RAMSEY TYPE PROPERTIES OF IDEALS

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Abstract. We study several combinatorial properties concerning (mostly definable) ideals on countable sets. In several cases, we identify *critical* ideals for such properties in the Katětov order. In particular, the ideal $\mathcal{R}$ generated by the homogeneous sets with respect to the random graph is critical with respect to the Ramsey property. The question as to whether there is a tall definable Ramsey ideal is raised and studied. It is shown that no tall F_σ ideal is Ramsey, while there is a tall co-analytic Ramsey ideal.

Introduction

The classical Ramsey's theorem asserts that for every coloring φ of unordered n-tuples of natural numbers in m colors, there is an infinite subset X of ω such that X is φ-homogeneous (i.e. $|\varphi([X]^n)| = 1$). In this paper we study ideals on ω satisfying Ramsey's theorem in the sense that the homogeneous set can be found positive with respect to the ideal (i.e. a set not belonging to I). More precisely, an ideal I is said to be *Ramsey*(ω) if for every coloring $\varphi : [\omega]^2 \to 2$ there is an I-positive set X which is φ-homogeneous, this property is also denoted by $\omega \longrightarrow (\mathsf{I}^+)^2_2$. A stronger property is the following: We say that I is *Ramsey* if for every I-positive set X and every coloring $\varphi : [X]^2 \to 2$ there is an I-positive subset Y of X which is φ-homogeneous. This property is denoted by $\mathsf{I}^+ \longrightarrow (\mathsf{I}^+)^2_2$.

The most common examples of Ramsey ideals are the selective ideals (or Mathias' happy families) [22] and, more generally, Farah's semiselective ideals [7]. In fact, it is well known that selective implies semiselective and this implication is proper.

We will be interested in combinatorial properties of tall ideals (i.e. those ideals satisfying that every infinite set contains an infinite subset belonging to the ideal) and more specifically of definable tall ideals (definable as subsets of the Cantor cube 2^ω). As we shall see, for many combinatorial properties of ideals there are ideals (usually Borel of a low complexity) which are critical in the Katětov order $\leq_K$ with respect to the given property. For instance, we show that $\omega \longrightarrow (\mathsf{I}^+)^2_2$ if and only if $\mathcal{R} \not\leq_K \mathsf{I}$, where $\mathcal{R}$ is the ideal generated by cliques and free sets in the random graph.

We will focus on definable Ramsey tall ideals because it is known that tall semiselective ideals are not definable (see section §1). The principal question considered here is the following: *Is there a Borel (analytic) tall Ramsey ideal?* We construct examples of definable

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(even F_σ) ideals satisfying $\omega \longrightarrow (\mathsf{I}^+)^2_2$, but failing $\mathsf{I}^+ \longrightarrow (\mathsf{I}^+)^2_2$. We also present an example of a co-analytic Ramsey tall ideal.

It is well known (probably due to Kunen) that Ramsey ultrafilters are exactly the dual filters of maximal ideals which are both P and Q-ideals. Kunen's proof actually shows that if I is both a P^+ and Q^+-ideal then $\mathsf{I}^+ \longrightarrow (\mathsf{I}^+)^2_2$. It is easy to see that a Ramsey ideal is necessarily Q^+ (Proposition 2.1) but not necessarily P^+ (Example 3.1). In fact, no definable tall ideal is both P^+ and Q^+ (Proposition 1.2). We will investigate properties like P^+, Q^+ and some natural variants of them.

There is a close connection between combinatorial and descriptive theoretic properties of an ideal I and the forcing properties of the quotient $\mathcal{P}(\omega)/\mathsf{I}$. In particular, in several places of the text we shall take advantage of this connection and consider properties of $\mathcal{P}(\omega)/\mathsf{I}$ such as: proper, σ-closed, does not add reals. We believe it would be useful to investigate this connection in more depth.

The paper is organized as follows. In the first section we briefly revisit the notion of semiselectivity.

Section 2 is dedicated to $\mathrm{Q}^+(\omega)$ and Q^+-ideals and related properties.

In section 3 we study P^+-ideals. We include a characterization of P-points using Katětov order (Theorem 3.6), which extends a result of Zapletal (Claim 2.4 in [28]). We use this to characterize definable P^+-ideals (Theorem 3.10) as those definable ideals which are indecomposable and locally F_σ. We study several natural variants of the P^+-property there.

Section 4 deals with variations of the Ramsey and Ramsey(ω) properties, their connections and their critical ideals in Katětov order.

In section 5 we compare the Ramsey-like properties with those introduced by Filipów, Mrożek, Recław and Szuca in [8], and answer two of their questions. We finish by listing some open questions.

We end the introduction by fixing some terminology. An *ideal* on a set X is a non-empty family of subsets of X closed under subsets and finite unions, which does not contain the set X. A subset of X that is not in I is a called I-positive. The collection of all I-positive sets is denoted by I^+. We will always assume that an ideal contains all finite subsets of X. When X is countable, an ideal I over X can be regarded as an ideal over ω via a bijection between X and ω. Thus we will state our results for ideals on ω.

An ideal I is *tall* if every infinite subset A of ω contains an infinite subset which is in I.

An ideal I on ω is a P-*ideal* if for every countable family $\{I_n : n \in \omega\} \subseteq \mathsf{I}$, there is $I \in \mathsf{I}$ such that $I_n \setminus I$ is finite, for all n. I is a P^+-*ideal* if for every decreasing sequence $\{X_n : n \in \omega\}$ of I-positive sets there is an I-positive set X such that $X \subseteq^* X_n$, for all n. When this property holds only for decreasing sequences $(X_n)_n$ of I-positive sets such that $X_n \setminus X_{n+1} \in \mathsf{I}$ for all n, we say that I is a $\mathrm{P}^+_{\mathsf{tower}}$-*ideal*. If, morover, the property holds only when $X_0 = \omega$ we will say that the ideal is $\mathrm{P}^+_{\mathsf{tower}}(\omega)$.

I is a Q-*ideal* if for every partition $\{J_n : n \in \omega\}$ of ω into finite sets, there is $I \in \mathsf{I}$ such that $|J_n \setminus I| \leq 1$ for all n. I is a $\mathrm{Q}^+(\omega)$-*ideal* if for every partition $\{F_n : n \in \omega\}$ of ω into finite sets there is an I-positive set Y such that $|Y \cap F_n| \leq 1$, for all n. I is a Q^+-*ideal* if for every I-positive set X and every partition $\{F_n : n \in \omega\}$ of X into finite sets there is an I-positive set $Y \subseteq X$ such that $|Y \cap F_n| \leq 1$, for all n. Such sets Y are called *selectors*.

One of the main tools for our study of combinatorial properties of ideals is the *Katětov order* $\leq_K$. Given ideals I and J on ω we say that $\mathsf{I} \leq_K \mathsf{J}$ if there is a function $f : \omega \to \omega$

such that $f^{-1}[I] \in \mathsf{J}$ for all $I \in \mathsf{I}$. If f is finite-to-one then we write $\mathsf{I} \leq_{KB} \mathsf{J}$ and say that I is *Katětov-Blass* below J.

1. Selectivity and semiselectivity

The purpose of this short section is to make precise something we said in the introduction about definable tall ideals. In particular, we want to explain why when dealing with definable tall ideals one has to investigate the Ramsey property directly, rather than the more common notions of selectivity and semiselectivity.

We start by recalling a result of Mathias

Theorem 1.1 (Mathias [22], Theorem 2.12)**.** *Let $\mathcal{U}$ be an ultrafilter on ω. Then, $\mathcal{U}$ is selective*[1] *if and only if $\mathcal{U} \cap \mathsf{I} \neq \emptyset$ for every analytic tall ideal I on ω.*

The following fact is probably due to Mathias (see [7, 22, 27]). We present a short proof for the convenience of the reader.

Proposition 1.2. *There are no tall analytic ideals which are both P^+ and Q^+.*

Proof. Let I be an analytic ideal on ω, and suppose that I is both a P^+ and a Q^+-ideal. Then, being P^+, the forcing $\mathcal{P}(\omega)/\mathsf{I}$, forcing equivalent with $(\mathsf{I}^+, \subseteq)$, is σ-closed, so it does not add new real numbers. Let G be a $\mathcal{P}(\omega)/\mathsf{I}$-generic ultrafilter. We shall show now, that in $V[G]$, $\mathcal{U} = \{X \subseteq \omega : [X]_{\mathsf{I}} \in G\}$ is a selective ultrafilter. To that end let $\varphi : [\omega]^2 \to 2$ be a coloring in $V[G]$. Again, as $\mathcal{P}(\omega)/\mathsf{I}$ is σ-closed, $\varphi \in V$. As P^+ together with Q^+ entail $\mathsf{I}^+ \longrightarrow (\mathsf{I}^+)^2_2$, the set of I-positive φ-homogeneous sets is dense in $\mathcal{P}(\omega)/\mathsf{I}$, hence the generic filter contains a φ-homogeneous set. Note that a selective filter is necessarily an ultrafilter.

To finish the proof it suffices to note that the generic selective ultrafilter is contained in I^+, hence is disjoint from I which remains tall and analytic as no reals were added, contradicting (in $V[G]$) the theorem 1.1. $\square$

Recall that an ideal I is *semiselective* [7], if whenever $\mathcal{D}_i$ are dense open subsets of the partial order $(\mathsf{I}^+, \subseteq)$, for $i \in \omega$, and A is a I-positive set, there is $D \subseteq A$ in I^+ such that $D/n \in \mathcal{D}_n$ for all $n \in D$. A subset $\mathcal{C} \subseteq [\omega]^\omega$ is said to be I-*Ramsey*, if for all $A \in \mathsf{I}^+$ there is $B \subseteq A$ in I^+ such that $[B]^\omega \cap \mathcal{C} = \emptyset$ or $[B]^\omega \subseteq \mathcal{C}$ [2]. It follows directly from the definition that an ideal I is itself I-Ramsey if and only if $\mathsf{I} \restriction X$ is not tall for all $X \in \mathsf{I}^+$. Farah [7] showed that every analytic set is I-Ramsey for any semiselective ideal I. In particular, this shows that there are no tall analytic semiselective ideals. In fact, there are stronger results. The following result implies that, under suitable set theoretic assumptions, every subset of $[\omega]^\omega$ is I-Ramsey for any semiselective ideal I, and therefore I is not tall.

Theorem 1.3 ([5])**.**
1. *Assume $AD_{\mathbb{R}}$. If I is a semiselective ideal, then every subset of $[\omega]^\omega$ is I-Ramsey.*
2. *If ZFC is consistent with the existence of a weakly compact cardinal, then so is the statement that for every semiselective ideal I every subset of $[\omega]^\omega$ from $L(\mathbb{R})$ is I-Ramsey.*

[1] Recall that an ultrafilter is *selective* if and only if the dual ideal is Ramsey, if and only if the dual ideal is both a P-ideal and a Q-ideal.

[2] Usually, a stronger condition is required but this is enough for our purposes.

In [20] a result similar to (1) was proved. We observe that part (1), together with the comments above, answers question 0.3 of [20].

2. $\mathrm{Q}^+(\omega)$ and Q^+-ideals

The following result was also mentioned in the introduction.

Proposition 2.1. *If* I *satisfies* $\mathsf{I}^+ \longrightarrow (\mathsf{I}^+)^2_2$ *then* I *is a* Q^+*-ideal.*

Proof. Suppose that I is not Q^+. Let X be an I-positive set and let $\{F_n : n \in \omega\}$ be a partition of X into finite sets every selector of which is in I. Define a coloring $\varphi : [X]^2 \to 2$ by $\varphi\{k,m\} = 0$ if and only if there is n such that $k, m \in F_n$. Any φ-homogeneous set is finite or is a selector for $\langle F_n : n \in \omega\rangle$ and thus φ-homogeneous sets are in I. $\square$

Consider the following *eventually different* ideals:

$$\mathcal{ED} = \{A \subseteq \omega \times \omega : (\exists n)(\forall m \geq n)(|\{(m,k) : k < \omega\} \cap A| \leq n)\},$$

and $\mathcal{ED}_{fin}$ - the restriction of $\mathcal{ED}$ to the set $\Delta = \{(n,m) \in \omega \times \omega : m \leq n\}$. The ideal $\mathcal{ED}_{fin}$ is (up to Katětov-equivalence), the unique ideal generated by the selectors of some partition of ω into finite sets $\{I_n : n \in \omega\}$ such that $\limsup_n |I_n| = \infty$. The ideal $\mathcal{ED}_{fin}$ probably first appeared in [21] and is critical for the Q^+-property.

Theorem 2.2. *Let* I *be an ideal on* ω*. Then*

(1) I *is a* $Q^+(\omega)$*-ideal if and only if* $\mathsf{I} \not\geq_{KB} \mathcal{ED}_{fin}$*, and*

(2) I *is a* Q^+*-ideal if and only if* $\mathsf{I} \restriction X \not\geq_{KB} \mathcal{ED}_{fin}$ *for all* I*-positive set* X*.*

Proof. (1) Let us suppose that $\mathcal{ED}_{fin} \leq_{KB} \mathsf{I}$, and let $f : \omega \to \Delta$ be the witnessing function. Let $\Delta_n = \Delta \cap (\{n\} \times \omega)$. Then, the family $\{f^{-1}(\Delta_n) : n \in \omega\}$ is a partition of ω into finite sets and every selector of the partition belongs to I.

On the other hand, if I is not a $\mathrm{Q}^+(\omega)$-ideal then there is a partition $\langle F_n : n \in \omega\rangle$ of ω into finite sets such that every selector belongs to I. Note that such a partition must have elements of arbitrarily large finite cardinality, otherwise I would be improper. Take a sequence of sets $\langle G_n : n \in \omega\rangle$ such that $G_n \in [F_{i_n}]^n$ for some $i_{n-1} < i_n \in \omega$ and $X = \bigcup_n G_n \in \mathsf{I}^+$. Then there is a copy of $\mathcal{ED}_{fin}$ inside of $\mathsf{I} \restriction X$.

(2) Analogous to (1). $\square$

A family $\mathcal{A}$ of infinite subsets of ω is ω*-hitting* if for every infinite partition $\langle X_n : n \in \omega\rangle$ of ω into infinite sets, there is $A \in \mathcal{A}$ such that $A \cap X_n$ is infinite, for all $n \in \omega$. Similarly, $\mathcal{A}$ is ω*-splitting* if for every countable family $\mathcal{X}$ of infinite subsets of ω there is $A \in \mathcal{A}$ such that $|A \cap X| = |X \setminus A| = \aleph_0$ for all $X \in \mathcal{X}$. It is easy to see that an ideal on ω is ω*-hitting* if and only if it is ω*-splitting*. The following was proved as theorem 3.3 in [11] for Borel ideals:

Corollary 2.3. *For any analytic ideal* I *the following conditions are equivalent*

(1) I *is a* $Q^+(\omega)$*-ideal,*

(2) $\mathcal{ED}_{fin} \not\leq_{KB} \mathsf{I}$*,*

(3) I *is not* ω*-hitting ideal,*

The extension to analytic ideals can be deduced from a theorem of Otmar Spinas: Theorem 1.2 in [26] claims that every analytic ω-splitting family contains a closed ω-splitting family. Let I be an analytic ω-splitting ideal, and let K be a closed ω-splitting family contained in the ideal I. The funcion φ given by $\varphi(X) = \min\{|B| : B \subseteq K \ \&\ X \subseteq \bigcup B\}$ is a lower semicontinuous submeasure (lscsm)[3] and then, $\mathsf{I}' = Fin(\varphi)$ is F_σ, ω-splitting and $\mathsf{I}' \leq_{KB} \mathsf{I}$. By the Borel case, $\mathcal{ED} \leq_{KB} \mathsf{I}$.

3. P$^+$-ideals

We begin this section by showing that $\mathsf{I}^+ \longrightarrow (\mathsf{I}^+)^2_2$ does not imply P$^+$.

Example 3.1. *A tall non* P$^+$*-ideal satisfying* $\mathsf{I}^+ \longrightarrow (\mathsf{I}^+)^2_2$.

Let $\{\mathcal{A}_n : n \in \omega\}$ be a sequence of MAD families, such that for each $A \in \mathcal{A}_n$, $\mathcal{A}_{n+1} \restriction A$ is an infinite MAD family in $\mathcal{P}(A)$. Let $\mathsf{I}_n = \{I \subseteq \omega : |\{A \in \mathcal{A}_n : |I \cap A| = \aleph_0\}| < \aleph_0\}$, and let $\mathsf{I} = \bigcap_{n\in\omega} \mathsf{I}_n$.

As countable intersection of tall ideals is tall, I is tall. Let us prove that I satisfies $\mathsf{I}^+ \longrightarrow (\mathsf{I}^+)^2_2$. First, note that $\mathsf{I}^+ = \bigcup_{n\in\omega} \mathsf{I}^+_n$. By a result of Mathias (MAD families are happy, see [22]), $\mathsf{I}^+_n \longrightarrow (\mathsf{I}^+_n)^2_2$ holds for each n. Let $\varphi : [X]^2 \to 2$ be a coloring. If $X \in \mathsf{I}^+$ then there is $n \in \omega$ such that $X \in \mathsf{I}^+_n$ and so, there is $Y \in \mathsf{I}^+_n \cap \mathcal{P}(X)$ (and so in I^+) which is φ-homogeneous.

To see that I is not a P$^+$-ideal take a sequence $\langle A_n : n \in \omega\rangle$ such that $A_n \in \mathcal{A}_n$, $A_n \supseteq A_{n+1}$, and $A_n \in \mathsf{I}_n \setminus \mathsf{I}_{n+1}$ for all $n \in \omega$. Hence, if $A \subseteq^* A_n$ for all $n \in \omega$ then $A \in \mathsf{I}_n$ for all n and so, $A \in \mathsf{I}$. $\square$

A somewhat similar example appears in Farah [7, Example 2.8].

Remark 3.2. Recall that a MAD family $\mathcal{A}$ is *completely separable* if for every $\mathsf{I}(\mathcal{A})$-positive set X there is an element A of $\mathcal{A}$ contained in X.[4] If the MAD families used in previous example are completely separable then $\mathcal{P}(\omega)/\mathsf{I} \cong Coll(\omega, 2^\omega)$.

Proof. By complete separability, $\bigcup_{n\in\omega} \mathcal{A}_n$ is dense in I^+. $\square$

In this section we will investigate the relation between P$^+$, F_σ and maximal P-ideals. The following lemma was probably first proved by Just and Krawczyk [14]. We include a proof for the sake of completness. We recall Mazur's characterization of F_σ-ideals [23, Lemma 1.2] which claims that for every F_σ-ideal I, there is a lscsm φ on ω such that $\mathsf{I} = Fin(\varphi)$.

Lemma 3.3 ([14]). *Every F_σ-ideal is a P^+-ideal.*

Proof. Let φ be a lscsm on ω such that $\mathsf{I} = Fin(\varphi)$, and let $\langle X_n : n \in \omega\rangle$ be a decreasing sequence of I-positive sets. For every $n \in \omega$, pick a finite set $F_n \subseteq X_n$ such that $\varphi(F_n) \geq n$. Then $X = \bigcup_{n\in\omega} F_n$ is a pseudointersection of $\langle X_n : n \in \omega\rangle$ and $\varphi(X) = \infty$. $\square$

Corollary 3.4. *If I is a tall F_σ-ideal then there is an I-positive set X such that $\mathsf{I} \restriction X$ is ω-hitting (on X).*

Proof. By 1.2, a tall F_σ-ideal I is not Q$^+$, so, by 2.2, there is an I-positive set X such that $\mathsf{I} \restriction X \geq_{KB} \mathcal{ED}_{fin}$, i.e, $\mathsf{I} \restriction X$ is ω-splitting on X. $\square$

[3] A lscsm φ is a function from $\mathcal{P}(\omega)$ to $[0, \infty]$ so that $\varphi(\emptyset) = 0$, $\varphi(A) \leq \varphi(A \cup B) \leq \varphi(A) + \varphi(B)$ and $\varphi(A) = \lim_{n\to\infty} \varphi(A \cap n)$. It is well known that $Fin(\varphi) = \{A \subseteq \omega : \varphi(A) < \infty\}$ is always an F_σ-ideal.

[4] The question whether completely separable MAD families exist in ZFC is an old open problem of Erdös and Shelah [6]. Shelah [24] recently proved that there is a completely separable MAD family assuming $2^{\aleph_0} < \aleph_\omega$.

It is perhaps worth noticing that a similar result is not true for ideals of higher complexity. In particular, the ideal nwd of nowhere dense subsets of the rationals is an $F_{\sigma\delta}$ ideal, no positive restriction of which is ω-hitting.

On the other hand, Farah [7] showed that every analytic ideal which is Q^+ and whose quotient is ω-distributive is not tall. Thus an analogous argument, as in the proof of Corollary 3.4, shows that there is an I-positive set X such that $\mathsf{I} \upharpoonright X$ is ω-hitting (on X) for every tall analytic ideal I such that the quotient $\mathcal{P}(\omega)/\mathsf{I}$ is ω-distribute (does not add reals).

3.1. **$\mathbf{P}^+$, extendability to F_σ, and P-points.** In this segment we investigate which Borel ideals can be extended to F_σ-ideals. Claude Laflamme [17] showed that this is sufficient for destructibility of the given ideal by an ω^ω-bounding forcing.[5] It turns out this problem gas a close connection to the P^+-property.

An easy result on extendability is the following lemma.

Lemma 3.5 (CH)**.** *If I is a P^+-ideal then there is a maximal P-ideal J containing I.*

Proof. We will find a P-point $\mathcal{U}$ such that $\mathsf{I}^* \subseteq \mathcal{U}$. Let $\langle X_\alpha : \alpha < \omega_1\rangle$ be an enumeration of $\mathcal{P}(\omega)$. For each $\alpha < \omega_1$, choose $U_\alpha \in \mathsf{I}^+$ such that $U_\alpha \subseteq^* U_\beta$ for all $\beta < \alpha$, and either $U_\alpha \subseteq X_\alpha$ or $U_\alpha \cap X_\alpha =^* \emptyset$. This is not difficult because if $\langle U_\beta : \beta < \alpha\rangle$ is a decreasing sequence in I^+ then there is $V \in \mathsf{I}^+$ such that $V \subseteq^* U_\beta$ for all $\beta < \alpha$ and then we can define $U_\alpha = V \cap X_\alpha$ if this set belongs to I^+, or $U_\alpha = V \setminus X_\alpha$ if not. Hence, $\langle U_\alpha : \alpha < \omega_1\rangle$ is a base for an ultrafilter contained in I^+, and consequently such ultrafilter contains I^*, and since this sequence is $\subseteq^*$-decreasing, the filter generated is a P-point. □

An important ideal for our considerations is the ideal $\mathbf{Fin} \times \mathbf{Fin}$:

$$\mathbf{Fin} \times \mathbf{Fin} = \{A \subseteq \omega \times \omega : \exists n \forall m \geq n |A \cap \{m\} \times \omega| < \aleph_0\}.$$

Laczkovich and Recław [19] proved that there are no Borel P^+-ideals (in particular, there are no F_σ-ideals) Katětov above $\mathbf{Fin} \times \mathbf{Fin}$. The following characterization of P-points is essentially due to Zapletal [28]:

Theorem 3.6 ([28])**.** *Let $\mathcal{U}$ be a free ultrafilter on ω. Then the following are equivalent:*

(1) $\mathcal{U}$ is a P-point.
(2) For every analytic tall ideal I disjoint from $\mathcal{U}$ there is an F_σ-ideal J disjoint from $\mathcal{U}$ containing I.

Proof. Suppose that $\mathcal{U}$ is not a P-point. Let $\langle U_n : n \in \omega\rangle$ be a strictly $\subseteq$-decreasing sequence of elements of $\mathcal{U}$ without a pseudointersection in $\mathcal{U}$. Letting $X_n = U_n \setminus U_{n+1}$ defines a partition $\{X_n : n \in \omega\}$ of ω into $\mathcal{U}$-small sets such that every set A satisfying $|A \cap X_n| < \aleph_0$ for all n is in $\mathcal{U}^*$. By tallness of I for any $n \in \omega$ there is an infinite set $I_n \in \mathsf{I}$ contained in X_n. Let J be the ideal generated by $\mathsf{I} \cup \{A \subseteq \omega : (\forall n \in \omega)(|A \cap I_n| < \aleph_0)\}$. Hence J contains a copy of $\mathbf{Fin} \times \mathbf{Fin}$ and is contained in $\mathcal{U}^*$, so J cannot be extended to an F_σ-ideal disjoint from $\mathcal{U}$. The other direction is Claim 2.4 in [28]. □

Notice the similarity of the result with Theorem 1.1. The theorem has the following immediate corollary:

Corollary 3.7. *For any analytic ideal I the following conditions are equivalent:*

[5] A forcing $\mathbb{P}$ *destroys* an ideal I on ω if $\mathbb{P}$ adds a new real number x such that $x \cap I$ is finite for all ground model element I of I.

(1) there is an F_σ-ideal J containing I,

(2) there is a P^+-ideal $\mathcal{K}$ containing I

and assuming there are enough P-points (e.g. assuming CH), (1) and (2) are equivalent to

(3) there is a maximal P-ideal $\mathcal{L}$ containing I.

Proof. (1) implies (2) follows from Lemma 3.3, (2) implies (3) from Lemma 3.5 and the rest follows from Theorem 3.6. □

The P^+ property as such does not have a critical ideal in the Katětov order since the ideal $\mathsf{I} = (\mathbf{Fin} \times \mathbf{Fin}) \oplus \mathcal{ED}$[6] is Katětov-equivalent to $\mathcal{ED}$ and the second one is a P^+-ideal while the first one is not.

We will say that an ideal I on ω is *decomposable* if there is an infinite partition $\{X_n : n \in \omega\}$ of ω into I-positive sets such that for every $A \subseteq \omega$,

$$A \in \mathsf{I} \text{ if and only if } (\forall n \in \omega)(A \cap X_n \in \mathsf{I}).$$

Such a partition will be called an I-*decomposition*. We will say that I is *hereditarily decomposable* if for every I-positive set X, $\mathsf{I} \restriction X$ is decomposable. We call an ideal I *indecomposable* if it is not decomposable. We will say that an ideal I on ω is σ-*closed*, if the quotient forcing $\mathcal{P}(\omega)/\mathsf{I}$ is σ-closed.

Theorem 3.8. *Let I be an ideal. Then*

(1) I is a P^+-ideal if and only if I is $\mathrm{P}^+_{\mathsf{tower}}$ and σ-closed.

(2) I is σ-closed if and only if I is indecomposable.

(3) I is not $\mathrm{P}^+_{\mathsf{tower}}(\omega)$ if and only if $\mathbf{Fin} \times \mathbf{Fin} \leq_K \mathsf{I}$.

(4) I is not $\mathrm{P}^+_{\mathsf{tower}}$ if and only if $\mathbf{Fin} \times \mathbf{Fin} \leq_K \mathsf{I} \restriction X$ for some $X \in \mathsf{I}^+$.

Proof. (1) and (2) are straightforward and left to the reader.

(3) Suppose I is not $\mathrm{P}^+_{\mathsf{tower}}(\omega)$ and let X_n be a decreasing sequence of I^+-positive sets such that $X_0 = \omega$ and $X_n \setminus X_{n+1} \in I$ for all n and without a I^+-positive diagonalization. Let (a_n) be an enumeration of X_0. Define $f : \omega \to \omega \times \omega$ by $f(a_n) = (m, n)$, if $a_n \in X_m \setminus X_{m+1}$. Then f shows that $\mathbf{Fin} \times \mathbf{Fin} \leq_K \mathsf{I}$. The other direction is easy as $\mathbf{Fin} \times \mathbf{Fin}$ is clearly not $\mathrm{P}^+_{\mathsf{tower}}(\omega)$.

Clause (4) follows directly from (3). □

Building on the work contained here, the first author and J. Verner in [13] showed that

Theorem 3.9 ([13, Theorem 2.5])**.** *Let I be an analytic ideal such that $\mathcal{P}(\omega)/\mathsf{I}$ does not add reals. Then the $\mathcal{P}(\omega)/\mathsf{I}$ generic filter is a P-point if and only if ideal I is* locally F_σ*, i.e. for every $X \in \mathsf{I}^+$ there is a $Y \subseteq X$, $Y \in \mathsf{I}^+$ such that $\mathsf{I} \restriction Y$ is F_σ.*

A similar argument can be used to characterize definable P^+ filters as follows:

Theorem 3.10. *An analytic ideal I is P^+ if and only if it is indecomposable and locally F_σ.*

Proof. First, assume I is analytic and P^+. Then it is indecomposable by theorem 3.8. We shall show that it is locally F_σ. Let X be an I-positive set, and let $\mathcal{U}$ be the I^+-generic ultrafilter on ω containing X. Then, in $V[\mathcal{U}]$, $\mathcal{U}$ is a P-point disjoint from I. This follows from the fact that I is P^+ and, in particular, $\mathcal{P}(\omega)/\mathsf{I} \simeq \mathsf{I}^+$ is σ-closed. By theorem 3.6 there

[6]For two ideals I and J, the sum is defined by $\mathsf{I} \oplus \mathsf{J} = \{A \subseteq (\omega \times \{0\}) \cup (\omega \times \{1\}) : \{n : (n, 0) \in A\} \in \mathsf{I}$ and $\{m : (m, 1) \in A\} \in \mathsf{J}\}$.

is, in $V[\mathcal{U}]$ an F_σ ideal J such that $\mathsf{I} \subseteq \mathsf{J}$ and $\mathsf{J} \cap \mathcal{U} = \emptyset$. Again as $\mathcal{P}(\omega)/\mathsf{I}$ is σ-closed, and F_σ ideals are coded by reals, J is in V, and there is a $Y \in \mathcal{U}$, without loss of generality $Y \subseteq X$, and $Y \in \mathsf{I}^+$, such that $Y \Vdash$ "$\mathsf{I} \subseteq \mathsf{J}$ and $\mathsf{J} \cap \mathcal{U} = \emptyset$". To finish the argument it suffices to see that $\mathsf{I} \restriction Y = \mathsf{J} \restriction Y$. If not there is a $Z \subseteq Y$, $Z \in \mathsf{J} \setminus \mathsf{I}$. Then, however, $Z \in \mathsf{I}^+$ and $Z \Vdash$ "$Z \in \mathcal{U} \cap \mathsf{J}$", which is a contradiction.

Now assume that I is indecomposable and locally F_σ. Let $\langle X_n : n \in \omega \rangle$ be a decreasing sequence of I-positive sets. As I is indecomposable, there is a $Y \in \mathsf{I}^+$ such that $Y \setminus X_n \in \mathsf{I}$ for all $n \in \omega$. Now, let $Z \subseteq Y$ be I-positive and such that $\mathsf{I} \restriction Z$ is F_σ. Then $\langle Z \cap X_n : n \in \omega \rangle$ is a decreasing sequence of $\mathsf{I} \restriction Z$-positive sets, which has a $\mathsf{I} \restriction Z$-positive pseudointersection $X \subseteq Z$ by lemma 3.3. Then X is an I-positive pseudointersection of $\langle X_n : n \in \omega \rangle$, hence I is P^+. $\square$

While there are locally F_σ ideals which are not P^+ (consider e.g. the ideal $\emptyset \times \mathsf{I}$, for any tall F_σ ideal I), in [13, Example 2.6] the authors construct tall Borel ideals of arbitrarily high Borel complexity which are locally F_σ. It can be easily checked that ideals presented there are, in fact, P^+ ideals.

The close relationship between F_σ ideals and the P^+ property will be further examined in the next section.

3.2. **More P^+-type properties.** Following [18], given a family $\mathcal{X}$ of infinite subsets of ω, we call a tree $T \subseteq ([\omega]^{<\omega})^{<\omega}$ a *$\mathcal{X}$-tree of finite sets* if for each $s \in T$ there is an $X_s \in \mathcal{X}$ such that $s^\frown a \in T$ for each $a \in [X_s]^{<\omega}$. An ideal I on ω is a *$\mathrm{P}^+_{\mathsf{tree}}$-ideal* if every I^+-tree of finite sets has a branch whose union is in I^+.

The next result ([12]) shows that for definable ideals this strengthening of the P^+-property coincides with being F_σ.

Theorem 3.11. *An analytic ideal I is $\mathrm{P}^+_{\mathsf{tree}}$ if and only if it is F_σ.*

It is easy to verify that for all ideals (Borel or not) $\mathrm{P}^+_{\mathsf{tree}}$ implies P^+ implies σ-closed. The reverse implications are not true. $\mathbf{Fin} \times \mathbf{Fin}$ is a σ-closed non-P^+-ideal and the ideal I_0 defined below is a P^+-ideal that is not a $\mathrm{P}^+_{\mathsf{tree}}$-ideal.

For each $f \in 2^\omega$, let us denote $A_f = \{f \restriction n : n \in \omega\}$. Then, I_0 is defined as the ideal on $2^{<\omega}$ generated by the family of sets A_f where f is not eventually zero. The ideal I_0 is $F_{\sigma\delta}$, non-F_σ and is not tall.

Going back to the problem of characterizing when an ideal I can be extended to an F_σ-ideal, it seems that the following ideal may play a critical role. Let conv be the ideal on $\mathbb{Q} \cap [0,1]$ generated by the convergent (in $[0,1]$) sequences of rational numbers[7]. It is easy to see that $\mathsf{conv} \leq_K \mathbf{Fin} \times \mathbf{Fin}$, in fact, every conv-positive set X contains a conv-positive set Y such that $\mathsf{conv} \restriction Y \simeq \mathbf{Fin} \times \mathbf{Fin}$.

The following theorem characterizes those ideals which are Katětov above the ideal conv.

Theorem 3.12. *For any ideal I on ω the following are equivalent:*

(1) $\mathsf{I} \geq_K \mathsf{conv}$,

(2) there is a linear order $\sqsubseteq$ for ω such that $(\omega, \sqsubseteq)$ is order-isomorphic to $\mathbb{Q} \cap [0,1]$ and every increasing and every decreasing sequence with respect to $\sqsubseteq$ is in I,

[7]In other words, we can define conv as the ideal of all subsets of $\mathbb{Q} \cap [0,1]$ such that the Cantor-Bendixson's derivative of its closure in $[0,1]$ is finite.

(3) *there is a topology τ on ω such that (ω, τ) is homeomorphic to $\mathbb{Q} \cap [0,1]$ and every τ-convergent sequence (in $[0,1]$) is in I, and*
(4) *there is a countable family $\mathcal{X} \subseteq [\omega]^\omega$ such that for every $Y \in \mathsf{I}^+$ there is $X \in \mathcal{X}$ such that $|X \cap Y| = |Y \setminus X| = \aleph_0$.*

Proof. $(2 \to 1)$ Such an isomorphism from ω into $\mathbb{Q} \cap [0,1]$ is a Katětov reduction between I and conv since every convergent sequence can be written as the union of an increasing and a decreasing sequence.

$(3 \to 2)$ Such homeomorphism between ω and $\mathbb{Q} \cap [0,1]$ induces an order $\sqsubseteq$ on ω isomorphic to the order of $\mathbb{Q} \cap [0,1]$; and all the $\sqsubseteq$-increasing and all the $\sqsubseteq$-decreasing sequences are in I as they are τ-convergent in $[0,1]$.

$(4 \to 3)$ Let $\mathcal{X}$ be such family. We can suppose that $\mathcal{X}$ separates points (i.e., for each pair $\{n, m\}$ there is $X \in \mathcal{X}$ such that $|X \cap \{m, n\}| = 1$), is closed under complements, and every non-empty Boolean combination of its elements is infinite; if not, we can recursively replace $\mathcal{X} = \{X_n : n \in \omega\}$ with $\{X'_n : n \in \omega\}$, where $X'_0 = X_0$ and X'_{n+1} is an infinite set D which is a Boolean combination of $\{X'_k : k \leq n\}$ such that $X_{n+1} =^* D$, if such D exists, otherwise, $X'_{n+1} = X_{n+1}$.

The topology generated by $\mathcal{X}$ is then homeomorphic to the subspace topology of $\mathbb{Q} \cap [0,1]$, and no $Y \in \mathsf{I}^+$ can be covered by finitely many τ-convergent sequences; given $Y \in \mathsf{I}^+$ one can recursively construct a family of τ-clopen sets $\{C_s : s \in 2^{<\omega}\}$ such that $\{C_s : s \in 2^n\}$ is pairwise disjoint for all n, $C_{s^\frown 0} \cup C_{s^\frown 1} = C_s$ and $Y \cap C_s$ is infinite for all $s \in 2^{<\omega}$.

$(1 \to 4)$ Let $\mathcal{C}$ be a countable base of the topology of $\mathbb{Q} \cap [0,1]$ consisting of clopen sets, and let f be a witness to $\mathsf{conv} \leq_K \mathsf{I}$. Let $\mathcal{X} = \{f^{-1}[C] : C \in \mathcal{C}\}$. Let $I \subseteq \omega$ be such that for every $X \in \mathcal{X}$, $I \cap X$ is finite or $I \setminus X$ is finite. Then, for every basic set $C \in \mathcal{C}$, $f''I$ is almost-contained in C, or it is almost-contained in $\mathbb{Q} \cap [0,1] \setminus C$. Then $f''I$ is a convergent sequence of $\mathbb{Q}$, hence $I \subseteq f^{-1}[f''I] \in \mathsf{I}$. ☐

Recall that a boolean algebra $\mathbb{B}$ is $(\omega, 2)$-*distributive*, if for every sequence $\{b_n : n \in \omega\} \subseteq \mathbb{B}$ and every $b \in \mathbb{B}^+$, there is $0 < a < b$ such that $a < b_n$ or $a < b_n^c$, for all n. Thus $\mathcal{P}(\omega)/\mathsf{I}$ is $(\omega, 2)$-distributive, if given I-positive sets $Y, \{X_n : n \in \omega\}$, there is an I-positive set $X \subseteq Y$ such that $Y \setminus X_n \in \mathsf{I}$ or $Y \cap X_n \in \mathsf{I}$, for all n. As in Theorem 3.12 (4), the following holds.

Theorem 3.13. $\mathsf{conv} \not\leq_K \mathsf{I} \restriction X$ *for all* I*-positive sets* X *if and only if* I *is* $\mathrm{P}^+_{\mathsf{tower}}$ *and* $\mathcal{P}(\omega)/\mathsf{I}$ *is* $(\omega, 2)$*-distributive.*

Proof. Suppose that $\mathsf{I} \restriction X \not\geq_K \mathsf{conv}$ for all $X \in \mathsf{I}^+$, and let $\{X_n : n \in \omega\}$ be a family of I-positive sets. By theorem 3.12(4), there is an I-positive set $Y \subseteq X$ such that for all n, $|Y \cap X_n| < \aleph_0$, or $|Y \setminus X_n| < \aleph_0$. In particular, $Y \cap X_n \in \mathsf{I}$, or $Y \setminus X_n \in \mathsf{I}$ for all n, proving that $\mathcal{P}(\omega)/\mathsf{I}$ is $(\omega, 2)$-distributive. By theorem 3.8 and the fact that $\mathsf{conv} \leq_K \mathbf{Fin} \times \mathbf{Fin}$, it follows that I is $\mathrm{P}^+_{\mathsf{tower}}$.

Now suppose that $\mathcal{P}(\omega)/\mathsf{I}$ is $(\omega, 2)$-distributive and I is $\mathrm{P}^+_{\mathsf{tower}}$. Let X be an I-positive set, and let $\{X_n : n \in \omega\}$ be a family of infinite subsets of X. Define Y_n, $n \in \omega$, by $Y_n = X_n$, if $X_n \in \mathsf{I}^+$, otherwise let $Y_n = \omega \setminus X_n$. By the $(\omega, 2)$-distributivity, there is an I-positive Z subset of X such that for all $n \in \omega$, $Z \setminus Y_n \in \mathsf{I}$ or $Z \cap Y_n \in \mathsf{I}$. For $n \in \omega$, let $Z_n = Z \cap Y_n$ if $Z \setminus Y_n \in \mathsf{I}$, and $Z_n = Z \setminus Y_n$ otherwise. Note that $Z_n \setminus Z_m \in \mathsf{I}$ for all $n < m$, hence we can assume that the sequence $\{Z_n : n \in \omega\}$ is decreasing. Since I is $\mathrm{P}^+_{\mathsf{tower}}$, there is an I-positive set W such that $W \subseteq^* Z_n$ for all n. Notice that $Z_n \subseteq X_n$ or $Z_n \subseteq \omega \setminus X_n$ for all n, then (4) in 3.12 fails for $\mathsf{I} \restriction X$ and we are done. ☐

Corollary 3.14. *If* I *is an* F_σ*-ideal on* ω *then* $\mathsf{I} \not\geq_K \mathsf{conv}$. $\square$

In the light of the previous results we conjecture the following

Conjecture 3.15. *If* I *is a Borel ideal then either there is an* I*-positive set* X *such that* $\mathsf{I} \restriction X \geq_K \mathsf{conv}$ *or there is an* F_σ*-ideal* J *containing* I.

Equivalently, Let I *be a Borel* $\mathrm{P}^+_{\mathsf{tower}}$*-ideal such that* $\mathcal{P}(\omega)/\mathsf{I}$ *is* $(\omega, 2)$*-distributive. Can* I *be extended to an* F_σ *ideal?*

Decomposability of ideals gives a criterion for ideals to be Katětov-above conv.

Lemma 3.16. *If there is a family* $\{\mathcal{X}_n : n \in \omega\}$ *of* I*-decompositions such that (1)* $\mathcal{X}_{n+1}$ *refines* $\mathcal{X}_n$ *and (2) all pseudointersections of decreasing chains* $\langle A_n : n \in \omega\rangle$ *such that* $A_n \in \mathcal{X}_n$ *are in* I*, then* $\mathsf{I} \geq_K \mathsf{conv}$.

Proof. We will prove that $\mathcal{X} = \bigcup_{n\in\omega} \mathcal{X}_n$ is a family as in 3.12(4). Let Y be an I-positive set. As each $\mathcal{X}_n$ is an I-decomposition, for each n, there is an $A \in \mathcal{X}_n$ such that either $Y \subseteq^* A$, or $|Y \cap A| = |Y \setminus A| = \aleph_0$. Note that for some $n \in \omega$ the second possibility holds, as otherwise Y would be a pseudointersection of the sequence $\langle A_n : n \in \omega\rangle$, where A_n is the (unique) element of $\mathcal{X}_n$ almost containing Y, which is a contradiction. $\square$

Lemma 3.17. *Let* I *be an ideal on* ω *such that* $\mathcal{P}(\omega)/\mathsf{I}$ *is proper and adds a new real. Then there is an* I*-positive set* X *such that* $\mathsf{I} \restriction X \geq_K \mathsf{conv}$.

Proof. We will work with the forcing I^+ instead of $\mathcal{P}(\omega)/\mathsf{I}$ again. Let $\dot{r}$ be an I^+ name for a new element of 2^ω, and pick a family $\{\mathcal{A}_n : n \in \omega\}$ of maximal antichains in I^+ so that $\mathcal{A}_{n+1}$ refines $\mathcal{A}_n$ and, for all $n \in \omega$, any condition in $\mathcal{A}_n$ decides $\dot{r} \restriction n$. Properness of I^+ implies that there is an $X \in \mathsf{I}^+$ such that $\mathcal{B}_n = \{A \in \mathcal{A}_n : A \cap X \in \mathsf{I}^+\}$ is countable for all $n \in \omega$.

Recursively, one can refine each $\mathcal{B}_n$ into a pairwise disjoint family of I-positive sets $\mathcal{X}_n$ which is a family of decompositions as in previous lemma, as follows: Enumerate $\mathcal{B}_0 = \{B_j : j < \omega\}$, and let $X_j = B_j \setminus \bigcup_{i<j} B_i$ for all $j \in \omega$. Then $\mathcal{X}_0 = \{X_j : j < \omega\}$ is a refinement of $\mathcal{B}_0$ which is an $\mathsf{I} \restriction X$-decomposition. Construct $\mathcal{X}_{n+1}$ by using the same argument for an enumeration of the family $\{A \cap X \in \mathsf{I}^+ : A \in \mathcal{B}_{n+1}, X \in \mathcal{X}_n\}$. Note that if $\mathcal{X}_n$ is an I^+-decomposition then $\mathcal{X}_{n+1}$ is an I^+-decomposition. Finally, let Y be an I-positive subset of X. We claim that there is $n \in \omega$ and there are $A_0 \neq A_1 \in \mathcal{X}_n$ such that $|Y \cap A_0| = |Y \cap A_1| = \aleph_0$. Since $\dot{r}$ is a name for a new real, there is an n such that Y does not decide $\dot{r} \restriction n$. That is Y is compatible with at least two distinct elements of $\mathcal{X}_n$ and the conclusion follows. $\square$

Finally, our best approximation to the Conjecture 3.15 is the following result.

Theorem 3.18. *Let* I *be a Borel ideal such that* $\mathcal{P}(\omega)/\mathsf{I}$ *is proper. Then, either there is an* I*-positive set* X *such that* $\mathsf{conv} \leq_K \mathsf{I} \restriction X$*, or there is a* F_σ*-ideal* J *containing* I.

Proof. By the previous lemma, if $\mathcal{P}(\omega)/\mathsf{I}$ adds a new real then we are done. Suppose that $\mathcal{P}(\omega)/\mathsf{I}$ does not add new reals. Let $\mathcal{U}$ be the I^+-generic ultrafilter. If $\mathcal{U}$ were a P-point (in $V[\mathcal{U}]$) then by Claim 2.4 in [28] (see Theorem 3.7) there is an F_σ-ideal $\mathsf{J} \supseteq \mathsf{I}$ disjoint from $\mathcal{U}$. As the forcing $\mathcal{P}(\omega)/\mathsf{I}$ does not add new reals, J is in V, and $\mathsf{I} \subseteq \mathsf{J}$.

If $\mathcal{U}$ is not a P-point then there is a strictly $\subseteq^*$-decreasing sequence $\mathcal{X} = \langle U_n : n \in \omega\rangle$ of elements of $\mathcal{U}$ without pseudointersections in $\mathcal{U}$. Define

$$D = \{Y \in \mathsf{I}^+ : (\forall n \in \omega)(Y \subseteq^* U_n) \vee (\exists n \in \omega)(Y \cap U_n = \emptyset)\}.$$

Since $\mathcal{X}$ has no pseudointersections in $\mathcal{U}$, D is not dense, and so, there is $Z \in \mathsf{I}^+$ such that every I-positive subset Y of Z is not in D. Let us see that $\mathsf{I} \restriction Z \geq_K \mathbf{Fin} \times \mathbf{Fin}$. Note that for any $n \in \omega$, $Z \setminus U_n \in \mathsf{I}$, but $Z \not\subseteq^* U_n$. For every $n \in \omega$ define a set $I_n \in \mathsf{I}$ as follows: $I_0 = Z \setminus U_0$, $I_{n+1} = Z \cap U_n \setminus U_{n+1}$. Each I_n is an infinite element of $\mathsf{I} \restriction Z$, $\bigcup_{n\in\omega} I_n = Z$ and if $A \subseteq Z$ is such that $A \cap I_n$ is finite for all $n \in \omega$ then $A \in \mathsf{I}$. Hence $\mathsf{I} \restriction Z \geq_K \mathbf{Fin} \times \mathbf{Fin}$. Since $\mathbf{Fin} \times \mathbf{Fin} \geq_K \mathsf{conv}$, we are done. $\square$

It should be remarked that in the proof of the theorem, we did not use properness in the case when $\mathcal{P}(\omega)/\mathsf{I}$ does not add new reals, so the question remains open for Borel I such that $\mathcal{P}(\omega)/\mathsf{I}$ is not proper and adds new reals, in other words, when the ideal I is $\mathrm{P}^+_{\mathsf{tower}}$, the algebra $\mathcal{P}(\omega)/\mathsf{I}$ is $(\omega, 2)$-distributive but not ω-distributive.

4. Idealized Ramsey Theorem

We now turn our attention to the study of the Ramsey(ω) and Ramsey properties.

4.1. **A critical ideal for Ramsey(ω).** The *random graph* on ω can be defined as follows: Let $\{X_n : n \in \omega\}$ be an independent family of subsets of ω such that $n \in X_m$ if and only if $m \in X_n$, for all $n, m \in \omega$ (getting such a family is easy by making finite changes to elements of any countable independent family). The set $E = \{\{n, m\} : m \in X_n\}$ is the set of vertices of the random graph. The random graph satisfies the following property: Given F and G disjoint finite subsets of ω there is $k < \omega$ such that $\{\{k, l\} : l \in F\} \subseteq E$ and $\{\{k, l\} : l \in G\} \cap E = \emptyset$. As an easy consequence, given a countable graph $\langle \omega, G\rangle$, there is a subset $X \subseteq \omega$ such that $\langle \omega, G\rangle \cong \langle X, E \restriction X\rangle$. This can be proved recursively using the abvoe property.

The *Random graph ideal* $\mathcal{R}$ is the ideal generated by the cliques and free sets in the random graph $\langle \omega, E\rangle$.

Clearly $\mathcal{R}$ is a tall F_σ-ideal. It is critical with respect to the $\omega \longrightarrow (\mathsf{I}^+)^2_2$ property in the Katětov order, as the following result shows.

Theorem 4.1. *Let I be an ideal on ω. Then, $\omega \longrightarrow (\mathsf{I}^+)^2_2$ if and only if $\mathsf{I} \not\geq_K \mathcal{R}$.*

Proof. Suppose $\mathcal{R} \leq_K \mathsf{I}$. Let $f \in \omega^\omega$ be such that for every $R \in \mathcal{R}$, $f^{-1}(R) \in \mathsf{I}$. Define a coloring φ by

$$\varphi(\{n, m\}) = \begin{cases} 0 & \text{if } \{f(n), f(m)\} \in E \\ 1 & \text{otherwise.} \end{cases}$$

If $A \subseteq \omega$ is φ-homogeneous in color 0 (color 1 is analogous) then $\{f(n), f(m)\} \in E$ for all $n \neq m \in A$. That is $f''A$ is a clique and consequently $f''A \in \mathcal{R}$, and so, $A \subseteq f^{-1}(f''A) \in \mathsf{I}$.

Now, suppose that $\omega \longrightarrow (\mathsf{I}^+)^2_2$ fails, and let $\varphi :\to 2$ be the witnessing coloring. Let $G = \varphi^{-1}(1)$, and consider the graph (ω, G). By the universality of the random graph mentioned above, there is a set $X \subseteq \omega$ and a function $f : \omega \to X$ such that $f : (\omega, G) \cong (X, E \cap [X]^2)$. It is easy to see that f is a Katětov reduction. $\square$

4.2. **Local minimality of $\mathcal{R}$.** The following is an immediate corollary of the last theorem:

Corollary 4.2. $\mathsf{I}^+ \longrightarrow (\mathsf{I}^+)^2_2$ *if and only if $\mathcal{R} \not\leq_K \mathsf{I} \restriction X$ for all $X \in \mathsf{I}^+$.*

This result has immediate connection with the question of existence of a locally K-minimal tall Borel ideal[8] (see [10]). In fact, it is not known whether $\mathcal{R}$ is locally K-minimal. This is, of course, equivalent asking whether no tall Borel ideal is Ramsey. We have been able to check this manually for all tall Borel ideals known to us but the general question remains open. We shall show that $\mathcal{R}$ is locally K-minimal among F_σ ideals, while there is a coanalytic Ramsey ideal.

We shall need the fact that conv is not Ramsey.

Lemma 4.3. $\omega \not\rightarrow (\mathsf{conv}^+)^2_2$.

Proof. We will use a trick due to W. Sierpiński. Let $\sqsubseteq$ be an order of $\mathbb{Q}$ of type ω, and define $\psi : [\mathbb{Q}]^2 \to 2$ by $\psi(\{q, r\}) = 0$ if and only if $q \sqsubseteq r \leftrightarrow q < r$ (here $<$ denotes the usual order of $\mathbb{Q}$). Every ψ-homogeneous set is a $<$-monotone sequence and so it is in conv. □

It is immediate from the previous lemma and Theorem 4.1 that $\mathsf{conv} \geq_K \mathcal{R}$.

Theorem 4.4. *Let I be a tall ideal on ω which is analytic, and $\mathcal{P}(\omega)/\mathsf{I}$ does not add reals, or $\mathcal{P}(\omega)/\mathsf{I}$ is proper and adds a real. Then there is an I-positive set X such that $\mathsf{I} \restriction X \geq_K \mathcal{R}$.*

Proof. Suppose that I is a tall analytic ideal such that $\mathcal{P}(\omega)/\mathsf{I}$ does not add new reals. Aiming toward a contradiction assume furthermore that for every I-positive set X and for every $f : [\omega]^2 \to 2$ there is a $Y \subseteq X$ which is I-positive and f-homogeneous.

Let $\mathcal{U}$ be an I^+-generic ultrafilter. Then $\mathcal{U}$ is a selective ultrafilter with respect to V (that is, for every coloring of $[\omega]^2$ in V there is $U \in \mathcal{U}$ which is homogeneous for such partition). Since I^+ does not add new colorings of $[\omega]^2$, $\mathcal{U}$ is also selective in $V[G]$. However, in $V[G]$, $\mathcal{U} \cap \mathsf{I} = \emptyset$, contradicting Mathias' theorem 1.1.

Now, if $\mathcal{P}(\omega)/\mathsf{I}$ is proper and adds a real, then by Lemma 3.17, there is $X \in \mathsf{I}^+$ such that $I \restriction X \geq_K \mathsf{conv}$, and then, by Lemma 4.3 we are done. □

A large class of ideals having $\mathcal{R}$ as a local minimum in the Katětov order is the class of F_σ ideals.

Theorem 4.5. *For every F_σ tall ideal I there is an I-positive X such that $\mathsf{I} \restriction X \geq_K \mathcal{R}$.*

Proof. By Proposition 1.2, I is not Q^+, then, by Theorem 2.2, there is an I-positive set X such that $\mathsf{I} \restriction X \geq_{KB} \mathcal{ED}_{\mathbf{Fin}} \geq_{KB} \mathcal{R}$. □

Next we provide an example of a co-analytic Ramsey tall ideal. The basic idea is similar to the one used in [7, Example 2.8]. The following lemma says that Ramsey's theorem holds uniformly, in the sense that there is a Borel function that selects an homogeneous set for a given coloring. This result may be known, but we did not find it in the literature.

Lemma 4.6. *There is a Borel function $F : [\omega]^\omega \times 2^{[\omega]^2} \to [\omega]^\omega$ such that $F(A, \varphi)$ is a φ-homogeneous infinite subset of A.*

Proof. We will verify that the standard proof of Ramsey's theorem produces the required function. Let A be an infinite subset of ω, and let φ be a 2-coloring of $[\omega]^2$.

(i) For each $i \in 2$ and $m \in A$ consider the following subset of A:

$$A^i_m = \{n \in A : \ \varphi\{m, n\} = i\}.$$

[8] A tall Borel ideal J is *locally K-minimal* if for any tall Borel ideal I there is an I-positive set X such that $\mathsf{J} \leq_K \mathsf{I} \restriction X$.

Define by recursion a sequence $\langle i(m) : m \in A\rangle$ and sets B_m, for $m \in A$, as follows: Let $\{m_k : k \in \omega\}$ be the increasing enumeration of A. Let $i(m_0) = 0$, if $A^0_{m_0}$ is infinite; otherwise, put $i(m_0) = 1$. Let $B_{m_0} = A^{i(m_0)}_{m_0}$. For the inductive step, let $i(m_{k+1}) = 0$ if $A^0_{m_{k+1}} \cap B_{m_k}$ is infinite; otherwise, let $i(m_{k+1}) = 1$. Let

$$B_{m_{k+1}} = A^{i(m_{k+1})}_{m_{k+1}} \cap B_{m_k}.$$

Notice that if $r \in B_{m_k}$, then $\varphi\{m_k, r\} = i(m_k)$. The function that maps (A, φ) to the sequence $\langle B_m : m \in A\rangle$ is Borel.

(ii) There is a Borel map that assigns a pseudointersection to each decreasing sequence $\langle B_m : m \in A\rangle$ of infinite subsets of A, i.e., it assigns an infinite subset B of A such that $B \subseteq^* B_m$ for all $m \in A$.

(iii) Given a decreasing sequence $\langle B_m : m \in A\rangle$ of infinite subsets of A and its pseudointersection B, define an increasing sequence $\langle a_k : k \in \omega\rangle \subseteq \omega$ as follows: Let $m_0 = \min A$, and $a_0 = \min\{p \in B : \ B \setminus [0, p) \subseteq B_{m_0}\}$. Then recursively on $k \in \omega$

$$a_{k+1} = \min\{p \in B : \ p > a_k \ \& \ B \setminus [0, p) \subseteq B_{a_k}\}.$$

Notice that $a_{k+2} \in B_{a_k}$ for all $k \in \omega$. The function that maps B and $\langle B_m : m \in A\rangle$ to the sequence $\langle a_k : k \in \omega\rangle$ is Borel.

Now we construct an homogeneous set for φ. Let $D = \{a_{2k} : \ k \in \omega\}$. Then

$$D \setminus m \subseteq B_m$$

for all $m \in D$. Thus for $p, q \in D$ with $p < q$, we have that $q \in B_p$ and therefore $\varphi\{p, q\} = i(p)$. Finally, let D_j be the set of all $p \in D$ such that $i(p) = j$. If D_0 is infinite, let $H = D_0$, otherwise put $H = D_1$. Then H is homogeneous for φ. Let $F(A, \varphi) = H$. Then F is the required function.

□

Lemma 4.7. *There is a continuous function $\psi : [\omega]^\omega \times 2^\omega \to [\omega]^\omega$ such that for every infinite $A \subseteq \omega$, the collection $\{\psi(A, x) : \ x \in 2^\omega\}$ is an almost disjoint family of infinite subsets of A. Moreover, for all infinite $A \subseteq \omega$ there is an infinite $B \subseteq A$ such that $B \cap \psi(A, x) = \emptyset$ for all $x \in 2^\omega$.*

Proof. Consider $D_x = \{x \restriction n : \ n \in \omega\}$ for each $x \in 2^\omega$. Then $\{D_x : \ x \in 2^\omega\}$ is an almost disjoint family of subsets of $2^{<\omega}$. Notice that the map that sends x to D_x is continuous. Any bijection between ω and $2^{<\omega}$ allows one to get the almost disjoint family over ω, which will be also denoted $\{D_x : \ x \in 2^\omega\}$. For each infinite $A \subseteq \omega$ let $\{e_A(m) : m \in \omega\}$ be the increasing enumeration of A. Let $\psi(A, x) = \{e_A(2m) : \ m \in D_x\}$. Let $B = \{e_A(2m+1) : \ m \in D_x\}$. □

Theorem 4.8. *There is a co-analytic Ramsey tall ideal.*

Proof. Let $\varphi : 2^\omega \to 2^{[\omega]^2}$ be a continuous surjection. Note that the range of φ is the collection of all 2-colorings of $[\omega]^2$. Let ψ and F be the functions given by lemma 4.7 and lemma 4.6, respectively. Define A_s for each $s \in (2^\omega)^{<\omega}$ as follows. Let $x \in 2^\omega$,

$$\begin{aligned} A_{\langle x\rangle} &= F(\psi(\omega, x), \varphi(x)), \\ A_{s^\frown x} &= F(\psi(A_s, x), \ \varphi(x)). \end{aligned}$$

Then for each s, the collection $\{A_{s^\frown x} : x \in 2^\omega\}$ is an almost disjoint family of subsets of A_s such that for each coloring $\varphi(x)$ of A_s, $A_{s^\frown x}$ is $\varphi(x)$-homogeneous. Let

$$\mathcal{C}_1 = \{F(\psi(\omega, x), \varphi(x)) : \ x \in 2^\omega\} = \{A_{\langle x\rangle} : \ x \in 2^\omega\}$$

$$\mathcal{C}_{n+1} = \{F(\psi(A, x), \varphi(x)) : (A, x) \in \mathcal{C}_n \times 2^\omega\} = \{A_{s^\frown x} : |s| = n, x \in 2^\omega\}.$$

Since F is Borel, then each $\mathcal{C}_n$ is analytic. Finally, let

$$G \in \mathcal{H} \Leftrightarrow \ (\exists n \in \omega)\ (\exists D \in \mathcal{C}_n)\ D \subseteq^* G.$$

We will show that $\mathsf{I} = \mathcal{P}(\omega) \setminus \mathcal{H}$ is a co-analytic Ramsey tall ideal. It is clear that $\mathcal{H}$ is analytic, hence I is co-analytic.

Now we show that I is an ideal. Suppose $G \cup K \in \mathcal{H} = \mathsf{I}^+$. Let $n \in \omega$ and $D \in \mathcal{C}_n$ be such that $D \subseteq^* G \cup K$. Consider the following coloring: $c\{n, m\} = 1$ if and only if $\{n, m\} \subseteq G$. Let $x \in 2^\omega$ be such that $\varphi(x) = c$. Let $E = F(\psi(D, x), \varphi(x))$. Then $E \in \mathcal{C}_{n+1}$, and it is c-homogeneous. If E is 1-homogeneous, then $E \subseteq G$ and if E is 0-homogeneous, then $E \cap G$ has at most one point. Since $E \subseteq D \subseteq^* G \cup K$, then $E \subseteq^* G$ or $E \subseteq^* K$. That is either G or K is I-positive.

To see that I is Ramsey, let $G \in \mathcal{H}$ and $D \in \mathcal{C}_n$ be such that $D \subseteq G$. Let c be a coloring of $[\omega]^2$. Then $c = \varphi(x)$, for some x, and $F(\psi(D, x), \varphi(x))$ is a c-homogeneous infinite subset of G in $\mathcal{H}$.

Finally, let us see that I is tall. Fix $A \in \mathcal{H}$. Then there is $n \in \omega$ and $D \in \mathcal{C}_n$ such that $D \subseteq A$. By lemma 4.7, there is $B \subseteq D$ infinite such that $\psi(D, x) \cap B = \emptyset$ for all $x \in 2^\omega$. We claim that $B \notin \mathcal{H}$. In fact, let $n < m$ and $E \in \mathcal{C}_m$. Towards a contradiction, suppose $E \subseteq^* B$. As $\mathcal{C}_m$ is a.d., then there is x such that $E \subseteq^* \psi(D, x)$, which is impossible. $\square$

Remark 4.9. We do not know whether an ideal constructed as in the proof of theorem 4.8 can be Borel.

4.3. **Weaker partition properties.** The partition properties defined below are weak versions of the Ramsey properties defined previously. We study them for their own sake, but also as tools to describe Borel ideals satisfying $\omega \longrightarrow (\mathsf{I}^+)^2_2$.

Definition 4.10. Let I be an ideal on ω. We will say that I satisfies:

(a) $\omega \longrightarrow (\omega, \mathsf{I}^+)^2_2$ if for every coloring $\varphi : [\omega]^2 \to 2$ there is an infinite 0-homogeneous set, or there is an I-positive 1-homogeneous set.
(b) $\omega \longrightarrow (< \omega, \mathsf{I}^+)^2_2$ if for every coloring $\varphi : [\omega]^2 \to 2$, either for every $m < \omega$ there is a 0-homogeneous set X of size m, or there is an I-positive 1-homogeneous set.
(c) $\mathsf{I}^+ \longrightarrow (\omega, \mathsf{I}^+)^2_2$ if for any I-positive set Y and any coloring $\varphi : [Y]^2 \to 2$ there is an infinite 0-homogeneous set, or there is an I-positive 1-homogeneous set.
(d) $\mathsf{I}^+ \longrightarrow (< \omega, \mathsf{I}^+)^2_2$ if for any I positive set Y and any coloring $\varphi : [Y]^2 \to 2$, either for every $m < \omega$ there is a 0-homogeneous set X of size m, or there is an I-positive 1-homogeneous set.

The following diagram shows the trivial implications between these properties:

$$\begin{array}{ccccc}
\omega \to (\mathsf{I}^+)^2_2 & \Longrightarrow & \omega \to (\omega, \mathsf{I}^+)^2_2 & \Longrightarrow & \omega \to (<\omega, \mathsf{I}^+)^2_2 \\
\Uparrow & & \Uparrow & & \Uparrow \\
\mathsf{I}^+ \to (\mathsf{I}^+)^2_2 & \Longrightarrow & \mathsf{I}^+ \to (\omega, \mathsf{I}^+)^2_2 & \Longrightarrow & \mathsf{I}^+ \to (<\omega, \mathsf{I}^+)^2_2
\end{array}$$

In order to study these properties, we introduce the following definition:

Definition 4.11. Given an I-positive set X and a sequence $\langle X_i : i \in X\rangle$ of subsets of X we say that $\langle X_i : i \in X\rangle$ is an I-*tower* if $X \setminus X_i \in \mathsf{I}$ for all $i \in X$.

A set $D \subseteq X$ is a *diagonal* of an I-tower $\langle X_i : i \in X\rangle$ if $D \setminus (i+1) \subseteq X_i$ for all $i \in D$.

We say that an ideal I *contains an* I-*tower* if there is an I-tower such that all of its diagonals are in the ideal.

Theorem 4.12. *If an ideal* I *on* ω *does not contain an* I-*tower then* $\mathsf{I}^+ \longrightarrow (<\omega, \mathsf{I}^+)^2_2$.

Proof. Suppose the conclusion fails, and let $X \in \mathsf{I}^+$ and let $\psi : [X]^2 \to 2$ be a coloring such that for some $m < \omega$ there is no $Y \in [X]^m$ homogeneous in color 0, and there is no $Y \in \mathsf{I}^+$ homogeneous in color 1. Take an I-positive $\overline{X} \subseteq X$ such that this m becomes minimal. Define $X_a = \{y \in \overline{X} : \psi(\{a, y\}) = 1\}$ for $a \in \overline{X}$. We claim that $\overline{X} \setminus X_a \in \mathsf{I}$ for all $a \in \overline{X}$. Suppose $Z = \overline{X} \setminus X_a \in \mathsf{I}^+$. By minimality of m, there is a $Z' \in [Z]^{m-1}$ homogeneous in color 0, then $Z' \cup \{a\}$ is also homogeneous of color 0 and has m elements, a contradiction. Therefore, $\overline{X} \setminus X_a \in \mathsf{I}$ for all $a \in \overline{X}$. Let $\{a_i\}$ be an enumeration of $\overline{X}$ and $\overline{X}_{a_i} = \bigcap_{j \leq i} X_{a_j}$. Then $\langle \overline{X}_a : a \in \overline{X}\rangle$ is an I-tower, and any diagonal of it is homogeneous in color 1, so by our assumption is in I. Thus the ideal contains an I-tower. $\square$

An example of an ideal I that does not contain a I-tower is nwd, the ideal of nowhere dense subsets of $\mathbb{Q}$. Hence, by 4.12, $\mathsf{nwd}^+ \longrightarrow (<\omega, \mathsf{nwd}^+)^2_2$.

Lemma 4.13. nwd *does not contain a* nwd-*tower.*

Proof. Suppose that $\langle X_i : i \in X\rangle$ is a nwd-tower. Without loss of generality $X_{i+1} \subseteq X_i$. Let $\{U_i : i < \omega\}$ be a base for the topology of $\mathbb{Q}$.

We will construct a positive diagonal $D = \{d_i : i < \omega\}$ recursively. Let $d_0 \in X$ be arbitrary, and suppose d_i was found. Let $\sqsubset$ be an order for $\mathbb{Q}$ of type ω. If there is $d \sqsupset d_i$ with $d \in X_{d_i} \cap U_i$ then take $d_{i+1} = d$, otherwise take arbitrary $d_{i+1} \sqsupset d_i$ with $d_{i+1} \in X_{d_i}$. By the construction D is a diagonal of the tower. As X is somewhere dense, say dense in an open set U, then D is also dense in U, hence $D \in \mathsf{nwd}^+$. $\square$

However, there are also ideals I satisfying $\mathsf{I}^+ \to (<\omega, \mathsf{I}^+)^2_2$ which do contain I-towers, as we shall see next. For the rest of the paper we will addopt the following notation: given a set $A \subseteq \omega \times \omega$ and $n \in \omega$ we denote $(A)_n = \{m \in \omega : (n, m) \in A\}$.

Theorem 4.14. *The ideal* $\mathcal{ED}_{fin}$ *has the following properties.*

(1) It contains an $\mathcal{ED}_{fin}$-*tower.*
(2) $\mathcal{ED}^+_{fin} \to (<\omega, \mathcal{ED}^+_{fin})^2_2$.
(3) $\omega \not\to (\omega, \mathcal{ED}^+_{fin})^2_2$.

Proof. (1) Let $X_{(n,m)} = \{(k,l) \in \Delta : n < k\}$ for $(n,m) \in \Delta$. Then $\{X_{(n,m)} : (n,m) \in \Delta\}$ is a $\mathcal{ED}_{fin}$-tower. Let $f : \omega \to \Delta$ be a bijection such that if $f(n) = (k,l)$ and $f(n') = (k',l')$ and $n < n'$, then $k \leq k'$. If $D \setminus \{f(0), \cdots, f(n)\} \subseteq X_{f(n+1)}$, then D is a selector and thus belongs to $\mathcal{ED}_{fin}$.

(2) Let a coloring $\psi : [\Delta]^2 \to 2$ be given and suppose that there is a bound for the size of the 0-homogeneous sets. Let $\Delta_n = \Delta \cap (\{n\} \times \omega)$.

Claim. *Let B be an infinite subset of ω, $m_0 \notin B$ and $A_m \subset \Delta_m$, for $m \in B$, with $|A_m|$ unbounded. Let $C \subset \Delta_{m_0}$ non empty. For all $n \in \omega$ there are a set $a \subset C$ of size at least $|C|/2$, an infinite set $B' \subseteq B$, an $l < 2$ and $A'_m \subseteq A_m$, for $m \in B'$, with $|A'_m|$ unbounded such that $\psi(\{x,y\}) = l$ for all $x \in a$ and $y \in \bigcup_{m \in B'} A'_m$.*

Proof of the Claim: Let $\{x_i : i < p\}$ be an enumeration of C. For each $m \in B$, choose $\epsilon_m < 2$, $A^0_m \subseteq A_m$ finite of size at least $|A_m|/2$ such that $\psi(\{x_0, y\}) = \epsilon_m$ for all $y \in A^0_m$. Now choose $B_0 \subseteq B$ infinite and l_0 such that $\epsilon_m = l_0$ for all $m \in B_0$. We can repeat this construction and recursively find sets B_i, A^i_m for $m \in B_i$ and $l_i < 2$ such that for all $i \leq p$

(i) $B_{i+1} \subseteq B_i$ are infinite sets,
(ii) $A^{i+1}_m \subseteq A^i_m$ and A^{i+1}_m has size at least $|A^i_m|/2$ for all $m \in B_{i+1}$,
(iii) $\psi(\{x_i, y\}) = l_i$ for all $y \in \bigcup_{m \in B_i} A^i_m$.

Let $l < 2$ and $D \subseteq \{0, 1, \cdots, p-1\}$ be a set of size at least $p/2$ such that $l_i = l$ for all $i \in D$. Then take $a = \{x_i : i \in D\}$, $B' = B_{p-1}$ and $A'_m = A^{p-1}_m$ for $m \in B'$. By the construction $|A'_m| \geq |A_m|/2^p$ for $m \in B'$, and the proof of the claim is complete.

Now, using the claim recursively we construct:

(i) An increasing sequence $\{m_i : i \in \omega\} \subseteq \omega$,
(ii) for each $i \in \omega$ a finite set $a_i \subset \Delta_{m_i}$ of size $i+1$,
(iii) a color $l_i < 2$ and
(iv) a decreasing sequence $\{D_i : i \in \omega\}$ of subsets of Δ each satisfying that the sequence $\{|(D_i)_m| : m \in \omega\}$ is unbounded.

such that $\psi(\{x,y\}) = l_i$ for all $x \in a_i$ and $y \in D_i$.

For each $m \in \omega$, pick a subset $A_m \subseteq \Delta_m$ which is ψ-homogeneous and of the largest possible size. By Ramsey's theorem, $|A_m|$ is unbounded, and by our assumption, we can find an infinite subset $B \subset \omega$ such that A_m is homogeneous of color 1 for all $m \in B$. Now we will use the claim. Let m_0 be the least element of B, $C = A_{m_0}$ and $n = 1$, by the claim there is $a_0 \subset A_{m_0}$ of size 1, $l_0 < 2$, $B' \subseteq B \setminus \{m_0\}$ infinite and $A'_m \subseteq A_m$ for each $m \in B'$ with $|A'_m|$ unbounded such that $\psi(\{x,y\}) = l_0$ for all $x \in a_0$ and $y \in \bigcup_{m \in B'} A'_m$. Let $D_0 = \bigcup_{m \in B'} A'_m$

In step $i+1$, let $m_{i+1} = \min\{m : (D_i)_m \neq \emptyset\}$, and apply the claim to $C = (D_i)_{m_{i+1}}$, $B = \{m > m_{i+1} : (D_i)_m \neq \emptyset\}$, $A_m = (D_i)_m$ for $m \in B$ and $n = i+1$, to obtain $B' \subseteq B$, $a_{i+1} \subseteq C$, $A'_m \subseteq A_m$ for $m \in B'$ and $l_i < 2$ as in the conclusion of the claim. Put $D_{i+1} = \bigcup_{m \in B'} A'_m$.

Finally, let $E \in [\omega]^\omega$ and $l < 2$ be such that $l_i = l$ for all $i \in E$. If $l = 1$, then $\bigcup_{i \in E} a_i$ is an $\mathcal{ED}_{fin}$-positive ψ-homogeneous set of color 1. Now we observe that l cannot be 0, in fact, suppose not and let $z_i \in a_i$ for $i \in E$. Then $\{z_i : i \in E\}$ is a ψ-homogeneous infinite set of color 0 which contradicts our assumption.

(3) Let $\varphi : \Delta \to 2$ be given by $\varphi(\{(n_1, m_1), (n_2, m_2)\}) = 0$ if and only if $n_1 = n_2$. It is clear that every 0-homogeneous set is contained in a column, and every 1-homogeneous is a partial selector of columns and so 1-homogeneous are in $\mathcal{ED}_{fin}$. $\square$

In fact, every ideal $\mathsf{I} \geq_{KB} \mathcal{ED}_{fin}$ fails to satisfy the property $\omega \longrightarrow (\omega, \mathsf{I}^+)^2_2$ since every such ideal admits a partition of ω into finite sets such that every selector of the partition is in I, and so, a coloring analogous to the one for $\mathcal{ED}_{fin}$ witnesses that I does not satisfy $\omega \longrightarrow (\omega, \mathsf{I}^+)^2_2$. In other words, $\omega \longrightarrow (\omega, \mathsf{I}^+)^2_2$ implies $\mathrm{Q}^+(\omega)$. However, the reverse implication does not hold, since $\mathbf{Fin} \times \mathbf{Fin}$ is a $\mathrm{Q}^+(\omega)$-ideal but $\omega \not\rightarrow (\omega, \mathbf{Fin} \times \mathbf{Fin}^+)^2_2$.

Now we will present a method for constructing Ramsey(ω) ideals. Given an ideal I on ω, the ideal $\widetilde{\mathsf{I}}$ is defined as follows.

$$\widetilde{\mathsf{I}} = \{A \subseteq \omega \times \omega : \exists k \in \omega \; (\forall i < k \; (A)_i \in \mathsf{I}) \;\&\; (\forall i > k \; |(A)_i| < k))\}.$$

It is clear that if I is a Borel ideal then $\widetilde{\mathsf{I}}$ is a Borel ideal too. In fact, if I is F_σ then so is $\widetilde{\mathsf{I}}$.

Lemma 4.15. *Suppose $\mathsf{I}^+ \longrightarrow (<\omega, \mathsf{I}^+)^2_2$. Then for every $X \in \mathsf{I}^+$ and every coloring $\varphi : [X]^2 \to 2$, there is an I-positive φ-homogeneous $Y \subseteq X$, or for every $n \in \omega$ and each $i < 2$ there is a φ-homogeneous set of color i of size n.*

Proof. Apply the hypothesis to both φ and $1 - \varphi$. $\square$

Theorem 4.16. *If $\mathsf{I}^+ \longrightarrow (<\omega, \mathsf{I}^+)^2_2$ then $\omega \longrightarrow (\widetilde{\mathsf{I}}^+)^2_2$.*

Proof. Let a coloring $\psi : [\omega \times \omega]^2 \to 2$ be given. Aiming toward a contradiction assume that there is no homogeneous $X \in \widetilde{\mathsf{I}}^+$.

Claim. *Given $C \in \mathsf{I}^+$, $B \subseteq \omega$ infinite, $A_m \in \mathsf{I}^+$ for $m \in B$, $m_0 \notin B$ and $n \in \omega$, there are a finite set $a \subset \{m_0\} \times C$ of size n, $B' \subseteq B$ infinite, $l < 2$ and $A'_m \subseteq A_m$ in I^+ for $m \in B'$ such that $\psi(\{x, y\}) = l$ for all $x \in a$ and $y \in a \cup \bigcup_{m \in B'} \{m\} \times A'_m$ with $y \neq x$.*

Proof of the claim: Let $\{c_i : i \in \omega\}$ be an enumeration of C. Put $x_0 = (m_0, c_0)$. For each $m \in B$, choose $\epsilon_m < 2$, and $A^0_m \subseteq A_m$ in I^+ such that $\psi(\{x_0, y\}) = \epsilon_m$ for all $y \in \{m\} \times A^0_m$. Now choose $B_0 \subseteq B$ infinite and $l_0 \in 2$ such that $\epsilon_m = l_0$ for all $m \in B_0$. We can repeat this construction and recursively find sets B_i, A^i_m for $m \in B_i$ and $l_i < 2$ such that for all $i \in \omega$

(i) $B_{i+1} \subseteq B_i$ are infinite sets,
(ii) $A^i_m \in \mathsf{I}^+$ for $m \in B_i$ and $A^{i+1}_m \subseteq A^i_m$ for all $m \in B_{i+1}$,
(iii) $\psi(\{x_i, y\}) = l_i$ for all $y \in \bigcup_{m \in B_i} \{m\} \times A^i_m$.

Let $l < 2$ and $D \subseteq \omega$ be an infinite set such that $l_i = l$ for all $i \in D$ and let $C' = \{c_i : i \in D\} \in \mathsf{I}^+$. Since $\{m_0\} \times C'$ is in $\widetilde{\mathsf{I}}^+$ and there are no such ψ-homogeneous sets, then by Lemma 4.15 (applied to the coloring of C' given by the restriction of ψ to $\{m_0\} \times C'$), there is $a \subseteq \{m_0\} \times C'$ of size n which is ψ-homogeneous of color l. Finally, let k be the maximum of the j's such that $x_j \in a$, then put $B' = B_k$ and $A'_m = A^k_m$ for $m \in B'$. We have completed the proof of the claim.

Now, using the claim recursively we construct:

(i) An increasing sequence $\{m_i : i \in \omega\} \subseteq \omega$,

(ii) for each $i \in \omega$ a finite set $a_i \subset \{m_i\} \times \omega$ of size $i+1$,

(iii) a color $l_i < 2$, and

(iv) a decreasing sequence $\{D_i : i \in \omega\}$ of subsets of $\omega \times \omega$ each satisfying that $|\{m : (D_i)_m \in \mathsf{I}^+\}| = \aleph_0$.

such that $\psi(\{x, y\}) = l_i$ for all $x \in a_i$ and $y \in D_i$.

Let $m_0 = 0$. By the claim applied to $B = \omega \setminus \{0\}$, $A_m = \omega$ for $m \in B$, $C = \omega$, and $n = 1$, there is $a_0 \subset \{m_0\} \times \omega$ of size 1, $l_0 < 2$, $B' \subseteq B$ infinite and an I positive set $A'_m \subseteq A_m$ for each $m \in B'$ such that $\psi(\{x, y\}) = l_0$ for all $x \in a_0$ and $y \in \bigcup_{m\in B'}\{m\} \times A'_m$. Let $D_0 = \bigcup_{m\in B'}\{m\} \times A'_m$.

In step $i+1$, let $m_{i+1} = \min\{m : (D_i)_m \in \mathsf{I}^+\}$ and apply the claim to $C = (D_i)_{m_{i+1}}$, $B = \{m > m_{i+1} : (D_i)_m \in \mathsf{I}^+\}$, $A_m = (D_i)_m$ for $m \in B$ and $n = i + 1$, to obtain $B' \subseteq B$, a_{i+1}, $A'_m \subseteq A_m$ for $m \in B'$ and $l_i < 2$ as in the conclusion of the claim. Put $D_{i+1} = \bigcup_{m\in B'}\{m\} \times A'_m$.

Finally, let $E \in [\omega]^\omega$ be such that $l_i = l_j$ for all $i, j \in E$. Then $\bigcup_{i\in E} a_i$ is an $\widetilde{\mathsf{I}}$-positive homogeneous set for ψ, which contradicts our assumption. $\square$

An immediate consequence of the 4.12, 4.13, 4.14 and 4.16 is the following.

Corollary 4.17. $\omega \to (\widetilde{\mathsf{nwd}}^+)^2_2$ *and* $\omega \to (\widetilde{\mathcal{ED}_{fin}}^+)^2_2$.

Next we list the partition properties of the ideal $\mathcal{ED}$.

Theorem 4.18. *The ideal $\mathcal{ED}$ has the following properties.*

(1) $\omega \not\to (\mathcal{ED}^+)^2_2$.

(2) $\mathcal{ED}^+ \not\to (\omega, \mathcal{ED}^+)^2_2$.

(3) $\omega \to (\omega, \mathcal{ED}^+)^2_2$.

(4) $\mathcal{ED}^+ \to (< \omega, \mathcal{ED}^+)^2_2$.

Proof. For (1) and (2), use a coloring analogous to the one appearing in the proof of part (3) of Theorem 4.14, moreover, for (2) use the fact that $\Delta \in \mathcal{ED}^+$.

(3) The argument is similar to the one used in the proof of Theorem 4.16. Let a coloring $\psi : [\omega \times \omega]^2 \to 2$ be given, and suppose that there are no infinite 0-homogeneous sets.

Claim. *Let C and B be infinite subsets of ω, $m_0 \notin B$ and A_m, with $m \in B$, a collection of infinite subsets of ω. For all $n \in \omega$ there are a finite set $a \subset \{m_0\} \times C$ of size n, $B' \subseteq B$ infinite, $l < 2$ and $A'_m \subseteq A_m$ infinite for $m \in B'$ such that $\psi(\{x, y\}) = l$ for all $x \in a$ and $y \in \bigcup_{m\in B'}\{m\} \times A'_m$.*

Proof of the claim: Let $\{c_i : i \in \omega\}$ be an enumeration of C. Put $x_0 = (m_0, c_0)$. For each $m \in B$, choose $\epsilon_m < 2$, and $A^0_m \subseteq A_m$ infinite such that $\psi(\{x_0, y\}) = \epsilon_m$ for all $y \in \{m\} \times A^0_m$. Now choose $B_0 \subseteq B$ infinite and $l_0 < 2$ such that $\epsilon_m = l_0$ for all $m \in B_0$. Repeat this construction, and recursively find sets B_i, A^i_m for $m \in B_i$ and $l_i < 2$ such that for all $i \in \omega$

(i) $B_{i+1} \subseteq B_i$ are infinite sets,

(ii) A^i_m infinite for all $m \in B_i$ and $A^{i+1}_m \subseteq A^i_m$ for all $m \in B_{i+1}$, and

(iii) $\psi(\{x_i, y\}) = l_i$ for all $y \in \bigcup_{m\in B_i}\{m\} \times A^i_m$.

Let $l < 2$ and $D \subseteq \omega$ be an infinite set such that $l_i = l$ for all $i \in D$. The take $a \subset \{m_0\} \times \{c_i : i \in D\}$ of size n. Finally, let k be the maximum of the j's such that $x_j \in a$, then put $B' = B_k$ and $A'_m = A^k_m$ for $m \in B'$. This completes the proof of the claim.

Again, using the claim recursively we construct:

(i) An increasing sequence $\{m_i : i \in \omega\} \subseteq \omega$,
(ii) for each $i \in \omega$ a finite set $a_i \subset \{m_i\} \times \omega$ of size $i + 1$,
(iii) a color $l_i < 2$, and
(iv) a decreasing sequence $\{D_i : i \in \omega\}$ of subsets of $\omega \times \omega$ each satisfying that $|\{m : |(D_i)_m| = \aleph_0\}| = \aleph_0$.

such that $\psi(\{x, y\}) = l_i$ for all $x \in a_i$ and $y \in D_i$.

Let $m_0 = 0$ and $B = \omega \setminus \{0\}$. By Ramsey's theorem and our assumption that there are no infinite ψ-homogeneous sets of color 0, we can pick an infinite set $A_m \subseteq \omega$ for $m \in B$ such that $\{m\} \times A_m$ is ψ-homogeneous of color 1. Let $C = A_0$ and $n = 1$, by the claim there is $a_0 \subset \{m_0\} \times A_0$ of size 1, $l_0 < 2$, $B' \subseteq B$ infinite and $A'_m \subseteq A_m$ infinite for each $m \in B'$ such that $\psi(\{x, y\}) = l_0$ for all $x \in a_0$ and $y \in \bigcup_{m \in B'} \{m\} \times A'_m$. Let $D_0 = \bigcup_{m \in B'} \{m\} \times A'_m$

In step $i+1$, let $m_{i+1} = \min\{m : (D_i)_m \text{ is infinite}\}$, and apply the claim to $C = (D_i)_{m_{i+1}}$, $B = \{m > m_{i+1} : (D_i)_m \text{ is infinite}\}$, $A_m = (D_i)_m$ for $m \in B$ and $n = i + 1$, to obtain $B' \subseteq B$, a_{i+1}, $A'_m \subseteq A_m$ for $m \in B'$ and $l_i < 2$ as in the conclusion of the Claim. Put $D_{i+1} = \bigcup_{m \in B'} \{m\} \times A'_m$.

Finally, let $E \in [\omega]^\omega$ and $l < 2$ be such that $l_i = l$ for all $i \in E$. If $l = 1$, then $\bigcup_{i \in E} a_i$ is an $\mathcal{ED}$-positive ψ-homogeneous set of color 1. Now we observe that l cannot be 0, in fact, suppose not and let $z_i \in a_i$ for $i \in E$. Then $\{z_i : i \in E\}$ is a ψ-homogeneous infinite set of color 0 which contradicts our assumption.

(4) If A is in $\mathcal{ED}^+$, then there is $B \subseteq A$ such that $\mathcal{ED} \restriction B$ is isomorphic to $\mathcal{ED}_{fin}$. Now the result follows from Theorem 4.14. $\square$

For every $n \in \omega$ we will say that an ideal I is *n-Ramsey* if for any coloring $\varphi : [\omega]^2 \to 2$ either there is a 0-homogeneous set A with cardinality n or there is an I-positive 1-homogeneous set A. We denote this property by

$$\omega \longrightarrow (n, \mathsf{I}^+)^2_2 .$$

Let $\mathcal{K}_n$ denote the complete graph with n vertices. We will say that a graph H *contains* $\mathcal{K}_n$ if H has an induced subgraph isomorphic to $\mathcal{K}_n$.

Lemma 4.19. [4] *For every $3 \leq n \in \omega$ there is a unique graph $G_n = \langle \omega, E_n \rangle$ (up to an isomorphism) such that for every graph H on ω, if H does not contain $\mathcal{K}_n$ then there is a subset A of ω such that $H \cong \langle A, E_n \restriction [A]^2 \rangle$.* $\square$

Definition 4.20. We let $\mathcal{R}_n$ be the ideal on ω generated by the free sets in G_n.

In order to show that the ideal $\mathcal{R}_n$ is proper, we recall an old result of P. Erdös (see [4]). Given a graph G, the *girth* of G is defined as the minimal length of a cycle contained in G. Erdös' lemma claims that *for any $k < \omega$ there is a graph H with girth greater than k and chromatic number greater than k.*

Now aiming for a contradiccion, suppose that $A_1, A_2, \ldots, A_k$ are mutually disjoint free sets with respect to E_n such that $\bigcup_{i=0}^{k} A_i = \omega$, and let H be a graph with girth greater than

$l = \max\{k, n\}$. By the property of E_n, there is $X \subseteq \omega$ such that $\langle \omega, H\rangle \cong \langle X, E_n \restriction X\rangle$. However, there is a coloring for E_n in k colors, and then, the chromatic number of H is not greater than l. Since this happens to all graphs with girth greater than l, we contradict Erdös' lemma.

The ideal $\mathcal{R}_n$ is critical with respect to the n-Ramsey property.

Theorem 4.21. *Let* I *be an ideal on* ω*. Then* I *is n-Ramsey if and only if* $\mathsf{I} \not\geq_K \mathcal{R}_n$*.*

Proof. If I is not n-Ramsey then there is a coloring $\varphi : [\omega]^2 \to 2$ such that there is no 0-homogeneous set of cardinality n and every 1-homogeneous set is in I. Let $H = \varphi^{-1}\{0\}$. Note that H does not have complete subgraphs of cardinality n, and so, there is an isomorphism $f : \langle \omega, H\rangle \to \langle A, E_n \restriction A\rangle$ for some $A \subseteq \omega$. The function f then witnesses $\mathsf{I} \geq_K \mathcal{R}_n$, since for every E_n-free set B, $f^{-1}(B)$ is 1-homogeneous respect to φ, and so, $f^{-1}(B) \in \mathsf{I}$.

On the other hand, let f be a witness to $\mathsf{I} \geq_K \mathcal{R}_n$, i.e., a function such that for every G_n-free set $f^{-1}(A) \in \mathsf{I}$, and define $\varphi : [\omega]^2 \to 2$ by $\varphi(\{m, n\}) = 0$ iff $f(m) \neq f(n)$ and $\{f(m), f(n)\} \in E_n$. If A is a 0-homogeneous set with $|A| = n$ then $f[A]$ is a complete subgraph of G_n, a contradiction. If A is a 1-homogeneous set then $f[A] \in \mathcal{R}_n$ and so, $A \subseteq f^{-1}(f[A]) \in \mathsf{I}$. $\square$

We do not know of any ideal satisfying $\mathsf{I}^+ \to (\omega, \mathsf{I}^+)^2_2$ and $\mathsf{I}^+ \not\to (\mathsf{I}^+)^2_2$. This is the only arrow in the diagram at the beginning of this section of which we do not know that it does not reverse. Let us mention that in [3] it was shown the two partition relations are equivalent for maximal ideals (i.e. ultrafilters). The proof essentially amounts to showing that $\mathsf{I}^+ \to (\omega, \mathsf{I}^+)^2_2$ implies that the ideal is $\mathrm{P}^+_{\mathsf{tower}}$ and clearly also Q^+.

5. Monotonicity and Ramsey property

In [8] the authors gave the following definitions:

Definition 5.1. Let I be an ideal on ω. Then

1. I is *Mon* (*monotone*) if for any sequence $S = \langle x_n : n \in \omega\rangle$ of real (equivalently rational) numbers there is an I-positive set X such that $S \restriction X$ is monotone.
2. I is *h-Mon* (*hereditarily monotone*) if $\mathsf{I} \restriction A$ is *Mon* for all $A \in \mathsf{I}^+$.
3. I is *Fin-BW* (*Fin-Bolzano-Weierstrass*) if for any bounded sequence $S = \langle x_n : n \in \omega\rangle$ of real numbers there is an I-positive set X such that $S \restriction X$ is convergent.
4. I is *locally selective* if for every partition $\{A_n : n \in \omega\}$ of ω into sets from I there is an I-positive set S such that $|A_n \cap S| \leq 1$.

It is easy to see that Ramsey(ω) implies Mon and Mon implies $Fin - BW$. Moreover,

1. I is $Fin - BW$ if and only if $\mathsf{I} \not\geq_K \mathsf{conv}$ and
2. I is locally selective if and only if $\mathsf{I} \not\geq_K \mathcal{ED}$.

Lemma 3.9 in [8] claims that if I is Mon then is locally selective. The authors of [8] remark that the sumable ideal $\mathsf{I}_{\frac{1}{n}}$ is $Fin - BW$ but is not Mon and they ask if every Mon ideal satisfies $\omega \longrightarrow (\mathsf{I}^+)^n_2$, for all $n \in \omega$. The following is an easy consequence of their results.

Theorem 5.2. *If* I *is* Mon *then* $\mathsf{I} \not\geq_K \mathsf{conv}$ *and* $\mathsf{I} \not\geq_K \mathcal{ED}$*.* $\square$

On the other hand we have a criterion for monotonicity of ideals.

Theorem 5.3. *Let I be an ideal on ω. If $\mathsf{I} \not\geq_K \mathsf{conv}$ and $\mathsf{I} \restriction X \not\geq_K \mathcal{ED}$ for all I-positive set X then I is Mon.*

Proof. Given $f : \omega \to \mathbb{Q}$ there is $A \in \mathsf{I}^+$ such that $f''A$ is a convergent sequence, since $\mathsf{I} \not\geq_K \mathsf{conv}$. Consider the partition $\{f^{-1}\{m\} : m \in f''A\}$ of A. If there is $m \in f''A$ such that $f^{-1}\{m\} \in \mathsf{I}^+$ we are done. If not, there is an I-positive subset B of A such that $f \restriction B$ is one to one, since $\mathsf{I} \not\geq_K \mathcal{ED}$. Let l be the limit of $f \restriction B$. Then either $B_0 = B \cap f^{-1}(-\infty, l) \in \mathsf{I}^+$ or $B_1 = B \cap f^{-1}(l, \infty) \in \mathsf{I}^+$. Let us suppose the first case (the other case is analogous). Let $\{b_k : k < \omega\}$ be the increasing enumeration of B_0. Let $k_0 = 0$ and k_1 be such that $f(b_k) > f(b_0)$ for all $k \geq k_1$; and for every $j \geq 1$ let k_{j+1} be such that $f(b_k) > \max\{f(b_i) : i < k_j\}$ for all $k \geq k_{j+1}$. For any $i < \omega$ define the family $C_i = \{b_k : k_i \leq k \leq k_{i+1}\}$. Then $\{C_i : i < \omega\}$ is a partition of B in finite sets. Since $\mathsf{I} \restriction B \not\geq_K \mathcal{ED}_{fin}$ there is an I-positive subset D of B such that $|C \cap C_i| \leq 1$ for all $i < \omega$. Now, either $D_0 = \bigcup_{i<\omega}(C \cap C_{2i}) \in \mathsf{I}^+$ or $D_1 = \bigcup_{i<\omega}(C \cap C_{2i+1}) \in \mathsf{I}^+$, and $f \restriction D_0$ and $f \restriction D_1$ are both increasing sequences. $\square$

In [8] the authors ask about colorings with more than two colors. An immediate corollary of the theorem is the following:

Corollary 5.4. *If I is $(\omega, 2)$-distributive and $\mathsf{I} \restriction X \not\geq_K \mathcal{ED}$ for all I-positive X then $\omega \longrightarrow (\mathsf{I}^+)^2_k$ for all k.*

Recently Kwela in [16] described a monotone F_σ ideal I such that $\omega \not\rightarrow (\mathsf{I}^+)^2_2$. We present an example of an F_σ-ideal for which the number of colors matters. This answers a question from [8]).

Theorem 5.5. $\omega \longrightarrow (\widetilde{\mathcal{ED}}^+)^2_2$ *but* $\omega \not\rightarrow (\widetilde{\mathcal{ED}}^+)^2_3$.

Proof. By corollary 4.18, $\omega \longrightarrow (<\omega, \mathcal{ED}^+)^2_2$, and by 4.16, $\omega \longrightarrow (\widetilde{\mathcal{ED}}^+)^2_2$. Let us consider the following coloring $\varphi : [\omega \times \omega \times \omega]^2 \to 3$ given by

$$\varphi\{(n_1, n_2, n_3), (m_1, m_2, m_3)\} = \begin{cases} 0 & \text{if } n_1 = m_1 \text{ and } n_2 = m_2 \\ 1 & \text{if } n_1 = m_1 \text{ and } n_2 \neq m_2 \\ 2 & \text{if } n_1 \neq m_1 \end{cases}$$

Let A be a subset of $\omega \times \omega \times \omega$. If A is 0-homogeneous set then the first projection of A is in $\mathcal{ED}$ because second projection of A is contained in a column of $\omega \times \omega$; if A is 1-homogeneous then the first projection of A is in $\mathcal{ED}$ because second projection of A is contained in a selector of columns in $\omega \times \omega$. Finally, if A is 2-homogeneous, then A is contained in a selector of the columns of the product $\omega \times (\omega \times \omega)$. In the three cases, $A \in \widetilde{\mathcal{ED}}$. $\square$

6. Questions

The main question considered but not answered in the paper is the following

Question 6.1. *Is there a Borel tall ideal I on ω satisfying $\mathsf{I}^+ \longrightarrow (\mathsf{I}^+)^2_2$?*

The question is equivalent to:

Question 6.2. *Is $\mathcal{R}$ locally $\leq_K$-minimal among Borel tall ideals?*

A positive answer to the second question would be a positive answer to half of the following more general question:

Question 6.3. *Is there a (locally) $\leq_K$-minimal ideal I among Borel ideals?*

A related question is the following:

Question 6.4. *Does every tall Borel ideal I contain an F_σ tall ideal? Does this hold locally, i.e. Is there for every tall Borel ideal I a set $X \in \mathsf{I}^+$ such that $\mathsf{I} \restriction X$ contains a tall F_σ ideal (on X)?*

A result of M. Laczkovich and I. Recłav [19] shows that for every Borel ideal I either $\mathsf{I} \geq_K \mathbf{Fin} \times \mathbf{Fin}$ or there is a F_σ set E such that $\mathsf{I} \subseteq E$ and $E \cap \mathsf{I}^* = \emptyset$. We wish to know if the F_σ hypothesis could be weakened to $F_{\sigma\delta}$ in order to replace "set" with "ideal". Solecki [25, Corollary 1.5] has shown that for any $F_{\sigma\delta}$ ideal I there is a F_σ set F such that $\mathsf{I} \subset F$ and $F \cap \mathsf{I}^* = \emptyset$.

Question 6.5. *Does every Borel ideal I satisfy that either $\mathsf{I} \geq_K \mathbf{Fin} \times \mathbf{Fin}$ or there is an $F_{\sigma\delta} -$ ideal J such that $\mathsf{I} \subseteq \mathsf{J}$?*

Question 6.6. *Is there an ideal such that $\mathsf{I}^+ \to (\omega, \mathsf{I}^+)^2_2$ and $\mathsf{I}^+ \not\to (\mathsf{I}^+)^2_2$? Is there Borel such ideal?*

Another promising line of research deals with higher dimensions, that is Ramsey type properties of ideals considering colorings of n-tuples for $n > 2$. A standard "stepping up" type argument shows that $\mathsf{I}^+ \to (\mathsf{I}^+)^2_2$ implies the stronger statement $\mathsf{I}^+ \to (\mathsf{I}^+)^n_k$ for all $n, k > 0$. In [3], Baumgartner and Taylor in effect showed that $\mathsf{I}^+ \to (4, \mathsf{I}^+)^3_2$ is equivalent to $\mathsf{I}^+ \to (\mathsf{I}^+)^2_2$, which shows that higher dimensions do behave differently from dimension 2.

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