma \in \Lambda_1$

$$\bar{p}(\gamma)(m) = \begin{cases} p(\gamma)(m) & \text{if } m \in dom(p(\gamma)), \\ 1 & \text{if } m = n \text{ and } F(\gamma) = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and let $\bar{s} = s \cup \{n\}$.)

Denote by $\bar{D}$ the downward closure of D and let

$$D^* = \{q \in (\mathbb{P}_\beta/\mathbb{P}_0) \cap M : \{\dot{q}' \in \dot{\mathbb{Q}}_\beta : \langle q, \dot{q}'\rangle \in \bar{D}\} \text{ is dense}\}.$$

The set D^* is a dense subset of $(\mathbb{P}_\beta/\mathbb{P}_0) \cap M$ so, by the inductive hypothesis, D^* is pre-dense in $\mathbb{P}_\beta/\mathbb{P}_0$. Choose $\bar{q} \leq q$ such that $\bar{q} \leq r$ for some $r \in D^*$ and $(b, \Gamma) \in \dot{\mathbb{Q}}_\beta$ such that $p \Vdash$ "$\langle \bar{q}, (b, \Gamma)\rangle \leq \langle q, (s, \Lambda_0)\rangle$". Now, using that $\max(s) = \max(dom(p(\gamma)))$ for every $\gamma \in \Lambda_1$, extend p to a $\bar{p}$ so that

$$\bar{p} \Vdash \text{“}b \setminus s \subseteq \bigcap_{\gamma \in \Lambda_1} \dot{A}_\gamma^{F(\gamma)}\text{”}$$

by letting

$$\bar{p}(\gamma)(m) = \begin{cases} p(\gamma)(m) & \text{if } m \in dom(p(\gamma)), \\ 1 & \text{if } m \in b \setminus s \text{ and } F(\gamma) = 0, \\ 0 & \text{otherwise,} \end{cases}$$

for $\gamma \in \Lambda_0 \cup \Lambda_1$ and $m \leq \max(b)$. Then

$$\bar{p} \Vdash \text{“}\langle \bar{q}, (b, \Gamma)\rangle \leq \langle q, (s, \Lambda_0 \cup \Lambda_1)\rangle\text{”}$$

which is what we needed to prove.

β limit. Fix D and let $p \Vdash$ "$\dot{q} \in \mathbb{P}_\beta/\mathbb{P}_0$". Then there is a p' and $\gamma < \beta$ such that $p' \Vdash$ "$\dot{q} \in \mathbb{P}_\gamma$". The conclusion then follows easily as the set $D^* = \{q \in$

$(\mathbb{P}_\gamma/\mathbb{P}_0)\cap M : \{\dot{q}' \in M\cap(P_\beta/\mathbb{P}_\gamma) : \langle q,\dot{q}'\rangle \in \bar{D}\}$ is dense$\}$ is dense in $M\cap(\mathbb{P}_\gamma/\mathbb{P}_0)$, hence by the induction hypothesis pre-dense in $\mathbb{P}_\gamma/\mathbb{P}_0$. $\square$

The same conclusion probably also holds in a model constructed in [FM]. Note that the model constructed in Theorem II.3 shares many of the properties of the Cohen model (model used in Theorem II.1). For instance, $cov(\mathcal{M}) = \mathfrak{c}$, and $non(\mathcal{M}) = \omega_1$ in fact there is a Luzin set, in both models.

Next we will show, that not only $\mathfrak{ss} < \mathfrak{is}$ is consistent (as shown by Theorem II.2) but it is possible to lower $\mathfrak{ss}$ while leaving "most" cardinal invariants large. Recall that an ω_1-tree T is a *Suslin tree* if it has no uncountable chains or antichains. When used as a forcing notion T (turned up-side-down) is a c.c.c partial order which is does not add any new reals. We will show that if $V \models$"$MA_{\sigma-centered}$ + There is a Suslin tree T" and if G is T-generic over V then $V[G] \models \mathfrak{ss} = \omega_1$. Note that such a model can be obtained e.g. by adding a single Cohen real to a model of MA (see [BJ]). In this model, however, $add(\mathcal{N}) = cov(\mathcal{N}) = \omega_1$ (see [BJ]). There is a way, however, to construct a model of $MA_{\sigma-centered}$ + There is a Suslin tree where, moreover, $add(\mathcal{N}) = \mathfrak{c}$.

Theorem II.4. *$Con(\mathfrak{ss} < add(\mathcal{N}) = \mathfrak{s} = \mathfrak{c})$.*

Proof. Let $V \models MA_{\sigma-centered}$, $add(\mathcal{N}) = \mathfrak{c}$ + There is a Suslin tree T. Let G be T-generic over V.

First note that as T is c.c.c. all cardinalities and cofinalities of ordinals are preserved. Moreover, as $|T| = \omega_1$, $V[G] \models \mathfrak{c} = \mathfrak{c}^V$. To see that $\mathfrak{s} = \mathfrak{c}$ note that, since T adds no reals, if $\dot{\mathcal{S}}$ were a name for a splitting family of size less than $\mathfrak{c}$ then the family

$$\{S \in [\omega]^\omega : \quad \exists t \in T \quad t \Vdash \text{"}S \in \dot{\mathcal{S}}\text{"}\}$$

would be a splitting family of size less than $\mathfrak{c}$ in V, which contradicts $MA_{\sigma-centered}$. A similar proof gives $V[G] \models add(\mathcal{N}) = \mathfrak{c}$.

So the only thing left to prove is that $\mathfrak{ss} = \omega_1$, in other words, no countable dense subset of 2^{ω_1} is sequentially dense. Without loss of generality we can assume that $T \subseteq 2^{<\omega_1}$ and that it is well-pruned and everywhere branching (i.e. $\forall t \in T$ $t^\frown 0, t^\frown 1 \in T$). While T adds no new reals it does add new elements of 2^{ω_1}. Aiming towards a contradiction let $\dot{\Phi}$ be a T-name for a function from ω to 2^{ω_1} and assume that $\Vdash_T$"$rng(\dot{\Phi})$ is sequentially dense". Let θ be a regular cardinal such that $T \in H(\theta)$. Put

$$\mathcal{C} = \{M \cap \omega_1 : M \prec H(\theta) \& \dot{\Phi}, T \in M\}$$

and increasingly enumerate $\mathcal{C}$ as $\{\delta_\alpha : \alpha < \omega_1\}$. Now let $\dot{f}$ be a T-name for an element of 2^{ω_1} defined by $p \Vdash$"$\dot{f}(\alpha) = i$" if and only if $\delta_\alpha \in dom(p)$ and $p(\delta_\alpha) = i$. To finish the proof it is sufficient to show that:

Claim. $\Vdash_T$ *"There is no sequence in $rng(\dot{\Phi})$ converging to $\dot{f}$".*

Assume the contrary, i.e. there is a $p \in T$ and an $h \in \omega^\omega$ such that $p \Vdash$"$\dot{\Phi} \circ h \to \dot{f}$". Note that this is without loss of generality as T does not add any new reals. Now, however, pick $M \prec H(\theta)$ such that $p, T, \dot{\Phi} \in M$ and let $\rho = M \cap \omega_1$. Then $\rho = \delta_\alpha$ for some $\alpha \leq \rho$ and without loss of generality $\alpha < \rho$. Let $q \in T_\rho$, $q \supset p$. Then q decides the value of $\dot{\Phi}(n)(\alpha)$ for every $n \in \omega$, yet it does not decide the value of $\dot{f}(\alpha)$ as $q^\frown i \Vdash$ "$\dot{f}(\alpha) = i$". Now, it is easy to reach a contradiction; If there are infinitely many n's such that $q \Vdash$ "$\dot{\Phi}(h(n))(\alpha) = 0$" let $q' = q^\frown 1$ otherwise let $q' = q^\frown 0$. In either case $q' \Vdash$ "$\dot{\Phi} \circ h \not\to \dot{f}$" which is absurd. $\square$

Let us point out the curious nature of the cardinal invariant $\mathfrak{ss}$. It is proved (though not stated) in [JS] that it is consistent that $\mathfrak{ss} = \omega_2$ yet all invariants from the Cichoń diagram are small. On the other hand, Theorem II.4 shows that consistently $\mathfrak{ss} < add(\mathcal{N})$, so there is no relation between $\mathfrak{ss}$ and most "standard" cardinal invariants of the continuum.

References

[BJ] T. Bartoszyński and H. Judah, *Set Theory, On the Structure of the real line*, A K Peters, 1995.

[Bl] A. Blass, *Combinatorial Cardinal Characteristics of the Continuum*, to appear in Handbook of Set theory.

[Br] J. Brendle, *Dow's principle and Q-sets*, Canad. Math. Bull. **42** (1999), 13-24.

[Br2] J. Brendle, *Around splitting and reaping*, Comment. Math. Univ. Carolin. **39** (1998), 269-279.

[vD] E. van Douwen, *The integers and topology*, in Handbook of Set Theoretic Topology (ed. K. Kunen and J. Vaughan), 1984, pp. 111–167, North-Holland, Amsterdam.

[D] A. Dow, *On compact separable radial spaces*, Canad. Math. Bull. **40** (1997), 422-432.

[DN] A. Dow and P. Nyikos, *Representing free Boolean algebra*, Fund. Math. **141** (1992), 21-30.

[FM] W. G. Fleissner and A. W. Miller, *On Q-sets*, Proc. Amer. Math. Soc. **78** (1980), 280-284.

[GKL] P. M. Gartside, R. W. Knight and J. T. H. Lo, *Sequential density*, (preprint).

[JS] H. Judah and S. Shelah, *Q-sets, Sierpiński sets and rapid filters*, Proc. Amer. Math. Soc. **111** (1991), 821-832.

[KW] A. Kamburelis and B. Weglorz, *Splittings*, Arch. Math. Logic **35** (1996), 263-277.

[Ku] K. Kunen, *Set Theory. An Introduction to Independence Proofs* (1980), North Holland, Amsterdam.

[Ma] M. V. Matveev, *Cardinal* $\mathfrak{p}$ *and a theorem of Pelczynski*, (preprint).

[Ny] P. Nyikos, *Answer to a question of Matveev*, (privat communication).

[Ta] F. Tall, *How separable is a space? That depends on your set theory!*, Proc. Amer. Math. Soc. **46** (1974), 310-314.

[Va] J. Vaughan, *Small uncountable cardinals and topology*, In J. van Mill and G. M. Reed, editors, Open problems in topology (1990), North-Holland, 196-216.

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