Local Theory in Tilings, Delone Sets, and Polytopes

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Basic Problem

Characterize certain global properties of a geometric or combinatorial object in terms of its local arrangements (neighborhoods, coronas, vertex-links, etc)!

Characterize global properties locally!

Structures: Delone sets (uniformly discrete point sets) in $\mathbb{E}^d$, tilings in $\mathbb{E}^d$, abstract polytopes, graphs, ...

Detecting transitivity properties locally!

How can a given space be tiled by copies of one or more shapes?

- $\mathbb{E}^d$ — **euclidean d-space** (or spherical d-space $\mathbb{S}^d$)

- **copies** — translates, congruent copies, isomorphic polytopes, ...

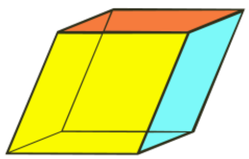
- **shapes** — convex polytopes, non-convex polytopes, topological polytopes, ...

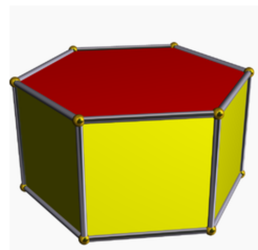
- **Long History!**

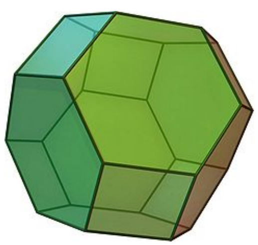
- **geometry, crystallography, geometry of numbers, arts, ...**

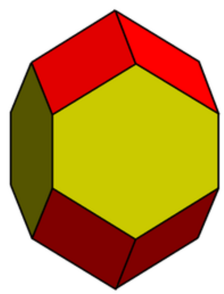
The five parallelohedra in $\mathbb{E}^3$.

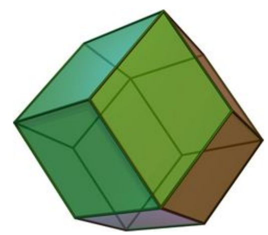
(Convex polyhedra that tile $\mathbb{E}^3$ by translation. Admit a unique face-to-face lattice tiling!)


Rhombohedron


Hexagonal Prism


Truncated Octahedron


Elongated Dodecahedron


Rhombic Dodecahedron

Tilings of Euclidean Space $\mathbb{E}^d$

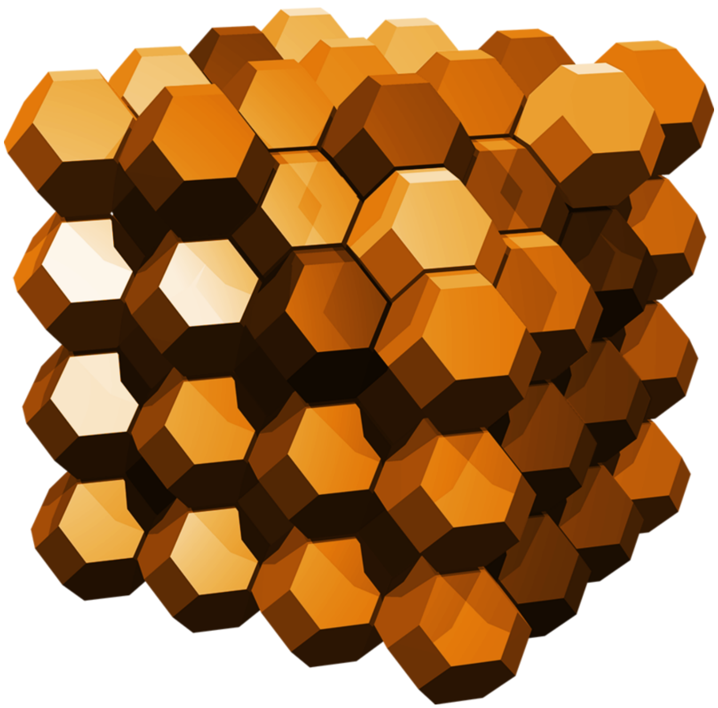
Tiling $\mathcal{T}$ — family of *convex* polytopes, called the tiles of $\mathcal{T}$, which cover $\mathbb{E}^d$ without gaps and overlaps.
(Minkowski: If the tiles in a tiling are convex and compact, then each tile is a convex polytope.)

- **$\mathcal{T}$ is locally finite if each point in $\mathbb{E}^d$ has a neighborhood meeting only finitely many tiles.**

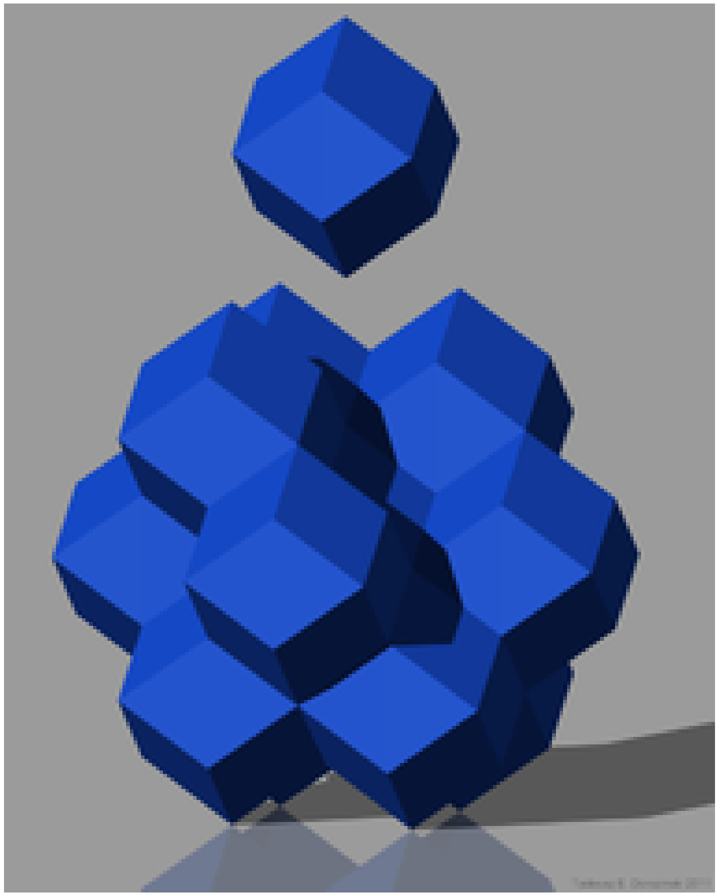
- **$\mathcal{T}$ is face-to-face if the intersection of any two tiles is a face of each (possibly the empty face).** (Then there is a *face-lattice*!)

- **$\mathcal{T}$ is normal if the tiles are uniformly bounded in size.**

- **$\mathcal{T}$ is monohedral if any two tiles are congruent.**



Truncated Octahedron



Rhombic Dodecahedron

Fundamental Open Problem

Space Fillers in $\mathbb{E}^d$

Classify all convex d-polytopes that admit a monohedral tiling of $\mathbb{E}^d$.
Variants require face-to-face or isohedral.

Planar case open! Pentagonal plane fillers not fully classified! All triangles and quadrangles, and certain pentagons and hexagons, tile $\mathbb{E}^2$ by congruent copies.
If $\mathcal{T}$ is a normal tiling of the plane by n-gons, then $n \leq 6$.

- **Space filler problem rephrased: What are the analogues of the triangles, quadrangles, pentagons and hexagons in d-space?**

Hilbert's 18th Problem (1900)

- **(a) Are there only finitely many crystallographic groups in $\mathbb{E}^d$?** Yes! Bieberbach

- **(b) Is there a polyhedral tile which admits a monohedral tiling in $\mathbb{E}^d$ but is not the fundamental region for a crystallographic group?** Yes! Reinhardt 1928, Heesch 1935, Kershner, Penrose

- **(c) What is the densest sphere packing in $\mathbb{E}^3$?** Kepler Conjecture — Hales

Closely related to Part (b): Is there a polyhedral shape in $\mathbb{E}^d$ that admits a monohedral tiling but not an isohedral tiling in $\mathbb{E}^d$? Yes! *Anisohedral* tiles do exist! (Reinhardt, Heesch)

Local Theorems (Dolbilin & S., 2004)

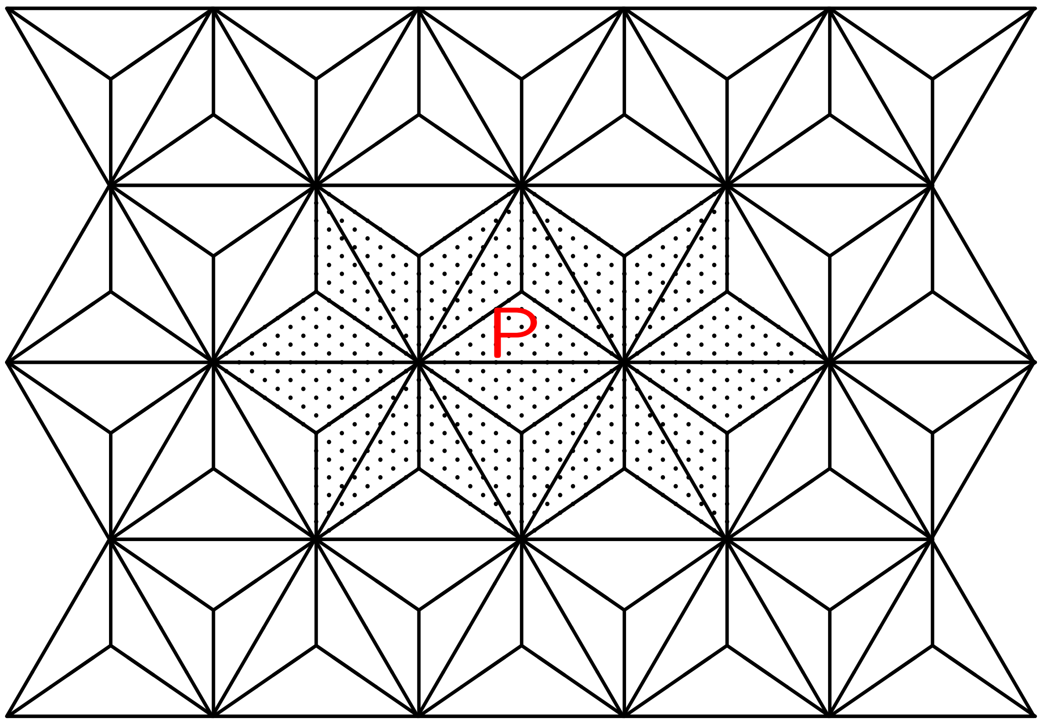
Generalize the classical Local Theorem for Monohedral Tilings (due to Delone, Dolbilin, Shtogrin, Galiulin)!

- **Goal: Detect tile-transitivity (isohedrality) locally!**

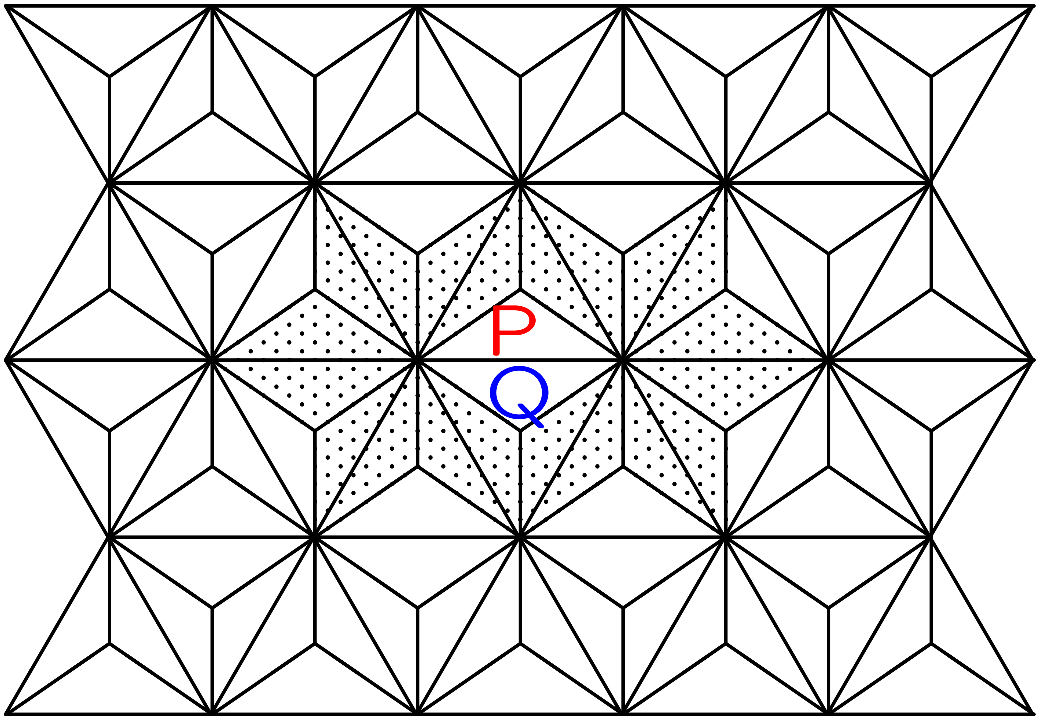
Centered k^{th} coronas: $\mathcal{T}$ a tiling, P a tile of $\mathcal{T}$

$$C^0(P) := \{P\}$$
$$C^1(P) := \{S \mid S \cap P \neq \emptyset\}$$
$$C^k(P) := \{S \mid S \cap R \neq \emptyset \text{ for an } R \in C^{k-1}(P)\}$$

The pair $(P, C^k(P))$ is the centered k^{th} corona of P.



The 1^{st} corona of tile P.



P **and** Q **have the same** 1^{st} **corona, consisting of the dotted tiles as well as** P **and** Q**.**

- Pairwise congruence of $(P, C^k(P))$ and $(Q, C^k(Q))$: there exists an isometry γ of $\mathbb{E}^d$ such that

$$\gamma(P) = Q,\ \gamma(C^k(P)) = C^k(Q).$$

- Symmetry group of $(P, C^k(P))$

$$S^k(P) := \{\gamma \in S(P) \mid \gamma(C^k(P)) = C^k(P)\}.$$

NOTE $S(P) = S^0(P) \supseteq S^1(P) \supseteq S^2(P) \supseteq \ldots\ldots\ldots,$

all finite groups. Finitely many proper descents!

Local Theorem for Monohedral Tilings

(Delone, Dolbilin, Shtogrin, Galiulin, 1974)

$\mathcal{T}$ is isohedral iff there exists $k > 0$ such that

- **any two centered k^{th} coronas are pairwise congruent, and**
- **$S^k(P) = S^{k-1}(P)$ for some tile (hence all tiles) P.**

Notes

- Then $S^k(P) = S_P(\mathcal{T})$.
- P asymmetric: $\mathcal{T}$ is isohedral iff $k = 1$.
- $d = 2$, P polygonal tile: then $k = 1$ already implies isohedrality. (Dolbilin, Schattschneider)

Local Theorem for Monotypic Tilings

- **Goal: Detect combinatorial tile-transitivity locally!** Now face-to-face tilings!

Centered k^{th} corona complex: $\mathcal{T}$ a face-to-face tiling by polytopes, P a tile of $\mathcal{T}$

$\mathcal{C}^0(P) := \mathcal{F}(P)$ (face-lattice of P)

$C^k(P)$ – complex consisting of all faces of tiles that meet a tile from $\mathcal{C}^{k-1}(P)$ in a face of dimension $\geq d-2$.

The pair $(P, C^k(P))$ is the centered k^{th} corona complex of P.

Automorphism group $\Gamma(\mathcal{C}^k(P))$ of $(P, \mathcal{C}^k(P))$ consists of the automorphisms of $\mathcal{C}^k(P)$ that fix P.

NOTE

$$\Gamma(P) = \Gamma(\mathcal{C}^0(P)) \supseteq \Gamma(\mathcal{C}^1(P)) \supseteq \Gamma(\mathcal{C}^2(P)) \supseteq \ldots\ldots,$$

all finite groups. Finitely many proper descents!

Local Theorem $\mathcal{T}$ a locally finite monotypic face-to-face tiling by polytopes in $\mathbb{E}^d$ (or $\mathbb{H}^d$ of $\mathbb{S}^d$).

$\mathcal{T}$ is combinatorially tile-transitive iff there exists $k > 0$ such that

- **any two centered k^{th} corona complexes are isomorphic**
- $\Gamma(\mathcal{C}^k(P)) = \Gamma(\mathcal{C}^{k-1}(P))$ **for some tile (hence all tiles) P.**

Notes

- Then $\Gamma(\mathcal{C}^k(P)) = \Gamma_P(\mathcal{T})$.
- Much harder to prove than classical Local Theorem! Exploits simply-connectedness! Extension of local isomorphisms to global isomorphisms along paths.

- P combinatorially asymmetric: then $\mathcal{T}$ is combinatorially tile-transitive iff any two centered first corona complexes are isomorphic ($k = 1$).

- Bounds for k depend on P.

  In the original Local Theorem, a bound k_d only depending on d can be derived from the Delone-bound for the number of facets ($k_2 = 1$, $k_3 = 5$, Dolbilin, Schattschneider, Shtogrin).

Delone Sets

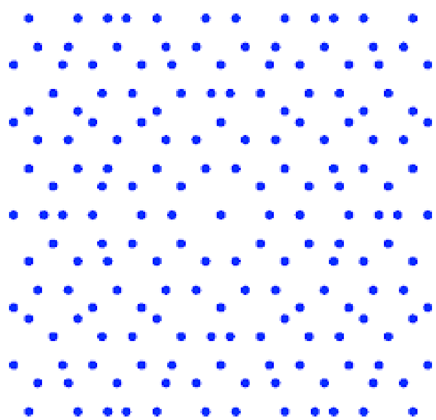
Delone set $X \subset \mathbb{E}^d$ of type (r, R), or (r, R)-*system*

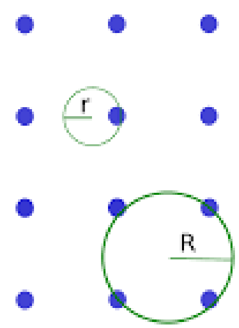
– Every open ball in $\mathbb{E}^d$ of some radius $r > 0$ contains at most one point of X.

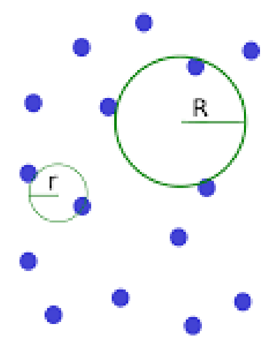
(Any two points of X at least $2r$ apart.)

– Every closed ball in $\mathbb{E}^d$ of some radius $R > 0$ contains at least one point of X.

(Each point of $\mathbb{E}^d$ at most R away from a point of X.)







Delone sets used in the modeling of crystals (location of atoms)

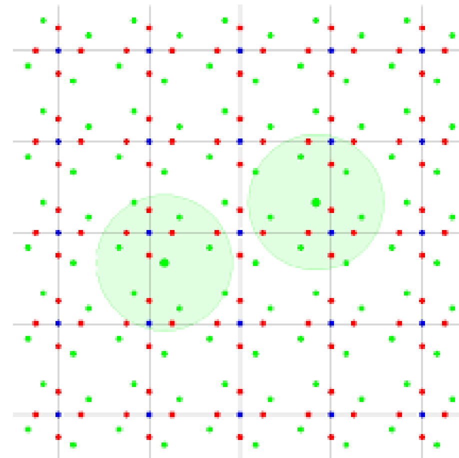
Delone set X called an *ideal crystal* if its symmetry group $S(X)$ has finitely many orbits on X. (Fedorov)

Delone set X called a *regular system of points* if $S(X)$ acts transitively on X.

(Orbits of crystallographic groups! For example, lattices. Bieberbach's Theorem.)

Basic question: How can one tell from local data that X is a regular system?
Answer: Local Theorem for Delone Sets!

X a Delone set



ρ-cluster at $x \in X$: $\quad C_x(\rho) := X \cap B_x(\rho)$

(all points at distance at most ρ from x)

Equivalence of clusters $C_x(\rho)$ and $C_{x'}(\rho)$: there exists an isometry g of $\mathbb{E}^d$ such that

$$g(x) = x', \;\; g(C_x(\rho)) = C_{x'}(\rho).$$

Cluster counting function

$N(\rho) :=$ #(equivalence classes of ρ-clusters)

Note: $N(\rho)$ increases or stays the same as ρ increases.

X regular system $\longrightarrow N(\rho) = 1 \; \forall \rho > 0$

Cluster group (local symmetry group)

$S_x(\rho) :=$ stabilizer of x in the symmetry group of $C_x(\rho)$

Note: $S_x(\rho)$ gets smaller or stays the same as ρ increases.

Local Theorem for Delone Sets (Delone, Dolbilin, Stogrin, Galiulin, 1976)

X a Delone set of type (r, R)

X is a regular system of points iff there exists a positive ρ_0 such that

- $N(\rho_0 + 2R) = 1$

  (any two clusters of X of radius $\rho_0 + 2R$ are equivalent)

- $S_x(\rho_0) = S_x(\rho_0 + 2R)$ (cluster groups of X begin stabilizing at radius ρ_0)

Consequence: If these two conditions hold, then actually $N(\rho) = 1\, \forall \rho > 0$ and $S_x(\rho)$ is the stabilizer of x in the full symmetry group of X $\forall \rho \geq \rho_0$.

Regularity radius – important problem

Find small positive numbers ρ such that each Delone set X of type (r, R) with mutually equivalent ρ-clusters is a regular system.

Regularity radius

$$\widehat{\rho_d} = \widehat{\rho_d}(r, R) = \text{smallest such } \rho$$

(depends on d, r, R)

Known:

$\widehat{\rho}_1 = 2R$, $\widehat{\rho}_2 = 4R$, $\widehat{\rho}_3 \leq 10R$ (Dolbilin, Stogrin)

R is the radius of the largest “empty ball” (ball in $\mathbb{E}^d$ without points of X in its *interior*).

Previously not known: dependence on d

Theorem (Baburin, Bouniaev, Dolbilin, Erokhovets, Garber, Krivovichev & S., 2017)

$$\hat{\rho}_{\mathbf{d}} \geq \mathbf{2dR}, \text{ for } \mathbf{d} \geq \mathbf{1}.$$

Bound grows linearly in d!

Case $d = 3$: $6R \leq \widehat{\rho}_3 \leq 10R$

Proof: Construction of layered Delone sets in $\mathbb{E}^d$, called *Engel sets*, in which the $(d-1)$-dimensional layers are fine grids shifted relative to each other in a rather sophisticated way.

How about local theorems for abstract polytopes (or other complexes)?

Something like: If all coronas of a certain size k are isomorphic, and the corona groups stabilize at level k, then transitivity on the “centers” of the coronas follows.

“Corona complexes” at vertices, at facets, at faces of rank i, or at flags. Notion of “Center” of a corona!

Several notions of corona (leading to larger of smaller coronas)!

Example: facet coronas of n-polytope $\mathcal{P}$

1^{st}-corona $\mathcal{C}^1(F)$ of facet F of $\mathcal{P}$: all facets of $\mathcal{P}$ meeting F in a face of rank at least $n-3$ (or some other number), and all their faces.

2^{nd}-corona $\mathcal{C}^2(F)$ of facet F of $\mathcal{P}$: all facets of $\mathcal{P}$ meeting a facet of $\mathcal{C}^1(F)$ in a face of rank at least $n-3$, and all their faces.

Corona groups $\Gamma(\mathcal{C}^k(P))$ of centered corona $(\mathcal{C}^k(F), F)$: all automorphisms of the complex $\mathcal{C}^k(F)$ that fix F.

Local Theorem prototype .....

An n-polytope $\mathcal{P}$ of "some kind" is facet-transitive iff there exists $k > 0$ such that

– any two centered k^{th} corona are isomorphic as complexes,

– $\Gamma(\mathcal{C}^k(P)) = \Gamma(\mathcal{C}^{k-1}(P))$ for some facet (hence all facets) of $\mathcal{P}$,

– and some other conditions (depending on the kind) hold.

Keep smiling when tiling !!

..... The End .....

Thank you

Abstract

Local detection of a global property in a geometric or combinatorial structure is usually a challenging problem. The Local Theorem for Tilings says that a tiling of Euclidean d-space is tile-transitive (isohedral) if and only if the large enough neighborhoods of tiles (coronas) satisfy certain conditions. This is closely related to the Local Theorem for Delone Sets, which locally characterizes those sets among uniformly discrete sets in d-space which are orbits under a crystallographic group. Both results are of great interest in crystallography. We discuss old and new results from the local theory of tilings and Delone sets, and point to interesting open problems for polytopes.