# Quantum gravity via spin foams in the general boundary formulation

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# Outline

# Limitations of the standard formulation

Usually, a quantum system is encoded through a Hilbert space $\mathcal{H}$ of states and an operator algebra $\mathcal{A}$ of observables.

Historically, this **standard formulation** of quantum theory was developed in analogy to non-relativistic classical mechanics. Resulting limitations preclude its application when spacetime is dynamical.

## Its operational meaning is tied to a background time:

States in $\mathcal{H}$ encode information on the system at fixed times and between measurements. A measurement is idealized as instantaneous. The temporal composition of measurements is encoded in the product of the corresponding observables.

## Spatial locality is not manifest:

States are extended over all of space. Locality only arises dynamically, depending on a background metric.

# Reactions

1. We keep a classical background in parts of spacetime, where the observers are located (usually at “infinity”). We can only describe quantum gravitational phenomena “far away” and approximately. [Perturbative Quantum Gravity, String Theory, AdS/CFT]
2. We keep the formalism, but throw away the background metric and with it (part of) the physical interpretation. We then have to construct a new physical interpretation of the formalism. If we are unlucky there may be none. [Quantum Geometrodynamics, LQG]
3. Quantum theory as we know it is really fundamentally limited and must be replaced by something new. Known physics is modified. [Causal sets, Gravity induced collapse models]

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3. Quantum theory as we know it is really fundamentally limited and must be replaced by something new. Known physics is modified. [Causal sets, Gravity induced collapse models]

OR

4. There is a more suitable formulation of quantum theory, free of these limitations. This is what we should use instead.

# GBF

The general boundary formulation (GBF) is based on:

- **Topological Quantum Field Theory** [Witten, Segal, Atiyah etc. 1980's]: generalizing notions of spacetime, state, transition amplitude, observable
- an extension of the **Born rule**, generalizing notions of transition probability and expectation value

In this talk we will not have time to consider observables and expectation values.

# GBF: Basic ingredients

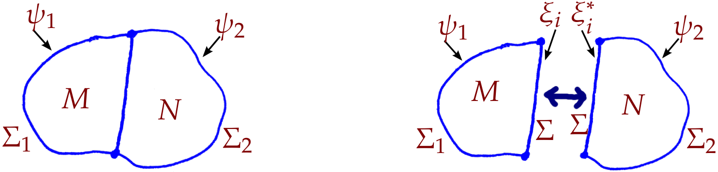
The elimination of an absolute notion of time comes at the cost of introducing a weak notion of spacetime. This consists in specifying a collection of **regions** (of dimension $d$) and **hypersurfaces** (of dimension $d-1$). This setting does not require a metric background. Regions and hypersurfaces merely need to carry a topological structure. However, depending on the model to be considered, they may be equipped with additional structure such as a metric.

- To each hypersurface $\Sigma$ associate a Hilbert space $\mathcal{H}_\Sigma$ of **generalized states**.
- To each region $M$ with boundary $\partial M$ associate a linear **amplitude map** $\rho_M : \mathcal{H}_{\partial M} \to \mathbb{C}$.

# GBF: Core axioms

The structures are subject to a number of **axioms**:

- Let $\overline{\Sigma}$ denote $\Sigma$ with opposite orientation. Then $\mathcal{H}_{\overline{\Sigma}} = \mathcal{H}_\Sigma^*$.
- **(Decomposition rule)** Let $\Sigma = \Sigma_1 \cup \Sigma_2$ be a disjoint union of hypersurfaces. Then $\mathcal{H}_\Sigma = \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\Sigma_2}$.
- **(Gluing rule)** If $M$ and $N$ are adjacent regions, then:

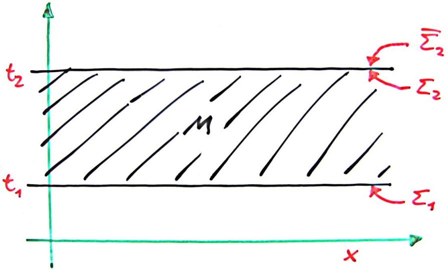



$$\rho_{M\cup N}(\psi_1 \otimes \psi_2) := \sum_{i\in\mathbb{N}} \rho_M(\psi_1 \otimes \xi_i)\rho_N(\xi_i^* \otimes \psi_2)$$

Here, $\psi_1 \in \mathcal{H}_{\Sigma_1}$, $\psi_2 \in \mathcal{H}_{\Sigma_2}$ and $\{\xi_i\}_{i\in\mathbb{N}}$ is an ON-basis of $\mathcal{H}_\Sigma$.

# GBF: Recovering transition amplitudes

Consider special regions in Minkowski space.




- region: $M = [t_1, t_2] \times \mathbb{R}^3$
- boundary: $\partial M = \Sigma_1 \cup \overline{\Sigma}_2$
- state space:
  $\mathcal{H}_{\partial M} = \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\overline{\Sigma}_2} = \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}^*_{\Sigma_2}$

Via time-translation symmetry identify $\mathcal{H}_{\Sigma_1} \cong \mathcal{H}_{\Sigma_2} \cong \mathcal{H}$. Then:

$$\boxed{\rho_{[t_1,t_2]}(\psi_1 \otimes \psi_2^*) = \langle \psi_2, U(t_1, t_2)\psi_1 \rangle}$$

# GBF: Probabilities

Consider a spacetime region $M$. The associated amplitude $\rho_M$ allows to extract probabilities for measurements in $M$.

Probabilities in quantum theory are generally conditional probabilities. They depend on two pieces of information. Here these are:

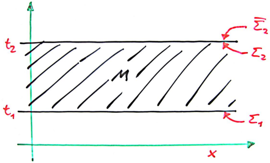
- $\mathcal{S} \subset \mathcal{H}_{\partial M}$ representing **preparation** or **knowledge**
- $\mathcal{A} \subset \mathcal{H}_{\partial M}$ representing **observation** or the **question**

The probability that the system is described by $\mathcal{A}$ given that it is described by $\mathcal{S}$ is:

$$P(\mathcal{A}|\mathcal{S}) = \frac{\sum_{i\in I} \left|\rho_M\left(P_{\mathcal{S}}\left(P_{\mathcal{A}}(\xi_i)\right)\right)\right|^2}{\sum_{i\in I} \left|\rho_M\left(P_{\mathcal{S}}(\xi_i)\right)\right|^2}$$

- $P_{\mathcal{S}}$ and $P_{\mathcal{A}}$ are the orthogonal projectors onto the subspaces, $\{\xi_i\}_{i\in I}$ an ON-basis of $\mathcal{H}_{\partial M}$.

# GBF: Recovering standard probabilities




To compute the probability of measuring $\psi_2$ at $t_2$ given that we prepared $\psi_1$ at $t_1$ we set

$$\mathcal{S} = \psi_1 \otimes \mathcal{H}^*, \quad \mathcal{A} = \mathcal{H} \otimes \psi_2^*.$$

The resulting expression yields correctly

$$P(\mathcal{A}|\mathcal{S}) = \frac{|\rho_{[t_1,t_2]}(\psi_1 \otimes \psi_2^*)|^2}{1} = |\langle \psi_2, U(t_1, t_2)\psi_1 \rangle|^2.$$

# The quantization problem

Quantum theories are often constructed by applying a **quantization scheme** to a classical theory. Standard quantization schemes are designed to output the ingredients of the standard formulation, i.e., a Hilbert space and operators on it. Instead we need quantization schemes that output the ingredients of the GBF: a Hilbert space per hypersurface, an amplitude map per region, and possibly observable maps (not considered in this talk).

There is one popular quantization scheme that is naturally adapted to the GBF: **Schrödinger-Feynman quantization**.

# Schrödinger-Feynman quantization I

Given a classical field theory we consider the following data:

- $K_\Sigma$ – Field configurations on the hypersurface $\Sigma$
- $K_M$ – Field configurations in the region $M$
- $S_M$ – Action evaluated in the region $M$

In the **Schrödinger representation** states are wave functions on field configurations. The state space $\mathcal{H}_\Sigma$ for the hypersurface $\Sigma$ is the space of complex functions on $K_\Sigma$ with inner product,

$$\langle\psi',\psi\rangle_\Sigma = \int_{K_\Sigma} \mathcal{D}\varphi\, \overline{\psi'(\varphi)}\psi(\varphi).$$

# Schrödinger-Feynman quantization II

The **Feynman path integral** serves to define the **field propagator** $Z_M : K_{\partial M} \to \mathbb{C}$ in a spacetime region $M$,

$$Z_M(\varphi) = \int_{\phi \in K_M, \phi|_{\partial M} = \varphi} \mathcal{D}\phi \, e^{\mathrm{i} S_M(\phi)}.$$

The **amplitude map** $\rho_M$ is then,

$$\rho_M(\psi) = \int_{K_{\partial M}} \mathcal{D}\varphi \, \psi(\varphi) Z_M(\varphi).$$

These data “automatically” satisfy the axioms of the GBF.

# Towards quantizing gravity

We start with the Palatini action of gravity,

$$S_M^{\text{Palatini}}(e, A) = \int_M \text{tr}(e \wedge e \wedge F).$$

- $A$ – connection with gauge group $\text{Spin}(1,3) = \text{SL}(2, \mathbb{C})$
- $F$ – curvature 2-form of the connection $A$
- $e$ – 4-bein frame field

To simplify this theory we replace $e \wedge e$ with the Lie algebra valued 2-form field $B$. This yields BF theory,

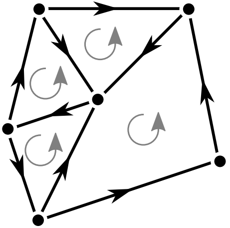
$$S_M^{\text{BF}}(B, A) = \int_M \text{tr}(B \wedge F).$$

This is not gravity, but becomes gravity if we add **constraints** that restrict the classical solutions to be those of gravity.

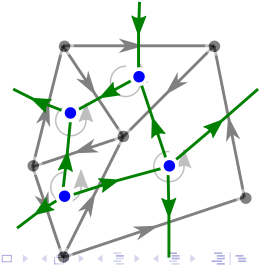
# Discretized connections I

BF theory is much simpler than gravity and can be quantized explicitly. It tuns out that the $B$-field can be integrated out so we only need to consider configurations of the connection field $A$.

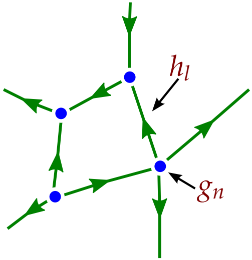
To make the “space of connections” on the hypersurface $\Sigma$ more manageable, we discretize $\Sigma$ via a **cellular decomposition**.



Given a “gauge” (local trivialization), connections give rise to **holonomies** along paths. We choose paths dual to the cellular decomposition. We call them **links** (green lines). Their end points are **nodes** (blue dots).

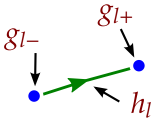


# Discretized connections II




The holonomies associate one element $h_l$ of the structure group $G$ to each link $l$. We denote this space by $K^1_\Sigma = G^L$, where $L$ is the number of links in $\Sigma$.

A gauge transformation consists of the assignment of one element $g_n$ of $G$ to each node $n$. The gauge group is thus $K^0_\Sigma = G^N$, where $N$ is the number of nodes.




A gauge transformation $g \in K^0_\Sigma$ acts on $h \in K^1_\Sigma$ via $(g \triangleright h)_l := g_{l+} h_l g^{-1}_{l-}$. The **configuration space** is the quotient $K_\Sigma := K^1_\Sigma / K^0_\Sigma$.

# State space

Supposing that $G$ is compact for simplicity, there is a unique normalized biinvariant measure on $G$, the **Haar measure** $\mu$. This allows to define a Hilbert space $\mathrm{L}^2(G)$ of complex functions on $G$ with the inner product,

$$\langle \psi, \eta \rangle = \int_G \overline{\psi(g)} \eta(g) \, \mathrm{d}\mu(g).$$

By putting the same inner product on each copy of $G$, we obtain a Hilbert space $\mathcal{H}^1_\Sigma := \mathrm{L}^2(K^1_\Sigma)$. The action of the gauge group $K^0_\Sigma$ on $K^1_\Sigma$ induces an action on $\mathcal{H}^1_\Sigma$. The invariant subspace $\mathcal{H}_\Sigma$ is a space of functions on $K_\Sigma$. This Hilbert space is our **state space**.

# Propagator

Recall that in SF quantization amplitudes are determined by propagators.

$$\rho_M(\psi) = \int_{K^1_\Sigma} \psi(h)\, Z_M(h^{-1})\, \mathrm{d}\mu(h)$$

Here, it is simpler to think of the **propagator** as a function $Z_M : K^1_{\partial M} \to \mathbb{C}$ rather than a function $K_{\partial M} \to \mathbb{C}$.

For BF theory the propagator turns out to be,

$$\tilde{Z}^{\mathrm{BF}}_M(h) = \prod_{l \in \partial M} \delta(h_l).$$

In gauge invariant form this is,

$$Z^{\mathrm{BF}}_M(h) = \int_{K^0_{\partial M}} \prod_{l \in \partial M} \delta(g_{l-} h_l g_{l+}^{-1})\, \mathrm{d}\mu(g).$$

# Other models

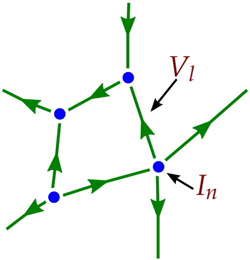
If we want to get closer to gravity and implement constraints it is useful to discretize also the interior of $M$ via a cellular decomposition. We may then think of each cell in the interior as an "elementary" spacetime region, all glued together according to the gluing axioms of the GBF. That is, to specify a model we only need to specify the **cell propagator** for one single cell.

A famous model for implementing the constraints is the **Barrett-Crane model**. In this model $G = \mathrm{SU}(2) \times \mathrm{SU}(2)$ and we write $g = (g^{\mathrm{L}}, g^{\mathrm{R}})$. The cell propagator for (a version of) this model is,

$$Z_C^{\mathrm{BC}}(h) = \int_{K^0_{\partial C}} \prod_{l \in \partial C} \left( \int_{\mathrm{SU}(2)\times\mathrm{SU}(2)} \delta(g^{\mathrm{L}}_{l-} k h^{\mathrm{L}}_l k' (g^{\mathrm{L}}_{l+})^{-1}) \delta(g^{\mathrm{R}}_{l-} k h^{\mathrm{R}}_l k' (g^{\mathrm{R}}_{l+})^{-1}) \, \mathrm{d}\mu(k) \mathrm{d}\mu(k') \right) \mathrm{d}\mu(g).$$

# The dual picture: spin networks

Elements of the Hilbert space $\mathcal{H}_\Sigma$ on the discretized hypersurface $\Sigma$ can be constructed explicitly in terms of **spin networks**.




- Associate to each link $l$ a finite-dimensional irreducible representation $V_l$ of $G$.
- Associate to each node $n$ an intertwiner $I_n \in \mathrm{Inv}\left(\bigotimes_{l\in\partial n} V_l^{\pm}\right)$ between the representations of the adjacent nodes.

Spin networks yield a complete description of $\mathcal{H}_\Sigma$:

$$\mathcal{H}_\Sigma = \bigoplus_{V_l} \bigotimes_{n\in\Sigma} \mathrm{Inv}\left(\bigotimes_{l\in\partial n} V_l^{\pm}\right).$$

# The dual picture: spin foams

In order to obtain the amplitude for a region $M$ composed of many elementary regions (cells) we need to sum over a complete ON-basis for each hypersurface where cells are glued together. (Recall GBF gluing rule.) Taking basis consisting of spin networks, each summand will by labeled by an assignment of a spin network to each of these interior hypersurfaces. We can think of those spin networks as extended through all the interior of $M$. Links then become surfaces and nodes become lines where the surfaces meet. Surfaces are labeled by irreducible representations and lines by intertwiners. This picture is what is usually called a **spin foam**. The vertices where the lines meet are dual to the cells. Thus, the cell amplitudes $\rho_C$, evaluated on spin networks, are usually called **vertex amplitudes**.

# Summary

- The standard formulation of quantum theory is not suitable for theories with dynamical spacetime structure. The GBF is.
- The GBF has lead to a new perspective on and new insights in QFT in curved spacetime. (see talks of Daniele Colosi and Max Dohse)
- There is a quantization scheme naturally adapted to the GBF: Schrödinger-Feynman quantization.
- Loop quantization (see talk of José Antonio Zapata) and spin foam models have a natural home in the GBF.
- The GBF yields a solid framework for furnishing a physical interpretation to the spin foam approach to quantum gravity.
- Various aspects of the spin foam approach are clarified in the GBF (not this talk): "face factors", renormalization . . .

## GBF motivation:

R. O., *Reverse Engineering Quantum Field Theory*, arXiv:1210.0944.

## GBF foundations:

R. O., *General boundary quantum field theory: Foundations and probability interpretation*, Adv. Theor. Math. Phys. **12** (2008) 319-352, hep-th/0509122.

## GBF and spin foam models:

R. O., *The general boundary approach to quantum gravity*, Proceedings of the First International Conference on Physics, Amirkabir University, Tehran, 2004, pp. 257-265, gr-qc/0312081.

## Spin foam models:

A. Perez, *The Spin Foam Approach to Quantum Gravity*, to appear in Living Reviews in Relativity, arXiv:1205.2019.