The structure of loop quantization

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Quantization

Elements of quantization

- Family of classical theories
- Family of quantum theories
- Quantization map:

$$\mathrm{Obs}_{\mathrm{class}} \xrightarrow{Q} \mathrm{Obs}_{\mathrm{quant}}$$

Loop quantization

Elements of loop quantization

- Classical gauge theories

dof: Connection A, possibly a soldering 1-form e, and possibly other dof.
g+symm: Internal gauge invariance, possibly spatial diffeomorphism invariance, possibly spacetime diff. inv., possibly space (or spacetime) isometries
examples: Yang-Mills, BF, Chern-Simons, ..., GR

- Loop quantum gauge theories
- Quantization map:

$$\text{Obs}_{\text{class}} \supset \text{L-Obs}_{\text{class}} \xrightarrow{Q} \text{L-Obs}_{\text{quant}}$$

Kinematical canonical loop quantization

Kinematics of classical canonical gauge theory

- Phase sp. $T^\star \mathcal{A}_\pi$, where $\pi = (E, \pi, \Sigma)$ a G-bundle
- "Variables": $\text{Hol-Flux}_\Sigma \subset \text{Obs}_{\text{class}}$
  - PT_c = Parallel transport along the curve
  - $E_{S,f} = \int_S E \cdot f$
  - Algebraic structure
    $\{PT_{c1}, PT_{c2}\} = 0$, $\{PT_c, E_{S,f}\}$ = derivative operator on G,
    $\{E_{S1,f1}, E_{S2,f2}\}$ = commutator, etc
- Gauge transfs. and symmetries
  Internal g. transfs.: $g : \Sigma \to G$ induces $T_g : T^\star \mathcal{A}_\pi \to T^\star \mathcal{A}_\pi$
  $PT_c(A) \stackrel{g}{\longmapsto} g(t(c))^{-1} PT_c(A) g(s(c))$, $\qquad E_{S,f} \stackrel{g}{\longmapsto} E_{S,R_1(g)[f]}$

  Diffs. or isometries: $\phi : \Sigma \to \Sigma$ induces $T_\phi : T^\star \mathcal{A}_\pi \to T^\star \mathcal{A}_\pi$
  $PT_c \stackrel{\phi}{\longmapsto} PT_{\phi(c)}$, $\qquad E_{S,f} \stackrel{\phi}{\longmapsto} E_{\phi(S), f\circ\phi^{-1}}$

Kinematical canonical loop quantization

Kinematics of classical canonical gauge theory (cont.)

- Gauge invariant observables and reduced phase space
  $O : T^\star \mathcal{A}_\pi \to \mathbb{R}$ such that
  $O \circ T_g = O$ for all g, and (possibly) $O \circ T_\phi = O$ for all ϕ

$$\tilde{O}([(p, q)]) \doteq [O(p, q)]$$

where $[(p, q)] \in T^\star \mathcal{A}_\pi|_{\text{constr.}} / \sim_{g.symm}$

Kinematical canonical loop quantization

Kinematics of canonical loop quantum gauge theory: Connection representation

$$\begin{aligned} \psi_\gamma(A) &= f_\gamma(PT_{e1}(A), ..., PT_{en}(A)) \\ &= f_{\gamma'}(PT_{e'1}(A), ..., PT_{e'm}(A)) \end{aligned}$$

for any $\gamma' \geq gamma$ and some $f_{\gamma'}$.

$$\mathcal{H}_\Sigma = L^2(\mathcal{A}_\Sigma, d\mu_{\mathrm{AL}})$$

$$\begin{aligned} (\psi^1_{\gamma 1}, \psi^2_{\gamma 2}) &= \int_{\mathcal{A}_\Sigma} \bar{\psi^1_{\gamma 1}} \psi^2_{\gamma 2} d\mu_{\mathrm{AL}} \\ &= \prod_{e \in Edges(\gamma)} \int_{G_e} d\mu_{\mathrm{Haar},e} \quad \bar{f^1_\gamma} f^2_\gamma \end{aligned}$$

Kinematical canonical loop quantization

Kinematics of canonical loop quantum gauge theory: Extended spin network basis

$$\psi_\gamma(A) = \sum_j \psi_\gamma(j) j(A)$$

where j is a coloring of graphs assigning:
(i) irreps of G to edges, (ii) elements of appropriate basis to vertices

Example (coloring of Θ graph)
j=[e1:1, e2:2, e3:1; v1:(m(e1)=-1/2, m(e2)=1, m(e3)=1/2),
v2:(m(e1)=-1/2, m(e2)=0, m(e3)=-1/2)]

$$j(A) = R_1(PT_{e1}(A))^{-1/2}_{-1/2} R_2(PT_{e2}(A))^{1}_{0} R_1(PT_{e3}(A))^{1/2}_{-1/2}$$

Orthogonality: $(j,k) = \prod_e \delta(j(e), k(e)) \prod_{v,e} \delta(j(v,e)k(v,e))$

Kinematical canonical loop quantization

Kinematics of canonical loop quantum gauge theory

- Kinematical observables
  $\widehat{PT_c}\psi_\gamma = PT_c \cdot \psi_\gamma, \qquad \widehat{E_{S,f}}\psi_\gamma =$ Derivative operator(ψ_γ)
- Gauge transfs. and symmetries
  Internal g. transfs.: $g : \Sigma \to G$ induces $U_g : \mathcal{H}_\Sigma \to \mathcal{H}_\Sigma$
  $U_g j(A) = j(A) \cdot [\prod_{v(j)} R_{j(v,e1)}(g(v)^{sgn(v,e})...R_{j(v,en)}(g(v)^{sgn(v,e})]$

  Diffs. or isometries: $\phi : \Sigma \to \Sigma$ induces $U_\phi : \mathcal{H}_\Sigma \to \mathcal{H}_\Sigma$
  $U_\phi j = (\phi^{-1})^\star j$
- Gauge inv. observables
  $\widehat{O} : Cyl_\Sigma \to Cyl_\Sigma, \qquad \widehat{O}^\dagger : Cyl_\Sigma \to Cyl_\Sigma$,
  $U_g^{-1}\widehat{O}U_g = \widehat{O}$ for all g, and (possibly) $U_\phi^{-1}\widehat{O}U_\phi = \widehat{O}$ for all ϕ

$$\widehat{\tilde{O}}[j] = [\widehat{O}j] \quad , \qquad [j] \in \mathcal{H}_{\rm inv} = \overline{Cyl_\Sigma / \sim}$$

equiv. is wrt the orbit inner product $\eta(j,k) =$ " $(j, \int_{\mathfrak{G}} D\mathfrak{g} U_{\mathfrak{g}} k)$ "

Kinematical canonical loop quantization

Uniqueness theoremS

"Consider the algebra of kinematical observables of loop quantization, Hol-Flux$_\Sigma$. There is a unique representation of Hol-Flux$_\Sigma$ with a cyclic invariant (internal gauge inv. and diff inv.) state."

Kinematical canonical loop quantization

Quantization / regularization $O \stackrel{Q}{\longmapsto} \widehat{O}$

- We know Q for holonomies and fluxes
- We mentioned that they are "enough" kinematical observables. What does it mean?
  $O \mapsto \{O_\Delta(\{PT_c\}_\Delta, \{E_{S,f}\}_\Delta\}$ which converges as $\Delta \to \Sigma$
- Does $\widehat{O}_\Delta$ converge?
  (i) In general there is no convergence of any type
  (ii) Thiemann: For quantum gravity
  $\eta_{\mathrm{diff}}(j, \lim_{\Delta\to\Sigma} \widehat{C_{\mathrm{scalar}}(\underset{\sim}{N})}_\Delta k)$ exists for $[j] \in \mathcal{H}_{\mathrm{diff}}$

Loop quantization

How to look for a propagator / inner product defining $\mathcal{H}_{\rm phys}$

- Canonical: (i) regularize hamiltonian (convergence ???), or (ii) regularize constraints and look for their kernel (Thiemann conv. in QG may be OK)
- Covariant: path integral methods "spin foams" (convergence ??) Again in QG conv. may be OK due to spatial diff inv. This is how spin foam models are used y Reisenberger and Rovelli.
- However, **recall lattice field theory** where path int. and transfer matrix methods make sense in the continuum limit of **Wilsonian renormalization**.

"Loop quantization as a continumm limit?"

What do we know about covariant loop quantization? ask Robert Oeckl

Elements of covariant loop quantization

- Covariant classical gauge theories
  (i) general algebraic structure, (ii) loop quant. variables
- Covariant loop quantum gauge theories: spin foam models
  (i) They can be defined indep. of auxiliary discretizations (if some limits exist)
  (ii) What is their physical meaning?
- Quantization map:

$$\text{Obs}_{\text{class}} \supset \text{L-Obs}_{\text{class}} \xrightarrow{Q} \text{L-Obs}_{\text{quant}}$$

Which observables?