Complete regular dessins and skew-morphisms of cyclic groups

Martin Škoviera
(joint work with Yan-Quan Feng, Kan Hu, Roman Nedela, and Naer Wang)

Comenius University, Bratislava, Slovakia

SIGMAP 2018
Centro de Ciencias Matematicas UNAM, Morelia, Mexico

29th June, 2018

Dessins d'enfant

Definition (Grothendieck, Esquisse d'un programme, 1984)

A dessin d'enfant (dessin, for short) is a 2-cell embedding of a connected bipartite graph into an oriented surface with a fixed vertex 2-colouring.

Dessins d'enfant

Definition (Grothendieck, Esquisse d'un programme, 1984)

A dessin d'enfant (dessin, for short) is a 2-cell embedding of a connected bipartite graph into an oriented surface with a fixed vertex 2-colouring.

Grothendieck was inspired by the following remarkable theorem of Belyi:

Theorem (Belyi, 1979)

A compact Riemann surface S, regarded as a projective algebraic curve, can be defined by a polynomial $P(x, y)$ with coefficients from the algebraic number field $\bar{\mathbb{Q}} \iff$ there exists a meromorphic function $\beta\colon S \to \mathbb{P}^1(\mathbb{C})$ unbranched outside $\{0, 1, \infty\}$.

Dessins d'enfant

Definition (Grothendieck, Esquisse d'un programme, 1984)

A dessin d'enfant (dessin, for short) is a 2-cell embedding of a connected bipartite graph into an oriented surface with a fixed vertex 2-colouring.

Grothendieck was inspired by the following remarkable theorem of Belyi:

Theorem (Belyi, 1979)

A compact Riemann surface S, regarded as a projective algebraic curve, can be defined by a polynomial $P(x, y)$ with coefficients from the algebraic number field $\bar{\mathbb{Q}}$ $\iff$ there exists a meromorphic function $\beta\colon S \to \mathbb{P}^1(\mathbb{C})$ unbranched outside $\{0, 1, \infty\}$.

The preimage $\beta^{-1}([0, 1])$ of the unit interval is a connected bipartite graph drawn on S $\implies$ dessin d'enfant

Regular dessins

Regular dessins

- dessin = 2-cell embedding of a connected bipartite graph into an oriented surface with a fixed vertex 2-colouring

Regular dessins

- dessin = 2-cell embedding of a connected bipartite graph into an oriented surface with a fixed vertex 2-colouring
- automorphism of a dessin = orientation and colour-preserving automorphism of the embedding

Regular dessins

- dessin = 2-cell embedding of a connected bipartite graph into an oriented surface with a fixed vertex 2-colouring
- automorphism of a dessin = orientation and colour-preserving automorphism of the embedding
- regular dessin = dessin whose automorphism group acts regularly on the edge set

  Note: Vertices with different colour may not have the same valency

Regular dessins

- dessin = 2-cell embedding of a connected bipartite graph into an oriented surface with a fixed vertex 2-colouring
- automorphism of a dessin = orientation and colour-preserving automorphism of the embedding
- regular dessin = dessin whose automorphism group acts regularly on the edge set

  Note: Vertices with different colour may not have the same valency
- complete regular dessin = regular dessin whose underlying graph is a complete bipartite graph
- (m, n)-complete regular dessin = regular dessin whose underlying graph is $K_{m,n}$

Combinatorial representation of dessins

Every dessin $\mathcal{D}$ can be identified with a triple $(\Omega; \rho, \lambda)$ where

(1) Ω is a non-empty finite set

(2) ρ and λ are permutations of Ω

(3) the permutation group $G = \langle \rho, \lambda \rangle$ is transitive on Ω.

Combinatorial representation of dessins

Every dessin $\mathcal{D}$ can be identified with a triple $(\Omega; \rho, \lambda)$ where

(1) Ω is a non-empty finite set

(2) ρ and λ are permutations of Ω

(3) the permutation group $G = \langle \rho, \lambda \rangle$ is transitive on Ω.

Given such a triple $(\Omega; \rho, \lambda)$

- edges of $\mathcal{D} \longleftrightarrow$ elements of Ω
- black vertices $\longleftrightarrow$ cycles of ρ
- white vertices $\longleftrightarrow$ cycles of λ
- $\rho =$ rotation around black vertices
- $\lambda =$ rotation around white vertices

Representing regular dessins

Every regular dessin $\mathcal{D}$ can be identified with a triple $(G; a, b)$ where G is a finite group such that $G = \langle a, b \rangle$.

- edges of $\mathcal{D} \longleftrightarrow$ elements of G
- black vertices $\longleftrightarrow$ left cosets $g\langle a \rangle$
- white vertices $\longleftrightarrow$ left cosets $g\langle b \rangle$
- an edge g joins $s\langle a \rangle$ to $t\langle b \rangle \iff g \in s\langle a \rangle \cap t\langle b \rangle$
- the underlying graph is simple $\iff \langle a \rangle \cap \langle b \rangle = 1$
- rotation around a black vertex $s\langle a \rangle$: $sa^i \mapsto sa^{i+1}$
- rotation around a white vertex $t\langle b \rangle$: $tb^i \mapsto tb^{i+1}$

From complete regular dessins to exact bicyclic groups

Definition

A finite group G is bicyclic if there exist $\langle a \rangle \leq G$ and $\langle b \rangle \leq G$ such that $G = \langle a \rangle \langle b \rangle$. A bicyclic group is exact if $\langle a \rangle \cap \langle b \rangle = 1$.

Bicyclic groups have been extensively studied by group theorists since 1950's: Huppert, Douglas, Ito, Blackburn, etc.

From complete regular dessins to exact bicyclic groups

Definition

A finite group G is bicyclic if there exist $\langle a \rangle \leq G$ and $\langle b \rangle \leq G$ such that $G = \langle a \rangle \langle b \rangle$. A bicyclic group is exact if $\langle a \rangle \cap \langle b \rangle = 1$.

Bicyclic groups have been extensively studied by group theorists since 1950's: Huppert, Douglas, Ito, Blackburn, etc.

Theorem (Jones, Nedela & S., 2007)

(i) *A regular dessin $\mathcal{D} = (G; a, b)$ is complete $\iff$ $G = \langle a \rangle \langle b \rangle$ is an exact bicyclic group.*

(ii) *The isomorphism classes of (m, n)-complete regular dessins with are in a 1-1 correspondence with the equivalence classes of exact bicyclic triples $(G; a, b)$ where $|a| = m$ and $|b| = n$.*

Two triples $(G; a, b)$ and $(G; a', b')$ are equivalent if there is an automorphism of G such that $a \mapsto a'$ and $b \mapsto b'$.

From complete regular dessins to exact bicyclic groups

Theorem (Jones, Nedela & S., 2007)

(i) *A regular dessin* $\mathcal{D} = (G; a, b)$ *is complete* $\iff$ $G = \langle a \rangle \langle b \rangle$ *is an exact bicyclic group.*

Proof.

From complete regular dessins to exact bicyclic groups

Theorem (Jones, Nedela & S., 2007)

(i) *A regular dessin* $\mathcal{D} = (G; a, b)$ *is complete* $\Longleftrightarrow$ $G = \langle a\rangle\langle b\rangle$ *is an exact bicyclic group.*

Proof.

$\Longrightarrow$:

From complete regular dessins to exact bicyclic groups

Theorem (Jones, Nedela & S., 2007)

(i) *A regular dessin* $\mathcal{D} = (G; a, b)$ *is complete* $\iff G = \langle a \rangle \langle b \rangle$ *is an exact bicyclic group.*

Proof.

$\Longrightarrow$:

- If $\mathcal{D} = (G; a, b)$ is (m, n)-complete, then $\mathcal{D}$ has mn edges, so $|G| = mn$.

From complete regular dessins to exact bicyclic groups

Theorem (Jones, Nedela & S., 2007)

(i) *A regular dessin $\mathcal{D} = (G; a, b)$ is complete $\iff$ $G = \langle a \rangle \langle b \rangle$ is an exact bicyclic group.*

Proof.

$\Longrightarrow$:

- If $\mathcal{D} = (G; a, b)$ is (m, n)-complete, then $\mathcal{D}$ has mn edges, so $|G| = mn$.
- Since black vertices are m-valent and white vertices are n-valent, we have $|\langle a \rangle| = m$ and $|\langle b \rangle| = n$.

From complete regular dessins to exact bicyclic groups

Theorem (Jones, Nedela & S., 2007)

(i) *A regular dessin* $\mathcal{D} = (G; a, b)$ *is complete* $\iff G = \langle a \rangle \langle b \rangle$ *is an exact bicyclic group.*

Proof.

$\Longrightarrow$:

- If $\mathcal{D} = (G; a, b)$ is (m, n)-complete, then $\mathcal{D}$ has mn edges, so $|G| = mn$.
- Since black vertices are m-valent and white vertices are n-valent, we have $|\langle a \rangle| = m$ and $|\langle b \rangle| = n$.
- The underlying graph is simple, therefore $\langle a \rangle \cap \langle b \rangle = 1$.

From complete regular dessins to exact bicyclic groups

Theorem (Jones, Nedela & S., 2007)

(i) *A regular dessin* $\mathcal{D} = (G; a, b)$ *is complete* $\iff G = \langle a \rangle \langle b \rangle$ *is an exact bicyclic group.*

Proof.

$\Longrightarrow$:

- If $\mathcal{D} = (G; a, b)$ is (m, n)-complete, then $\mathcal{D}$ has mn edges, so $|G| = mn$.
- Since black vertices are m-valent and white vertices are n-valent, we have $|\langle a \rangle| = m$ and $|\langle b \rangle| = n$.
- The underlying graph is simple, therefore $\langle a \rangle \cap \langle b \rangle = 1$.
- Clearly, $|\langle a \rangle \langle b \rangle| = |\langle a \rangle|.|\langle b \rangle|/|\langle a \rangle \cap \langle b \rangle| = mn/1 = |G|$.

From complete regular dessins to exact bicyclic groups

Theorem (Jones, Nedela & S., 2007)

(i) *A regular dessin $\mathcal{D} = (G; a, b)$ is complete $\iff G = \langle a \rangle \langle b \rangle$ is an exact bicyclic group.*

Proof.

$\Longrightarrow$:

- If $\mathcal{D} = (G; a, b)$ is (m, n)-complete, then $\mathcal{D}$ has mn edges, so $|G| = mn$.
- Since black vertices are m-valent and white vertices are n-valent, we have $|\langle a \rangle| = m$ and $|\langle b \rangle| = n$.
- The underlying graph is simple, therefore $\langle a \rangle \cap \langle b \rangle = 1$.
- Clearly, $|\langle a \rangle \langle b \rangle| = |\langle a \rangle|.|\langle b \rangle|/|\langle a \rangle \cap \langle b \rangle| = mn/1 = |G|$.
- Therefore $G = \langle a \rangle \langle b \rangle$ with $\langle a \rangle \cap \langle b \rangle = 1$, so G is an exact bicyclic group.

From complete regular dessins to exact bicyclic groups

Theorem (Jones, Nedela & S., 2007)

(i) *A regular dessin* $\mathcal{D} = (G; a, b)$ *is complete* $\Longleftrightarrow$ $G = \langle a \rangle \langle b \rangle$ *is an exact bicyclic group.*

Proof.

$\Longrightarrow$:

- If $\mathcal{D} = (G; a, b)$ is (m, n)-complete, then $\mathcal{D}$ has mn edges, so $|G| = mn$.
- Since black vertices are m-valent and white vertices are n-valent, we have $|\langle a \rangle| = m$ and $|\langle b \rangle| = n$.
- The underlying graph is simple, therefore $\langle a \rangle \cap \langle b \rangle = 1$.
- Clearly, $|\langle a \rangle \langle b \rangle| = |\langle a \rangle|.|\langle b \rangle| / |\langle a \rangle \cap \langle b \rangle| = mn/1 = |G|$.
- Therefore $G = \langle a \rangle \langle b \rangle$ with $\langle a \rangle \cap \langle b \rangle = 1$, so G is an exact bicyclic group.

$\Longleftarrow$: Similar.

Skew-morphisms

A skew-morphism is a generalisation of a group automorphsism introduced by Jajcay and Širáň (2002) in the context of regular Cayley maps

Definition

A skew-morphism of G is a bijection $\varphi\colon G \to G$ such that

$$\varphi(1) = 1 \quad \text{and} \quad \varphi(xy) = \varphi(x)\varphi^{\pi(x)}(y)$$

for some function $\pi\colon A \to \mathbb{Z}$, a power function for φ.

If $\pi(x) = 1$ for all $x \in G$, then φ is an automorphism of G.

Skew-morphisms

A skew-morphism is a generalisation of a group automorphsism introduced by Jajcay and Širáň (2002) in the context of regular Cayley maps

Definition

A skew-morphism of G is a bijection $\varphi\colon G \to G$ such that

$$\varphi(1) = 1 \quad \text{and} \quad \varphi(xy) = \varphi(x)\varphi^{\pi(x)}(y)$$

for some function $\pi\colon A \to \mathbb{Z}$, a power function for φ.

If $\pi(x) = 1$ for all $x \in G$, then φ is an automorphism of G.

Theorem (Jajcay & Širáň, 2002)

A Cayley map $\mathcal{M} = CM(G, X, \rho)$ is regular $\iff$ there is a skew-morphism φ of G such that $\varphi|X = \rho$.

Skew-morphisms

A skew-morphism is a generalisation of a group automorphsism introduced by Jajcay and Širáň (2002) in the context of regular Cayley maps

Definition

A skew-morphism of G is a bijection $\varphi\colon G \to G$ such that

$$\varphi(1) = 1 \quad \text{and} \quad \varphi(xy) = \varphi(x)\varphi^{\pi(x)}(y)$$

for some function $\pi\colon A \to \mathbb{Z}$, a power function for φ.

If $\pi(x) = 1$ for all $x \in G$, then φ is an automorphism of G.

Theorem (Jajcay & Širáň, 2002)

A Cayley map $\mathcal{M} = CM(G, X, \rho)$ is regular $\iff$ there is a skew-morphism φ of G such that $\varphi|X = \rho$.

$\mathrm{Aut}(\mathcal{M}) \cong G\langle\varphi\rangle$ with multiplication defined by the rule $\varphi.g = \varphi(g)\varphi^{\pi(g)}$.

Skew-morphisms and cyclic subgroups

Let G be a finite group expressible as a product AB of two subgroups A and B where $B = \langle b \rangle$ is cyclic and $A \cap B = 1$.

Skew-morphisms and cyclic subgroups

Let G be a finite group expressible as a product AB of two subgroups A and B where $B = \langle b \rangle$ is cyclic and $A \cap B = 1$.

- Every element $x \in G$ can be uniquely written as $x = ab^i$.

Skew-morphisms and cyclic subgroups

Let G be a finite group expressible as a product AB of two subgroups A and B where $B = \langle b \rangle$ is cyclic and $A \cap B = 1$.

- Every element $x \in G$ can be uniquely written as $x = ab^i$.
- Since $AB = BA$, for every $x \in A$ we have

$$bx = yb^k$$

where $y \in A$ and $0 \leq k < |b|$.

Skew-morphisms and cyclic subgroups

Let G be a finite group expressible as a product AB of two subgroups A and B where $B = \langle b \rangle$ is cyclic and $A \cap B = 1$.

- Every element $x \in G$ can be uniquely written as $x = ab^i$.
- Since $AB = BA$, for every $x \in A$ we have

  $$bx = yb^k$$

  where $y \in A$ and $0 \le k < |b|$.
- Both y and $k \in \mathbb{Z}$ are uniquely determined by x.

Skew-morphisms and cyclic subgroups

Let G be a finite group expressible as a product AB of two subgroups A and B where $B = \langle b \rangle$ is cyclic and $A \cap B = 1$.

- Every element $x \in G$ can be uniquely written as $x = ab^i$.
- Since $AB = BA$, for every $x \in A$ we have

  $$bx = yb^k$$

  where $y \in A$ and $0 \leq k < |b|$.
- Both y and $k \in \mathbb{Z}$ are uniquely determined by x.
- We define $\varphi_b(x) = y$ and $\pi_b(x) = k$.

Skew-morphisms and cyclic subgroups

Let G be a finite group expressible as a product AB of two subgroups A and B where $B = \langle b \rangle$ is cyclic and $A \cap B = 1$.

- Every element $x \in G$ can be uniquely written as $x = ab^i$.
- Since $AB = BA$, for every $x \in A$ we have

$$bx = yb^k$$

  where $y \in A$ and $0 \leq k < |b|$.
- Both y and $k \in \mathbb{Z}$ are uniquely determined by x.
- We define $\varphi_b(x) = y$ and $\pi_b(x) = k$.
- $\varphi_b \colon A \to A$ is a skew-morphism and π_b is the associated power function. [Conder, Jajcay & Tucker (2006)].

We call φ_b the skew-morphism of H induced by $b \in B$.

From bicyclic groups to skew-morphisms

Let $G = AB$ be an exact bicyclic group, $A = \langle a \rangle \cong \mathbb{Z}_m$ and $B = \langle b \rangle \cong \mathbb{Z}_n$.

From bicyclic groups to skew-morphisms

Let $G = AB$ be an exact bicyclic group, $A = \langle a \rangle \cong \mathbb{Z}_m$ and $B = \langle b \rangle \cong \mathbb{Z}_n$.

- The triple $(G; a, b)$ induces a pair of skew-morphisms $\varphi_b \colon A \to A$ and $\varphi_a \colon B \to B$.

From bicyclic groups to skew-morphisms

Let $G = AB$ be an exact bicyclic group, $A = \langle a \rangle \cong \mathbb{Z}_m$ and $B = \langle b \rangle \cong \mathbb{Z}_n$.

- The triple $(G; a, b)$ induces a pair of skew-morphisms $\varphi_b \colon A \to A$ and $\varphi_a \colon B \to B$.
- Since $(G; a, b) =$ complete regular dessin $\implies$

Every (m, n)-complete regular dessin $\mathcal{D}$ gives rise to a pair of skew-morphisms (σ, τ) where $\sigma \colon \mathbb{Z}_n \to \mathbb{Z}_n$ and $\tau \colon \mathbb{Z}_m \to \mathbb{Z}_m$.

From bicyclic groups to skew-morphisms

Let $G = AB$ be an exact bicyclic group, $A = \langle a \rangle \cong \mathbb{Z}_m$ and $B = \langle b \rangle \cong \mathbb{Z}_n$.

- The triple $(G; a, b)$ induces a pair of skew-morphisms $\varphi_b \colon A \to A$ and $\varphi_a \colon B \to B$.
- Since $(G; a, b)$ = complete regular dessin $\implies$

Every (m, n)-complete regular dessin $\mathcal{D}$ gives rise to a pair of skew-morphisms (σ, τ) where $\sigma \colon \mathbb{Z}_n \to \mathbb{Z}_n$ and $\tau \colon \mathbb{Z}_m \to \mathbb{Z}_m$.

- The pair (σ, τ) is a complete invariant of $\mathcal{D}$:

$\mathcal{D}$ can be reconstructed from (σ, τ) by letting σ and τ act on the disjoint union $\mathbb{Z}_m \cup \mathbb{Z}_n$ and producing the triple $(G; a, b)$ where

$$a = \tau\rho_m \quad \text{and} \quad b = \sigma\rho_n,$$

ρ_k being the cyclic shift in $\mathbb{Z}_k$.

Example: The standard (m, n)-complete regular dessin

- For every pair of integers $m \geq 2$ and $n \geq 2$, the group $\mathbb{Z}_m \times \mathbb{Z}_n$ gives rise to the triple $(\mathbb{Z}_m \times \mathbb{Z}_n; 1_m, 1_n)$ which represents a regular dessin with underlying graph $K_{m,n}$.

Example: The standard (m, n)-complete regular dessin

- For every pair of integers $m \geq 2$ and $n \geq 2$, the group $\mathbb{Z}_m \times \mathbb{Z}_n$ gives rise to the triple $(\mathbb{Z}_m \times \mathbb{Z}_n; 1_m, 1_n)$ which represents a regular dessin with underlying graph $K_{m,n}$.
- The resulting dessin $\mathcal{D}$ is uniquely determined by $\mathbb{Z}_m \times \mathbb{Z}_n$ up to isomorphism.
- We call $\mathcal{D}$ the standard (m, n)-complete dessin.

Example: The standard (m,n)-complete regular dessin

- For every pair of integers $m \geq 2$ and $n \geq 2$, the group $\mathbb{Z}_m \times \mathbb{Z}_n$ gives rise to the triple $(\mathbb{Z}_m \times \mathbb{Z}_n; 1_m, 1_n)$ which represents a regular dessin with underlying graph $K_{m,n}$.

- The resulting dessin $\mathcal{D}$ is uniquely determined by $\mathbb{Z}_m \times \mathbb{Z}_n$ up to isomorphism.

- We call $\mathcal{D}$ the standard (m,n)-complete dessin.

- The pair of skew-morphisms induced by $\mathcal{D}$ is $(\mathrm{id}_m, \mathrm{id}_n)$.

Example: The standard (m, n)-complete regular dessin

- For every pair of integers $m \geq 2$ and $n \geq 2$, the group $\mathbb{Z}_m \times \mathbb{Z}_n$ gives rise to the triple $(\mathbb{Z}_m \times \mathbb{Z}_n; 1_m, 1_n)$ which represents a regular dessin with underlying graph $K_{m,n}$.
- The resulting dessin $\mathcal{D}$ is uniquely determined by $\mathbb{Z}_m \times \mathbb{Z}_n$ up to isomorphism.
- We call $\mathcal{D}$ the standard (m, n)-complete dessin.
- The pair of skew-morphisms induced by $\mathcal{D}$ is $(\mathrm{id}_m, \mathrm{id}_n)$.
- The algebraic curve associated with $\mathcal{D}$ is the generalised Fermat curve $x^m + y^n = 1$.

Reciprocal pairs of skew-morphisms of cyclic groups

Definition

A pair of skew-morphisms $\varphi\colon \mathbb{Z}_n \to \mathbb{Z}_n$ and $\varphi^*\colon \mathbb{Z}_m \to \mathbb{Z}_m$ with power functions π and π^*, respectively, is called reciprocal if the following conditions are satisfied:

(i) $|\varphi|$ divides m and $|\varphi^*|$ divides n

(ii) $\pi(x) = \varphi^{*x}(1_m)$ and $\pi^*(y) = \varphi^y(1_n)$.

Reciprocal pairs of skew-morphisms of cyclic groups

Definition

A pair of skew-morphisms $\varphi\colon \mathbb{Z}_n \to \mathbb{Z}_n$ and $\varphi^*\colon \mathbb{Z}_m \to \mathbb{Z}_m$ with power functions π and π^*, respectively, is called reciprocal if the following conditions are satisfied:

(i) $|\varphi|$ divides m and $|\varphi^*|$ divides n

(ii) $\pi(x) = \varphi^{*x}(1_m)$ and $\pi^*(y) = \varphi^y(1_n)$.

Theorem

There exists a one-to-one correspondence between any pair of the following three types of objects:

(1) *isomorphism classes of (m, n)-complete regular dessins,*

(2) *equivalence classes of exact (m, n)-bicyclic triples, and*

(3) *(m, n)-reciprocal pairs of skew-morphisms.*

Example: Regular dessins on $K_{27,9}$

Example: Regular dessins on $K_{27,9}$

We determine all regular dessins on $K_{27,9}$ by deriving all $(27, 9)$-reciprocal pairs of skew-morphisms $\varphi\colon \mathbb{Z}_9 \to \mathbb{Z}_9$ and $\varphi^*\colon \mathbb{Z}_{27} \to \mathbb{Z}_{27}$.

There are exactly 27 of them, falling into two types.

Example: Regular dessins on $K_{27,9}$

We determine all regular dessins on $K_{27,9}$ by deriving all $(27, 9)$-reciprocal pairs of skew-morphisms $\varphi\colon \mathbb{Z}_9 \to \mathbb{Z}_9$ and $\varphi^*\colon \mathbb{Z}_{27} \to \mathbb{Z}_{27}$.

There are exactly 27 of them, falling into two types.

Type I: Both φ and φ^* are group automorphisms.

In this case $\varphi(x) = ex$ and $\varphi^*(x) = fx$ where

(i) $e = 1$ and $f \in \{1, 4, 7, 10, 13, 16, 19, 22, 25\}$, or

(ii) $e \in \{4, 7\}$ and $f \in \{1, 10, 19\}$.

Hence, there are $9 + 6 = 15$ reciprocal pairs of Type I.

Example: Regular dessins on $K_{27,9}$

We determine all regular dessins on $K_{27,9}$ by deriving all $(27,9)$-reciprocal pairs of skew-morphisms $\varphi\colon \mathbb{Z}_9 \to \mathbb{Z}_9$ and $\varphi^*\colon \mathbb{Z}_{27} \to \mathbb{Z}_{27}$.

There are exactly 27 of them, falling into two types.

Type I: Both φ and φ^* are group automorphisms.

In this case $\varphi(x) = ex$ and $\varphi^*(x) = fx$ where

(i) $e = 1$ and $f \in \{1, 4, 7, 10, 13, 16, 19, 22, 25\}$, or

(ii) $e \in \{4, 7\}$ and $f \in \{1, 10, 19\}$.

Hence, there are $9 + 6 = 15$ reciprocal pairs of Type I.

Type II: φ is a group automorphism but φ^* is not.

In this case $\varphi(x) = ex$ and $\varphi^*(y) = y + 3t\sum_{i=1}^{y}\sigma(s, e^{i-1})$ where
$e \in \{4, 7\}$ and $\sigma(s, e^{i-1}) = \sum_{j=1}^{e^{i-1}} s^{j-1}$ where
$(s, t) = (4, 1), (7, 2), (4, 4), (7, 5), (4, 7)$ or $(7, 8)$.

These give rise to $2 \times 6 = 12$ reciprocal skew-morphism pairs of Type II.

Uniqueness theorem

For what pairs (m, n) of positive integer does there exist a unique (m, n)-complete regular dessin, the standard one?

Uniqueness theorem

For what pairs (m, n) of positive integer does there exist a unique (m, n)-complete regular dessin, the standard one?

Definition

Call a pair (m, n) of positive integers m and n singular if $\gcd(m, \phi(n)) = \gcd(n, \phi(m)) = 1$.

Uniqueness theorem

For what pairs (m, n) of positive integer does there exist a unique (m, n)-complete regular dessin, the standard one?

Definition

Call a pair (m, n) of positive integers m and n singular if $\gcd(m, \phi(n)) = \gcd(n, \phi(m)) = 1$.

Theorem

The following statements are equivalent.

(i) *The pair (m, n) is singular.*

(ii) *Every finite group factorisable as a product of two cyclic groups of orders m and n is isomorphic to $\mathbb{Z}_m \times \mathbb{Z}_n$.*

(iii) $(\mathrm{id}_n, \mathrm{id}_m)$ *is the only (m, n)-reciprocal pair of skew-morphisms.*

(iv) *There is a unique isomorphism class of regular dessins on $K_{m,n}$.*

(v) *There exists a unique orientable edge-transitive embedding of $K_{m,n}$.*

The symmetric case

The symmetric case

A skew-morphism $\varphi\colon \mathbb{Z}_n \to \mathbb{Z}_n$ is symmetric if the pair (φ, φ) is reciprocal.

The symmetric case

A skew-morphism $\varphi\colon \mathbb{Z}_n \to \mathbb{Z}_n$ is symmetric if the pair (φ, φ) is reciprocal.

A skew-morphism $\varphi\colon \mathbb{Z}_n \to \mathbb{Z}_n$ with power function $\pi\colon G \to \mathbb{Z}$ is symmetric $\iff$

- $|\varphi|$ divides n
- $\pi(x) = \varphi^x(1)$

The symmetric case

A skew-morphism $\varphi\colon \mathbb{Z}_n \to \mathbb{Z}_n$ is symmetric if the pair (φ, φ) is reciprocal.

A skew-morphism $\varphi\colon \mathbb{Z}_n \to \mathbb{Z}_n$ with power function $\pi\colon G \to \mathbb{Z}$ is symmetric $\iff$

- $|\varphi|$ divides n
- $\pi(x) = \varphi^x(1)$

Theorem

The isomorphism classes of orientably regular embeddings of complete bipartite graphs $K_{n,n}$ are in a 1-1 correspondence with symmetric skew-morphisms of $\mathbb{Z}_n$.

Example: Regular embedings of $K_{8,8}$ via skew-morphisms

The cyclic group $\mathbb{Z}_8$ has the total of 6 symmetric skew-morphisms. They correspond to 6 different orientably regular embeddings of $K_{8,8}$:

- four automorphisms
- two proper symmetric skew-morphisms:

$$\varphi_1 = (0)(1\,3\,5\,7)(2)(4)(6), \qquad \pi_1 = [1]\ [3\,3\,3\,3]\ [1]\,[1]\ [1],$$
$$\varphi_2 = (0)(1\,7\,5\,3)(2)(4)(6), \qquad \pi_2 = [1]\ [3\,3\,3\,3]\ [1]\,[1]\ [1].$$

Classification of orientably regular embeddings of $K_{n,n}$

Theorem

The isomorphism classes of orientably regular embeddings of complete bipartite graphs $K_{n,n}$ are in a 1-1 correspondence with symmetric skew-morphisms of $\mathbb{Z}_n$.

$\Longrightarrow$ It is possible to classify orientably regular embeddings of $K_{n,n}$ by classifying the corresponding symmetric skew-morphisms of $\mathbb{Z}_n$.

Classification of orientably regular embeddings of $K_{n,n}$

Theorem

The isomorphism classes of orientably regular embeddings of complete bipartite graphs $K_{n,n}$ are in a 1-1 correspondence with symmetric skew-morphisms of $\mathbb{Z}_n$.

$\Longrightarrow$ It is possible to classify orientably regular embeddings of $K_{n,n}$ by classifying the corresponding symmetric skew-morphisms of $\mathbb{Z}_n$.

A complete classification of orientably regular embeddings of complete bipartite graphs $K_{n,n}$ has been accomplished in a series of seven papers by Jones, Nedela, S., Du, Kwak, Zlatoš (2002–2013).

Classification of orientably regular embeddings of $K_{n,n}$

Theorem

The isomorphism classes of orientably regular embeddings of complete bipartite graphs $K_{n,n}$ are in a 1-1 correspondence with symmetric skew-morphisms of $\mathbb{Z}_n$.

$\Longrightarrow$ It is possible to classify orientably regular embeddings of $K_{n,n}$ by classifying the corresponding symmetric skew-morphisms of $\mathbb{Z}_n$.

A complete classification of orientably regular embeddings of complete bipartite graphs $K_{n,n}$ has been accomplished in a series of seven papers by Jones, Nedela, S., Du, Kwak, Zlatoš (2002–2013).

During 2008-2010 Yanquan Feng, Roman Nedela, and M.S. attempted to approach classification of orientably regular embeddings of $K_{n,n}$ via symmetric skew-morphisms of $\mathbb{Z}_n$.

Problems

Problem 1

Classify complete regular dessins via reciprocal skew-morphisms.
Determine the symmetric skew-morphisms by explicit formulae.

Problems

Problem 1

Classify complete regular dessins via reciprocal skew-morphisms.
Determine the symmetric skew-morphisms by explicit formulae.

Problem 2

Classify orientably regular embeddings of $K_{n,n}$ via symmetric skew-morphisms of $\mathbb{Z}_n$.

Problems

Problem 1

Classify complete regular dessins via reciprocal skew-morphisms.
Determine the symmetric skew-morphisms by explicit formulae.

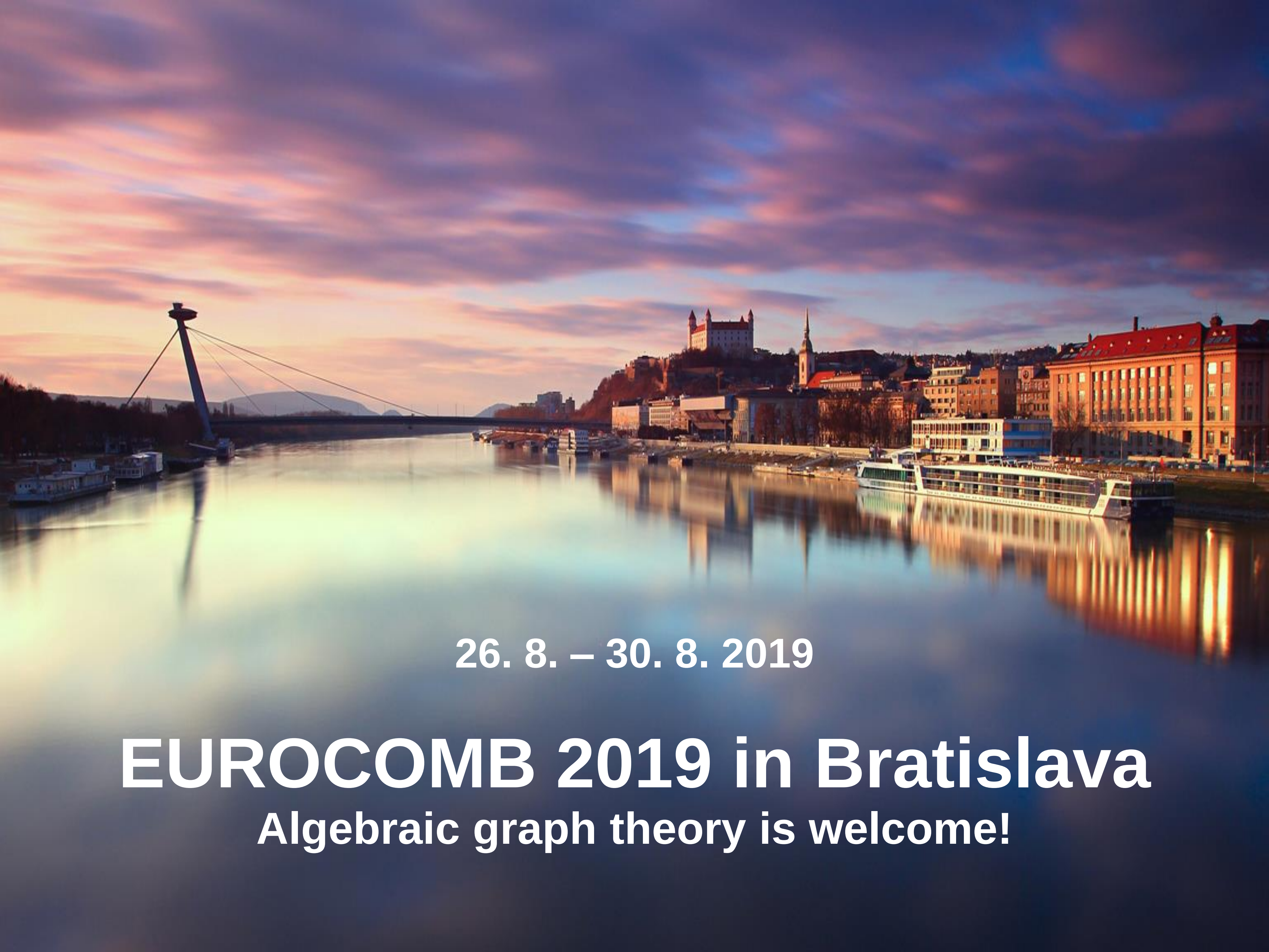
Problem 2

Classify orientably regular embeddings of $K_{n,n}$ via symmetric skew-morphisms of $\mathbb{Z}_n$.

Problem 3

Which complete regular dessins correspond to reciprocal pairs (φ, φ^*) of skew-morphisms where both φ and φ^* are automorphisms?

Thank you for listening



26. 8. – 30. 8. 2019

EUROCOMB 2019 in Bratislava

Algebraic graph theory is welcome!

Barcelona, Berlin, Bordeaux, Budapest, Bergen

Bratislava!