# Applications of Namba forcing to weak partition properties on trees

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**Abstract**

We answer several questions of Hrušák, Simon and Zindulka regarding weak partition relations on trees. In particular, we show that the Namba forcing on $add(\mathcal{N})$ and $cof(\mathcal{N})$ does not have the Sacks property. We also construct a model where there is a singular cardinal $\kappa$ such that $cof(\kappa)$ has the boundedness property but $\kappa$ does not.

In [5] Hrušák, Simon and Zindulka studied several partition relations on trees. With a different notation, they introduced the following concepts: (where $\mathbb{NB}(\kappa)$ denotes the Namba forcing on $\kappa$).

**Definition 1** *Let $\kappa$ be a cardinal and $g : \omega \longrightarrow \omega$.*

1. *We say $\kappa$ is a* Zindulka cardinal *if for every coloring $\chi : \kappa^{<\omega} \longrightarrow \omega$ there is a $T \in \mathbb{NB}(\kappa)$ with stem $\emptyset$ such that $T$ has only finitely many colors in each level.*
2. *We say $\kappa$ is a $g$-*Zindulka cardinal *if for every coloring $\chi : \kappa^{<\omega} \longrightarrow \omega$ there is a $T \in \mathbb{NB}(\kappa)$ with stem $\emptyset$ such that $|\chi[T_n]| \leq g(n)$ for every $n \in \omega$.*

In [5] the following questions were asked:

**Problem 2 (Hrušák, Simon, Zindulka)** *Let $g : \omega \longrightarrow \omega$ be an increasing function.*

1. *Is $\mathfrak{b}$ the first regular uncountable cardinal that is not Zindulka?*
2. *Could $\mathsf{cof}(\mathcal{N})$ be a $g$-Zindulka cardinal?*
3. *Could $\mathsf{add}(\mathcal{N})$ be a $g$-Zindulka cardinal?*

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*4. Could $\mathfrak{m}_{\sigma-linked}$ be a g-Zindulka cardinal?*

It is not hard to see (see the next section) that $\kappa$ is a Zindulka cardinal if and only the Namba forcing on $\kappa$ does not add unbounded reals. In [4] the preservation properties of Namba forcing were studied in detail and the first question was answered positively. In this note, we will prove that $\mathbb{NB}(\kappa)$ has the Sacks property if and only if $\kappa$ is a $g$-Zindulka cardinal for some (any) increasing function $g$. Then we will prove that $\mathsf{cof}(\mathcal{N})$ and $\mathsf{add}(\mathcal{N})$ can not be $g$-Zindulka cardinals while $\mathfrak{m}_{\sigma-linked}$ may consistently be.

In [5] the following property was also studied: a cardinal $\kappa$ has the *Boundedness Property* if for every sequence $\mathcal{A} = \langle f_\alpha \mid \alpha \in \kappa \rangle$ where $f_\alpha : \omega \longrightarrow \omega$, there is $g : \omega \longrightarrow \omega$ such that the set $\{\alpha \mid f_\alpha < g\}$ has size $\kappa$. In [5] it was proved that if $\kappa$ has the Boundedness property then $\mathsf{cof}(\kappa)$ also has the Boundedness property and it was asked if the converse is also true. We will answer negatively their question, and we study some variants of this property. The definitions of the cardinal invariants used in this paper may be consulted in [2].

# 1 Basic properties of Namba forcing

In this section we fix some notation and will quote the relevant results we need from [4]. Let $\kappa$ be a cardinal, a tree $T \subseteq \kappa^{<\omega}$ is called a *$\kappa$-Namba tree* (or just Namba tree if the cardinal $\kappa$ is clear by context) if there is $s \in T$ (called the *stem of* $T$) such that every $t \in T$ is comparable with $s$; furthermore if $t \sqsubset s$ then $t$ has just one immediate successor and if $s \sqsubseteq t$ then $t$ has $\kappa$ many immediate successors. By $\mathbb{NB}(\kappa)$ we will denote the set of all $\kappa$-Namba trees ordered by inclusion; in this way, $\mathbb{NB}(\omega)$ is the Laver forcing. A generic filter for $\mathbb{NB}(\kappa)$ may be coded as a sequence which we will denote by $\mathfrak{n}_{gen} : \omega \longrightarrow \kappa$. It is easy to see that $\mathbb{NB}(\kappa)$ forces $\kappa$ to have countable cofinality. Given $S$ and $T$ two $\kappa$-Namba trees, $S \leq_0 T$ will mean that $S \leq T$ and both $S$ and $T$ have the same stem. By $[T]$ we denote the set of branches of $T$ and if $s \in T$ then we define $T_s$ as the set of all $t \in T$ such that either $t \sqsubseteq s$ or $s \sqsubseteq t$ and $suc_T(s) = \{\alpha \in \kappa \mid s^\frown\alpha \in T\}$. By $B(T)$ we denote the set of nodes of $T$ that extend the stem. By $stem(T)$ we denote the stem of $T$ and $\mathbb{NB}_0(\kappa)$ will denote the set of all $\kappa$-Namba trees with empty stem.

Let $T$ be a tree, given $F : T \longrightarrow \omega$ define the function $\overline{F} : [T] \longrightarrow \omega^\omega$ such that if $x \in \kappa^\omega$ and $n \in \omega$ then $\overline{F}(x) \upharpoonright n = F(x)$. A function $H : [T] \longrightarrow \omega^\omega$ is called *Lipschitz* if there is a function $F : T \longrightarrow \omega$ such that $H = \overline{F}$. Clearly every Lipschitz function is continuous. If $G : \kappa^\omega \longrightarrow \mu\omega$ is a continuous function, define $G^* : \kappa^{<\omega} \longrightarrow \omega^{<\omega}$ where $G^*(s) = \left(\bigcup\{t \mid G[\langle s\rangle] \subseteq \langle t\rangle\}\right) \upharpoonright |s|$. The following result is probably well know (see [4] for a proof).

**Proposition 3** *Let $\kappa$ be a cardinal of uncountable cofinality, $T \in \mathbb{NB}(\kappa)$ and $\dot{y}$ a $\mathbb{NB}(\kappa)$-name such that $T \Vdash$ "$\dot{y} \in \omega^{\omega}$". Then there is $S \leq_0 T$ such that:*

1. *If $s \in S$ then $S_s$ decides $\dot{y} \upharpoonright (|s| + 1)$.*
2. *There is $F : S \longrightarrow \omega$ such that $S \Vdash$ "$\overline{F}(\mathfrak{n}_{gen}) = \dot{y}$".*

We will need the following result from [4]:

**Proposition 4** *Let $\kappa$ be a cardinal $\mu <$ cof$(\kappa)$ and let $\{A_\alpha \mid \alpha \in \mu\}$ be a family of Borel sets of $\kappa^{\omega}$ such that $\kappa^{\omega} = \bigcup_{\alpha<\mu} A_\alpha$. Then there is $T \in \mathbb{NB}_0(\kappa)$ and $\alpha < \mu$ such that $[T] \subseteq A_\alpha$.*

# 2 $g$-Zindulka cardinals

In this section we will prove that $\mathsf{cof}(\mathcal{N})$ and $\mathsf{add}(\mathcal{N})$ can not be $g$-Zindulka cardinals while $\mathfrak{m}_{\sigma-linked}$ may consistently be (for $g$ an incresing function). We will need the following definitons:

**Definition 5** *Let $\kappa$ be a cardinal and $g : \omega \longrightarrow \omega$.*

1. *We say $\kappa$ is a* weak Zindulka cardinal *if for every coloring $F : \kappa^{<\omega} \longrightarrow \omega$ there is a $T \in \mathbb{NB}(\kappa)$ such that $F[T_n]$ is finite for every $n \in \omega$.*
2. *We say $\kappa$ is a* weak $g$-Zindulka cardinal *if for every coloring $F : \kappa^{<\omega} \longrightarrow \omega$ there is a $T \in \mathbb{NB}(\kappa)$ such that $|F[T_n]| \leq g(n)$ is finite for every $n \in \omega$.*

In [5] it was proved that $\mathfrak{b}$ is not a Zindulka cardinal and that $\kappa$ is a Zindulka cardinal if and only if $\kappa$ is a weak Zindulka cardinal. By the Lipschitz reading of names of $\kappa$-Namba forcing, it is easy to prove that $\mathbb{NB}(\kappa)$ does not add unbounded reals if and only if $\kappa$ is a weak Zindulka cardinal.

**Proposition 6 ([4])**

1. *$\kappa$ is a Zindulka cardinal if and only if $\mathbb{NB}(\kappa)$ does not add an unbounded real.*
2. *$\mathfrak{b}$ is the first uncountable regular cardinal that is not a Zindulka cardinal.*

Let $\mathcal{C} = \{g \in \omega^{\omega} \mid lim(g(n)) = \infty \wedge \forall n (g(n) > 0)\}$. For any $g \in \mathcal{C}$ we define the $g$-*slaloms* as the set of all $S : \omega \longrightarrow [\omega]^{<\omega}$ such that $|S(n)| \leq g(n)$ for every $n \in \omega$. Denote by $\mathcal{SL}_g$ the set of all $g$-slaloms. If $f \in \omega^{\omega}$ and $S \in \mathcal{SL}_g$ then $f \sqsubseteq^* S$ means that $f(n) \in S(n)$ holds for almost every $n \in \omega$. Given $F : \kappa^{<\omega} \longrightarrow \omega$, $S : \omega \longrightarrow [\omega]^{<\omega}$ and $T \in \mathbb{NB}(\kappa)$ we will say that $S$ *captures*

$(F,T)$ if $F[T_n] \subseteq S(n)$ for every $n \in \omega$. In this way, $\kappa$ is $g$-Zindulka if and only if for every $F : \kappa^{<\omega} \longrightarrow \omega$ there is a $T \in \mathbb{NB}(\kappa)$ with empty stem and $S \in \mathcal{SL}_g$ such that $S$ captures $(F,T)$. In a similar way, we say $S$ *almost captures* $(F,T)$ if for every $x \in [T]$ it is the case that $\overline{F}(x) \sqsubseteq^* S$.

**Lemma 7** *Let $\kappa$ be a cardinal of uncountable cofinality, $g \in \mathcal{C}$, $F : \kappa^{<\omega} \longrightarrow \omega$ and $T \in \mathbb{NB}(\kappa)$. Then the following are equivalent:*

1. *There is $S \in \mathcal{SL}_g$ and $T' \leq_0 T$ such that $S$ captures $(F,T')$.*
2. *There is $S \in \mathcal{SL}_g$ and $T' \leq_0 T$ such that $S$ almost captures $(F,T')$.*

**Proof.** Clearly 1 implies 2 , we will show that 2 implies 1, let $S$ and $T'$ as in 2. Given $x \in [T']$ define $a_x = \{n \mid F(x \restriction n) \notin S(n)\}$ and we also define $b_x = \{(n, F(x \restriction n)) \mid n \in a_x\}$. Note that both $a_x$ and $b_x$ are finite sets. Given $a \in [\omega]^{<\omega}$ and $b \in [\omega \times \omega]^{<\omega}$ let $B(a,b) = \{x \in [T] \mid a_x = a \wedge b_x = b\}$. Clearly each $B(a,b)$ is a Borel set and $[T'] = \bigcup\limits_{a,b} B(a,b)$ and since every $B(a,b)$ is Borel and $\kappa$ has uncountable cofinality, there are $T'' \leq_0 T'$ and $a, b$ such that $[T''] \subseteq B(a,b)$. Let $a = \{n_i \mid i < l\}$ and $b = \{(n_i, m_i) \mid i \in l\}$ define $S' : \omega \longrightarrow [\omega]^{<\omega}$ such that $S'(k) = S(k)$ if $k \notin a$ and $S'(n_i) = \{m_i\}$ for every $i < l$. Then $S'$ captures $(F, T'')$. $\blacksquare$

In this way, if $\kappa$ has uncountable cofinality, then $\kappa$ is a $g$-Zindulka cardinal if and only if for every $F : \kappa^{<\omega} \longrightarrow \omega$ there is a $T \in \mathbb{NB}(\kappa)$ with empty stem and $S \in \mathcal{SL}_g$ such that $S$ almost captures $(F,T)$.

**Definition 8** *We say $(A, B, \longrightarrow)$ is an* invariant *if,*

1. *$\longrightarrow \subseteq A \times B$.*
2. *For every $a \in A$ there is a $b \in B$ such that $a \longrightarrow b$ (which means $(a,b) \in \longrightarrow$).*
3. *There is no $b \in B$ such that $a \longrightarrow b$ for all $a \in A$.*

The *evaluation of* $(A, B, \longrightarrow)$ (denoted by $\|A, B, \longrightarrow\|$) is defined as the minimum size a family $D \subseteq B$ such that for every $a \in A$ there is a $d \in D$ such that $a \longrightarrow d$. The invariant $(A, B, \longrightarrow)$ is called a *Borel invariant* if $A, B$ and $\longrightarrow$ are Borel subsets of some polish space. Most (but not all) of the usual invariants are in fact Borel invariants.

**Definition 9** *Let $(A^-, A^+, \longrightarrow_A)$ and $(B^-, B^+, \longrightarrow_B)$ be two Borel invariants. We define the following relations:*

1. $(A^-, A^+, \longrightarrow_A) \leq_{BT} (B^-, B^+, \longrightarrow_B)$ *($(A^-, A^+, \longrightarrow_A)$ is* Borel-Tukey below $(B^-, B^+, \longrightarrow_B)$*) if there are Borel functions* $F^- : A^- \longrightarrow B^-$ *and* $F^+ : B^+ \longrightarrow A^+$ *such that for every* $a \in A^-$ *and* $b \in B^+$ *the following holds:*

   *If* $F^-(a) \longrightarrow_B b$ *then* $a \longrightarrow_A F^+(b)$.

2. $(A^-, A^+, \longrightarrow_A) \simeq_{BT} (B^-, B^+, \longrightarrow_B)$ *($(A^-, A^+, \longrightarrow_A)$ is* Borel-Tukey equivalent to $(B^-, B^+, \longrightarrow_B)$*) if* $(A^-, A^+, \longrightarrow_A) \leq_{BT} (B^-, B^+, \longrightarrow_B)$ *and* $(B^-, B^+, \longrightarrow_B) \leq_{BT} (A^-, A^+, \longrightarrow_A)$.

It is easy to see that if $(A^-, A^+, \longrightarrow_A) \simeq_{\mathsf{BT}} (B^-, B^+, \longrightarrow_B)$ then $(A^+, A^-, {}_A\not\longleftarrow) \simeq_{\mathsf{BT}} (B^+, B^-, {}_B\not\longleftarrow)$. Given a Borel invariant $(A, B, \longrightarrow)$ and a forcing notion $\mathbb{P}$, we say that $\mathbb{P}$ *destroys* $(A, B, \longrightarrow)$ if there is a $\mathbb{P}$-name $\dot{r}$ such that $\mathbb{P} \Vdash$ "$\dot{r} \in A$" and if $b \in B$ (with $b \in V$) then $\mathbb{P} \Vdash$ "$\dot{r} \not\rightarrow b$". The following lemma is well known and easy to prove:

**Lemma 10** *Let* $(A^-, A^+, \longrightarrow_A)$, $(B^-, B^+, \longrightarrow_B)$ *be two Borel invariants such that* $(A^-, A^+, \longrightarrow_A) \leq_{BT} (B^-, B^+, \longrightarrow_B)$ *and* $\mathbb{P}$ *a forcing notion. If* $\mathbb{P}$ *destroys* $(A^-, A^+, \longrightarrow_A)$ *then* $\mathbb{P}$ *destroys* $(B^-, B^+, \longrightarrow_B)$.

The following proposition is well known but we included a proof for the convenience of the reader:

**Proposition 11** *Let* $f, g \in \mathcal{C}$ *then* $(\omega^\omega, \mathcal{SL}_f, \sqsubseteq^*) \simeq_{BT} (\omega^\omega, \mathcal{SL}_g, \sqsubseteq^*)$. *Moreover, there are* $R : \omega^\omega \longrightarrow \omega^\omega$, $H : \mathcal{SL}_f \longrightarrow \mathcal{SL}_g$ *and* $k \in \omega$ *such that the following holds:*

1. *$R$ and $H$ are continuous.*
2. *For every* $x \in \omega^\omega, S \in \mathcal{SL}_f$ *if* $R(x) \sqsubseteq^* S$ *then* $x \sqsubseteq^* H(S)$.
3. *For every* $x \in \omega^\omega, S \in \mathcal{SL}_f$ *if* $R(x) \sqsubseteq S$ *then* $x(m) \in H(S)(m)$ *for every* $m > k$.

**Proof.** We define an interval partition $\mathcal{P} = \{P_n \mid n \in \omega\}$ such that for every $n, m \in \omega$ if $P_n \subseteq m$ then $f(n) \leq g(m)$. Let $\{t_n \mid n \in \omega\}$ be an enumeration of all functions $p : s \longrightarrow \omega$ where $s \in [\omega]^{<\omega}$ and define $R : \omega^\omega \longrightarrow \omega^\omega$ such that if $m \in \omega$ and $x \in \omega^\omega$ then $x \restriction P_{m+1} = t_{R(x)(m)}$. Now define $H : \mathcal{SL}_f \longrightarrow \mathcal{SL}_g$ such that if $S \in \mathcal{SL}_f$ then the following holds:

**Case 12** $m \in P_0$. *Then* $H(S)(m) = \{0\}$.

**Case 13** $m \in P_{i+1}$. *Let* $S(i) = \left\{l_1, ..., l_{f(n)}\right\}$ *then define the set* $H(S)(m) = \{t_{l_1}(m), ..., t_{l_{f(n)}}(m)\}$.

It is easy to see that both $R$ and $H$ are continuous. Let $k = max(P_0)$, we will prove 3 and 2 will follow by the proof of 3. Let $x \in \omega^\omega, S \in \mathcal{SL}_f$ and $m > k$ such that $R(x) \sqsubseteq S$. Let $i \in \omega$ such that $m \in P_{i+1}$, since $R(x)(i) \in S(i)$ then $x(m) = (x \upharpoonright P_{i+1})(m) = t_{R(x)(i)}(m) \in H(S)(m)$. $\blacksquare$

The last proposition implies the well known result that a forcing notion $\mathbb{P}$ has the Sacks property if and only if it does not destroy $(\omega^\omega, \mathcal{SL}_f, \sqsubseteq^*)$ for every (any) $f \in \mathcal{C}$. It is easy to see that if $\kappa$ has uncountable cofinality, then $\mathbb{NB}(\kappa)$ has the Sacks property if and only if $\kappa$ is a weak $f$-Zindulka cardinal for every (any) $f \in \mathcal{C}$.

**Proposition 14** *Let $\kappa$ be a cardinal and $f, g \in \mathcal{C}$. Then $\kappa$ is a $f$-Zindulka cardinal if and only if $\kappa$ is a $g$-Zindulka cardinal.*

**Proof.** It is easy to see that if $\kappa$ has countable cofinality then $\kappa$ is not a Zindulka cardinal, so in particular, it is neither $f$-Zindulka or $g$-Zindulka. Now assume that $\kappa$ is an $f$-Zindulka cardinal of uncountable cofinality. Fix $R : \omega^\omega \longrightarrow \omega^\omega$, $H : \mathcal{SL}_f \longrightarrow \mathcal{SL}_g$ and $k$ as in the previous proposition. Let $F : \kappa^{<\omega} \longrightarrow \omega$, we can then find $T \in \mathbb{NB}_0(\kappa)$ and $G : B(T) \longrightarrow \omega$ such that $T \Vdash$ "$\overline{G}(\mathfrak{n}_{gen}) = R\overline{F}(\mathfrak{n}_{gen})$". Since $\kappa$ is $f$-Zindulka we can then find $S \in \mathcal{SL}_f$ and $T' \leq_0 T$ such $S$ captures $(G, T')$. We claim that $H(S)$ almost captures $(F, T')$.

Let $t \in T'$ such that $|t| = n > k$. We will show that $F(t) \in H(S)(n)$. Since $S$ captures $(G, T')$ we know that $G(t) \in S(n)$. Let $\mathfrak{n} : \omega \longrightarrow \kappa$ be a generic branch through $T'$ extending $t$. In this way, $R\overline{F}(\mathfrak{n})(n) = \overline{G}(\mathfrak{n}_{gen})(n) = G(t)$ so $R\overline{F}(\mathfrak{n})(n) \in S(n)$. In this way, $F(t) = \overline{F}(\mathfrak{n})(n) \in H(S)(n)$. $\blacksquare$

Let $f, g \in \mathcal{C}$ and $S_1 \in \mathcal{SL}_f$, $S_2 \in \mathcal{SL}_g$. Define $S_1 \leq S_2$ if $S_1(n) \subseteq S_2(n)$ for every $n \in \omega$ and $S_1 \leq^* S_2$ if $S_1(n) \subseteq S_2(n)$ holds for almost all $n \in \omega$. We now recursively build functions $\{f_n \mid n \in \omega\} \subseteq \omega^\omega$ as follows: $f_0(m) = m + 1$ for every $m \in \omega$ and $f_{n+1}(m) = (m+1)^2 f_n(m)$. Finally, let $f_\omega : \omega \longrightarrow \omega$ such that $f_n \leq^* f_\omega$ for every $n \in \omega$.

**Lemma 15** *Let $\kappa$ such that $\mathbb{NB}(\kappa)$ has the Sacks property and $n \in \omega$. If $\{S_\alpha \mid \alpha \in \kappa\} \subseteq \mathcal{SL}_{f_n}$ then there is $A \in [\kappa]^\kappa$ and $S \in \mathcal{SL}_{f_{n+1}}$ such that $S_\alpha \leq S$ for every $\alpha \in A$.*

**Proof.** Let $\mathfrak{n} : \omega \longrightarrow \kappa$ be a generic sequence for $\mathbb{NB}(\kappa)$. In $V[\mathfrak{n}]$ we define $Z : \omega \longrightarrow [\omega]^{<\omega}$ where $Z(m) = \bigcup\limits_{i \leq m} S_{\mathfrak{n}(i)}(m)$. Note that $|Z(m)| \leq (m+1) f_n(m)$. Let $[\omega]^{<\omega} = \{t_m \mid m \in \omega\}$ and (still in $V[\mathfrak{n}]$) we define $h : \omega \longrightarrow \omega$ such that $Z(m) = t_{h(m)}$ for every $m \in \omega$. Since $\mathbb{NB}(\kappa)$ has the Sacks property then there is $T \in \mathbb{NB}(\kappa)$ and $W : \omega \longrightarrow [\omega]^{<\omega}$ such that if $m \in \omega$ then $|W(m)| \leq m + 1$ and $T \Vdash$ "$h(m) \in W(m)$". Without losing generality, we may assume that if $i \in W(m)$ then $|t_i| \leq (m+1) f_n(m)$.

We now define $S : \omega \longrightarrow [\omega]^{<\omega}$ such that $S(m) = \bigcup\limits_{i \in W(m)} t_i$. Note that $|S(m)| \leq (m+1)^2 f_n(m)$ so $S \in \mathcal{SL}_{f_{n+1}}$. It follows by the definitions that $T \Vdash$ "$Z \leq S$". Let $s$ be the stem of $T$ and $A = suc_T(s)$. We claim that if $\alpha \in A$ and $m > |s| + 1$ then $S_\alpha(m) \subseteq S(m)$. Let $T' \leq T$ such that $s^\frown\alpha \subseteq st(T')$, in this way, $T' \Vdash$ "$S_\alpha(m) \subseteq Z(m)$" and then $T' \Vdash$ "$S_\alpha(m) \subseteq S(m)$" so $S_\alpha(m) \subseteq S(m)$.

Since $\kappa$ has uncountable cofinality, it is then easy to find $A' \in [A]^\kappa$ and $S'$ a finite modification of $S$ such that $S_\alpha \leq S'$ for every $\alpha \in A'$. ■

We have the following combinatorial characterization of the Sacks property for $\kappa$-Namba forcing:

**Proposition 16** *Let $\kappa$ be a cardinal. Then $\mathbb{NB}(\kappa)$ has the Sacks property if and only if $\kappa$ is a $g$-Zindulka cardinal for some (every) $g \in \mathcal{C}$.*

**Proof.** If $\kappa$ has countable cofinality then $\kappa$ is not $g$-Zindulka for some (every) $g \in \mathcal{C}$ and since Laver forcing $\mathbb{NB}(\omega)$ does not have the Sacks property, neither does $\mathbb{NB}(\kappa)$. We now assume that $\kappa$ has uncountable cofinality. Since every $f$-Zindulka cardinal is a weak $f$-Zindulka cardinal, it follows that if $\kappa$ is $f$-Zindulka then $\mathbb{NB}(\kappa)$ has the Sacks property.

Let $\kappa$ be a cardinal of uncountable cofinality such that $\mathbb{NB}(\kappa)$ has the Sacks property. We will prove that $\kappa$ is an $f_\omega$-Zindulka cardinal. Let $F : \kappa^{<\omega} \longrightarrow \omega$, define a rank function $rk : \kappa^{<\omega} \longrightarrow OR \cup \{\infty\}$ as follows:

1. $rk(s) = 0$ if there are $n \in \omega$, $T \in \mathbb{NB}(\kappa)$ with stem $s$ and $S \in \mathcal{SL}_{f_n}$ that captures $(T, F)$.
2. $rk(s) \leq \alpha$ if $|\{\xi \mid rk(s^\frown\xi) < \alpha\}| = \kappa$.
3. $rk(s) = \alpha$ if $rk(s) \leq \alpha$ and there is no $\beta < \alpha$ such that $rk(s) \leq \beta$.
4. $rk(s) = \infty$ if there is no $\alpha$ such that $rk(s) \leq \alpha$.

We will first prove that $rk(s) \neq \infty$ for every $s \in \kappa^{<\omega}$. Assume this is not the case, we can then recursively build $T \in \mathbb{NB}(\kappa)$ such that $stem(T) = s$ and $rk(t) = \infty$ for every $t \in B(t)$. We then arrive at a contradiction since $\kappa$ was a weak $f_0$-Zindulka cardinal.

We now claim that $rk(\emptyset) = 0$. Assume this is not the case, then we can find $s \in \kappa^{<\omega}$ such that $rk(s) = 1$ and let $A = \{\alpha \mid rk(s^\frown\alpha) = 0\}$. Note that $|A| = \kappa$ since $rk(s) = 1$. For every $\alpha \in A$ choose $n_\alpha \in \omega, T_\alpha \in \mathbb{NB}(\kappa)$ with stem $s^\frown\alpha$ and $S_\alpha \in \mathcal{SL}_{f_{n_\alpha}}$ such that $S_\alpha$ captures $(T_\alpha, F)$. We can then find $n \in \omega$ such that $B = \{\alpha \in A \mid n_\alpha = n\}$ has size $\kappa$. By the previous lemma, there are $C \in [B]^\kappa$ and $S \in \mathcal{SL}_{f_{n+1}}$ such that $S_\alpha \leq S$ for every $\alpha \in C$. Define

$T = \{s \restriction i \mid i \leq |s|\} \cup \bigcup_{\alpha \in C} T_\alpha$ then $S$ captures $(F, T)$, but this is a contradiction since $rk(s) = 1$.

In this way, there are $n \in \omega$, $T \in \mathbb{NB}(\kappa)$ and $S \in \mathcal{SL}_{f_n}$ such that $stem(T) = \emptyset$ and $S$ captures $(T, F)$. Since $f_n \leq^* f_\omega$ we can find $S_1 \in \mathcal{SL}_{f_\omega}$ such that $S_1$ almost captures $(T, F)$. ■

Fix a set $C = \{C^n_m \mid n, m \in \omega\}$ with the following properties:

1. Each $C^n_m \subseteq 2^\omega$ is a clopen set of Lebesgue measure at most $\frac{1}{2^n}$.
2. If $D \subseteq 2^\omega$ is a clopen set of measure at most $\frac{1}{2^n}$ then there is $m \in \omega$ such that $D = C^n_m$.

Given $f : \omega \longrightarrow \omega$ define $N(f) = \bigcap_{n \in \omega} \bigcup_{i > n} C^i_{f(i)}$ which clearly is a null set. It is known that $\{N(f) \mid f \in \omega^\omega\}$ is a cofinal subset of null sets (see [1] lemma 3.2). Given $f, g \in \omega^\omega$ define $f \leq_{\mathcal{N}} g$ if $N(f) \subseteq N(g)$. We will need the following important result:

**Proposition 17 (see [1])** $(\omega^\omega, \mathcal{SL}, \sqsubseteq^*) \simeq_{BT} (\omega^\omega, \omega^\omega, \leq_{\mathcal{N}})$.

In this way, a forcing $\mathbb{P}$ has the Sacks property if and only if every null set in an extension by $\mathbb{P}$ is contained in a ground model null set. We can then prove the following:

**Theorem 18** *Neither $\mathbb{NB}(\mathsf{add}(\mathcal{N}))$ nor $\mathbb{NB}(\mathsf{cof}(\mathcal{N}))$ have the Sacks property.*

**Proof.** We first show that $\mathbb{NB}(\mathsf{cof}(\mathcal{N}))$ does not have Sacks property. Let $D = \{N_\alpha \mid \alpha \in \mathsf{cof}(\mathcal{N})\}$ be a cofinal family of null sets. Given $\beta < \mathsf{cof}(\mathcal{N})$ define $D_\beta = \{x_\alpha \mid \alpha \leq \beta\}$. Since $D_\beta$ is not cofinal, there is $M_\beta \in \mathcal{N}$ such that $M_\beta \nsubseteq N_\alpha$ for every $\alpha \leq \beta$.

Let $\mathfrak{n} : \omega \longrightarrow \mathsf{cof}(\mathcal{N})$ be a generic sequence for $\mathbb{NB}(\mathsf{cof}(\mathcal{N}))$. In $V[\mathfrak{n}]$ let $M = \bigcup_{n \in \omega} M_{\mathfrak{n}(n)}$ which is clearly a null set. We claim that $M$ is not contained in any element of $\mathcal{N} \cap V$, it is enough to prove that if $\alpha \in \mathsf{cof}(\mathcal{N})$ then $M \nsubseteq N_\alpha$. By genericity, there is $m \in \omega$ such that $\alpha < \mathfrak{n}(m)$, since $M_{\mathfrak{n}(m)} \subseteq M$ while $M_\beta \nsubseteq N_\alpha$ we conclude that $M \nsubseteq N_\alpha$.

Now we prove that $\mathbb{NB}(\mathsf{add}(\mathcal{N}))$ does not have Sacks property, this is just the dual argument of the previous proof. Let $B = \{N_\alpha \mid \alpha \in \mathsf{add}(\mathcal{N})\} \subseteq \mathcal{N}$ such that $\bigcup B \notin \mathcal{N}$. Given $\beta < \mathsf{add}(\mathcal{N})$ define $B_\beta = \{x_\alpha \mid \alpha \leq \beta\}$. Since $\beta <$

$\mathsf{add}(\mathcal{N})$, then $M_\beta = \bigcup\limits_{\alpha \leq \beta} N_\alpha$ is a null set. Let $\mathfrak{n} : \omega \longrightarrow \mathsf{add}(\mathcal{N})$ be a generic sequence for $\mathbb{NB}(\mathsf{add}(\mathcal{N}))$. In $V[\mathfrak{n}]$ define the null set $M = \bigcup\limits_{n \in \omega} M_{\mathfrak{n}(n)}$, we claim that $M$ is not contained in any ground model null set. Let $A \in \mathcal{N} \cap V$ and since $\bigcup B \notin \mathcal{N}$ then there is $\alpha \in \mathsf{add}(\mathcal{N})$ such that $N_\alpha \nsubseteq A$. By genericity, there is $m \in \omega$ such that $\alpha < \mathfrak{n}(m)$. Note that $M_{\mathfrak{n}(m)} \subseteq M$ and on the other hand, $M_{\mathfrak{n}(m)} \nsubseteq A$ because $N_\alpha \nsubseteq A$ and $N_\alpha \subseteq M_{\mathfrak{n}(m)}$. ■

In [4] it was proved that if $\kappa$ is a regular cardinal such that $\kappa < \mathsf{add}(\mathcal{N})$ then $\mathbb{NB}(\kappa)$ has the Sacks property. We can then conclude the following:

**Corollary 19** *The cardinal invariant $\mathsf{add}(\mathcal{N})$ is the least regular cardinal $\kappa$ that is not a $g$-Zindulka cardinal for every (any) $g \in \mathcal{C}$.*

Since the inequality $\mathfrak{m}_{\sigma-linked} < \mathsf{add}(\mathcal{N})$ is consistent, we conclude that $\mathfrak{m}_{\sigma-linked}$ may consistently be a $g$-Zindulka cardinal.

# 3 The Boundedness property

We call a subtree $T \subseteq \kappa^{<\omega}$ a *broom tree* if there is $s \in T$ such that $s$ has $\kappa$ immediate sucessors and every other node has just one successor. The statement $\kappa \rightsquigarrow_{\mathsf{b}} (\kappa)^{<\omega}_\omega$ means that for every coloring $\chi : \kappa^{<\omega} \longrightarrow \omega$ there is a finitely colored broom tree. On the other hand, $\kappa \rightsquigarrow_{\mathsf{W}} (\kappa)^{<\omega}_\omega$ means that for any coloring $\chi : \kappa^{<\omega} \longrightarrow \omega$ there is a finitely colored tree $T \subseteq \kappa^{<\omega}$ of size $\kappa$. Obviously $\kappa \rightsquigarrow_{\mathsf{b}} (\kappa)^{<\omega}_\omega$ implies $\kappa \rightsquigarrow_{\mathsf{W}} (\kappa)^{<\omega}_\omega$. Furthermore, if $\kappa$ has uncountable cofinality, then $\kappa \rightsquigarrow_{\mathsf{b}} (\kappa)^{<\omega}_\omega$ if and only if $\kappa \rightsquigarrow_{\mathsf{W}} (\kappa)^{<\omega}_\omega$. However, this relations are not equivalent as the next result shows:

**Proposition 20** *If $\kappa = \mathfrak{c}^{+\omega}$ [1]then $\kappa \rightsquigarrow_{\mathsf{W}} (\kappa)^{<\omega}_\omega$ but $\kappa \not\rightsquigarrow_{\mathsf{b}} (\kappa)^{<\omega}_\omega$.*

**Proof.** We will first show that $\kappa \rightsquigarrow_{\mathsf{W}} (\kappa)^{<\omega}_\omega$. Let $\chi : \kappa^{<\omega} \longrightarrow \omega$ and let $S = \kappa^{<\omega}$. Since $\mathfrak{c}^+$ is a Zindulka cardinal (see [4])then we may find $T(0) \subseteq S_{\langle 0 \rangle}$ with $T(0) \in \mathbb{NB}(\mathfrak{c}^+)$ that is finitely colored. In the same way, we may find $T(1) \subseteq S_{\langle 1,0 \rangle}$ finitely colored with $T(1) \in \mathbb{NB}(\mathfrak{c}^{++})$ and then we find $T(2) \subseteq S_{\langle 1,1,0 \rangle}$ finitely colored with $T(2) \in \mathbb{NB}(\mathfrak{c}^{+++})$... After $\omega$ steps, we define $T = \bigcup\limits_{n \in \omega} T(n)$ and it is clear that it is finitely colored and of size $\kappa$.

Now we will show that $\kappa \not\rightsquigarrow_{\mathsf{b}} (\kappa)^{<\omega}_\omega$, actually we will prove that if $\mu$ has countable cofinality then $\mu \not\rightsquigarrow_{\mathsf{b}} (\mu)^{<\omega}_\omega$. Let $\chi : \mu^{<\omega} \longrightarrow \omega$ such that for every $s \in \mu^{<\omega}$ and every $n \in \omega$, the set $\{\alpha \mid \chi(s^\frown \alpha) = n\}$ is bounded, then clearly there can not be a finitely colored broom tree. ■

The following property was also introduced in [5]:

[1] i.e. $\kappa = \bigcup \{\mathfrak{c}, \mathfrak{c}^+, \mathfrak{c}^{++}, ...\}$

**Definition 21** *We say that* $\kappa$ *has the* Boundedness property *if for every sequence* $\mathcal{A} = \langle f_\alpha \mid \alpha \in \kappa \rangle$ *where* $f_\alpha : \omega \longrightarrow \omega$, *there is* $g : \omega \longrightarrow \omega$ *such that the set* $\{\alpha \mid f_\alpha < g\}$ *has size* $\kappa$. $BP(\kappa)$ *will abbreviate that* $\kappa$ *has the boundedness property.*

IAs pointed before $\aleph_\omega \not\rightsquigarrow_{\mathfrak{b}} (\aleph_\omega)^{<\omega}_{\omega}$ but we will see that it can not be decided in $\mathsf{ZFC}$ if the weak arrow holds or not. Given $S$ a set of ordinals, we will denote by $V[C_S]$ as the extension obtained by adding $S$ Cohen reals.

**Proposition 22** *The statement* $\aleph_\omega \rightsquigarrow_{\mathsf{W}} (\aleph_\omega)^{<\omega}_{\omega}$ *is independent from* $\mathsf{ZFC}$.

**Proof.** If $\mathfrak{c} < \aleph_\omega$ then $\aleph_\omega \rightsquigarrow_{\mathsf{W}} (\aleph_\omega)^{<\omega}_{\omega}$ by the previous result, so we just need to build a model where the relation does not hold. Assume $V \models \mathsf{GCH}$ and consider $V[C_{\aleph_\omega}]$ the forcing extension obtained by adding $\aleph_\omega$ Cohen reals. In $V[C_{\aleph_\omega}]$ define $\chi : \aleph_\omega^{<\omega} \longrightarrow \omega$ where $\chi(\langle \alpha_0, ..., \alpha_n \rangle) = \sum\limits_{i,j<n} c_{\alpha_i}(j)$ and assume there is $T$ a finitely colored tree of size $\aleph_\omega$ and $g : \omega \longrightarrow \omega$ such that $T \leq g$. Clearly, there must be $s \in T$ that has (at least) $\omega_1$ successors. Let $n = |s|$ and $\langle \beta_\xi \mid \xi \in \omega_1 \rangle \subseteq suc_T(s)$. Define $h : \omega \longrightarrow \omega$ given by $h(m) = g(n+m)$ then it follows that $c_{\beta_\xi} \leq h$ for every $\xi \in \omega_1$ which is clearly impossible since $h$ must had appeared in an intermediate extension where only $\omega$ Cohen reals has been added. ■

We will need the following well known lemma:

**Lemma 23** *If* $c$ *is a Cohen real over* $V$, *then* $\mathfrak{p}^V = \mathfrak{p}^{V[c]}$.

**Proof.** It is a result of Roitman that $\mathfrak{p}^V \leq \mathfrak{p}^{V[c]}$ (see [6]) furthermore, it is easy to see that Cohen forcing does not fill ground model towers, so $\mathfrak{t}^{V[c]} \leq \mathfrak{t}^V$ and by Malliaris and Shelah's theorem (see [7]) we concluded the desired result. ■

We may define a natural two cardinal variation of the boundedness property, given $\kappa, \lambda$ the statement $BP(\kappa, \lambda)$ will mean that for every sequence $\langle f_\alpha \mid \alpha \in \kappa \rangle$ of reals, there is $g : \omega \longrightarrow \omega$ such that the set $\{\alpha < \kappa \mid f_\alpha < g\}$ has size at least $\lambda$. Obviously, $BP(\kappa)$ is the same as $BP(\kappa, \kappa)$. We know that both $BP(\mathfrak{b})$ and $BP(\mathfrak{d})$ are false, however we have the following result:

**Proposition 24** *The statement* $BP(\mathfrak{d}, \mathfrak{b})$ *is independent from* $\mathsf{ZFC}$.

**Proof.** To get a model where $BP(\mathfrak{d}, \mathfrak{b})$ fails, assume $V \models \mathsf{GCH}$ and add $\omega_2$ Cohen reals, then it is clear that $\mathfrak{b} = \omega_1, \mathfrak{d} = \omega_2$ and $BP(\omega_2, \omega_1)$ fails because of the Cohen reals. To build a model where $BP(\mathfrak{d}, \mathfrak{b})$ holds, start with a model of $\mathfrak{p} = \mathfrak{c} = \omega_2$ and add $\omega_1$ Cohen reals, clearly in the extension $\mathfrak{b} = \omega_1$.

We now show that in $V[C_{\omega_1}]$ we get $\mathfrak{d} = \omega_2$. Assume this is not the case, so there must be a dominating family $\mathcal{F} = \{f_\alpha \mid \alpha \in \omega_1\} \in V[C_{\omega_1}]$. For every

$p \in \mathbb{C}_{\omega_1}$ and $\alpha \in \omega_1$ define $f^p_\alpha : \omega \longrightarrow \omega$ given by $f^p_\alpha(n) = \mathsf{min}\{m \mid \exists r \leq p(r \Vdash$ "$\dot{f}_\alpha(n) = m$")$\}$ and note that $\{f^p_\alpha \mid \alpha \in \omega_1 \wedge p \in \mathbb{C}_{\omega_1}\}$ belongs to the ground model. This is a family of size $\omega_1$, so there is $g \in V$ that is not dominated by any $f^p_\alpha$. However, since $\mathcal{F}$ is dominating in the extension, there must be $\alpha \in \omega_1$ and $p \in \mathbb{C}_{\omega_1}$ such that $p \Vdash$ "$g < f_\alpha$" which would then imply $g < f^p_\alpha$ which is a contradiction.

It only remains to prove $BP(\omega_2, \omega_1)$ so (in $V[C_{\omega_1}]$) take a sequence $A = \langle f_\alpha \mid \alpha \in \omega_2 \rangle$ and since every real appears in an intermediate extension, then there is a countable $\alpha$ such that $A \cap V[C_\alpha]$ has size $\omega_1$. Note that $W = V[C_\alpha]$ is equivalent to a single Cohen extension and since $\mathfrak{p}^W = \omega_2$, then $\omega_1$ has the boundedness property in $W$, so we may find a function that dominates uncountable many elements of $A \cap W$. ■

In [5] it was proved that if $\kappa \rightsquigarrow_{\mathsf{b}} (\kappa)^{<\omega}_\omega$ then $\mathsf{cof}(\kappa) \rightsquigarrow_{\mathsf{b}} (\mathsf{cof}(\kappa)^{<\omega}_\omega)$ and it was asked if the converse is also true. We will now answer this question negatively.

Given $\mathbb{P}$ and $\mathbb{Q}$ partial orders, we say $\mathbb{P}$ is a *regular (or complete) suborder of* $\mathbb{Q}$ (which we denote by $\mathbb{P} \leq_r \mathbb{Q}$) if $\mathbb{P} \subseteq \mathbb{Q}$, the order and incomparability relation of $\mathbb{P}$ are the order and incomparability relations of $\mathbb{Q}$ restricted to $\mathbb{P}$ and every maximal antichain (dense) of $\mathbb{P}$ is also a maximal antichain (predense) of $\mathbb{Q}$. This is equivalent that for every $q \in \mathbb{Q}$ there is $p \in \mathbb{P}$ such that if $p' \leq p$ then $p' \parallel q$, such $p$ (which in general is not unique) is called a *reduction of* $q$. If $\mathbb{P} \leq_r \mathbb{Q}$ and $G \subseteq \mathbb{Q}$ is generic, then $G \cap \mathbb{P}$ is generic for $\mathbb{P}$. For more details the reader may consult [6].

The key for our result is the next lemma (which we took from [3] but we proved it here for the sake of completeness).

**Lemma 25 ([3])** *Assume $V \subseteq W$, $\mathbb{P} \in V$ and $\mathbb{Q} \in W$. Moreover (in $W$) $\mathbb{P} \leq_r \mathbb{Q}$ and there is $c \in W$ which is unbounded for $V$. Let $G_\mathbb{Q} \subseteq \mathbb{Q}$ be a generic $(W, \mathbb{Q})$-generic filter and let $G_\mathbb{P} = G_\mathbb{Q} \cap \mathbb{P}$. Then $V[G_\mathbb{P}] \subseteq W[G_\mathbb{Q}]$ and $c$ is unbounded for $V[G_\mathbb{P}]$.*

**Proof.** Assume this is not the case, so there is $\dot{f} \in V$ and $q \in \mathbb{Q}$ such that $q \Vdash$ "$c < \dot{f}$". Let $p \in \mathbb{P}$ be a reduction of $q$. In $V$, define $h : \omega \longrightarrow \omega$ where $h(n) = min\{m \mid \exists r \leq p(r \Vdash$ "$\dot{f}(n) = m$")$\}$. Since $h \in V$ then there is $n \in \omega$ such that $h(n) < c(n)$. Find $r \leq p$ such that $r \Vdash$ "$\dot{f}(n) = h(n)$" and since $p$ is a reduction of $q$ there is $\overline{r} \in \mathbb{Q}$ such that $\overline{r} \leq r, q$. Note that $\overline{r}$ forces $\dot{f}(n) < c(n)$ and $\dot{f}(n) > c(n)$ which is a contradiction. ■

Given $\mathcal{F} = \langle f_\alpha \mid \alpha \in \omega_1 \rangle$ define $\mathbb{H}(\mathcal{F})$ (the *Hechler forcing restricted to* $\mathcal{F}$) as the set of all pairs of the form $(s, \mathcal{G})$ where $s \in \omega^{<\omega}$ and $\mathcal{G} \in [\omega_1]^{<\omega}$. If $(s_1, \mathcal{G}_1), (s_2, \mathcal{G}_2) \in \mathbb{H}(\mathcal{F})$ then define $(s_1, \mathcal{G}_1) \leq (s_2, \mathcal{G}_2)$ is $s_1 \subseteq s_2$, $\mathcal{G}_2 \subseteq \mathcal{G}_1$ and if $i \in dom(s_1) \setminus dom(s_2)$ and $\alpha \in \mathcal{G}_2$ then $s_1(i) > f_\alpha(i)$. Now we are ready for the announced consistency result:

**Proposition 26** *Assume* GCH *holds in $V$. There is $\mathbb{P}$ such that if $G \subseteq \mathbb{P}$ is generic, then in $V[G]$ the following hold,*

1. $\omega_1 < \mathfrak{b}$ *so $\omega_1$ has the boundedness property,*
2. $\mathfrak{d} = \aleph_{\omega_{1+1}}$,
3. $\aleph_{\omega_1}$ *does not have the boundedness property.*

**Proof.** Let $\mathbb{P} = \mathbb{C}_{\aleph_{\omega_1+1}} * \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha \in \aleph_{\omega_{1+1}} \rangle$ where $\mathbb{P}_\alpha \Vdash$ "$\exists \dot{\mathcal{F}}_\alpha \in [\omega^\omega]^{\omega_1}$ $(\dot{\mathbb{Q}}_\alpha = \mathbb{H}(\dot{\mathcal{F}}_\alpha))$" and we iterate with finite support. Moreover, (with a suitable book-keeping device) we arrange that every sequence of reals of lenght $\omega_1$ in the final model is used at some successor step. It is clear that $\omega_1 < \mathfrak{b}$ and when we prove that $\aleph_{\omega_1}$ does not have the boundedness property, it will follow that $\aleph_{\omega_1} \leq \mathfrak{d}$ but since $\mathfrak{b} \leq \mathsf{cof}(\mathfrak{d})$ then we may conclude that $\aleph_{\omega_1} < \mathfrak{d}$. We will now prove that $\aleph_{\omega_1}$ does not have the boundedness property.

Let $\langle c_\alpha \mid \alpha \in \aleph_{\omega_1} \rangle$ be the first $\aleph_{\omega_1}$ Cohen reals added by $\mathbb{P}$, we will show that if $g : \omega \longrightarrow \omega \in V[G]$ then the set $\{\alpha \mid c_\alpha \leq^* g\}$ has size less than $\aleph_{\omega_1}$. Let $\dot{g}$ be a name for $g$ and find $M$ and elementary submodel of $H(\theta)$ (for some big enough $\theta$) such that ${}^{\omega_1}M \subseteq M$, $\mathbb{P}, \dot{g} \in M$ and let $S = \aleph_{\omega_1} \cap M$. Obviously $V[C_S] \subseteq V[C_{\aleph_{\omega_1+1}}]$ and if $\alpha \notin S$ then $c_\alpha$ is unbounded for $V[C_S]$. Now we define another finite support iteration $\overline{\mathbb{P}} = \mathbb{C}_S * \langle \overline{\mathbb{P}}_\alpha, \dot{\overline{\mathbb{Q}}}_\alpha \mid \alpha \in \aleph_{\omega_{1+1}} \rangle$ where $\overline{\mathbb{P}}_\alpha \Vdash$ "$\dot{\overline{\mathbb{Q}}}_\alpha = \mathbb{H}(\dot{\mathcal{F}}_\alpha)$" if $\alpha \in M$ and $\overline{\mathbb{P}}_\alpha \Vdash$ "$\dot{\overline{\mathbb{Q}}}_\alpha = \{\emptyset\}$" in the other case. It is not evident that this is well defined, since although $\dot{\mathcal{F}}_\alpha$ is a $\mathbb{P}_\alpha$-name for a sequence of reals, at the moment it is not clear it is also a $\overline{\mathbb{P}}_\alpha$-name, however, the next claim will take care of this problem:

**Claim 27** *If $\alpha < \aleph_{\omega_1+1}$ then the following holds:*

$1_\alpha)$ $\mathbb{C}_S * \overline{\mathbb{P}}_\alpha \leq_r \mathbb{C}_{\aleph_{\omega_1+1}} * \mathbb{P}_\alpha$,

$2_\alpha)$ *If $\alpha \in M$ and $a \in (\mathbb{C}_{\aleph_{\omega_1+1}} * \mathbb{P}_\alpha) \cap M$ then there is $b \in \left(\mathbb{C}_S * \overline{\mathbb{P}}_\alpha\right) \cap M$ that is* equivalent to $a$ *(i.e. $a \leq b$ and $b \leq a$).*

$3_\alpha)$ *If $\alpha \in M$ and $\dot{f} \in M$ is a $\mathbb{C}_{\aleph_{\omega_1+1}} * \mathbb{P}_\alpha$ name for a real, then there is $\dot{h} \in M$ a $\mathbb{C}_S * \overline{\mathbb{P}}_\alpha$-name such that that $1 \Vdash$ "$\dot{f} = \dot{h}$".*

We first note that $1_\alpha$ and $2_\alpha$ imply $3_\alpha$. Given $\alpha, \dot{f} \in M$ then without loss of generality, we may assume $\dot{f} = \{\{n\} \times A_n \mid n \in \omega\}$ where $A_n \subseteq \mathbb{C}_{\aleph_{\omega_1+1}} * \mathbb{P}_\alpha$ is a maximal antichain deciding the value of $n$. Since ${}^{\omega}M \subseteq M$ and using $2_\alpha$ we can easily construct such an $h$. We will now prove $1_\alpha$ and $2_\alpha$ by induction.

Assume they hold for $\alpha$, we need to show they hold for $\alpha + 1$. We will first assume $\alpha \in M$, let $a \in (\mathbb{C}_{\aleph_{\omega_1+1}} * \mathbb{P}_\alpha) \cap M$ then we may assume $a = (s, p, z, \mathcal{G})$

where $(s,p) \in \mathbb{C}_{\aleph_{\omega_1+1}} * \mathbb{P}_\alpha$, $z \in \omega^{<\omega}$ and $\mathcal{G} \in [\omega_1]^{<\omega}$. Since $a \in M$ then $(s,p) \in (\mathbb{C}_{\aleph_{\omega_1+1}} * \mathbb{P}_\alpha) \cap M$ and by our hypothesis, there is $(s', p') \in \left(\mathbb{C}_S * \overline{\mathbb{P}}_\alpha\right)$ equivalent to $(s,p)$ and then $b = (s', p', z, \mathcal{G})$ is equivalent to $a$. In case $\alpha \notin M$ then $1_{\alpha+1}$ and $2_{\alpha+1}$ are trivially true. Now assume $\alpha$ is limit, then $1_\alpha$ follows by lemma of 10 of [3] and $2_\alpha$ follows since we are taking direct limit.

In this way, we may conclude that $g$ is in some forcing extension of $V[C_S]$ so we may conclude that if $\alpha \notin S$ then $c_\alpha \not\leq^* g$ by the previous lemma. ■

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