# Archimedean Toroids and their Almost Regular Covers

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## Introduction

This project began as part of the Fields-MITACS Undergraduate Summer Research Program 2011, where Kostiantyn Drach and Maksym Skoryk were participants in the "Symmetries of Euclidean Tessellations and their Covers" group, which was supervised by Isabel Hubard, Mark Mixer, Daniel Pellicer, and Asia Weiss. The students were given the following topic to consider.

> Let $\tau$ be a map on the torus. Describe other maps $\phi$ on the torus so that $\phi$ covers $\tau$ and $\phi$ has as few flag orbits as possible.

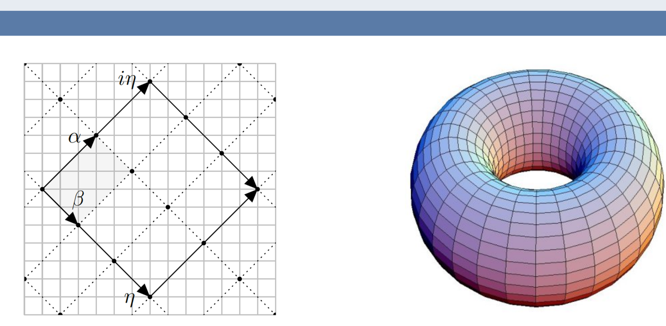
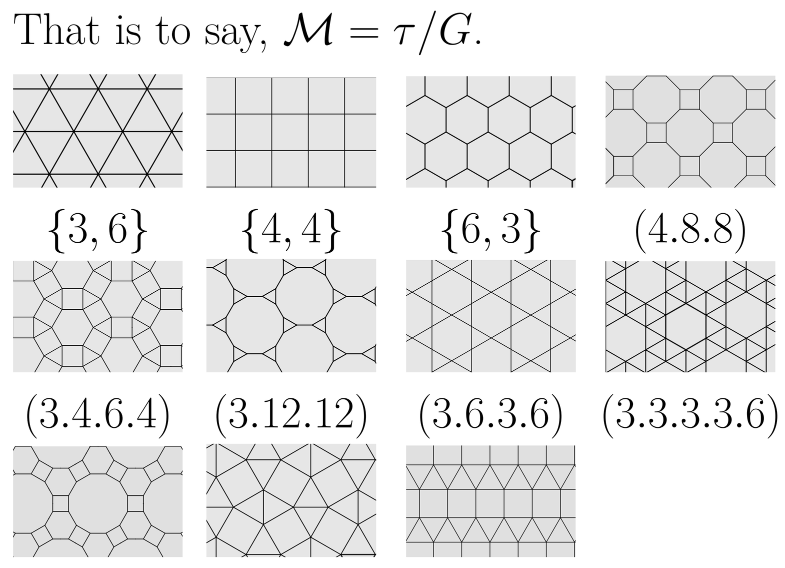


Figure 1: For the 2-orbit toroidal map $\{4,4\}_{\alpha,\beta}$, $\alpha = 3+3i$, $\beta = 2-2i$ the minimal rotary and regular covering toroidal maps coincide with the regular toroidal map $\{4,4\}_\eta$, $\eta = 6-6i$, which can be obtained by gluing together 6 fundamental regions of the initial map.

Their work on covers of the equivelar toroidal maps led to [3]. The project continued as undergraduate research for Yurii Haidamaka (who is currently a masters student), and results on covers of the vertex transitive non-equivelar toroidal maps will be found in [4].

## Archimedean Tilings of the Torus

The Archimedean tilings of the plane lead to eleven different types of tilings of the torus [5]. Let $\mathcal{M}$ be an Archimedean map on the torus. Then $\mathcal{M}$ can be obtained as a quotient of a tessellation $\tau$ of the Euclidean plane by some translation subgroup $G < T_\tau$ generated by two linearly independent vectors. That is to say, $\mathcal{M} = \tau/G$.




## Toroidal Covers

A surjective function from a map $\mathcal{N}$ to a map $\mathcal{M}$ that preserves adjacency and sends vertices to vertices, edges to edges, and faces to faces is called a *covering* of the map $\mathcal{M}$ by the map $\mathcal{N}$. This is denoted by $\mathcal{N} \searrow \mathcal{M}$. Here, we are considering the case where both maps are on the torus.

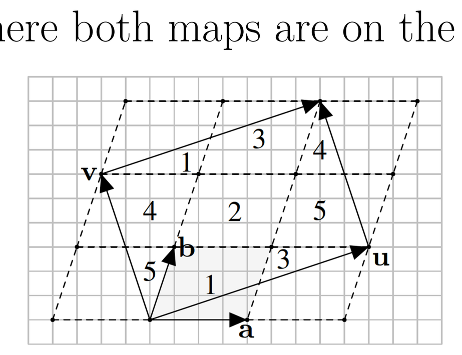


Figure 2: $\{4,4\}_{\mathbf{u},\mathbf{v}} \searrow \{4,4\}_{\mathbf{a},\mathbf{b}}$ is a 5-sheeted covering, and the covering map $\{4,4\}_{\mathbf{u},\mathbf{v}}$ is obtained by gluing together 5 fundamental regions of $\{4,4\}_{\mathbf{a},\mathbf{b}}$

## Flag orbits in the plane

An Archimedean toroidal map cannot have fewer flag orbits than the tessellation of the plane of the same type. For example, in the case of a map of type (4.8.8) a toroidal map will always have at least three flag orbits.

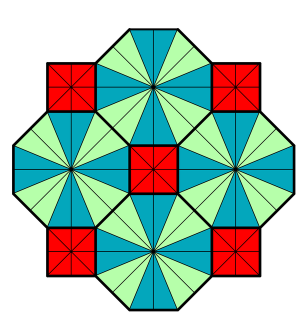


Figure 3: Three flag orbits in a map of type $(4.8.8)$

## Almost regular maps

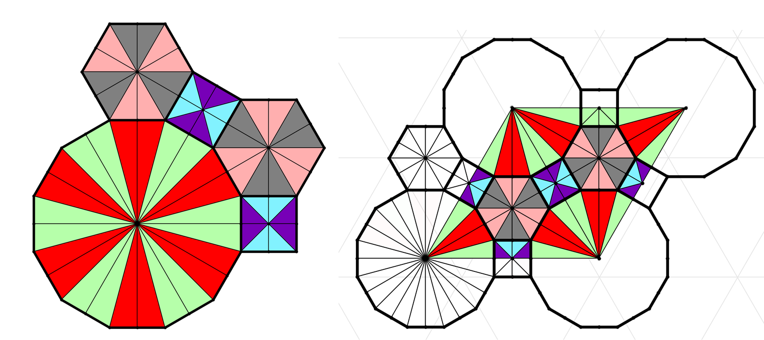


Figure 4: Six flag orbits in a map of type $(4,6,12)$

An Archimedean map on the torus will be called *almost regular* if it has as few flag orbits as possible for its type. In the case of tilings of type $\{4,4\}$, $\{3,6\}$, and $\{6,3\}$ the almost regular maps coincide with typical regular maps.

## Gaussian and Eisenstein Integers

The Gaussian and Eisenstein integers provide a tool for constructing minimal covers of equivelar toroidal maps. Plotting these sets on the complex plane produces the vertex set of a regular tessellation: $\tau = \{4,4\}$ for the Gaussian integers, and $\tau = \{3,6\}$ for the Eisenstein integers.
The *Gaussian integers* $\mathbb{Z}[i]$ are defined as $\{a+bi \mid a,b \in \mathbb{Z}\}$, where $i = \sqrt{-1}$. Similarly, the *Eisenstein integers* $\mathbb{Z}[\omega]$ are defined as $\{a+b\omega \mid a,b \in \mathbb{Z}\}$, where $\omega = \frac{1+i\sqrt{3}}{2}$. We write $\mathbb{Z}[\sigma]$, $\sigma = i, \omega$ to denote either of these two sets.

## Construction for the non-equivelar cases

For a tessellation $\tau$ of type (4.8.8) on the plane, let $\mathbf{e_1}$ and $\mathbf{e_2}$ be the shortest translational symmetries of the tessellation (at an angle of $\frac{\pi}{2}$). For the remaining non-equivelar tessellations, other than (3.3.3.4.4), we will use the vectors $\mathbf{e_1}$ and $\mathbf{e_2}$ to represent the shortest possible translational symmetries (at the angle of $\frac{\pi}{3}$).

Given a tessellation $\tau$, the points : $\{\lambda\mathbf{e_1} + \mu\mathbf{e_2} : \lambda, \mu \in \mathbb{Z}\}$ form the vertices of another tessellation which we denote $\tau^*$. For Archimedean toroidal maps $\tau_{u,v}$ - other than of type (3.3.3.4.4) - we can study the covers of $\tau_{u,v}$ by understanding the covers of $\tau^*_{u,v}$. Since $\tau^*_{u,v}$ is either of type $\{4,4\}$ or $\{3,6\}$, we must consider the covers of toroidal maps of these types.

## The equivelar cases

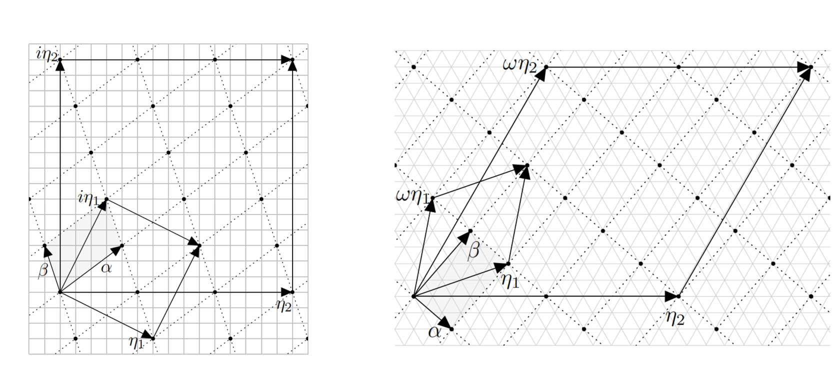


Figure 5: Examples of the minimal chiral and regular covers of equivelar maps

Given an equivelar toroidal map of $\{4,4\}$ and $\{3,6\}$, we can describe its minimal regular cover. The regular cover will be of the same type, and these regular maps have been classified [1, 2].

**Theorem:** Let $\tau_{\alpha,\beta}$ be an equivelar toroidal map, and let $\alpha, \beta \in \mathbb{Z}[\sigma]$. Let $c = \mathrm{GCD}(\operatorname{Re}\alpha, \operatorname{Im}\alpha, \operatorname{Re}\beta, \operatorname{Im}\beta)$. Then for $\tau_{\alpha,\beta}$ there exists a unique minimal regular covering map $\tau_{\eta_{min}}$ with

$$\eta_{min} = \begin{cases} \frac{\operatorname{Im}(\overline{\alpha}\beta)}{N(1+\sigma)c}(1+\sigma) : & \frac{\operatorname{Re}\alpha}{c} \equiv \frac{\operatorname{Im}\alpha}{c}, \frac{\operatorname{Re}\beta}{c} \equiv \frac{\operatorname{Im}\beta}{c} \bmod N(1+\sigma) \\ \frac{\operatorname{Im}(\overline{\alpha}\beta)}{c} & \text{otherwise.} \end{cases}$$

Moreover, the minimal number $K_{min}$ of fundamental regions of $\tau_{\alpha,\beta}$ glued together in order to obtain $\tau_{\eta_{min}}$ is equal to

$$K_{min} = \begin{cases} \frac{|\operatorname{Im}(\overline{\alpha}\beta)|}{N(1+\sigma)c^2} & \text{if } \frac{\operatorname{Re}\alpha}{c} \equiv \frac{\operatorname{Im}\alpha}{c} \text{ and } \frac{\operatorname{Re}\beta}{c} \equiv \frac{\operatorname{Im}\beta}{c} \bmod N(1+\sigma) \\ \frac{|\operatorname{Im}(\overline{\alpha}\beta)|}{c^2} & \text{otherwise.} \end{cases}$$

Given $\alpha = a + b\sigma \in \mathbb{Z}[\sigma]$, we define $\overline{\alpha} := a + b\overline{\sigma}$, where $\overline{\sigma}$ is the complex number conjugate to $\sigma \in \mathbb{C}$. Also we denote $\operatorname{Re}\alpha := a$ and $\operatorname{Im}\alpha := b$

## The special case of (3.3.3.4.4)

Due to the structure of the translational symmetries of this tessellation, we treat it separately.

**Theorem:** Let $\tau_{\mathbf{a},\mathbf{b}}$ be Archimedean toroidal map of type (3.3.3.4.4). Then for $\tau_{\mathbf{a},\mathbf{b}}$ there exist a unique minimal almost regular covering map $\tau_{\mathbf{u},\mathbf{v}}$ with

$$\langle \mathbf{u}, \mathbf{v} \rangle = \begin{cases} \langle c\vec{e}_1, c\vec{e}_2 + d(\vec{e}_2 - \vec{e}_1) \rangle, & \text{Case 1} \\ \langle c\vec{e}_1, d(2\vec{e}_2 - \vec{e}_1) \rangle, & \text{otherwise.} \end{cases}$$

Here $\vec{a} = (a_1, a_2), \vec{b} = (b_1, b_2), l_2 = \mathrm{GCD}(a_2, b_2)$, $d_1 = \mathrm{GCD}(2a_1 + a_2, 2b_1 + b_2), c = \frac{a_1b_2 - a_2b_1}{l_2}$, $d = \frac{a_2b_1 - b_2a_1}{2l_2} + \frac{a_2b_1 - b_2a_1}{2d_1}$.
Then we are in Case 1 if $\frac{a_2}{2l_2} \equiv \frac{2a_1+a_2}{d_1} \pmod 2$ and $\frac{b_2}{2l_2} \equiv \frac{2b_1+b_2}{d_1} \pmod 2$.
Moreover, the number $K_{min}$ of fundamentals regions of $\tau_{\mathbf{a},\mathbf{b}}$ glued together in order to obtain $\tau_{\mathbf{u},\mathbf{v}}$ is equal to:

$$K_{min} = \begin{cases} \frac{|a_1b_2 - a_2b_1|}{d_1l_2}, & \text{Case 1} \\ 2\frac{|a_1b_2 - a_2b_1|}{d_1l_2}, & \text{otherwise.} \end{cases}$$

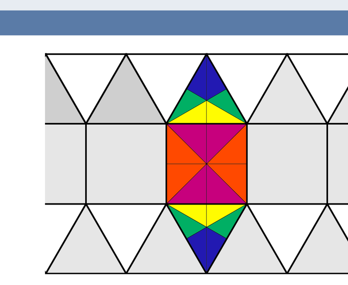


Figure 6: The minimum number of orbits of a map of type $\{3.3.3.4.4\}$

For this type of map we are using the basis $\vec{e}_1 = (1,0)$, $\vec{e}_2 = \left(\frac{1}{2}, \frac{1}{2}(2+\sqrt{3})\right)$.

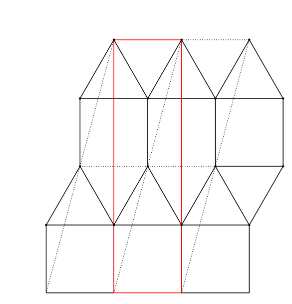


Figure 7: For the map $(3.3.3.4.4)_{a,b}$ where $a = \vec{e}_1$ and $b = \vec{e}_2$, the minimal almost regular covering toroidal map is $(3.3.3.4.4)_{u,v}$ where $u = \vec{e}_1$ and $v = \vec{e}_1 - 2\vec{e}_2$, which can be obtained by gluing together 2 fundamental regions of the initial map.

## Conclusion and Future Work

The topic of toroidal maps and their symmetries and covers was fruitful for two undergraduate research projects. While the question was solved in this specific case, there could be much more to do on other surfaces and with other types of maps.

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## Acknowledgements

As some of this work was done while most of the authors were involved in the Fields-MITACS 2011 Summer Undergraduate Research Program. We would like to express our deep gratitude to the Fields Institute for its hospitality and creative atmosphere. We also would like to thank Daniel Pellicer, Isabel Hubard, and Asia Weiss for their support, and for providing the questions addressed in this poster.

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