The Effective Equations for Bianchi IX Loop Quantum Cosmology

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Introduction

- Bianchi models are spatially homogeneous models such that the symmetry group $\mathcal{S}$ acts simply and transitively on a space manifold $\Sigma \cong \mathcal{S}$. The symmetry group for Bianchi IX model is generated by three spatial rotations on a 3-sphere. We identify this group with SU(2).
- The fiducial cell is a 3-sphere with radius a_o (=2) and its volume is $V_o = 2\pi^2 a_o^3$. We define $\ell_o = V_o^{1/3}$ and $\vartheta = \ell_o / a_o = (2\pi^2)^{1/3}$.
- The fiducial frame and co-frame satisfy

$$[{}^o e_i, {}^o e_j] = \frac{2}{a_o} {}^o\epsilon_{ij}{}^{k}\ {}^o e_k \ , \ \mathrm{d}{}^o\omega^k = -\frac{1}{a_o} {}^o\epsilon^k{}_{ij} {}^o\omega^i \wedge {}^o\omega^j \tag{1}$$

- The physical metric is given by

$$q_{ab} = a_i^2 {}^o\omega_a^i {}^o\omega_{bi} \tag{2}$$

- For diagonal Bianchi IX model

$$A_a^i = c^{i\,o}\omega_a^i/\ell_o \text{ and } E_i^a = p_i\sqrt{{}^o q}\;{}^o e_i^a/\ell_o^2$$

where p_i's in terms of scale factors a_i are $|p_i| = \ell_o^2 a_j a_k$, and the volume is $V = \sqrt{|p_1 p_2 p_3|}$. The nonzero Poisson brackets are given by $\{c_i, p_j\} = 8\pi G\gamma\delta_{ij}$.

- The classical hamiltonian constraint in terms of the phase space variables used in loop quantum gravity, a connection A_a^i and a densitized triad E_i^a, is

$$\mathcal{C}_H = \int_{\mathcal{V}} N\left[-\frac{\epsilon^{ij}{}_k E_i^a E_j^b}{16\pi G\gamma^2\sqrt{|q|}}\left(F_{ab}^k - (1+\gamma^2)\Omega_{ab}^k\right) + \mathcal{H}_{\mathrm{matter}}\right]\mathrm{d}^3x \tag{3}$$

Therefore the classical Hamiltonian constraint for Bianchi IX is

$$\begin{aligned}\mathcal{C}_H = &- \frac{1}{8\pi G\gamma^2\lambda^2}\left(\sqrt{\frac{p_1p_2}{p_3}}c_1c_2 + \sqrt{\frac{p_1p_3}{p_2}}c_1c_3 + \sqrt{\frac{p_2p_3}{p_1}}c_2c_3\right.\\ &+ 2\vartheta\left[\sqrt{\frac{p_1p_2}{p_3}}c_3 + \sqrt{\frac{p_2p_3}{p_1}}c_1 + \sqrt{\frac{p_1p_3}{p_2}}c_2\right]\\ &- \vartheta^2(1+\gamma^2)\left[2\frac{p_1^{3/2}}{\sqrt{p_2p_3}} + 2\frac{p_2^{3/2}}{\sqrt{p_1p_3}} + 2\frac{p_3^{3/2}}{\sqrt{p_1p_2}} - \frac{(p_1p_2)^{3/2}}{p_3^{5/2}}\right.\\ &\left.\left.- \frac{(p_1p_3)^{3/2}}{p_2^{5/2}} - \frac{(p_2p_3)^{3/2}}{p_1^{5/2}}\right]\right) + \rho V \approx 0\end{aligned} \tag{4}$$

Quantization

- To construct the quantum kinematics, we have to select a set of elementary observables such that their associated operators are unambiguous.
- In loop quantum gravity they are the holonomies h_e defined by the connection A_a^i along edges e and the fluxes of the densitized triad E_i^a across surfaces.
- For this model we choose holonomies and p_i.
- The first term in the classical Hamiltonian constraint, $\epsilon^{ij}_{\ \ k} E_i^a E_j^b / \sqrt{|q|}$, as in loop quantum gravity, can be treated by using Thiemann's strategy.

$$\epsilon_{ijk} \frac{E^{ai} E^{bj}}{\sqrt{|q|}} = \sum_i \frac{1}{2\pi\gamma G\mu} \ {}^{o}\epsilon^{abc} \ {}^{o}\omega_c^i \mathrm{Tr}(h_i^{(\mu)} \{h_i^{(\mu)-1}, V\} \tau_k) \tag{5}$$

To find an operator related to the curvature F_{ab}^k, for isotropic models and Bianchi I, one can consider a square $\Box_{ij}$ in the $i-j$ plane which is spanned by two of the fiducial triads (for the closed isotropic model since triads do not commute, to define this plane we use a triad and a right invariant vector ${}^o\xi_i^a$), with each of its sides having length μ_i'. Therefore, F_{ab}^k is given by

$$F_{ab}^k = 2 \lim_{Area_\Box \to 0} {}^o\epsilon_{ij}{}^k \mathrm{Tr}\left(\frac{h_{\Box_{ij}}^{\mu'} - \mathbb{I}}{\mu_i'\mu_j'}\tau^k\right) {}^o\omega_a^i \, {}^o\omega_b^j \,. \tag{6}$$

Since in loop quantum gravity, the area operator does not have a zero eigenvalue, one can take the limit of above equation to the point where the area is equal to the smallest eigenvalue of the area operator, $\lambda^2 = 4\sqrt{3}\pi\gamma l_p^2$, instead of zero. Then, $\mu_i' a_i = \lambda$. We take $\mu_i' = \bar{\mu}_i \ell_o$ where $\bar{\mu}_i$ is a dimensionless parameter and, by previous considerations, is equal to

$$\bar{\mu}_i = \lambda\sqrt{|p_i|}/\sqrt{|p_j p_k|}\ , \ (i \neq j \neq k)$$

.

For Bianchi IX, we cannot use this method because the resulting operator is not almost periodic, therefore we express the connection A_a^i in terms of holonomies and then use the standard definition of curvature F_{ab}^k.

$$A_a^i = \lim_{\ell_i \to 0} \frac{1}{2\ell_i}(h^{(\ell_i)} - h^{(\ell_i)-1})$$

To be consistent with other models, we choose

$$\ell_i = 2\mu_i'$$

The operators corresponding to the connection are given by

$$\hat{c}_i = \frac{\widehat{\sin \bar{\mu}_i c_i}}{\bar{\mu}_i} \,. \tag{7}$$

The terms related to the curvatures, F^k_{ab} and Ω^k_{ab}, contain some negative powers of p_i which are not well defined operators. To solve this problem we use the same idea as Thiemann's strategy.

$$|p_i|^{(\ell-1)/2} = -\frac{\sqrt{|p_i|}\ell_o}{16\pi G\gamma\tilde{\mu}_i\ell}\mathrm{Tr}(\tau_i h_i^{(\tilde{\mu}_i)}\{h_i^{(\tilde{\mu}_i)-1}, |p_i|^{\ell/2}\}) \,, \tag{8}$$

- For these three different operators we have three different curve lengths $(\mu, \mu', \tilde{\mu})$ where μ and $\tilde{\mu}$ can be some arbitrary functions of p_i. For simplicity we choose all of them to be equal to μ'.

With this choice, the eigenvalues for the operator $\widehat{|p_i|^{-1/4}}$ are given by

$$J_i(p_1, p_2, p_3) = \frac{h(V)}{V_c}\prod_{j\neq i} p_j^{1/4} \,, \tag{9}$$

with

$$h(V) = \sqrt{V + V_c} - \sqrt{|V - V_c|}, \quad \text{and} \quad V_c = 2\pi\gamma\lambda\ell_p^2. \tag{10}$$

- By using these results and choosing some factor ordering, we can construct the total constraint operator.
- By solving the constraint equation $\hat{\mathcal{C}}_H \cdot \Psi = 0$, we can obtain the physical states and the physical Hilbert space $\mathcal{H}_{\text{phys}}$.
- Working with full quantum theories of the models is difficul. It is shown that for isotropic models, the behavior of the effective or semiclassical equations, which are 'classical' equations with some quantum corrections, are good approximations to the numerical quantum evolutions even near the Planck scale.

Effective Equation

To obtain the effective equations, one way is calculating the expectation value of the Hamiltonian operator respect to a semiclassical state and keeping leading terms.

$$\begin{aligned}\mathcal{H}_{eff} =& -\frac{V}{8\pi G\gamma^2\lambda^2}(\sin\bar{\mu}_1 c_1 \sin\bar{\mu}_2 c_2 + \sin\bar{\mu}_1 c_1 \sin\bar{\mu}_3 c_3 + \sin\bar{\mu}_2 c_2 \sin\bar{\mu}_3 c_3) \\ &+ \frac{\vartheta}{8\pi G\gamma^2\lambda}\left(\frac{p_1p_2}{p_3}\sin\bar{\mu}_3 c_3 + \frac{p_2p_3}{p_1}\sin\bar{\mu}_1 c_1 + \frac{p_1p_3}{p_2}\sin\bar{\mu}_2 c_2\right) \\ &- \frac{\vartheta^2(1+\gamma^2)}{32\pi G\gamma^2}\left(2\frac{p_1^{3/2}}{\sqrt{p_2p_3}} + 2\frac{p_2^{3/2}}{\sqrt{p_1p_3}} + 2\frac{p_3^{3/2}}{\sqrt{p_1p_2}} - \frac{(p_1p_2)^{3/2}}{p_3^{5/2}} \right. \\ &\left. - \frac{(p_1p_3)^{3/2}}{p_2^{5/2}} - \frac{(p_2p_3)^{3/2}}{p_1^{5/2}}\right) + \rho V \approx 0\end{aligned} \tag{11}$$

To gain qualitative insights into the quantum effects for small volumes we add some corrections which come from the inverse triad terms.

$$\mathcal{H}_{\text{eff}} = -\frac{V^4 A(V) h^6(V)}{8\pi G V_c^6 \gamma^2 \lambda^2}\Big(\sin\bar{\mu}_1 c_1 \sin\bar{\mu}_2 c_2 + \sin\bar{\mu}_1 c_1 \sin\bar{\mu}_3 c_3$$
$$+ \sin\bar{\mu}_2 c_2 \sin\bar{\mu}_3 c_3\Big) + \frac{\vartheta A(V) h^4(V)}{4\pi G V_c^4 \gamma^2 \lambda}\Big(p_1^2 p_2^2 \sin\bar{\mu}_3 c_3 + p_2^2 p_3^2 \sin\bar{\mu}_1 c_1$$
$$+ p_1^2 p_3^2 \sin\bar{\mu}_2 c_2\Big) - \frac{\vartheta^2 (1+\gamma^2) A(V) h^4(V)}{8\pi G V_c^4 \gamma^2}\Big(2V\Big[p_1^2 + p_2^2 + p_3^2\Big]$$
$$- \Big[(p_1 p_2)^4 + (p_1 p_3)^4 + (p_2 p_3)^4\Big]\frac{h^6(V)}{V_c^6}\Big) + \rho V \approx 0$$

where

$$A(V) = \frac{1}{2V_c}(V + V_c - |V - V_c|) = \begin{cases} V/V_c & V < V_c \\ 1 & V \geq V_c \end{cases}$$

is a correction term which comes from $\epsilon_k^{ij} E_i^a E_j^b / \sqrt{|q|}$.

Thank You!