Lagrangian approach to the physical degree of freedom count

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Constrained Systems

Gauge Theory $\Longrightarrow$ Constrained Hamiltonian System.

Aside

A constraint is any relation between the coordinates, namely:

- **Hamiltonian:** $\phi(q,p) = 0$.
- **Lagrangian:** $\psi(q,\dot{q}) = 0$.

Procedures:

1. Hamiltonian: Dirac's Method. It is well studied but spoil some features of the theory(Ex. Explicit Covariance.)
2. Lagrangian: It is not widely developed. It can be achieved in a covariant way.

LAGRANGIAN APPROACH(IRREDUCIBLE THEORY)

The Lagrangian Action:

$$S_L[q] = \int_{t_0}^{t_1} L(q, \dot{q}) dt$$

E-L Equations of Motion:

$$\begin{aligned} E_i^{(0)} &\doteq \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} \qquad\qquad i = 1, \ldots, N \quad (1) \\ &\equiv W_{ij}^{(0)}(q, \dot{q})\ddot{q}^j + K_i^{(0)}(q, \dot{q}) \, [= 0 \; on - shell], \end{aligned}$$

where

$$W_{ij}^{(0)}(q, \dot{q}) \doteq \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \quad and \quad K_i^{(0)}(q, \dot{q}) \doteq \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j}\dot{q}^j - \frac{\partial L}{\partial q^i}, \qquad (2)$$

CONSISTENCY

Null vectors of $W^{(0)} \Longrightarrow$ Lagrangian constraints $\psi(q,\dot{q})$.
By means of a consistency methodology (similar to Dirac)

$$\vec{E}^{(1)} \doteq \begin{pmatrix} \vec{E}^{(0)} \\ \frac{d}{dt}\psi^{(0)}_{\bar{a}_1}(q,\dot{q}) \end{pmatrix} \tag{3}$$

it is possible to find relations in the form

$$G^{(k)}_{\hat{a}_{k+1}} = \sum_{s=0}^{k} \frac{d^s}{dt^s}\left[{M^{(k)}}^{i}_{s}(q,\dot{q})E^{(0)}_i\right] = 0(\mathit{off}-\mathit{shell}). \tag{4}$$

Multiplying by an arbitrary function $\epsilon^{(k)}$ of time and using the product law for derivatives the necessary times:

$$\epsilon^{(k)} \bullet G^{(k)}_{\hat{a}_{k+1}} \equiv \sum_{s=0}^{k} \left[(-1)^s \frac{d^s \epsilon^{(k)}}{dt^s} M^{(k)}{}^{i}_{s} \right] E^{(0)}_i + \frac{d}{dt} U = 0, \qquad (5)$$

where U is in general a function of $\epsilon^{(k)}$, the coordinates and time derivatives thereof.
Similar to *Noether's identity*

$$\boxed{E^{(0)}_j \underbrace{\left(\delta q^j - \dot{q}^j \delta t\right)}_{\tilde{\delta} q^j} + \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}^j}\delta q^j + \left(L - \frac{\partial L}{\partial \dot{q}^j}\dot{q}^j\right)\delta t - F\right] = 0}\,. \qquad (6)$$

If the procedure shows T relation in the form (5) it has been proved by *Shirzad* that the next transformation leaves the action invariant is:

$$\boxed{\delta q^i = \sum_{\alpha=1}^{T} \delta q^{i(k_\alpha)} = \sum_{\alpha=1}^{T} \sum_{s=0}^{k_\alpha} (-1)^s \frac{d^s \epsilon^{(k_\alpha)}}{dt^s} M^{(k_\alpha)}{}^i_s}$$

where the ϵ_a are arbitrary functions of time.
i.e.,*Gauge Transformations.*

DEGREE OF FREEDOM COUNT

Hamiltonian analysis

- Number of independent primary Hamiltonian constraints: $rank(W^{(0)})$.
- Evolution of constraints: $\frac{d}{dt}\phi(q,p) = 0 \Longrightarrow$ New constraints.
- Total number of constraints: $N_1 + N_2$.

Lagrangian analysis

- Number of independent Lagrangian constraints plus gauge identities: $rank(W^{(0)})$.
- Evolutions of constraints: $\frac{d}{dt}\psi(q,\dot{q}) = 0 \Longrightarrow$ New constraints plus gauge identities.
- Total number of independent Lagrangian constraints plus gauge identities: $l + g$

It make sense that

$$N_1 + N_2 = l + g \tag{7}$$

Let us label the number of effective parameters e. It was proved by *Henneaux et al* that $N_1 = e$. Coming back to (7): $N_2 = l + g - N_1 \equiv l + g - e$.

Theorem

In the irreducible case, the number of physical degrees of freedom (P.D.F.) is

$$P.D.F. = N - N_1 - \frac{N_2}{2} \equiv N - \frac{1}{2}(l + g + e) \tag{8}$$

where l is the number of Lagrangian Constraints, g is the number of independent Gauge Identities and e is the number of "effective" gauge parameters.

ALGORITHM

1. Find the Lagrangian constraints $\psi(q, \dot{q})$.
2. Find the relations G.
3. Achieve the contraction $\epsilon \bullet G$ and read the Lagrangian gauge transformation law δq.
4. Read the Lagrangian parameters l, g and e.
5. $P.D.F. = N - \frac{1}{2}(l + g - e)$.

EXAMPLE 1

The Lagrangian

$$L(q,\dot{q}) = \dot{q}_1 q_2 - \dot{q}_2 q_1 - (q_1 - q_2) q_3.$$

Because (2)

$$W^{(0)} = 0, \qquad K^{(0)} = (2\dot{q}_2 + q_3, -2\dot{q}_1 - q_3, q_1 - q_2).$$

So $left - null(W^{(0)}) = gen\{(1,0,0),(0,1,0),(0,0,1)\}$ and contracting with $\vec{E}^{(0)}$ we get the Lagrangian constraints:

$$\begin{aligned}
\psi_1^{(0)} &\doteq E_1^{(0)} = 2\dot{q}_2 + q_3 (= 0 \text{ ``}on - shell\text{''}),\\
\psi_2^{(0)} &\doteq E_2^{(0)} = -2\dot{q}_1 - q_3 (= 0 \text{ ``}on - shell\text{''}),\\
\psi_3^{(0)} &\doteq E_3^{(0)} = q_1 - q_2 (= 0 \text{ ``}on - shell\text{''}).
\end{aligned}$$

Constraints functionally independent $\Longrightarrow$ No "Gauge Identities" at this step.

Step 2.

Conservation in time of the constraints

$$\vec{E}^{(1)} = \begin{pmatrix} \vec{E}^{(0)} \\ \frac{d}{dt}\vec{\psi}^{(0)} \end{pmatrix} = \begin{pmatrix} 2\dot{q}_2 + q_3 \\ -2\dot{q}_1 - q_3 \\ q_1 - q_2 \\ 2\ddot{q}_2 + \dot{q}_3 \\ -2\ddot{q}_1 - \dot{q}_3 \\ \dot{q}_1 - \dot{q}_2 \end{pmatrix} = W^{(1)} \bullet \ddot{\vec{q}} + \vec{K}^{(1)},$$

where

$$W^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad K^{(1)} = \begin{pmatrix} 2\dot{q}_2 + q_3 \\ -2\dot{q}_1 - q_3 \\ q_1 - q_2 \\ \dot{q}_3 \\ -\dot{q}_3 \\ \dot{q}_1 - \dot{q}_2 \end{pmatrix}$$

and so

$$left - null(W^{(1)}) = gen\{(1,0,0,0,0,0),(0,1,0,0,0,0),\\ (0,0,1,0,0,0),(0,0,0,0,0,1)\}.$$

Contracting the last one with $\vec{E}^{(1)}$:

$$E_6^{(1)} = \frac{d}{dt}\psi_3^{(0)} = \dot{q}_1 - \dot{q}_2 \\ \equiv -\frac{1}{2}\left(\psi_1^{(0)} + \psi_2^{(0)}\right).$$

$$\Rightarrow G_1^{(1)} = \tfrac{d}{dt}E_3^{(0)} + \tfrac{1}{2}E_1^{(0)} + \tfrac{1}{2}E_2^{(0)} = 0.$$

Contracting with $\epsilon(t)$:

$$\epsilon(t)G_1^{(1)} = \epsilon(t)\frac{d}{dt}E_3^{(0)} + \epsilon(t)\frac{1}{2}E_1^{(0)} + \epsilon(t)\frac{1}{2}E_2^{(0)} = 0$$

$$\Longrightarrow \underbrace{\frac{\epsilon}{2}}_{\delta q_1} E_1^{(0)} + \underbrace{\frac{\epsilon}{2}}_{\delta q_2} E_2^{(0)} - \underbrace{\dot{\epsilon}}_{\delta q_3} E_3^{(0)} + \frac{d}{dt}\left(\epsilon E_3^{(0)}\right) = 0. \tag{9}$$

i.e.,

$$\delta q_1 = \frac{\epsilon}{2}, \qquad \delta q_2 = \frac{\epsilon}{2}, \qquad \delta q_3 = -\dot{\epsilon}.$$

Finally $l = 3$, $g = 1$, $e = 2$ and using (7):

$$P.D.F. = 3 - \frac{1}{2}(3 + 1 + 2) = 0$$

By the way, $N_1 = e = 2$ and $N_2 = l + g - e = 2$.

Hamiltonian analysis

- Hamiltonian constraints:

$$\phi_1^{(1)} = p_1 - q_2 \approx 0; \qquad \phi_2^{(1)} = p_2 + q_1 \approx 0;$$
$$\phi_3^{(1)} = p_3 \approx 0; \qquad \phi_1^{(2)} = q_2 - q_1 \approx 0.$$

- Classification:

$$\begin{aligned} First: \ & \Omega_1^{(1)} = \phi_3^{(1)} = p_3; \ \Omega_2^{(1)} = \phi_1^{(1)} + \phi_2^{(1)} + 2\phi_1^{(2)} \Rightarrow N_1 = 2 \\ Second: \ & \Omega_1^{(2)} = \phi_1^{(1)}; \qquad \Omega_2^{(2)} = \phi_2^{(1)} \Rightarrow N_2 = 2. \end{aligned}$$

Then $P.D.F. = N - N_1 - \frac{N_2}{2} = 0$ and the gauge transformation laws

$$\delta_T q_1 = \delta_T q_2 = \frac{\epsilon}{2}; \ \delta_T q_3 = -\dot{\epsilon}$$

Reducible theory, a field case

Recalling from the above analysis:

$$\lambda^i_{a_1}(x)E^{(0)}_i(\vec{x},t) = \lambda^i_{a_1}(x)K^{(0)}_i(\vec{x},t) \Longrightarrow \psi_{a_1} \tag{10}$$

The *free massless antisymmetric tensor gauge fields of second rank* are described by the Lagrangian action

$$\mathcal{L} = -\frac{1}{12}F^{\alpha\mu\nu}F_{\alpha\mu\nu},$$

where the totally antisymmetric field tensor $F_{\mu\nu\alpha}$ is defined in terms of the antisymmetric tensor potential $A_{\mu\nu}(x) = -A_{\nu\mu}$ as

$$F_{\mu\nu\alpha}(x) = \partial_\mu A_{\nu\alpha}(x) + \partial_\nu A_{\alpha\mu}(x) + \partial_\alpha A_{\mu\nu}(x).$$

Separating the spatial and temporal parts and taking into account the antisymmetry of the fields, the Lagrangian density is reduced to

$$\mathcal{L} = -\frac{1}{4}\dot{A}^{ij}\dot{A}_{ij} - (\partial_i A_{j0})\dot{A}^{ij} - \frac{1}{2}(\partial_i A_{j0})(\partial^i A^{j0}) + \frac{1}{2}(\partial_i A_{j0})(\partial^j A^{i0})$$
$$- \frac{1}{4}(\partial_i A_{jk})(\partial^i A^{jk}) + \frac{1}{2}(\partial_i A_{jk})(\partial^j A^{ik}).$$

i.e., The fundamental fields are $\{A_{ij}, A_{j0}\}$.
The Euler equation

$$E^{(0)pq} = -\frac{1}{2}\ddot{A}^{pq} + \frac{1}{2}(\partial^p \dot{A}^{0q} - \partial^q \dot{A}^{0p}) - \frac{1}{2}\partial_i F^{ipq}$$

and

$$E^{(0)0m} = \frac{1}{2}\partial_i F^{0im}$$

The constraints at level zero

$$\psi^{(0)0m} \doteq E^{(0)0m} = \frac{1}{2}\partial_i F^{0im}. \quad (11)$$

However,

$$\boxed{\partial_m \psi^{(0)0m} = 0} \; (\textit{Reducibility Condition}) \quad (12)$$

Next, we build the Euler derivative at the next order

$$\vec{E}^{(1)} = \begin{pmatrix} \vec{E}^{(0)} \\ \frac{\partial}{\partial t}\psi^{(0)0m} \end{pmatrix},$$

where

$$\frac{\partial}{\partial t}\psi^{(0)0m} = \frac{1}{2}\partial_i \left[\ddot{A}^{im} + \partial^i \dot{A}^{m0} + \partial^m \dot{A}^{0i}\right].$$

In vectorial form

$$\vec{E}^{(1)} = \begin{pmatrix} E^{(0)pq} = -\frac{1}{2}\ddot{A}^{pq} + \frac{1}{2}(\partial^p \dot{A}^{0q} - \partial^q \dot{A}^{0p}) - \frac{1}{2}\partial_i F^{ipq} \\ E^{(0)0m} = \frac{1}{2}\partial_i F^{0im} \\ \frac{\partial}{\partial t}\psi^{(0)0m} = \frac{1}{2}\partial_i \left[\ddot{A}^{im} + \partial^i \dot{A}^{m0} + \partial^m \dot{A}^{0i}\right] \end{pmatrix},$$

Thus,

$$\begin{aligned} \partial_i E^{(0)im} + \frac{\partial}{\partial t}\psi^{(0)0m} =& \partial_i \left[-\frac{1}{2}\ddot{A}^{im} + \frac{1}{2}(\partial^i \dot{A}^{0m} - \partial^m \dot{A}^{0i}) - \frac{1}{2}\partial_k F^{kim}\right] \\ &+ \frac{1}{2}\partial_i \left[\ddot{A}^{im} + \partial^i \dot{A}^{m0} + \partial^m \dot{A}^{0i}\right] = 0. \end{aligned}$$

Recalling that $\psi^{(0)0m} = E^{(0)0m}$:

$$G^m \doteq \partial_i E^{(0)im} + \frac{\partial}{\partial t} E^{(0)0m} = 0 \tag{13}$$

However,

$$\boxed{\partial_m G^m = 0} \; (\textit{Reducibility Condition}) \tag{14}$$

Multiplying (13) by the arbitrary parameters $\epsilon_m(\vec{x}, t)$ and using the product rule:

$$\epsilon_m G^m \equiv \underbrace{\dot{\epsilon}_m}_{\delta A_{0m}} E^{(0)0m} + \underbrace{\frac{1}{2}(\partial_i \epsilon_m - \partial_m \epsilon_i)}_{\delta A_{im}} E^{(0)im} - \frac{\partial}{\partial t}[\epsilon_m E^{(0)0m}] - \partial_i[\epsilon_m E^{(0)im}]$$

$$= 0.$$

i.e., The *gauge transformation laws* are

$$\boxed{\delta A_{0m} = \dot{\epsilon}_m} \, , \qquad \boxed{\delta A_{im} = \frac{1}{2}(\partial_i \epsilon_m - \partial_m \epsilon_i)} \, . \tag{15}$$

DEGREE OF FREEDOM COUNT

1. *Original Fields*: $A_{\mu\nu}$ is antisymmetric $\Longrightarrow$ N=6,
2. *Independent Lagrangian constraints*: $\left[\psi^{(0)}_{0m}\right] - [R.C.\,(12)]$ $\Longrightarrow$ l=3-1=2,
3. *Independent gauge identities*: $[G^m] - [R.C.\,(14)]$ $\Longrightarrow$ g=3-1=2,
4. *"Effective" parameters*: $[\epsilon_m] + [\dot{\epsilon}_m] \Longrightarrow$ e=3+3=6.

With the above information equation (8) gives us

$$P.D.F. = 6 - \frac{1}{2}(2 + 2 + 6) = 1 \tag{16}$$

DEGREE OF FREEDOM COUNT, REDUCIBLE CASE

This pointed out the following:

Theorem

The formula that gives the right count of the physical degrees of freedom in field theory in the reducible case is given by

$$\boxed{P.D.F. = N - \frac{1}{2}\left(\mathbf{l_I} + \mathbf{g_I} + e\right)}$$

where the sub-index I means independent.

BF THEORY PLUS COSMOLOGICAL CONSTANT

The Action

$$S[B,A]=\int_{\mathcal{M}} B^{IJ}\wedge F_{IJ}[A]-\Lambda\int_{\mathcal{M}} B^{IJ}\wedge\star B_{IJ}.$$

where $B^{IJ}=-B^{JI}$ is a set of real 2-forms and $F[A]$ is the curvature of the connection A.
Separating in spatial and temporal parts, and defining the variable $\Pi^{aIJ}=\frac{1}{2}\eta^{0abc}B_{bc}{}^{IJ}$ we reduce the Action to

$$S[B,A]=\int_{\mathbb{R}} dt\int_{\Omega} d^3x\left[\dot{A}_{aIJ}\Pi^{aIJ}+A_{0IJ}D_a\Pi^{aIJ}+B_{0a}{}^{IJ}\left(\frac{1}{2}\eta^{abc}F_{bcIJ}\right.\right.$$
$$\left.\left.-\ \Lambda\varepsilon_{IJKL}\Pi^{aKL}\right)\right].$$

Thus, the set of fundamental variables is $\{A_{cIJ},\Pi^{aIJ},A_{0IJ},B_{0a}{}^{IJ}\}$.

Lagrangian Analysis

The Euler's expressions

$$
\begin{aligned}
E^{(0)eMN} &= \dot{\Pi}^{eMN} + \left(\Pi^{eMJ} {A_0}^N{}_J - \Pi^{eNJ} {A_0}^M{}_J\right) + \eta^{abe} D_b {B_{0a}}^{MN}, \\
E^{(0)}{}_{fPR} &= -\dot{A}_{fPR} + D_f A_{0PR} + \Lambda \varepsilon_{IJPR} {B_{0f}}^{IJ}, \\
E^{(0)0KL} &= -D_a \Pi^{aKL}, \\
E^{(0)0a}{}_{ST} &= -\frac{1}{2}\eta^{abc} F_{bcST} + \Lambda \varepsilon_{STKL} \Pi^{aKL}.
\end{aligned}
$$

The Lagrangian constraints

$$
\begin{aligned}
\psi^{(0)eMN} &= E^{(0)eMN}, \qquad \psi^{(0)}{}_{fPR} = E^{(0)}{}_{fPR}, \\
\psi^{(0)0KL} &= E^{(0)0KL}, \qquad \psi^{(0)0a}{}_{ST} = E^{(0)0a}{}_{ST}.
\end{aligned}
$$

However,

$$
\boxed{D_a \psi^{(0)0a}{}_{ST} + \Lambda \varepsilon_{STKL} \psi^{(0)0KL} = 0} \quad (\textit{Six reducibility Conditions}) \tag{17}
$$

The "Gauge Identities":

$$
\begin{aligned}
G^{KL} =& D_0 E^{(0)0KL} - D_a E^{(0)aKL} - \left(\Pi^{aKP} E^{0}_{aP}{}^{L} - \Pi^{aLP} E^{0}_{aP}{}^{K}\right) \\
& - \left(B_{0b}{}^{KJ} E^{(0)0b}{}_{J}{}^{L} - B_{0b}{}^{LJ} E^{(0)0b}{}_{J}{}^{K}\right), \\
G^{a}{}_{ST} =& - D_0 E^{(0)0a}{}_{ST} + \eta^{abc} D_b E^{(0)}{}_{cST} + \Lambda \varepsilon_{STKL} E^{(0)aKL}.
\end{aligned}
$$

However,

$$
\boxed{D_a G^{a}{}_{ST} + \Lambda \varepsilon_{STKL} G^{KL} = 0} \;\; (\textit{Six reducibility conditions}) \qquad (18)
$$

Let ϵ_{KL} and ${\epsilon_a}^{ST}$ be arbitrary parameters. Thus $\epsilon_{KL}G^{KL}+{\epsilon_a}^{ST}{G^a}_{ST}$ give us

$$
-\underbrace{\left[\dot{\epsilon}_{KL}-\left({A_{0K}}^{J}\epsilon_{JL}-{A_{0L}}^{J}\epsilon_{JK}\right)\right]}_{\delta A_{0KL}}E^{(0)0KL}
$$

$$
-\underbrace{\left[-\left({\epsilon^K}_P\Pi^{aPL}-{\epsilon^L}_P\Pi^{aPK}\right)+\eta^{abc}\left(D_b{\epsilon_c}^{KL}\right)\right]}_{\delta\Pi^{aKL}}{E^{(0)}}_{aKL}
$$

$$
-\underbrace{\left[D_a\epsilon_{KL}+\Lambda\varepsilon_{STKL}{\epsilon_a}^{ST}\right]}_{\delta A_{aKL}}E^{(0)aKL}
$$

$$
-\underbrace{\left[\left({B_{0b}}^{KJ}{\epsilon_J}^{L}-{B_{0b}}^{LJ}{\epsilon_J}^{K}\right)+{\dot{\epsilon}_a}{}^{KL}-\left({\epsilon_a}^{KJ}{A_{0J}}^{L}-{\epsilon_a}^{LJ}{A_{0J}}^{K}\right)\right]}_{\delta {B_{0a}}^{KL}}{E^{(0)0b}}_{KL}
$$

$$
+\frac{\partial}{\partial t}\left[\epsilon_{KL}E^{(0)0KL}+{\epsilon_a}^{KL}{E^{(0)0a}}_{KL}\right]+\partial_a\left[\epsilon_{KL}E^{(0)aKL}+\eta^{abc}{\epsilon_c}^{KL}{E^{(0)}}_{aKL}\right]=0.
$$

Thus, the *gauge transformations laws* are

$$
\begin{aligned}
\delta_L A_{0KL} &= -\,\dot{\epsilon}_{KL} + \left(A_{0K}{}^{J}\epsilon_{JL} - A_{0L}{}^{J}\epsilon_{JK}\right), \\
\delta_L B_{0a}{}^{KL} &= -\,\dot{\epsilon}_{a}{}^{KL} + \left(\epsilon_{a}{}^{KJ}A_{0J}{}^{L} - \epsilon_{a}{}^{LJ}A_{0J}{}^{K}\right) - \left(B_{0b}{}^{KJ}\epsilon_{J}{}^{L} - B_{0b}{}^{LJ}\epsilon_{J}{}^{K}\right), \\
\delta_L \Pi^{aKL} &= \left(\epsilon^{K}{}_{P}\Pi^{aPL} - \epsilon^{L}{}_{P}\Pi^{aPK}\right) - \eta^{abc}D_b\epsilon_c{}^{KL}, \\
\delta_L A_{aKL} &= -\,D_a\epsilon_{KL} - \Lambda\varepsilon_{STKL}\epsilon_a{}^{ST}.
\end{aligned}
$$

DEGREE OF FREEDOM COUNT

1. *Original fields*: $[A_{aIJ}, \Pi^{aIJ}, A_{0IJ}, B_{0a}{}^{IJ}]$ that are antisymmetric in $\{I, J\} \Longrightarrow$ N=60,
2. *Independent Lagrangian constraints*: $\left[\psi^{(0)eMN}\right] + \left[\psi^{(0)}{}_{fPR}\right] + \left[\psi^{(0)0KL}\right] + \left[\psi^{(0)0a}{}_{ST}\right] - [R.C.\,(17)]$ $\Longrightarrow$ l=18+18+6+18-6=54,
3. *Independent gauge identities*: $[G_{KL}] + [G^a{}_{ST}] - [R.C.\,(18)]$ $\Longrightarrow$ g=6+18-6=18,
4. *"Effective" parameters*: $[\epsilon_{KL}] + \left[\epsilon_a{}^{ST}\right] + [\dot{\epsilon}_{KL}] + \left[\dot{\epsilon}_a{}^{ST}\right] \Longrightarrow$ e=6+18+6+18=48.

With the above information equation (8) gives us

$$P.D.F. = 60 - \frac{1}{2}(54 + 18 + 48) = 0. \tag{19}$$

BF plus Cosmological constant [arbitrary gauge group]

$$S[A^a, B^a] = \int_{\mathcal{M}} \left[F^a \wedge B_a - \frac{\alpha}{2} B^a \wedge B_a \right] \tag{20}$$

The equation of motion

$$E_1^a \doteq F^a - \alpha B^a \ , \qquad E_2^a \doteq DB^a \tag{21}$$

The gauge identities

$$\begin{aligned} G_1^a &= DE_1^a + \alpha E_2^a (= 0 \ off - shell) \\ G_2^a &= DE_2^a - C^a{}_{bc} E_1^b \wedge B^c (= 0 \ off - shell) \end{aligned} \tag{22}$$

The Noether Identities

$$\begin{aligned} &\lambda_a \bullet G_1^a \Longrightarrow D\lambda_a \wedge E_1^a + \alpha \lambda_a \wedge E_2^a = d\left(\lambda_a \wedge E_1^a\right) \\ &\Longrightarrow \delta B^a = D\lambda^a \ , \qquad \delta A^a = \alpha \lambda^a \end{aligned} \tag{23}$$

$$\begin{aligned} \epsilon_a \bullet G_1^a \Longrightarrow & D\epsilon_a \wedge E_2^a + \epsilon_a C^a{}_{bc} B^c \wedge E_1^b = d\left(\epsilon_a E_2^a\right) \\ \Longrightarrow & \delta A^a = D\epsilon^a \,, \qquad \delta B^a = -\epsilon_b C_c^{ab} B^c \end{aligned} \tag{24}$$

Physical degree of freedom count

1. Fields: $A^a, B^a \Longrightarrow N = \dim(g) \times 4 + dim(g) \times 6,$
2. Lag. constraints $E_1^a, E_2^a \Longrightarrow l = \dim(g) \times 6 + dim(g) \times 4,$
3. Gauge identities $G_1^a, G_2^a \Longrightarrow g = \dim(g) \times 4 + dim(g) \times 1,$
4. Gauge parameters $\lambda^a, \epsilon^a \Longrightarrow e = \dim(g) \times 4 + dim(g) \times 1,$

$$P.D.F. = N - \frac{1}{2}\left(l + g + e\right) = 0. \tag{25}$$

Thanks!

Thanks!

Any questions?