The propagators of the free particle and particle in a box in polymer quantum mechanics.

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Motivation

- General: Polymer quantum mechanics is a "toy model" which gives a friendly introduction to technical and conceptual issues of Loop Quantum Gravity by simple mechanics systems.

1. Polymer $\to$ Schrödinger: $\psi_{poly} \to \psi_{sch}$ and $E_{poly} \to E_{sch}$.
   A. Ashtekar, S. Fairhurst and J. L. Willis, *Classical Quantum Gravity* **20** (2003).
   A. Corichi, T. Vukasinac and J. A. Zapata, *Phys. Rev.D***76**, 044016 (2007).
2. Free Scalar Field Propagator: G.M. Hossain, V. Husain, and S.S. Seahra, *Phys. Rev.D***82**, 124032 (2010).

Schrödinger Representation

Canonical Quantization: $\{q, p\} = 1 \quad \mapsto \quad [\widehat{q}, \widehat{p}] = i\hbar\widehat{1}.$

Kinematics

$\hat{q}, \hat{p} \mapsto \mathcal{H}_{sch} = L^2 (\mathbb{R}, d\mu)$

Dynamics

$\widehat{H}(\hat{q}, \hat{p}) |\psi\rangle = E |\psi\rangle \mapsto \mathcal{H}_{sch}$

Weyl Algebra

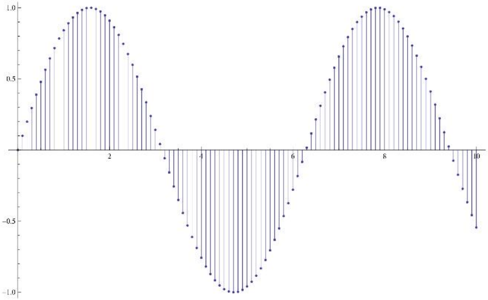
$$U(\lambda)V(\mu) = e^{-i\lambda\mu}V(\lambda)U(\mu), U(\lambda_1)U(\lambda_2) = U(\lambda_1 + \lambda_2),$$
$$V(\mu_1)V(\mu_2) = V(\mu_1 + \mu_2).$$

Stone-Von Neumann Theorem:

If $\widehat{U}(\lambda)$ and $\widehat{V}(\mu)$ are weakly continuous in λ and μ then the representation is unitarily equivalent to the Schrödinger representation and $\exists$ $\hat{q}$ and $\hat{p}$ self-adjoint such that

$$\widehat{U}(\lambda) = e^{i\lambda\hat{x}}, \qquad \widehat{V}(\mu) = e^{\frac{i}{\hbar}\hat{p}\mu}.$$

Kinematics (PQM).



$$x_j \to |x_j\rangle \,, \langle x_i|x_j\rangle = \delta_{i,j},$$

$$|\psi\rangle = \sum_{i=1}^{n} a_i \, |x_i\rangle \,,$$

$$E.V. = \left\{ |\psi\rangle = \sum_{j=1}^{n} a_j \, |x_j\rangle \,, \ \forall\, \gamma = \{x_j\} \subset \mathbb{R} \right\}$$

$$\psi(x) = \sum_{j=1}^{N} a_j \delta_{x_j}(x),$$

Polymer Hilbert Space

$$\mathcal{H}_{poly} = \left\{ \sum_{j=-\infty}^{\infty} a_j \delta_{x_j}(x) : \sum_{j=-\infty}^{\infty} |a_j|^2 < \infty \right\}, \quad \mathcal{H}_{poly} = L^2(\mathbb{R}, d\mu_c).$$

Representation of the Weyl Algebra.

$$\hat{U}(\lambda) \left| x_j \right\rangle = e^{i\lambda x_j} \left| x_j \right\rangle \quad \& \quad \hat{V}(\mu) \left| x_j \right\rangle = \left| x_j - \mu \right\rangle ;$$

but in this representation $\widehat{V}$ is not weakly continuous in μ:

$$\lim_{\mu \mapsto 0} \left\langle x_i \right| \hat{V}(\mu) \left| x_j \right\rangle \neq \left\langle x_i \right| \hat{V}(0) \left| x_j \right\rangle \Rightarrow \nexists\, \hat{p}, \quad \widehat{U}(\lambda) \text{ O.K. } \Rightarrow \exists\, \hat{x}.$$

POLYMER REPRESENTATION IS UNITARILY INEQUIVALENT TO THE SCHRÖDINGER ONE.

Dynamics (PQM)

How is $\widehat{H}$ introduced in $\mathcal{H}_{poly}$ if $\nexists\ \hat{p}$?

$$p^2 \sim \frac{\hbar^2}{\mu_0^2}\left[2 - e^{\frac{-i}{\hbar}\mu_0 p} - e^{\frac{i}{\hbar}\mu_0 p}\right] \quad \mapsto \quad \widehat{p^2} = \frac{\hbar^2}{\mu_0^2}\left[2 - \widehat{V}(\mu_0) - \widehat{V}(-\mu_0)\right].$$

The eigenvalue equation of the Hamiltonian becomes a difference equation:

$$\psi_{x_0}^{(j+1)} + \psi_{x_0}^{(j-1)} = \left(2 - \frac{2m\mu_0^2}{\hbar^2}\left(E - \mathcal{V}(x_0 + j\mu_0)\right)\right)\psi_{x_0}^{(j)}.$$

The free particle A Corichi et al

$$\psi_p^0(x_j) = e^{-\frac{i}{\hbar}x_j p},$$
$$E_{\mu_0,n} = \frac{\hbar^2}{m\mu_0^2}\left[1 - \cos\left(\frac{\mu_0 p}{\hbar}\right)\right].$$

The particle in a box G Chacón et al

$$\psi_n^0(j) = \sqrt{\frac{2}{N}}\sin\left(\frac{n\pi j}{N}\right),$$
$$E_{\mu_0,n} = \frac{\hbar^2}{m\mu_0^2}\left[1 - \cos\left(\frac{n\pi}{N}\right)\right].$$

Free particle propagator (PQM).

Propagator (SAKURAI)

$$k(x_j, t; x_r, t_0) := \langle x_j | \exp\left\{ -\frac{i}{\hbar} \widehat{H}_{poly}(t - t_0) \right\} | x_r \rangle ,$$
$$= i^l J_l(z)\, e^{-iz}, \quad z = \frac{\hbar(t - t_0)}{m\mu_0^2}, \; l = j - r = \frac{x_j - x_r}{\mu_0}.$$

Properties

- $$\psi_p(x_j, t) = \sum_{r=-\infty}^{\infty} k(x_j, t; x_r, t_0)\psi_p(x_r, t_0) = \exp\left\{ -\frac{i}{\hbar} E(t - t_0) \right\} \psi_p(x_j, t_0).$$
- "Composition": $k(x_s, t_2; x_p, t_0) = \sum_{j=\infty}^{\infty} k(x_s, t_2; x_j, t_1) k(x_j, t_1; x_p, t_0)$. o.k.
- $\text{lím}_{t \longrightarrow t_0} k(x_j, t; x_r, t_0) = i^{r-j} J_{r-j}(0) = \delta_{x_j, x_r}$. o.k.
- $K(x_s, t_2; x_p, t_0)$ is Green's function of the operator: $i\hbar \frac{\partial}{\partial t} - \widehat{H}_{poly}$.

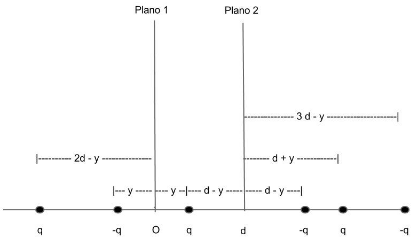
Continuum approximation: $k(x_j, t; x_r, t_0)_{poly} \longrightarrow K(x, t; x', t')_{Sch}$

1. We have $z = \frac{\lambda}{2\pi\mu_0} l$, if $\lambda \sim 10^{-10} m$, $\mu_0 \sim l_p \sim 10^{-35} m \rightarrow z \gg l$. Therefore $J_l(z)$ is the dominant factor in $k(x_i, x_j; t, t_0)_{poly}$, moreover we can use its asymptotic expansion.
2. $\mu_0 d\mu_c \rightarrow d\mu_{Schr} \Rightarrow \frac{k(x_i, x_j; t, t_0)_{poly}}{\mu_0} := k(x_j, t; x_r, t_0)_{Shr}$.
3. But it is insufficient to take only the limit $\mu_0 \rightarrow 0$, we need also to consider the limit $l \rightarrow \infty$, keeping $x_j - x_r = \mu_0 l$ fixed.

This yields

$$\Rightarrow \lim_{l \to \infty} \lim_{\mu_0 \to 0} \frac{k(x_j, t; x_r, t_0)}{\mu_0} = \sqrt{\frac{m}{2i\pi\hbar(t - t_0)}} \exp\left\{ \frac{im(x_j - x_r)^2}{2\hbar(t - t_0)} \right\}.$$

The method of images.



The potential for one point charge between two infinite plane conductors can be written in the form

$$\Phi = -q \sum_{n=-\infty}^{\infty} \{ G_f(\vec{r}, [2nd + y]\hat{x}) - G_f(\vec{r}, [2nd + (-y)]\hat{x}) \}$$

In agreement with the method of images we obtain

$$k_c^i(x_j, t; x_r + 2kN, t_0)$$

$$\begin{aligned} = \quad & k_p^i(x_j, t; x_r + 2kN, t_0) \\ & - k_p^i(x_j, t; -x_r + 2kN, t_0), \\ = \quad & e^{-iz} \sum_{k=-\infty}^{\infty} \left\{ i^{j-r-2kN} J_{j-r-2kN}(z) \right. \\ & \left. - i^{j+r-2kN} J_{j+r-2kN}(z) \right\}, \end{aligned}$$

Relationship between $k_c^i(x_j, t; x_r + 2kN, t_0)$ and $k_c^e(x_j, t; x_r, t_0)$.

$$k_c^e(x_j, t; x_r, t_0) = \frac{2}{N} \sum_{n=1}^{N-1} \sin\left(\frac{n\pi j}{N}\right) \sin\left(\frac{n\pi r}{N}\right) e^{-iz\left[1-\cos\left(\frac{n\pi}{N}\right)\right]}. \tag{1}$$

If we introduce the Jacobi-Anger expansion $e^{iz\cos\theta} = \sum_{k=-\infty}^{\infty} i^k J_k(Z) e^{ik\theta}$ in (1) it turns out

$$k_c^e(x_j, t; x_r, t_0) = \frac{e^{-iz}}{N} \sum_{n=1}^{N-1} \sum_{k=-\infty}^{\infty} i^k J_k(z) \left\{ \cos\left[\frac{n\pi(j-r-k)}{N}\right] - \cos\left[\frac{n\pi(j+r-k)}{N}\right] \right\}.$$

We can verify

$$\sum_{n=1}^{N-1} \cos\left(\frac{n\pi p}{N}\right) = -1 + \sum_{m-\infty}^{\infty} \left\{ N\delta_{p,2mN} + \frac{1}{2}[1-(-1)^p]\delta_{p,m} \right\},$$

and then obtain

$$k_c^e(x_j, t; x_r, t_0) = k_c^i(x_j, t; x_r + 2kN, t_0).$$

Summary and perspectives

1. We have calculated the propagators of the free particle and the particle in a box in the polymer representation and we have checked some consistency properties they satisfy.
2. We have put in contact both propagators with their continuum analogs.
3. Future: To do the analysis of the continuum limit of the propagators through the Wilson's renormalization group:
   A. Corichi, T. Vukasinac and J. A. Zapata, *Classical Quantum Gravity* **24** (2007).
4. Future: To calculate the propagator for the simple harmonic oscillator.