Lorentz-covariant Hamiltonian analysis of BF gravity with the Immirzi parameter

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Introduction

Gravity $\rightarrow$ manifestation of the curvature of spacetime. The dynamics of the gravitational field is governed by the Einstein's equations: $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$.

Action principles

- Metric: $S[g_{\mu\nu}] = \kappa \int_M d^4x \sqrt{-g} R$.
- First-order: $S[e, A] = \int_M \left[*(e^I \wedge e^J) - \frac{1}{\gamma} e^I \wedge e^J \right] \wedge F_{IJ}[A]$; ${F^I}_J = d{A^I}_J + {A^I}_K \wedge {A^K}_J$, $\gamma \rightarrow$ Immirzi parameter.
- BF: $S[B, A, \phi, \mu] = \int_M \left(B^{IJ} \wedge F_{IJ}[A] - \phi_{IJKL} B^{IJ} \wedge B^{KL} + \mu H(\phi) \right)$; $\phi_{IJKL} = -\phi_{JIKL} = -\phi_{IJLK} = \phi_{KLIJ}$ and $H(\phi) = \epsilon^{IJKL} \phi_{IJKL}$, ${\phi^{IJ}}_{IJ}$, $a_1 {\phi^{IJ}}_{IJ} + a_2 \epsilon^{IJKL} \phi_{IJKL}$.

- The Immirzi parameter has a topological nature (PRD **85** 024026 (2012)). It does not affect the classical equations of motion, but it shows up in the spectra of quantum operators.
- Since general relativity is a constrained theory $\longrightarrow$ use Dirac's method.
  - Metric $\longrightarrow$ ADM formulation.
  - First-order $\longrightarrow$ Barbero's formulation: one of the cornerstones of loop quantum gravity.
- Novel approach $\longrightarrow$ spin foam models $\longrightarrow$ path integral quantization of gravity based on the BF formulation. This approach supplements the loop approach and is Lorentz-covariant.

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Hamiltonian analysis of the CMPR action

The CMPR action (CQG 18 L49)

$$S[B,A,\phi,\mu]=\int_M \left[B^{IJ}\wedge F_{IJ}-\phi_{IJKL}B^{IJ}\wedge B^{KL}+\mu\left(a_1\phi_{IJ}{}^{IJ}+a_2\epsilon_{IJKL}\phi^{IJKL}\right)\right].$$

We assume that $M=\mathbb{R}\times\Omega$ ($\partial\Omega=0$). The $(3+1)$-decomposition of the action leads to

$$S[A,\Pi,\phi,\mu_0]=\int_{\mathbb{R}}dt\int_{\Omega}d^3x\Big\{\Pi^{aIJ}\dot{A}_{aIJ}+A_{0IJ}D_a\Pi^{aIJ}+\frac{1}{2}B_{0a}{}^{IJ}\tilde{\eta}^{abc}F_{bcIJ}$$
$$-\left[2B_{0a}{}^{IJ}\Pi^{aKL}-\mu_0\left(a_1\eta^{I[K|}\eta^{J|L]}+a_2\epsilon^{IJKL}\right)\right]\phi_{IJKL}\Big\},$$

where $\Pi^{aIJ}\equiv\frac{1}{2}\tilde{\eta}^{abc}B_{bc}{}^{IJ}$. The equation of motion corresponding to Φ_{IJKL} is

$$B_{0a}{}^{IJ}\Pi^{aKL} + B_{0a}{}^{KL}\Pi^{aIJ} - \mu_0\left[a_1\eta^{I[K|}\eta^{J|L]} + a_2\epsilon^{IJKL}\right] = 0,$$

which has the solution

$$B_{0a}{}^{IJ} = \frac{1}{8}Nh_{ab}\epsilon^{IJKL}\Pi^{b}{}_{KL} + \frac{1}{2}\underset{\sim}{\eta}_{abc}\Pi^{bIJ}N^{c} + \frac{1}{16h}Nh_{ac}h_{bd}\Pi^{bIJ}\left(\Phi^{cd} + \frac{a_1}{a_2}hh^{cd}\right),$$

$$\mu_0 = \sigma\mathcal{V}/4a_2, \quad \Phi^{ab} + \frac{a_1}{a_2}hh^{ab} = 0,$$

where we have introduced the quantities

$$\mathcal{V} \equiv \frac{1}{3}\epsilon_{IJKL}B_{0a}{}^{IJ}\Pi^{aKL}, \quad N^{a} \equiv \frac{\sigma}{2h}\tilde{\eta}^{abc}h_{bd}B_{0c}{}^{IJ}\Pi^{d}{}_{IJ}, \quad N \equiv \frac{\mathcal{V}}{h}$$

$$hh^{ab} \equiv \frac{\sigma}{2}\Pi^{aIJ}\Pi^{b}{}_{IJ}, \quad \Phi^{ab} \equiv -\sigma * \Pi^{a}{}_{IJ}\Pi^{bIJ}.$$

By substituting the expression for ${B_{0a}}^{IJ}$ in the action we obtain

$$S[A,\Pi] = \int_{\mathbb{R}} dt \int_{\Omega} d^3x \left[\Pi^{aIJ} \dot{A}_{aIJ} + A_{0IJ} \mathcal{G}^{IJ} + N\mathcal{H} + N^a \mathcal{H}_a + \lambda_{ab} \varphi^{ab} \right].$$

Primary constraints

$$\mathcal{G}^{IJ} \equiv D_a \Pi^{aIJ} \approx 0, \quad \mathcal{H}_a \equiv \frac{1}{2} \Pi^{bIJ} F_{baIJ} \approx 0,$$

$$\mathcal{H} \equiv \frac{1}{8} \tilde{\eta}^{abc} h_{ad} * \Pi^{dIJ} F_{bcIJ} \approx 0, \quad \varphi^{ab} \equiv \Phi^{ab} + \frac{a_1}{a_2} h h^{ab} \approx 0.$$

The Hamiltonian is $H = -\int_{\Omega} d^3x \left(A_{0IJ} \mathcal{G}^{IJ} + N\mathcal{H} + N^a \mathcal{H}_a + \lambda_{ab} \varphi^{ab} \right)$.
Now, according to Dirac's method, the primary constraints must be preserved in time $\Longrightarrow \{C, H\} \approx 0$ for each constraint C.
The canonical variables (A, Π) satisfy

$$\{A_{aIJ}(x), \Pi^{bKL}(y)\} = \delta_a^b \delta_I^{[K} \delta_J^{L]} \delta^3(x,y).$$

The primary constraint algebra has the following form:

- $\{\mathcal{G}^{IJ}, C\} \approx 0$, $\{\mathcal{H}_a, C\} \approx 0, \{\mathcal{H}, \mathcal{H}\} \approx 0$, $\{\varphi^{ab}, \varphi^{cd}\} = 0$.
- $\{\mathcal{H}(x), \varphi^{ab}(y)\} = \left[-\frac{a_1}{4a_2} h_{cf}(x) \tilde{\eta}^{(a|cd} \varphi^{f|b)}(x) \frac{\partial}{\partial x^d} + \Psi^{ab}\right] \delta^3(x,y)$, where
$$\Psi^{ab} \equiv \frac{1}{2} h_{cf} \left(-\Pi^f{}_{IJ} + \frac{\sigma a_1}{2a_2} * \Pi^f{}_{IJ}\right) \tilde{\eta}^{(a|cd} D_d \Pi^{|b)IJ}.$$

The evolutions of $\mathcal{G}^{IJ}$ and $\mathcal{H}_a$ generate neither new constraints nor conditions on the Lagrange multipliers. On the other hand, the evolution of φ^{ab} leads to

$$N\Psi^{ab} \approx 0,$$

whose solution is $\Psi^{ab} \approx 0$. Then Ψ^{ab} becomes a secondary constraint$\longrightarrow$ evolve Ψ^{ab}.

- $\{\Psi^{ab}, \mathcal{G}^{IJ}\} = 0$, $\{\Psi^{ab}, \mathcal{H}_a\} \approx 0$.
- $\{\Psi^{ab}(x), \mathcal{H}(y)\} \approx F^{ab}\delta^3(x,y)$ and $\{\Psi^{ab}(x), \varphi^{cd}(y)\} = M^{(ab)(cd)}\delta^3(x,y)$, where $M^{(ab)(cd)}$ defines a 6×6 non-singular matrix.

$$M^{(ab)(cd)} \equiv \sigma h_{ef}\left(-\Pi^f{}_{IJ} + \frac{\sigma a_1}{2a_2} * \Pi^f{}_{IJ}\right)\left[\left(*\Pi^{cI}{}_K - \frac{a_1}{2a_2}\Pi^{cI}{}_K\right)\tilde{\eta}^{(a|de}\Pi^{|b)KJ} + (c \leftrightarrow d)\right].$$

The evolution of Ψ^{ab} then fixes the Lagrange multipliers λ_{ab}: $\lambda_{ab} \approx \frac{1}{4} N F^{cd}(M^{-1})_{(cd)(ab)} \Longrightarrow$ the Dirac's method concludes here!

Classification of the constraints

- $\mathcal{G}^{IJ}, \mathcal{H}_a$ and $\bar{\mathcal{H}} \equiv \mathcal{H} + \frac{1}{4}F^{cd}(M^{-1})_{(cd)(ab)}\varphi^{ab}$ are first-class.
- φ^{ab} and Ψ^{ab} are second-class.

Degree of freedom count

$$\mathrm{DOF} = \frac{1}{2}[2\times \underbrace{18}_{A_{aIJ}} - 2\times(\underbrace{6}_{\mathcal{G}^{IJ}} + \underbrace{3}_{\mathcal{H}_a} + \underbrace{1}_{\bar{\mathcal{H}}}) - (\underbrace{6}_{\varphi^{ab}} + \underbrace{6}_{\Psi^{ab}})] = 2.$$

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Alternative BF principle

BF gravity plus cosmological constant (PRD **85** 064011)

$$S[B,A,\phi,\mu] = \int_M \left[\left(B^{IJ} + \frac{1}{\gamma} * B^{IJ} \right) \wedge F_{IJ} - \phi_{IJKL} B^{IJ} \wedge B^{KL} - \mu \phi_{IJKL} \epsilon^{IJKL} + \mu\lambda + l_1 B_{IJ} \wedge B^{IJ} + l_2 B_{IJ} \wedge * B^{IJ} \right]$$

Following a similar procedure as before, the action can be cast in the form

$$S[A, \overset{(\gamma)}{\Pi}] = \int_{\mathbb{R}} dt \int_{\Omega} d^3x \left[\overset{(\gamma)}{\Pi}{}^{aIJ} \dot{A}_{aIJ} + A_{0IJ} \mathcal{G}^{IJ} + N^a \mathcal{H}_a + N\mathcal{H} + \lambda_{ab} \Phi^{ab} \right],$$

where $\Pi^{aIJ} \equiv \frac{1}{2} \tilde{\eta}^{abc} B_{bc}{}^{IJ}$, $\overset{(\gamma)}{V}{}^{IJ} \equiv V^{IJ} + \frac{1}{\gamma} * V^{IJ}$, and we have introduced the same quantities N, N^a, h^{ab} and Φ^{ab}.

Primary constraints

$$\mathcal{G}^{IJ} \equiv D_a \overset{(\gamma)}{\Pi}{}^{aIJ} \approx 0, \quad \mathcal{H}_a \equiv \frac{1}{2} \overset{(\gamma)}{\Pi}{}^{bIJ} F_{baIJ} \approx 0,$$

$$\mathcal{H} \equiv \frac{1}{8} \tilde{\eta}^{abc} h_{ad} * \overset{(\gamma)}{\Pi}{}^{dIJ} F_{bcIJ} + \Lambda h \approx 0, \quad \Phi^{ab} \approx 0.$$

Here $\Lambda = 3l_2 - \sigma\lambda/4$. Moreover, we need to express h^{ab} and Φ^{ab} in terms of the new canonical variable $\overset{(\gamma)}{\Pi}$, i.e.,

$$hh^{ab} = \eta \left[\overset{(\gamma)}{(hh^{ab})} + \frac{\gamma^{-1}}{1+\sigma\gamma^{-2}} \overset{(\gamma)}{\Phi}{}^{ab} \right], \quad \Phi^{ab} = \eta \left[\overset{(\gamma)}{\Phi}{}^{ab} + \frac{4\sigma\gamma^{-1}}{1+\sigma\gamma^{-2}} \overset{(\gamma)}{(hh^{ab})} \right],$$

with $\eta \equiv \frac{\gamma^2(\gamma^2+\sigma)}{(\gamma^2-\sigma)^2}$.

It turns out that the Poisson brackets among the primary constraints are very similar to those of the CMPR action. The only non-(weakly)vanishing Poisson bracket is given by

$$\{\mathcal{H}(x), \Phi^{ab}(y)\} = \frac{1}{4}\bar{\Psi}^{ab}\delta^3(x,y)$$

where

$$\bar{\Psi}^{ab} \equiv -2\eta h_{cf}\left(-\overset{(\gamma)}{\Pi}{}^{f}{}_{IJ} + \frac{2\gamma^{-1}}{1+\sigma\gamma^{-2}} * \overset{(\gamma)}{\Pi}{}^{f}{}_{IJ}\right)\tilde{\eta}^{(a|cd}D_d \overset{(\gamma)}{\Pi}{}^{|b)IJ}.$$

Then $\bar{\Psi}^{ab}$ becomes a secondary constraint and its evolution leads to the fixing of the Lagrange multiplier λ^{ab}.

Finally, the classification of the constraints and the degree of freedom count are as follows:

- $\mathcal{G}^{IJ}, \mathcal{H}_a$, and $\bar{\mathcal{H}} \equiv \mathcal{H} + \frac{1}{4}\bar{F}^{cd}(\bar{M}^{-1})_{(cd)(ab)}\Phi^{ab}$ are first-class.
- Φ^{ab} and Ψ^{ab} are second-class.
- The number of physical degrees of freedom is 2.

Conclusions and perspectives

1. Both BF action principles of gravity possess 2 local degrees of freedom, the same number of general relativity.
2. Despite the Immirzi parameter enters in both action principles (do not consider the cosmological coupling in the second action principle) in different ways and the constraint algebras may differ a little from each other, we can make the algebras coincide by performing the changes $\overset{(\gamma)}{\Pi} \rightarrow \Pi$ and $\frac{4\sigma\gamma^{-1}}{1+\sigma\gamma^{-2}} \rightarrow \frac{a_1}{a_2}$ in the constraints of the alternative BF principle. It is also necessary to redefine suitably the constraint $\mathcal{H}$ for eliminating factors proportional to the constraints Φ^{ab}.

3. Manage the second-class constraints to make contact with the Lorentz-covariant formulations of gravity based on the first-order action.
4. Coupling of fermions.
5. Work out the quantum theories arising from these constrained systems.
6. BF gravity with boundary terms.

¡¡Thank You!!